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ON WELL-BEHAVED UNBOUNDED REPRESENTATIONS OF $*$-ALGEBRAS

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Abstract. A general approach to well-behaved unbounded $*$-representations of a $*$-algebra $\mathcal{X}$ is proposed. Let $\mathcal{B}$ be a normed $*$-algebra equipped with a left action $\triangleright$ of $\mathcal{X}$ on $\mathcal{B}$ such that $(x \triangleright a)^+ b = a^+(x^+ \triangleright b)$ for $a, b \in \mathcal{B}$ and $x \in \mathcal{X}$. Then the pair $(\mathcal{X}, \mathcal{B})$ is called a *compatible pair*. For any continuous non-degenerate $*$-representation ρ of $\mathcal{B}$ there exists a closed $*$-representation ρ' of $\mathcal{X}$ such that $\rho'(x)\rho(b) = \rho(x \triangleright b)$, where $x \in \mathcal{X}$ and $b \in \mathcal{B}$. The $*$-representations ρ' are called the *well-behaved $*$-representations* associated with the compatible pair $(\mathcal{X}, \mathcal{B})$. A number of examples illustrating this concept are developed in detail.

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0. INTRODUCTION

Unbounded representations of general $*$-algebras in Hilbert space occur in various branches of mathematics and mathematical physics such as representation theory of Lie algebras, algebraic quantum field theory, the theory of quantum groups and quantum algebras. One of the natural questions is to ask for a description or classification of *all* $*$-representations of the corresponding $*$-algebra. But it turns out that this is not a well-posed problem for general $*$-algebras. In order to explain this, let $\mathcal{X}$ be the $*$-algebra $\mathbb{C}[x, y]$ of all polynomials in two commuting hermitean indeterminates x and y or the $*$-algebra $A(\mathsf{p}, \mathrm{x})$ of two hermitean generators p and x satisfying the Heisenberg commutation relation $\mathsf{p}\mathrm{x} - \mathrm{x}\mathsf{p} = -\mathrm{i}$. In both cases it seems to be impossible to classify in a reasonable way all $*$-representations of $\mathcal{X}$ even if we assume that the images of the generators x, y respectively p, x are essentially self-adjoint (see, for instance, [14], Chapter 9, and [13]). The large variety of such $*$-representations of the $*$-algebra $\mathbb{C}[x, y]$ is illustrated by the following result ([14], Theorem 9.4.1): For any properly infinite von Neumann algebra $\mathcal{N}$ on a separable Hilbert space there exists a $*$-representation ρ of the polynomial algebra $\mathbb{C}[x,y]$ such that the operators $\rho(x)$ and $\rho(y)$ are essentially self-adjoint and the spectral projections of these operators generate the von Neumann algebra $\mathcal{N}$.

In most situations it suffices to know some class of "nice" $*$-representations of the $*$-algebra which is characterized by means of additional requirements in order to exclude pathological behaviour of operators. In what follows we shall call these $*$-representations *well-behaved*. In earlier papers (see [14], Chapters 9 and 10, and [15]) we have called them *integrable representations* because of the commonly used terminology in representation theory of Lie algebras. For the $*$-algebras $\mathbb{C}[x,y]$ and $A(\mathsf{p},\mathrm{x})$ it is easy to guess how to define well-behaved $*$-representations. A $*$-representation ρ of $\mathbb{C}[x,y]$ is called *well-behaved* if ρ is self-adjoint (see [9] or [14] for this notion) and if $\rho(x)$ and $\rho(y)$ are essentially self-adjoint operators such that their spectral projections commute. In the case $\mathcal{X}=A(p,q)$ the latter condition should be replaced by the requirement that $P:=\overline{\rho(\mathsf{p})}$ and $X:=\overline{\rho(\mathrm{x})}$ are self-adjoint operators satisfying the Weyl relation $\mathrm{e}^{\mathrm{i}tP}\mathrm{e}^{\mathrm{i}sX}=\mathrm{e}^{\mathrm{i}st}\mathrm{e}^{\mathrm{i}sX}\mathrm{e}^{\mathrm{i}tP}$, $s,t\in\mathbb{R}$.

