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A TENSOR NORM FOR Q-SPACES

ÉRIC RICARD

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ABSTRACT. An analogue of the Haagerup tensor norm is introduced for Q-spaces. Using the non commutative Grothendieck inequality, we compare it to the classical operator space tensor norms and deduce lower estimates for the completely bounded distance between Q-spaces and their duals.

KEYWORDS: *Operator spaces, Q-spaces, Q-algebras, tensor norms.*

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The basic and probably best known example of Banach algebra is the set of continuous functions on a compact space. Unfortunately this class of algebras is not closed under elementary algebraic operations such as taking quotient, subspace or direct sums (it does not form a variety). The question of the determination of the smallest variety of Banach algebras containing $\mathbb{C}$ was answered by Dixon, it is the class of Q-algebras; a Q-algebra is the quotient algebra of a subalgebra of continuous functions on a compact set. The study of these algebras in the 70's led to two interesting results (see [5], for instance). The first one, called Craw's lemma, characterizes Q-algebras among commutative Banach algebras as those satisfying the multivariable von Neumann inequality. The second one proved by Cole states that a Q-algebra can be realized as an operator algebra (a subalgebra of $B(H)$, the space of bounded endomorphisms on a Hilbert space H).

If E is a subspace of some $B(H)$, then for each integer n, $M_n(E)$, the vector space of square matrices of size n with entries in E, inherits a norm from its embedding into $M_n(B(H)) = B(H^n)$. The category of operator spaces consists of the class of all closed subspaces E of $B(H)$ for some Hilbert space H equipped with the family of norms on $M_n(E)$ we just described. The morphisms of this category are the completely bounded maps; a linear map f between two operator spaces E and F is completely bounded if the maps $\mathrm{Id}_{M_n} \otimes f$ induced by f from $M_n(E)$ to $M_n(F)$ are uniformly bounded for any integer n. Many concepts and results from the theory of Banach spaces can be extended to this setting, we refer to [16] and [6] about this subject. For instance, it is possible to define an ℓ_∞ sum; if $(E_i \subset B(H_i))_{i\in I}$ is a family of operator spaces, the Banach space $\ell_\infty(I, E_i)$ is naturally embedded into the space $\ell_\infty(I, B(H_i))$ which is the operator subspace of $B(\oplus_2 H_i)$ consisting of diagonal operators with respect to the decomposition $H = \oplus_2 H_i$. From this, for each n, we have the identification:

$$M_n(\ell_\infty(I, E_i)) = \ell_\infty(I, M_n(E_i)).$$

When $F \subset E$ are operator spaces, it is also possible to define an operator space structure on the quotient E/F which satifies

$$M_n(E/F) = M_n(E)/M_n(F).$$

Moreover, each Banach space X can be equipped with an operator space structure, two of them are distinguished; the smallest one called the minimal operator space structure, denoted by $\min(X)$, it is obtained by viewing X as a subspace of the set of continuous functions on the unit ball of X^*, it also has the property that any bounded linear map u from an operator space F into $\min(X)$ is automatically completely bounded and $\|u\|_{\mathrm{cb}} = \|u\|$. The other one is a kind of dual notion, $\max(X)$ is the biggest operator space structure on X, it is characterized by the property that any bounded map u from $\max(X)$ into another operator space is automatically completely bounded with $\|u\|_{\mathrm{cb}} = \|u\|$; see [11].

In this paper, we are interested in the class of subspaces of Q-algebras as operator spaces; they turn out to be exactly the class of quotients (in the operator space sense) of minimal operator spaces. In the first part, we use results of Blecher and Le Merdy on Q-algebras to associate to each operator space a kind of universal Q-algebra following the idea of [15]. After recalling some basic facts about Q-spaces in the second part, we use these universal Q-algebras to introduce a kind of Haagerup cross norm on the tensor product of Q-spaces, which turns out to be equivalent to the Haagerup cross norm at the Banach level.

All along this paper, $\mathcal{K}$ will stand for the space of compact operator on a separable Hilbert space, and M_n will be the space of square matrices over $\mathbb{C}$ of size n.

1. Q-ALGEBRAS

We start with some definitions and basic facts about Q-algebras.

DEFINITION 1.1. Q-algebras are quotients (in the operator space sense) of subalgebras of commutative C^*-algebras.

From now, Q-algebras will always be considered as operator spaces. The basic example is given by the disc algebra $A(\mathbb{D})$ and its quotient by the ideal generated by an inner function.

PROPOSITION 1.2. *Q-algebras form a variety of operator algebras; they are stable under taking quotient and subalgebras, and direct sums in ℓ^∞ sense.*

More informations about quotient operator algebras can be found in [4].

As a consequence of Dixon's ideas in [5], we have a characterization of Q-algebras as operator algebras satisfying a family of inequalities. Following [2], we denote by $\mathcal{P}_n$ the set of all polynomials in n^2 variables without constant term and with the family of norms over $M_N(\mathcal{P}_n)$:

$$\|P\|_N = \sup\{\|P(x_{i,j})\|_N \mid x_{i,j} \in \mathbb{C},\ \|(x_{i,j})\|_n \leqslant 1\}.$$

PROPOSITION 1.3. ([2]) *A commutative algebra A equipped with an operator space structure is a Q-algebra if and only if for all $P \in M_N(\mathcal{P}_n)$ and $(a_{i,j}) \in M_n(A)$ of norm less than one:*

$$\|P((a_{i,j}))\|_{M_N(A)} \leqslant \|P\|_N.$$

It is possible to associate to each operator space a Q-algebra in a functorial way. Recall that if E is a vector space, its tensor algebra $T(E) = \bigoplus_{i=1}^{\infty} E^{\otimes^i}$ has the following universal property: each map $\pi : E \to A$ from E to an algebra A uniquely extends to an algebra homomorphism $\widetilde{\pi} : T(E) \to A$.