Many papers of the mathematical physics literature dealing with unbounded $*$-representations claim to determine *all* $*$-representations of the $*$-algebra. However, a closer look at the proofs shows that often hidden additional assumptions are used and that only a particular class of well-behaved $*$-representations is investigated. For instance, for $*$-algebras related to the canonical commutation relations or for q-oscillator algebras it is often required that a vacuum vector exists or that certain operators have a complete set of eigenvectors.

There is no general method to select the well-behaved $*$-representations of a given $*$-algebra. Also it is important to stress that the choice of well-behaved $*$-representations may depend on the aim of considerations. The examples developed in Section 4 show that for the same $*$-algebra there are various natural candidates for the definition of well-behavedness. The additional conditions selecting well-behaved $*$-representations depend, generally speaking, on the underlying $*$-algebra.

In this paper we propose a general approach to the study of well-behaved $*$-representations. The idea is easily explained as follows: Let $\mathcal{X}$ be a $*$-algebra and let $\mathcal{B}$ be a normed $*$-algebra equipped with a left action, written $x\triangleright b$, of $\mathcal{X}$ on $\mathcal{B}$ satisfying the compatibility condition $(x\triangleright a)^+b=a^+(x^+\triangleright b)$ for $a,b\in\mathcal{B}$ and $x\in\mathcal{X}$. We shall call such a pair $(\mathcal{X},\mathcal{B})$ *compatible*. Then, for any continuous non-degenerate $*$-representation ρ of the normed $*$-algebra $\mathcal{B}$ there exists a closed $*$-representation ρ' of $\mathcal{X}$ such that $\rho'(x)\rho(b)=\rho(x\triangleright b)$, where $x\in\mathcal{X}$ and $b\in\mathcal{B}$. The $*$-representations ρ' of $\mathcal{X}$ obtained in this way are called the *well-behaved $*$-representations of $\mathcal{X}$ associated with the compatible pair* $(\mathcal{X},\mathcal{B})$.

The main purpose of this paper is to develop a number of important examples and show that they fit into this context. In Section 2 we discuss some compatible pairs $(\mathcal{X},\mathcal{B})$ for the polynomial algebra $\mathcal{X}=\mathbb{C}[x_1,\dots,x_n]$. In Section 3 we treat the G-integrable representations of the enveloping algebra $\mathcal{E}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of a Lie group G. Here $\mathcal{B}$ is the $*$-algebra $C_0^\infty(G)$ with convolution multiplication. In Section 4 we study various compatible pairs $(\mathcal{X},\mathcal{B})$ by using the Weyl calculus of pseudodifferential operators. Among others, various classes of well-behaved $*$-representations of the coordinate $*$-algebra $\mathcal{X}=\mathcal{O}(\mathbb{R}_q^2)$ of the real quantum plane are considered in this approach. In Section 5 we consider the quantum group $SU_q(1,1)$. The paper closes with a short outlook in Section 6.

Let us collect some definitions and facts on unbounded operator algebras and unbounded $*$-representations used in what follows. More details can be found in the monograph [14]; see also [5].

Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathcal{H}$. Then the vector space

$$\mathcal{L}^+(\mathcal{D}) = \{x \in \operatorname{End}\mathcal{D} : \mathcal{D} \subseteq \mathcal{D}(x^*),\ x^*\mathcal{D} \subseteq \mathcal{D}\}$$