Let E be an operator space, for each integer $n \geqslant 1$, we define a norm on $M_n(E)$ by the following:

$$\|x\|_{M_n(T(E))} = \sup_{\pi : E \to A} \{\|\mathrm{Id}_{M_n} \otimes \widetilde{\pi}(x)\| \mid \|\pi\|_{\mathrm{cb}} \leqslant 1,\ A\ Q\text{-algebra}\}$$

where the sup runs over all completely contractive maps π from E to any Q-algebra A.

We denote by $OA_Q(E)$ the Banach space obtained after taking quotient and completion for the norm on $T(E)$ (i.e. $n = 1$). Then for each n, we have a norm on $M_n(OA_Q(E))$. With these norms, $OA_Q(E)$ is an operator space. Indeed, we can restrain the sup to a set I of maps $\pi : E \to A_\pi$ (for each element x in $M_n(OA_Q(E))$, just choose a sequence of maps such that $\|\mathrm{Id}_{M_n} \otimes \widetilde{\pi}(x)\|$ converges to $\|x\|_{M_n(OA_Q(E))}$), then the map

$$\Pi : \begin{array}{ccc} E & \to & \ell_\infty(I, A_\pi) \\ x & \mapsto & (\pi(x))_{\pi \in I} \end{array}$$

satifies $\|\mathrm{Id}_{M_n} \otimes \widetilde{\Pi}(x)\| = \|x\|_{M_n(OA_Q(E))}$ for all $x \in M_n(T(E))$. So, $OA_Q(E)$ is just the completion of the range of $\widetilde{\Pi}$ which is a Q-algebra by Proposition 1.2.

DEFINITION 1.4. The Q-algebra $OA_Q(E)$ is called the *universal Q-algebra* associated to E.

This kind of construction in the operator space setting originates in [15] and [16].

Remarks 1.5. (i) As Q-algebras are commutative, we can use the symmetric algebra $S(E)$ of E instead of its tensor algebra to avoid taking quotient.

(ii) There is a natural isometric embedding of E into $OA_Q(E)$, it is also a complete contraction but it may fail to be a complete isometry. Actually, it is a complete isometry if and only if E is a Q-space (see next section).

It is rather easy to show that $OA_Q(\mathbb{C}) = A(\mathbb{D})$ completely isometrically.

Proposition 1.6. *The Q-algebra $OA_Q(E)$ has the following universal property: Each completely contractive map $u : E \to A$, where A is a Q-algebra, uniquely extends to a completely contractive algebra homomorphism $\widetilde{u} : OA_Q(E) \to A$.*

Obviously, the canonical map $E \to OA_Q(E)$ extends to the identity. Therefore if A is a commutative operator algebra, A is a Q-algebra if and only if the identity of A can be extended to a completely contractive morphism from $OA_Q(E)$ to A, where E denotes the underlying operator space. In this kind of situation, we always have categorical properties, the following one will be useful in the next section.

We recall that a map between operator spaces E and F is said to be a *complete metric surjection* if it maps the open unit ball of $M_n(E)$ onto the open unit ball of $M_n(F)$ for each integer n.

Proposition 1.7. *$OA_Q(\cdot)$ is projective, namely if $F \subset E$ then $OA_Q(E) \to OA_Q(E/F)$ is a complete metric surjection. More precisely, if $x \in M_n(S(E/F))$ has norm less than one then there exists $y \in M_n(S(E))$ of norm less than one such that $\mathrm{Id}_{M_n} \otimes \widetilde{q}(y) = x$.*

Proof. Let q be the quotient map from E to E/F, and i_X be the completely contractive inclusion $X \to OA_Q(X)$. We have the commutative diagram:

$$\begin{tikzcd}
 & OA_Q(E) \arrow[rr, "\tilde{q}"] \arrow[dr, "\rho"] & & OA_Q(E/F) \arrow[dl, shift left, "\sigma"] \\
 & & OA_Q(E)/\mathrm{Ker}\,\tilde{q} \arrow[ur, shift left, "\underline{\tilde{q}}"] & \\
F \arrow[r, hook] & E \arrow[uu, "i_E"] \arrow[ur] \arrow[rr, "q"] & & E/F \arrow[uu, "i_{E/F}"'] \arrow[ul, "j"']
\end{tikzcd}$$

where $\widetilde{q}$ is the completely contractive algebra homomorphism from $OA_Q(E)$ to $OA_Q(E/F)$ extending the contractive map $i_{E/F}q : E \to OA_Q(E/F)$ given by the universal property of $OA_Q(E)$, so we have $\widetilde{q}i_E = i_{E/F}q$. $OA_Q(E)/\mathrm{Ker}\,\widetilde{q}$ is a Q-algebra and $\widetilde{q}$ can be factorized through it. We denote by ρ the complete homomorphism from $OA_Q(E)$ to $OA_Q(E)/\mathrm{Ker}\,\widetilde{q}$, and by $\underline{\widetilde{q}}$ the complete homomorphism from $OA_Q(E)/\mathrm{Ker}\,\widetilde{q}$ to $OA_Q(E/F)$ such that $\widetilde{q} = \underline{\widetilde{q}}\rho$.

As $F \subset \mathrm{Ker}\,\widetilde{q}$, the map $j : E/F \to OA_Q(E)/\mathrm{Ker}\,\widetilde{q}$ is well defined and completely contractive. By the universal property of OA_Q, we can extend j to get a completely contractive morphism $\sigma : OA_Q(E/F) \to OA_Q(E)/\mathrm{Ker}\,\widetilde{q}$.

The restriction to E/F of the complete homomorphism $\underline{\widetilde{q}}\sigma$ is exactly $i_{E/F}$. To prove this fact since q is a quotient map it suffices to show that $\underline{\widetilde{q}}\sigma i_{E/F}q = i_{E/F}q$, but by construction we have $\sigma i_{E/F} = j$ and $\rho i_E = jq$ combining this we get

$$\underline{\widetilde{q}}\sigma i_{E/F}q = \underline{\widetilde{q}}\rho i_E = \widetilde{q}i_E = i_{E/F}q.$$

So by uniqueness of the extension of $i_{E/F}$, we obtain $\underline{\widetilde{q}}\sigma = \mathrm{Id}_{OA_Q(E/F)}$.