is a unital $*$-algebra with operator product as multiplication and the restriction $x^+ := x^*\lceil\mathcal{D}$ of the adjoint operator x^* to $\mathcal{D}$ as involution. The set $\mathbb{B}(\mathcal{D})$ of all bounded operator b on $\mathcal{H}$ such that $b\mathcal{H} \subseteq \mathcal{D}$ and $b^*\mathcal{H} \subseteq \mathcal{D}$ is obviously a $*$-algebra. An O^*-algebra on the domain $\mathcal{D}$ is a $*$-subalgebra of $\mathcal{L}^+(\mathcal{D})$ which contains the identity map of $\mathcal{D}$. By a $*$-representation of an abstract $*$-algebra A (without unit in general) on a domain $\mathcal{D}$ we mean a $*$-homomorphism ρ of A into the $*$-algebra $\mathcal{L}^+(\mathcal{D})$. The $*$-representation ρ of A is said to be closed if $\mathcal{D}$ is the intersection of all domains $\mathcal{D}(\overline{\rho(a)})$, $a \in A$, where the bar refers to the closure of the operator $\rho(a)$. The representation ρ is called *non-degenerate* if $\rho(A)\mathcal{D}$ is dense in the underlying Hilbert space $\mathcal{H}$. If A is a normed $*$-algebra (that is, A is equipped with a submultiplicative $*$-invariant norm $\|\cdot\|$), then a $*$-representation ρ is called *continuous* if all operators are $\rho(a)$, $a \in A$, are bounded and there exist a positive constant C such that $\|\rho(a)\| \leqslant C\|a\|$ for all $a \in A$, where $\|\rho(a)\|$ denotes the operator norm of $\rho(a)$.

1. COMPATIBLE PAIRS

Let $\mathcal{X}$ be a $*$-algebra with unit element 1 and let $\mathcal{B}$ be a $*$-algebra (without unit in general). The involutions of $\mathcal{X}$ and $\mathcal{B}$ are denoted by $x \to x^+$ and $b \to b^+$, respectively. Suppose that the vector space $\mathcal{B}$ is a left $\mathcal{X}$-module with left action denoted by $\triangleright$, that is, there exists a linear mapping $\phi : \mathcal{X} \otimes \mathcal{B} \to \mathcal{B}$, written as $\phi(x \otimes b) = x \triangleright b$, such that $(xy) \triangleright b = x \triangleright (y \triangleright b)$ and $1 \triangleright b = b$ for $x, y \in \mathcal{X}$ and $b \in \mathcal{B}$.

Proposition 1.1. *Suppose that the left action of the $*$-algebra $\mathcal{X}$ on the $*$-algebra satisfies the condition*

$$(x \triangleright a)^+ b = a^+(x^+ \triangleright b) \quad \textit{for all } x \in \mathcal{X} \textit{ and } a, b \in \mathcal{B}. \tag{1.1}$$

Then, for any non-degenerate $*$-representation ρ of the $*$-algebra $\mathcal{B}$ there exists a unique $*$-representation $\widetilde{\rho}$ of the $*$-algebra $\mathcal{X}$ on the domain $\mathcal{D}(\widetilde{\rho}) = \rho(\mathcal{B})\mathcal{D}(\rho)$ such that

$$\widetilde{\rho}(x)(\rho(b)\varphi) = \rho(x \triangleright b)\varphi, \quad x \in \mathcal{X},\ b \in \mathcal{B},\ \varphi \in \mathcal{D}(\rho). \tag{1.2}$$

Let ρ' denote the closure of the $*$-representation $\widetilde{\rho}$.

Proof. Let $\zeta = \sum_i \rho(a_i)\varphi_i$ and $\eta = \sum_j \rho(b_j)\psi_j$ be vectors of the domain $\mathcal{D}(\widetilde{\rho})$, where $a_i, b_j \in \mathcal{B}$ and $\varphi_i, \psi_j \in \mathcal{D}(\rho)$. Let $x \in \mathcal{X}$. Using condition (1.1) and the assumption that ρ is a $*$-representation of the $*$-algebra $\mathcal{B}$ we compute

$$\begin{aligned}(1.3)\qquad &\Big\langle \sum_i \rho(x \triangleright a_i)\varphi_i, \eta \Big\rangle \\ &= \Big\langle \sum_i \rho(x \triangleright a_i)\varphi_i, \sum_j \rho(b_j)\psi_j \Big\rangle = \sum_{i,j} \langle \varphi_i, \rho((x \triangleright a_i)^+ b_j)\psi_j \rangle \\ &= \sum_{i,j} \langle \varphi_i, \rho(a_i^+(x^+ \triangleright b_j))\psi_j \rangle = \Big\langle \sum_i \rho(a_i)\varphi_i, \sum_j \rho(x^+ \triangleright b_j)\psi_j \Big\rangle \\ &= \Big\langle \zeta, \sum_j \rho(x^+ \triangleright b_j)\psi_j \Big\rangle.\end{aligned}$$

First we shall use relation (1.3) in order to show that equation (1.2) defines unambignoulsy a linear operator $\widetilde{\rho}(x)$ on the domain $\mathcal{D}(\widetilde{\rho})$. In order to do so, it suffices to check that $\zeta \equiv \sum_i \rho(a_i)\varphi_i = 0$ implies that $\sum_i \rho(x \triangleright a_i)\varphi_i = 0$. Indeed, if $\zeta = 0$, then it follows from (1.2) that $\Big\langle \sum_i \rho(x \triangleright a_i)\varphi_i, \eta \Big\rangle = 0$ for all $\eta \in \mathcal{D}(\widetilde{\rho})$. Since ρ is non-degenerate, $\mathcal{D}(\widetilde{\rho})$ is dense in the underlying Hilbert space and so $\sum_i \rho(x \triangleright a_i)\varphi_i = 0$. Hence the operator $\widetilde{\rho}(x)$ is well-defined.

From the properties of a left action if follows at once that $\widetilde{\rho}$ is an algebra homomorphism of $\mathcal{X}$ into the linear operators acting on the domain $\mathcal{D}(\widetilde{\rho})$ and leaving $\mathcal{D}(\widetilde{\rho})$ invariant. In order to prove that $\widetilde{\rho}$ preserves the involution, we combine equations (1.2) and (1.3) and conclude that $\langle \widetilde{\rho}(x)\zeta, \eta \rangle = \langle \zeta, \widetilde{\rho}(x^+)\eta \rangle$ for all $\zeta, \eta \in \mathcal{D}(\widetilde{\rho})$. Thus, $\widetilde{\rho}$ is indeed a $*$-representation of the $*$-algebra $\mathcal{X}$ on the domain $\mathcal{D}(\widetilde{\rho})$. ∎

DEFINITION 1.2. A *compatible pair* is a pair $(\mathcal{X}, \mathcal{B})$ of a unital $*$-algebra $\mathcal{X}$ and a normed $*$-algebra $\mathcal{B}$ equipped with a left action of $\mathcal{X}$ on $\mathcal{B}$ satisfying condition (1.1).

Our guiding example of a compatible pair is the following

EXAMPLE 1.3. Let $\mathcal{X}$ be an O^*-algebra on a dense domain $\mathcal{D}$ of a Hilbert space and let $\mathcal{B}$ be a $*$-subalgebra of $\mathbb{B}(\mathcal{D})$ such that $xb \in \mathcal{B}$ for all $x \in \mathcal{X}$ and $x \in \mathcal{B}$. We equipp the $*$-algebra $\mathcal{B}$ with the operator norm. Then there is a left action $\triangleright$ of $\mathcal{X}$ on $\mathcal{B}$ defined by the operator product, that is, $x \triangleright b := bx,\ x \in \mathcal{X}, b \in \mathcal{B}$. It is not difficult to show that $(xb)^+ a = b^+ x^+ a$ for $a, b \in \mathcal{B}$ and $x \in \mathcal{X}$. Hence $(\mathcal{X}, \mathcal{B})$ is a compatible pair. We call such a pair $(\mathcal{X}, \mathcal{B})$ a *compatible O^*-pair* on the domain $\mathcal{D}$. In particular, $(\mathcal{L}^+(\mathcal{D}), \mathbb{B}(\mathcal{D}))$ is a compatible O^*-pair, because $xb \in \mathbb{B}(\mathcal{D})$ for $x \in \mathcal{L}^+(\mathcal{D})$ and $b \in \mathbb{B}(\mathcal{D})$.