If $x \in M_n(OA_Q(E/F))$ is in the unit ball, then $\mathrm{Id}_{M_n} \otimes \sigma(x)$ is also in the unit ball of $M_n(OA_Q(E)/\mathrm{Ker}\,\widetilde{q})$. Since ρ is a complete metric surjection, we can find a lifting y of $\mathrm{Id}_{M_n} \otimes \sigma(x)$ in the unit ball of $OA_Q(E)$, and we have $\mathrm{Id}_{M_n} \otimes q(y) = x$, whence the result.

For the second part, it suffices to notice that one can replace $OA_Q(E)/\mathrm{Ker}\,\widetilde{q}$ by the closure of $S(E)/(\mathrm{Ker}\,\widetilde{q} \cap S(E))$ for the naturally induced norm coming from the inclusion $S(E) \subset OA_Q(E)$, this is indeed a Q-algebra by Proposition 1.3. So if $x \in S(E)$ has norm less than 1, its image in $\overline{S(E)/S(E) \cap \mathrm{Ker}\,\widetilde{q}}$ is actually in $S(E)/S(E) \cap \mathrm{Ker}\,\widetilde{q}$, so that we can find $y \in S(E)$ with the same property and such that $\widetilde{q}(y) = x$. (The proof at the matrix level is the same). ∎

DEFINITION 1.8. Let x be in $OA_Q(E)$; x is called *homogeneous of degree* $d \in \mathbb{N}$, if $\pi_z(x) = z^d x$ for all complex numbers z with $|z| = 1$, where π_z is the extension to $OA_Q(E)$ of the multiplication by z on E composed with i_E. We denote by $OA_Q^d(E)$ the subspace of such elements.

If $F \subset E$ are operator spaces, F is said 1-completely complemented if there is a completely contractive projection from E onto F.

$OA_Q^d(E)$ is the same as the closure of the image of $E^{\otimes^d}$ in $OA_Q(E)$: it readily follows from

PROPOSITION 1.9. *$OA_Q^d(E)$ is 1-completely complemented in $OA_Q(E)$.*

Indeed, we have an explicit formula for the projection:

$$P_d(x) = \int_{\mathbb{T}} \overline{z}^d \pi_z(x) \frac{\mathrm{d}z}{2z\mathrm{i}\pi}.$$

$OA_Q^1(E)$ can be thought as the universal Q-space envelope of E (see below for the definition).

We will need a slight generalization of $OA_Q(\cdot)$ for several operator spaces simultaneously. Let $E_1, \ldots, E_n$ be n operator spaces, one can construct a Q-algebra $OA_Q((E_i))$ which admits an n-tuple of complete contractions $E_i \to OA_Q((E_j))$ satisfying the following universal property:

PROPOSITION 1.10. *Each n-tuple of completely contractive maps $u_i : E_i \to A$, where A is a Q-algebra, extends uniquely to a completely contractive algebra homomorphism $\widetilde{u} : OA_Q((E_i)) \to A$.*

Let $S((E_i))$ be the formal commutative algebra generated by the (E_i), that is

$$S((E_i)) = \bigoplus_{(i_1,\ldots,i_n)\in\mathbb{N}^n} E_1^{\otimes_s^{i_1}} \otimes \cdots \otimes E_n^{\otimes_s^{i_s}}$$

where $\otimes_{\mathrm{s}}$ denotes the symmetrization of the tensor product, that is, $E^{\otimes_{\mathrm{s}}^d}$ is the subspace of $E^{\otimes^d}$ spanned by the elements of the form $e \otimes e \otimes \cdots \otimes e$. It is the space of elements of degree d in $S(E)$ and so there are natural multiplication maps $m_{d,p} : E^{\otimes_{\mathrm{s}}^d} \times E^{\otimes_{\mathrm{s}}^p} \to E^{\otimes_{\mathrm{s}}^{(d+p)}}$ such that if $x \in E^{\otimes_{\mathrm{s}}^d}$ and $y \in E^{\otimes_{\mathrm{s}}^p}$, then $m_{d,p}(x,y) = m_{p,d}(y,x)$.

The commutative multiplication on $S((E_i))$ is defined in the following way for elementary tensors: if $x = x_1 \otimes \cdots \otimes x_n \in E_1^{\otimes_{\mathrm{s}}^{i_1}} \otimes \cdots \otimes E_n^{\otimes_{\mathrm{s}}^{i_n}}$ and $y = y_1 \otimes \cdots \otimes y_n \in E_1^{\otimes_{\mathrm{s}}^{j_1}} \otimes \cdots \otimes E_n^{\otimes_{\mathrm{s}}^{j_s}}$ then

$$x \cdot y = y \cdot x = m_{i_1,j_1}(x_1, y_1) \otimes \cdots \otimes m_{i_n,j_n}(x_n, y_n),$$

and it is extended by biliniarity.

$S((E_i))$ has a universal property: each n-tuple of maps (u_i) from E_i to a given commutative algebra A uniquely extends to a morphism $\widetilde{u}$ from $S((E_i))$ to A.

To define $OA_Q((E_i))$, we follow the construction of $OA_Q(E)$. If $x \in M_n(S((E_i)))$, its norm is then defined by $\|x\| = \sup\{\|\mathrm{Id}_{M_n} \otimes \widetilde{u}(x)\|_A\}$ where the supremum runs over all n-tuples of complete contractions (u_i) from E_i to any Q-algebra A. $OA_Q((E_i))$ is the completion of $S((E_i))$ for this norm, with $n = 1$. $OA_Q(E)$ with this family of norms is then a Q-algebra.

$OA_Q((\cdot))$ shares the same properties as $OA_Q(\cdot)$.

PROPOSITION 1.11. *$OA_Q((\cdot))$ is projective, namely if $F_i \subset E_i$ then $OA_Q((E_i)) \to OA_Q((E_i/F_i))$ is a complete metric surjection. More precisely, if $x \in M_n(S((E_i/F_i)))$ has norm less than one then there exists $y \in M_n(S((E_i)))$ of norm less than one such that $\mathrm{Id}_{M_n} \otimes \widetilde{q}(y) = x$.*

The proof is essentially the same.