Now let $(\mathcal{X}, \mathcal{B})$ be a compatible pair and let ρ be a continuous $*$-representation of the normed $*$-algebra $\mathcal{B}$ on a Hilbert space $\mathcal{H}$. Then it is clear that $\widetilde{\rho}(\mathcal{X})$ is an O^*-algebra on the domain $\mathcal{D} = \rho(\mathcal{B})\mathcal{H}$ such that $(\widetilde{\rho}(\mathcal{X}), \rho(\mathcal{B}))$ is a compatible O^*-pair on the domain $\mathcal{D}$. That is, any continuous $*$-representation ρ of $\mathcal{B}$ gives raise to a homomorphism of the (abstract) compatible pair $(\mathcal{X}, \mathcal{B})$ to the compatible O^*-pair $(\widetilde{\rho}(\mathcal{X}), \rho(\mathcal{B}))$.

Remark 1.4. Let $\overline{\mathbb{B}}(\mathcal{D})$ be the completion of the normed $*$-algebra $(\mathbb{B}(\mathcal{D}), \|\cdot\|)$. Obviously, the closure of the finite rank operators in $\overline{\mathbb{B}}(\mathcal{D})$ is the $*$-algebra $\mathcal{C}(\mathcal{H})$ of compact operators on $\mathcal{H}$. Thus $\mathcal{C}(\mathcal{H})$ is contained in the C^*-algebra $\overline{\mathbb{B}}(\mathcal{D})$. We call the quotient C^*-algebra

$$C^*(\mathcal{D}) := \overline{\mathbb{B}}(\mathcal{D})/\mathcal{C}(\mathcal{H})$$

the *C^*-algebra associated with the domain $\mathcal{D}$*. It carries important information about the infinite dimensional closed subspaces of $\mathcal{H}$ contained in $\mathcal{D}$. As a sample, we mention the following result which is stated here without proof:

Suppose that $\mathcal{D}$ is a commutatively dominated Frechet domain (see [14], p. 108, for the definition)*. Then the domain $\mathcal{D}$ contains an infinite dimensional closed linear subspace of $\mathcal{H}$ if and only if $C^*(\mathcal{D})$ is non-trivial, that is, $C^*(\mathcal{D}) \neq \{0\}$.*

2. WELL-BEHAVED REPRESENTATIONs OF THE POLYNOMIAL ALGEBRA $\mathbb{C}[x_1, \dots, x_n]$

In this section $\mathcal{X}$ denotes the $*$-algebra $\mathbb{C}[x_1, \dots, x_n]$ of all polynomials with complex coefficients in n commuting hermitean indeterminates $x_1, \dots, x_n$.

Example 2.1. Let M be a closed subset of $\mathbb{R}^n$ and let $\mathcal{B}_1$ be the $*$-algebra $C_0(\mathbb{R}^n)$ of all compactly supported continuous functions on $\mathbb{R}^n$ with pointwise multiplication $(fg)(t) = f(t)g(t)$ and involution $f^+(t) = \overline{f(t)}$. Let $\|f\|$ be the supremum of the function $|f(t)|$ over M. It is obvious that the multiplication

$$(2.1) \qquad (p \triangleright f)(t_1, \dots, t_n) = p(t_1, \dots, t_n) f(t_1, \dots, t_n), \quad p \in \mathcal{X},\ f \in \mathcal{B}_1,$$

defines a left action of $\mathcal{X}$ on $\mathcal{B}_1$ such that $(\mathcal{X}, \mathcal{B}_1)$ is a compatible pair.

Now let ρ be a non-degenerate continuous $*$-representation of $\mathcal{B}_1$ on a Hilbert space $\mathcal{H}$. It is well-known that there exists a spectral measure $E(\lambda), \lambda \in \mathbb{R}^n$, on $\mathcal{H}$ supported in M such that $\rho(f) = \int f(\lambda)\, \mathrm{d}E(\lambda), f \in \mathcal{B}_1$. Then

$$A_1 := \int \lambda_1 \, \mathrm{d}E(\lambda), \dots, A_n := \int \lambda_n \, \mathrm{d}E(\lambda)$$

are self-adjoint operators with commuting spectral projections and

$$\rho'(x_j)\rho(f) = \rho(x_j \triangleright f) = \int \lambda_j f(\lambda)\, \mathrm{d}E(\lambda) = \int \lambda_j \, \mathrm{d}E(\lambda) \int f(\lambda)\, \mathrm{d}E(\lambda) = A_j \rho(f)$$

for $j = 1, \dots, n$. Conversely, any spectral measure on $\mathcal{H}$ with support contained in M gives a $*$-representation ρ' as above.