We can also define for any n-tuple of integers (d_i), the 1-complemented subspace $OA_Q^{(d_i)}((E_i))$ of (d_i) homogeneous elements as the closure of $\otimes E_i^{\otimes_{\mathrm{s}}^{d_i}}$ in $OA_Q((E_i))$. They are exactly the elements in $OA_Q((E_i))$ which satify $\pi_z(x) = z_1^{d_1} \cdots z_n^{d_n} x$ for all $z = (z_1, \ldots, z_n)$, where π_z is the unique extension to $OA_Q((E_i))$ of the multiplication maps π_{z_i} by z_i from E_i to $OA_Q((E_i))$. The projection onto $OA_Q^{(d_i)}((E_i))$ is given by:

$$P_d(x) = \int_{\mathbb{T}^n} \overline{z}_1^d \cdots \overline{z}_n^d \pi_z(x) \frac{\mathrm{d}z_1}{2z_1 \mathrm{i}\pi} \cdots \frac{\mathrm{d}z_n}{2z_n \mathrm{i}\pi}.$$

2. Q-SPACES

2.1. Definition and examples.

Definition 2.1. A Q-space is an (operator) quotient of a minimal operator space.

By [2], Remark 4.6, any Q-space is a Q-algebra with null product. As a simple consequence of the definition :

Corollary 2.2. *The category of Q-spaces is stable under taking quotient and subspace.*

If E and F are completely isomorphic operator spaces, it is possible to define an analogue of the Banach-Mazur distance between E and F

$$d_{\mathrm{cb}}(E,F) = \inf\{\|u\|_{\mathrm{cb}}\,\|u^{-1}\|_{\mathrm{cb}} \mid u : E \to F \text{ isomorphism}\}.$$

Junge found a very simple characterization of Q-spaces. We can measure the distance of a given operator space to the class of Q-spaces in the following way:

$$d_Q(E) = \inf\{d_{\mathrm{cb}}(E,F) \mid F \text{ is a } Q\text{-space}\}.$$

As Q-spaces are stable by taking ultraproducts, E is a Q-space if and only if $d_Q(E) = 1$.

Theorem 2.3. ([8]) *Let E be an operator space, then $d_Q(E)$ is the smallest C such that for any $n \geqslant 1$ and any map $T : M_n \to M_n$,*

$$\|T \otimes \mathrm{Id}_E : M_n(E) \to M_n(E)\| \leqslant C\|T\|.$$

Remark 2.4. Another way to state this theorem is: E is a Q-space if and only if for all n, we have $\max(M_n) \otimes_{\min} E = M_n \otimes_{\min} E$ isometrically. In fact, we can replace M_n by any injective operator space (e.g. $B(H)$); to prove this, we can assume E is finite dimensional. Because E is a quotient of a min space, its dual E^* is a subspace of a max space, say $\max(X)$. Let T be a bounded map from $B(H)$ into itself and v be an element of the unit ball of $E \otimes_{\min} B(H)$, then v can be viewed as a completely contractive map from $E^* \to B(H)$ and so admits a completely bounded extension $\widetilde{v}$ to $\max(X)$. The universal property of max spaces implies that the map $T\widetilde{v}$ is completely bounded with completely bounded norm less than $\|T\|$, so it is its restriction to E^* which is exactly $\mathrm{Id}_E \otimes T(v)$.

This kind of argument can be found in [10], where submaximal spaces (duals of Q-spaces) are studied. The other implication will be shown in the next section.

Using this theorem, one can show:

Corollary 2.5. ([2]) *The class of Q-spaces is stable under complex interpolation. More precisely, if E_0 and E_1 form a compatible couple of operator spaces, we have the estimate:*

$$\forall\, 0<\theta<1, \quad d_Q(E_\theta) \leqslant d_Q(E_0)^{1-\theta} d_Q(E_1)^\theta.$$

This theorem is a pratical tool to see whether an operator space is Q-space or not. We denote by R_n and C_n the classical n-dimensional row and column operator spaces and by OH_n the n-dimensional Hilbertian operator space of [14]. Φ_n will stand for the operator space spanned by Clifford matrices $U_1, \dots, U_n$; these are self-adjoint unitaries in M_{2^n} satisfying:

(i) $\forall\, \alpha_i \in \mathbb{C} \quad \Big(\sum\limits_{i=1}^n |\alpha_i|^2\Big)^{\frac12} \leqslant \Big\| \sum\limits_{i=1}^n \alpha_i U_i \Big\|_{M_{2^n}} \leqslant \sqrt{2} \Big(\sum\limits_{i=1}^n |\alpha_i|^2\Big)^{\frac12}$,

(ii) $\Big\| \sum\limits_{i=1}^n U_i \otimes U_i \Big\|_{M_{2^n \times 2^n}} = n$.

Remark 2.6. Q-spaces are easily seen to be symmetric in the sense that $E = E^{\mathrm{op}}$ for all such spaces, more precisely, the transposition of matrices is isometric on $M_n(E)$. Consequently, any non-symmetric space cannot be a Q-space.

Corollary 2.7. *We have the following estimates:*

(i) $d_Q(M_n) = n$;

(ii) $d_Q(OH_n) \approx n^{\frac14}$;

(iii) $d_Q(R_n) = d_Q(C_n) = n^{\frac12}$;

(iv) $d_Q(R_n \cap C_n) \leqslant \sqrt{2}$;

(v) $d_Q(R_n + C_n) \approx n^{\frac12}$;

(vi) $d_Q(\Phi_n) \approx n^{\frac12}$.

Proof. Let $(e_{i,j})$ be the canonical basis for M_n. For M_n, we have

$$d_{\mathrm{cb}}(M_n, \min(M_n)) \leqslant n.$$

We just consider the formal identity:

$$\begin{aligned} \|(b_{i,j})\|_{M_n(B(H))} &\leqslant \Big(\sum_{i,j=1}^n \|b_{i,j}\|^2_{B(H)} \Big)^{\frac12} \\ &\leqslant n \max_{i,j} \{\|b_{i,j}\|_{B(H)}\} \leqslant n \|(b_{i,j})\|_{\min(M_n)\otimes B(H)}. \end{aligned}$$

For the other estimate, we use Theorem 2.3 for T the transposition of M_n. We compute norms to find a minoration of C:

$$\Big\| \sum_{i,j} e_{i,j} \otimes e_{j,i} \Big\|_{M_n(M_n)} = 1 \quad \text{and} \quad \Big\| \sum_{i,j=1}^n e_{i,j} \otimes e_{i,j} \Big\|_{M_n(M_n)} = n,$$

so $d_Q(M_n) \geqslant n$, and $d_{\mathrm{cb}}(M_n, \min(M_n)) = n$. The estimate for OH_n can be found in [2], the upper bound follows from interpolation, and the lower bound is obtained using Clifford matrices. The proof for C_n and R_n is nearly the same as the one for M_n. The transposition of M_n on $\sum_{i=1}^{n} e_{i,1} \otimes e_{1,i}$ gives $d_Q(R_n) \geqslant n^{\frac{1}{2}}$, and for any contraction $T : M_p \to M_p$:

$$\Big\| \sum_{i=1}^{n} T(m_i) \otimes e_{1,i} \Big\|_{M_p(R_n)} = \Big(\Big\| \sum_{i=1}^{n} T(m_i) T(m_i)^* \Big\|_{M_p} \Big)^{\frac{1}{2}} \leqslant \Big(\sum_{i=1}^{n} \|m_i\|_{M_p}^2 \Big)^{\frac{1}{2}}$$
$$\leqslant n^{\frac{1}{2}} \Big\| \sum_{i=1}^{n} m_i \otimes e_{1,i} \Big\|_{M_p(R_n)}.$$

So $d_Q(R_n) \leqslant n^{\frac{1}{2}}$, in fact we have $d_{\mathrm{cb}}(\min(\ell_2), R_n) = n^{\frac{1}{2}}$; $d_Q(R_n \cap C_n) \leqslant \sqrt{2}$ is exactly a weak form of the non commutative Grothendieck inequality (see [12] and [7]): For any bounded linear map v between C^*-algebras A and B, we have:

$$\|\mathrm{Id}_{R_n \cap C_n} \otimes v\|_{(R_n \cap C_n) \otimes_{\min} A \to (R_n \cap C_n) \otimes_{\min} B} \leqslant \sqrt{2} \|v\|_{A \to B}.$$

The minoration of $d_Q(R_n + C_n)$ follows from the interpolation formula; indeed $OH_n = [R_n + C_n, R_n \cap C_n]_{\frac{1}{2}}$, so the above results yield $d_Q(R_n + C_n) \gtrsim n^{\frac{1}{2}}$. On the other hand, $d_{\mathrm{cb}}(R_n + C_n, R_n \cap C_n) \leqslant \sqrt{n}$.

The map from C_n to Φ_n which sends $e_{i,1}$ to U_i is bounded by $\sqrt{2}$, and since C_n is complemented in M_n, we get a bounded map $v : M_n \to M_{2^n}$. It satisfies:

$$\Big\| \sum_i e_{i,1} \otimes U_i \Big\| = n^{\frac{1}{2}} \quad \text{and} \quad \Big\| \sum_i v(e_{i,1}) \otimes U_i \Big\| = n.$$

So $d_Q(\Phi_n) \geqslant \left(\frac{n}{2}\right)^{\frac{1}{2}}$. Moreover, we have $d_{\mathrm{cb}}(\Phi_n, R_n \cap C_n) \approx n^{\frac{1}{2}}$, which completes the proof for Φ_n. ∎

Remark 2.8. In fact, we can exhibit a Q-space completely isomorphic to $R_n \cap C_n$. By [13], $R_n + C_n$ is completely isomorphic to the space spanned in L_1 by n Gaussian random variables, so its dual which is a Q-space is such an example. More generally, it is shown in [7] that $(R_n \cap C_n)^{\otimes^i_{\min}}$ is completely isomorphic to a Q-space. $R_n \cap C_n$ can also be realized as a subspace of $L^\infty(\mathbb{T})/\overline{H_0^\infty}(\mathbb{T})$ corresponding to the linear span of the class of the sequence $\{\exp(2\mathrm{i}\pi 2^k t)\}_{n \geqslant k \geqslant 1}$ where $\overline{H_0^\infty}(\mathbb{T})$ is the closed subspace of bounded functions with vanishing non negative Fourier coefficients. By the Nehari-Sarason-Page theorem, $L^\infty(\mathbb{T})/\overline{H_0^\infty}(\mathbb{T})$ (which is a Q-space) can be completely isometrically identified with Hankelian matrices, and in this particular case Junge's theorem can be strengthened, one can replace M_n by any von Neumann algebra.

$R_n \cap C_n$ plays a central role among Q-spaces, this is illustrated by the following theorem from [14] (Theorem 9.7 applied for the weight associated to $R + C$):

Theorem 2.9. *For every n-dimensional operator space E, we have $d_{\mathrm{cb}}(E, R_n \cap C_n) \leqslant \sqrt{n}$.*

Corollary 2.10. *There is a constant C such that for any n-dimensional operator space, $d_Q(E) \leqslant C\sqrt{n}$.*

The previous examples show that this is optimal.