We now specialize to the case where $M = \mathbb{R}^n$. It is obvious that the operator $\rho'(p)$ is essentially self-adjoint on $\rho(\mathcal{B}_1)\mathcal{H}$ for any $p = p^+ \in \mathcal{X}$. Therefore, the $*$-representation ρ' (which is by definition the closure of its restriction to $\rho(\mathcal{B}_1)\mathcal{H}$) is integrable in the sense of [14], Chapter 9 (see Theorem 9.12). Conversely, any integrable $*$-representation of $\mathcal{X} = \mathbb{C}[x_1, \dots, x_n]$ is of this form. Thus, the $*$-representations ρ' associated with the compatible pair $(\mathcal{X}, \mathcal{B}_1)$ for $M = \mathbb{R}^n$ are precisely the integrable $*$-representations of the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$.

EXAMPLE 2.2. Suppose that K is a fixed compact subset $\mathbb{R}^n$. Let $\mathcal{B}_2 = \mathcal{X} = \mathbb{C}[x_1, \ldots, x_n]$ equipped with the supremum norm over the compact set K. With the left action (2.1) of $\mathcal{X}$ on $\mathcal{B}_2$, $(\mathcal{X}, \mathcal{B})$ is a compatible pair.

Let ρ be a continuous $*$-representation of $\mathcal{B}_2$ on a Hilbert space $\mathcal{H}$. Since ρ is $\|\cdot\|$-continuous, ρ extends to a $*$-representation of the C^*-algebra $C(K)$. Hence there exists a spectral measure E an $\mathcal{H}$ supported on the set K such that $\rho(p) = \int p(\lambda)\,\mathrm{d}E(\lambda)$ for $p \in \mathcal{B}_2$. As in the preceding example, we obtain

$$\overline{\rho'(p)} = \int_K p(\lambda)\,\mathrm{d}E(\lambda), \quad p \in \mathcal{X}.$$

Thus, the $*$-representations ρ' of $\mathcal{X}$ are precisely those bounded $*$-representation of $\mathbb{C}[x_1, \ldots, x_n]$ for which the joint spectrum of the self-adjoint operators $\overline{\rho'(x_1)}, \ldots$ $\ldots, \overline{\rho'(x_n)}$ is contained in the set K. Among others, this example shows that the class of $*$-representations ρ' essentially depends on the choice of the norm $\|\cdot\|$.

3. INTEGRABLE REPRESENTATIONS OF ENVELOPING ALGEBRAS

Throughout this example, G is a finite dimensional real Lie group with left Haar measure μ_l and Lie algebra $\mathfrak{g}$ and $\mathcal{E}(\mathfrak{g})$ is the complex universal enveloping algebra of $\mathfrak{g}$.

The algebra $\mathcal{E}(\mathfrak{g})$ is a $*$-algebra with involution determined by $x^+ = -x$ for $x \in \mathfrak{g}$. Let $\mathcal{X}$ denote the $*$-algebra $\mathcal{E}(\mathfrak{g})$. The vector space $B = C_0^\infty(G)$ is a $*$-algebra with respect to the convolution multiplication

$$(a \cdot b)(g) = \int_G a(h) b(h^{-1}g)\,\mathrm{d}_{\mu_l}(h), \quad a, b \in C_0^\infty(G), \tag{3.1}$$

and the involution

$$a^+(g) = m(g)^{-1}\overline{a(g^{-1})}, \quad a \in C_0^\infty(G), \tag{3.2}$$

where m denotes the modular function of the Lie group G. Since m is a C^∞-function on G (see [18]), a^+ is again in $C_0^\infty(G)$. We equip the $*$-algebra $\mathcal{B}$ with the $*$-invariant submultiplicative norm

$$\|a\| = \int_G |a(g)|\,\mathrm{d}\mu_l(g).$$

All these facts are well-known and can be found, for instance, in [7], Section 28. The completion of $(\mathcal{B}, \|\cdot\|)$ is nothing but the Banach $*$-algebra $L^1(G)$.

Now we define the left action of $\mathcal{X}$ on $\mathcal{B}$. Let $x \to \mathrm{e}^x$ denote the exponential map of $\mathfrak{g}$ into G. Each element x of $\mathcal{E}(\mathfrak{g})$ acts as a right-invariant differential operator $\widetilde{x}$ of G. For $x \in \mathfrak{g}$, the operator $\widetilde{x}$ is given by

$$(\widetilde{x}a)(g) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} a(\mathrm{e}^{-tx}g), \quad a \in C_0^\infty(G). \tag{3.3}$$

Using the formulas (3.1) and (3.3) and the left invariance of the measure μ one easily verifies that

$$\widetilde{x}(a \cdot b) = (\widetilde{x}a) \cdot b, \quad a, b \in C_0^\infty(G), \tag{3.4}$$

for $x \in \mathfrak{g}$. Since the map $x \to \widetilde{x}$ of $\mathcal{E}(\mathfrak{g})$ into the differential operators on G is an algebra homomorphism, (3.4) is valid for all $x \in \mathcal{E}(\mathfrak{g})$. From the preceding we conclude at once that

$$x \triangleright a := \widetilde{x} a, \quad a \in C_0^\infty(G),\, x \in \mathcal{E}(\mathfrak{g}), \tag{3.5}$$

defines a left action of the $*$-algebra $\mathcal{X} = \mathcal{E}(\mathfrak{g})$ on the $*$-algebra $\mathcal{B} = C_0^\infty(G)$.

LEMMA 3.1. *$(\mathcal{X}, \mathcal{B})$ is a compatible pair.*

Proof. Clearly, it suffices to verify the compatibility condition (1.1) for elements x of the Lie algebra $\mathfrak{g}$. From the analysis on locally compact groups it is well-known (see [7], p. 376) that there is a *right* Haar measure μ_{r} on G such that

$$\mathrm{d}\mu_{\mathrm{l}}(g) = m(g)\, \mathrm{d}\mu_{\mathrm{r}}(g), \quad g \in G. \tag{3.6}$$

Using formulas (3.1), (3.2), (3.3), (3.5) and (3.6) and

$$\begin{aligned}
(b^+ \cdot (x^+ \triangleright a))(g) &= -\int b^+(h)(\widetilde{x}a)(h^{-1}g)\, \mathrm{d}\mu_{\mathrm{l}}(h) \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int m(h)^{-1} \overline{b}(h^{-1}) a(\mathrm{e}^{-tx} h^{-1} g)\, \mathrm{d}\mu_{\mathrm{l}}(h) \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int \overline{b}(h^{-1}) a(\mathrm{e}^{-tx} h^{-1} g)\, \mathrm{d}\mu_{\mathrm{r}}(h) \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int \overline{b}(\mathrm{e}^{tx} k^{-1}) a(k^{-1} g)\, \mathrm{d}\mu_{\mathrm{r}}(k\mathrm{e}^{\cdot tx}) \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int \overline{b}(\mathrm{e}^{tx} k^{-1}) a(k^{-1} g)\, \mathrm{d}\mu_{\mathrm{r}}(k) \\
&= \int (\overline{\widetilde{x}b})(k^{-1}) a(k^{-1} g)\, \mathrm{d}\mu_{\mathrm{r}}(k) \\
&= \int m(k^{-1}(\overline{\widetilde{x}b})(k^{-1} a(k^{-1} g)\, \mathrm{d}\mu_{\mathrm{l}}(k) \\
&= \int (x \triangleright b)^+(k) a(k^{-1} g)\, \mathrm{d}\mu_{\mathrm{l}}(k) = ((x \triangleright b)^+ \cdot a)(g)
\end{aligned}$$

for $x \in \mathfrak{g}$, $a, b \in C_0^\infty(G)$ and $g \in G$. This proves that condition (1.1) is satisfied. ∎