2.2. The tensor product $\otimes_{f_Q}$ of two Q-spaces. It is an old open problem to know whether the minimal tensor product of Q-algebras is still a Q-algebra (in the Banach sense) or not. Consequently, it is also unclear whether the minimal tensor product is well defined in the category of Q-spaces, nevertheless in the same way as the Haagerup tensor product is defined for operator spaces, we introduce a tensor product on Q-spaces as follows:

Definition 2.11. Let E_1, E_2 be two operator spaces. We define $E_1 \otimes_{f_Q} E_2$ as the operator space $OA_Q^{1,1}(E_1, E_2)$. So if $x \in \mathcal{K} \otimes E_1 \otimes E_2$, $x = \sum_i k_i \otimes a_i \otimes b_i$,

$$\|x\|_{f_Q} = \sup_{\sigma_i : E_i \to A} \Big\| \sum_i k_i \otimes \sigma_1(a_i)\sigma_2(b_i) \Big\|_{\mathcal{K} \otimes_{\min} A}$$

where the supremum is taken over all complete contractions $\sigma_i : E_i \to A$ with A Q-algebra, and $E_1 \otimes_{f_Q} E_2$ is just the operator space obtained after completion.

As $\mathbb{C}$ is a Q-algebra, we have a norm one map $E_1 \otimes_{f_Q} E_2 \to E_1 \overset{\vee}{\otimes} E_2$, the Banach injective tensor product of E_1 and E_2. Since $OA_Q^{1,1}(E_1, E_2)$ is a Q-space:

Proposition 2.12. *$E_1 \otimes_{f_Q} E_2$ is a Q-space.*

$E_1 \otimes_{f_Q} E_2$ can degenerate in the sense that if the E_i are not Q-spaces, then there is no isometric copy of them in $E_1 \otimes_{f_q} E_2$; we have that $E = E \otimes_{f_Q} \mathbb{C}$ if and only if E is a Q-space. In a similar way, one can define $E_1 \otimes_{f_Q} \cdots \otimes_{f_Q} E_n$ for any n-tuple of Q-spaces $E_1, \dots, E_n$ as the operator space $OA_Q^{1,\dots,1}(E_1, \dots, E_n)$.

Corollary 2.13. *If E_i, F_i are Q-spaces and $\sigma_i : E_i \to F_i$ are completely bounded maps, then $\sigma_1 \otimes \sigma_2$ extends from $E_1 \otimes_{f_Q} E_2$ to $F_1 \otimes_{f_Q} F_2$ with completely bounded norm at most $\|\sigma_1\|_{\mathrm{cb}} \cdot \|\sigma_2\|_{\mathrm{cb}}$.*

Corollary 2.14. *The Haagerup cross norm, and a fortiori the projective cross norm dominate the norm f_Q.*

Proof. From its very definition the norm f_Q is dominated by the Haagerup norm, since the latter can be defined in the same way with the supremum runing on all completely contractive maps without any restriction on their range.

To prove directly the second assertion, according to [3] we only need to show that if $(a_{i,j}) \in M_n(E_1)$ and $(b_{k,l}) \in M_m(E_2)$ with norms less than one, then $\|(a_{i,j} \otimes b_{k,l})\|_{f_Q} < 1$. Thanks to the previous proposition, it suffices to show that, in a Q-algebra A, if $\|(a_{i,j})\| < 1$ and $\|(b_{k,l})\| < 1$, then $\|(a_{i,j} b_{k,l})\|_{mn} < 1$. This holds for $A = \mathbb{C}$, and therefore for all algebras of continuous functions as well as their subspaces and quotients; this is the easy part of Proposition 1.3. ∎

Proposition 2.15. *The norm $\otimes_{f_Q}$ is a completely projective cross norm. If $F_i \subset E_i$, then the following is a complete surjection:*

$$q : E_1 \otimes_{f_Q} E_2 \to E_1/F_1 \otimes_{f_Q} E_2/F_2.$$

More precisely if $x \in M_n(E_1/F_1 \otimes E_2/F_2)$ is of norm less than one, then there exists a lifting $y \in M_n(E_1 \otimes E_2)$ of x with norm less than one.

Proof. This is a consequence of the projectivity of $OA_Q(\cdot)$. Since $E_1 \otimes_{f_Q} E_2 = OA_Q^{1,1}(E_1, E_2)$, we will identify them. The latter is 1-completely complemented in $OA_Q(E_1, E_2)$, the result is then a consequence of projectivity of $OA_Q(\,\cdot\,,\,\cdot\,)$. We denote by $\widetilde{q} : OA_Q(E_1, E_2) \to OA_Q(E_1/F_1, E_2/F_2)$, the extension of $E_i \to E_i/F_i$, then $\widetilde{q}_{|E_1 \otimes_{f_Q} E_2} = q$.

Let us take $x \in E_1/F_1 \otimes E_2/F_2$ with norm less than one, we can lift x in $S(E_1, E_2)$ to get some y with norm less than one. Now, we just have to check that $P^{1,1}(y) \in E_1 \otimes E_2$ is again a lifting of x:

$$q(P^{1,1}(y)) = \int\limits_{\mathbb{T}\times\mathbb{T}} \overline{z}_1\, \overline{z}_2 \widetilde{q}(\pi_{z_1,z_2}(y))\, \frac{\mathrm{d}z_1}{2\pi \mathrm{i} z_1}\, \frac{\mathrm{d}z_2}{2\pi \mathrm{i} z_2};$$

but, it is easy to see that $\widetilde{q}$ commutes with the multiplications. So, $q(P^{1,1}(y)) = P^{1,1}(\widetilde{q}y) = x$. The proof for matrices with entries in $E_1/F_1 \otimes_{f_Q} E_2/F_2$ is the same. ∎

DEFINITION 2.16. An operator space E has the (completely contractive) lifting property, if for each map $u : E \to F/G$ such that $\|u\|_{\mathrm{cb}} < 1$, there is a completely bounded lifting $\widetilde{u} : E \to F$ with $\|\widetilde{u}\|_{\mathrm{cb}} < 1$.

This property was introduced and studied by Ruan and Kye ([9]). The basic exemple S_n^1 comes from the definition of the operator space stucture of a quotient.

PROPOSITION 2.17. *If E_1, E_2 have the completely contractive lifting property, then the completely isometrically identification holds*

$$E_1 \otimes_{f_Q} E_2 = \min(E_1 \overset{\vee}{\otimes} E_2).$$

Proof. In this case, we can lift each σ_i and we are reduced to consider only maps into commutative C^*-algebras, so $E_1 \otimes_{f_Q} E_2$ is a min space, which is easily seen to be $E_1 \overset{\vee}{\otimes} E_2$ as Banach space. ∎

NOTATION 2.18. If w is a multilinear map from M_n^N to M_d, we denote by $\widetilde{w}$ the multilinear map from $M_n(E_1) \times \cdots \times M_n(E_N)$ to $M_d(E_1 \otimes \cdots \otimes E_N)$ defined by:

$$\widetilde{w}(m_1 \otimes e_1, \ldots, m_N \otimes e_N) = w(m_1 \ldots, m_N) \otimes e_1 \otimes \cdots \otimes e_N.$$

THEOREM 2.19. *Let $E_1, \ldots, E_N$ be operator spaces, let $x \in M_d(E_1 \otimes \cdots \otimes E_N)$, then*

$$\|x\|_{M_d(E_1 \otimes_{f_Q} \cdots \otimes_{f_Q} E_N)} = \inf\{\|w\|\, \|x_1\|_{M_n(E_1)} \cdots \|x_N\|_{M_n(E_N)} \mid x = \widetilde{w}(x_1, \ldots, x_N)\}.$$

In this infimum n is arbitrary and w is any N-linear map from $M_n \times \cdots \times M_n$ (N-times) to M_d.

Proof. For simplicity let us take $N = 2$ and $d = 1$. As operator spaces, E_1, E_2 are quotients of spaces of the form $\ell_1^I(S_1^{n_i})$. This was observed by Blecher in [1], for example just take $I = \bigcup_p \{\phi \in M_p(E_1) \mid \|\phi\| \leqslant 1\}$; each element i in I is a completely contractive map from some S_1^p to E. Let $n_i = p$ and define

$$\rho_1 : \begin{array}{ccc} \ell_1^I((S_1^{n_i})_{i\in I}) & \to & E_1 \\ (s_i) & \mapsto & \sum\limits_{i\in I} i(s_i) \end{array}$$

and similarly for ρ_2. These spaces have the lifting property.

Let $x \in E_1 \otimes E_2$, $\|x\|_{f_Q} < 1$, then by Proposition 2.15, we can lift x to

$$z = \sum_t a_t \otimes b_t \in \ell_1^I(S_1^{n_i}) \otimes \ell_1^J(S_1^{n_j}) \quad \text{with} \quad \|z\|_{f_Q} < 1.$$

Each a_t can be approximated by elements in $\ell_1^I(S_1^{n_i})$ of smaller norm with finite support, say a_t'. Moreover if we let

$$\widehat{a}_t = a_t' + \|\rho_1(a_t - a_t')\| 1_{\rho_1(a_t - a_t')/\|\rho_1(a_t - a_t')\|},$$

where 1_x is the element of $\ell_1^I(S_1^{n_i})$, the components of which are all zero but the one corresponding to $i = x \in M_1(E_1)$ which equals 1. Then $\rho_1(\widehat{a}_t) = \rho_1(a_t)$ and $\widehat{a}_t$ has a finite support; define in the same way $\widehat{b}_t$, then $y = \sum_t \widehat{a}_t \otimes \widehat{b}_t$ is another lifting of x with finite support $I' \times J'$ and by the triangular inequality:

$$\|y\| < \|z\| + \sum_t \big(\|\rho_1(a_t - a_t')\| \, \|b_t\| + \|a_t\| \, \|\rho_2(b_t - b_t')\| + \|\rho_1(a_t - a_t')\| \, \|\rho_1(b_t - b_t')\|\big),$$

so choosing a_t', b_t' close enough to the a_t, b_t, forces this norm to be less than one.

By Proposition 2.17, and the fact that $\ell_1^{I'}(S_1^{n_i})$ is 1-completely complemented in $\ell_1^I(S_1^{n_i})$, we obtain $1 > \|y\|_{f_Q} = \|w\|$ if w is the bilinear form on $\ell_{I'}^\infty(M_{n_i}) \times \ell_{J'}^\infty(M_{n_j})$ associated to y. If x_i are the restrictions of ρ_i to $\ell_1^{I'}(S_1^{n_i})$ and $\ell_1^{J'}(S_1^{n_i})$, by choosing n big enough, we can assume that $x_i \in M_n(E_i)$ and w is defined on $M_n \times M_n$, so that $x = \rho_1 \otimes \rho_2(y) = \widetilde{w}(x_1, x_2)$ with the right norm estimate.

Reversing the last part of this argument and Corollary 2.13 give the other inequality; it can also be directly checked using Proposition 1.3. ∎

Remark 2.20. This result is also available for $N = 1$, it gives a way to compute the norm of elements in $M_n(OA_Q^1(E))$ as

$$\|x\|_{M_n(OA_Q^1(E))} = \inf\{\|w\| \, \|x_1\|_{M_k(E_1)} : x = w \otimes \mathrm{Id}_E(x_1)\}$$

where the infimum runs over all maps $w : M_k \to M_n$ and all k.

The statement: an operator space E is a Q-space if and only if $E = OA^1(E)$, is exactly Junge's theorem.

In the case $N = 2$, we can interpret it as a factorization result, using that $\max(M_m) \otimes_{\min} E = M_m(E)$, for any Q-space E:

PROPOSITION 2.21. *Let E_1, E_2 be Q-spaces, and $u \in M_n(E_1 \otimes E_2)$ such that $\|u\|_{f_Q} < 1$, then viewed as a map from E_1^* to $M_n(E_2)$, u admits for some m a factorization:*

$$\begin{tikzcd} \max(M_m) \arrow[r, "w"] & M_n(\min(S_1^m)) \arrow[d, "\mathrm{Id}_{M_n} \otimes v_2"] \\ E_1^* \arrow[u, "v_1^*"] \arrow[r, "u"] & M_n(E_2) \end{tikzcd}$$

with v_1, v_2 completely contractive and $\|w\| < 1$. Conversely, if such a factorization holds with $\|w\|_{M_n(\min(S_1^m) \otimes_{\min} \min(S_1^m))} < 1$, then $\|u\|_{f_Q} < 1$.

COROLLARY 2.22. *If E_i are Q-spaces then, for any $u \in M_n(E_1 \otimes E_2)$:*

$$\|u\|_{\min} \leqslant \|u\|_{f_Q}.$$

Indeed, if $\|u\|_{f_Q} < 1$ the diagram proves that u is completely bounded with norm less than 1.

REMARK 2.23. This fails in general for operator spaces since $E_1 \otimes_{f_Q} E_2$ is always a Q-space.

2.3. $E_1 \otimes_{f_Q} E_2$ AS A BANACH SPACE.

We denote by $\otimes^\wedge$ the operator space projective tensor product. It is possible to characterize $E_1 \otimes_{f_Q} E_2$ at the Banach level.

THEOREM 2.24. *If E_1 and E_2 are Q-spaces then $E_1 \otimes_{f_Q} E_2$ is K-isomorphic to $E_1 \otimes_h E_2$ and Kk-isomorphic to $E_1 \otimes^\wedge E_2$, where K is the non commutative Grothendieck constant appearing in* [12] *page 119, and k is the constant in* [11], $k = \lim\limits_{n\to\infty} n / \|\min(\ell_2^n) \to \max(\ell_2^n)\|_{\mathrm{cb}}$.

Proof. Since all the three tensor norms appearing are projective, it suffices to show it for minimal operator spaces. The main theorem in [11] can be extended to Q-spaces: $E_1 \otimes^\wedge E_2$ is k-isomorphic to $E_1 \otimes_h E_2$.

Grothendieck's non commutative theorem states that any contractive map w from a C^*-algebra to a dual of C^*-algebra factorize through an Hilbert space with factorization norm less than K.

If $u \in E_1 \otimes E_2$ satisfies $\|u\|_{f_Q} < 1$, combining Grothendieck's theorem and Proposition 2.21 gives that u factors through a Hilbert space with factorization norm less than K. Then by [11], $\|u\|_{E_1 \otimes_h E_2} \leqslant K$ and $\|u\|_{E_1 \otimes^\wedge E_2} \leqslant Kk$. ∎

REMARK 2.25. The equivalence of these norms at the matrices level does not hold in general. For instance, answering a question of [11], LeMerdy used the non symmetry of the Haagerup cross norm to show that the identity $\min(\ell_2^n) \otimes_h \min(\ell_2^n) \to \min(\ell_2^n) \widehat{\otimes} \min(\ell_2^n)$ has completely bounded norm greater than $\sqrt{n}$.

The norms f_Q and min are not always equivalent on $E_1 \otimes E_2$ for Q-spaces E_1 and E_2. Just take $E_1 = \min(\ell_1), E_2 = \min(\ell_\infty)$. By the above theorem, if those norms were equivalent, the identity of ℓ_∞ would factor through a Hilbert space.

The tensor norm f_Q does not seem to be associative, at least we can show that $\min(\ell_2) \otimes_{f_Q} \min(\ell_2) \otimes_{f_Q} \min(\ell_1)$ differs from $(\min(\ell_2) \otimes_{f_Q} \min(\ell_2)) \otimes_{f_Q} \min(\ell_1)$. Indeed from the fact that C and $\max(\ell_1)$ have the lifting property, one can deduce that $\min(\ell_2)\otimes_{f_Q}\min(\ell_2)\otimes_{f_Q}\min(\ell_1) = \min(\ell_2 \overset{\vee}{\otimes} \ell_2)\otimes_{\min}\min(\ell_1)$ and $(\min(\ell_2)\otimes_{f_Q} \min(\ell_2)) \otimes_{f_Q} \min(\ell_1) = \min(\ell_2 \overset{\vee}{\otimes} \ell_2) \otimes_{f_Q} \min(\ell_1)$. As ℓ_∞ is complemented in $\ell_2 \overset{\vee}{\otimes} \ell_2$, this remark reduces to the previous one.

Corollary 2.26. *If E_1, E_2 are Q-spaces of dimension n, then $d_{\mathrm{cb}}(E_1, E_2^*) \geqslant \frac{1}{k}\sqrt{n}$.*

Proof. First remark that any Q-space or submaximal space E is symmetric and satisfies $E \otimes_{\min} C = E \otimes_{\min} R = E \otimes_{\min} (R \cap C)$. Let $u : E_2 \to E_1^*$ be such that $\|u\|_{\mathrm{cb}}\, \|u^{-1}\|_{\mathrm{cb}} = d_{\mathrm{cb}}(E_1, E_2^*)$.

By dualizing Paulsen's results, we obtain $\|u\|_{\mathrm{cb}} \geqslant \frac{1}{k}\|u\|_h$, so we have a factorization of u through R_n, using the symmetry of E_1^* and E_2, we get:

$$\begin{tikzcd} E_1^* \arrow[r, "u^{-1}"] & E_2 \arrow[r, "u"] \arrow[d, "v"] & E_1^* \\ & C_n \arrow[r, "\mathrm{Id}"] & R_n \arrow[u, "w"'] \end{tikzcd}$$

with $\|w\|_{\mathrm{cb}}\|v\|_{\mathrm{cb}} \leqslant k\|u\|_{\mathrm{cb}}$, we get $\|w\|_{\mathrm{cb}}\|u\|_{\mathrm{cb}}\|v\|_{\mathrm{cb}} \geqslant \|\mathrm{Id}\|_{\mathrm{cb}} = \sqrt{n}$. ∎

In fact, in the preceding proof, we show:

Corollary 2.27. *Any map from an n-dimensional Q-space to a submaximal space factors completely boundedly through the identity $R_n \cap C_n \to R_n + C_n$.*

In the preceeding diagram, using symmetry, we could have considered that $v : E_2 \to (R_n \cap C_n)$ and $w : R_n + C_n \to E_1^*$ with the same norm estimates.

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ÉRIC RICARD
Université Paris VI
Équipe d'Analyse
4 place Jussieu
75252 Paris Cedex 05
FRANCE

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