Next let us look at the corresponding $*$-representations. In order to do so, we need to recall some facts on representation theory of Lie groups and Lie algebras which can be found in [14], Chapter 10, or in [18]. Let U be a unitary representation of the Lie group G on a Hilbert space $\mathcal{H}$, that is, $g \to U(g)$ is a homomorphism of G into the group of unitaries of $\mathcal{H}$ such that the map $g \to U(g)\varphi$ of G into $\mathcal{H}$ is continuous for each vector $\varphi \in \mathcal{H}$. Then there exists a unique $*$-representation $\mathrm{d}U$ of the $*$-algebra $\mathcal{E}(\mathfrak{g})$ on the domain $\mathcal{D}^\infty(U)$ of C^∞-vectors for U. For $x \in \mathfrak{g}$, the operator $\mathrm{d}U(x)$ acts as

$$\mathrm{d}U(x)\varphi = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} U(\mathrm{e}^{tx})\varphi, \quad \varphi \in \mathcal{D}^\infty(U). \tag{3.7}$$

The linear span $\mathcal{D}_G(U)$ of vectors

$$U_a\varphi := \int a(g) U(g)\varphi\, \mathrm{d}\mu_{\mathrm{l}}(g), \quad a \in C_0^\infty(G),\, \varphi \in \mathcal{H},$$

is contained in the space $\mathcal{D}^\infty(U)$ of C^∞-vectors. The vector space $\mathcal{D}_G(U)$ is called the *Gårding space* of the unitary representation U. It was proved in [1] that the Gårding space $\mathcal{D}_G(U)$ is equal to $\mathcal{D}^\infty(U)$, but we shall not need this deep result here. For our purposes it is sufficient to know (see [14], Corollary 10.1.16) that the Gårding space is a core for all operators $\mathrm{d}U(x)$, $x\in\mathcal{E}(\mathfrak{g})$. This implies that the $*$-representation $\mathrm{d}U$ on the domain $\mathcal{D}^\infty(U)$ is the closure of its restriction to $\mathcal{D}_G(U)$.

Suppose now that ρ is a non-degenerate $\|\cdot\|$-continuous $*$-representation of the $*$-algebra $\mathcal{B}$ on the Hilbert space $\mathcal{H}$. Note that all operators $\rho(x), x\in\mathcal{B}$, are bounded and defined on the whole Hilbert space $\mathcal{H}$. Since $\mathcal{B}$ is $\|\cdot\|$-dense in $L^1(G)$ and ρ is $\|\cdot\|$-continuous, ρ extends by continuity to a non-degenerate $*$-representation, denoted again by ρ, of the Banach $*$-algebra $L^1(G)$. It is well-known ([7], Section 29, Theorem 1) that there exists a unique unitary representaiton U of the Lie group G on the Hilbert space $\mathcal{H}$ such that

$$U_a=\rho(a),\quad a\in L^1(G). \tag{3.8}$$

Using formulas (3.7), (3.5) and (3.8), we obtain that

$$\rho'(x)U_a=\rho'(x)\rho(a)=\rho(x\triangleright a)=\rho(\tilde{x}a)=U_{\tilde{x}a}=\mathrm{d}U(x)U_a \tag{3.9}$$

for $x\in\mathcal{E}(\mathfrak{g})$ and $a\in C_0^\infty(G)$, where used the relation $\mathrm{d}U(x)U_a=U_{\tilde{x}a}$ ([14], Lemma 10.1.12). Therefore, the $*$-representations ρ' and $\mathrm{d}U$ of $\mathcal{X}=\mathcal{E}(\mathfrak{g})$ coincide on the Gårding space. Since ρ' and $\mathrm{d}U$ are both the closures of their restriction to $\rho(\mathcal{B})\mathcal{H}=\mathcal{D}_G(U)$, we conclude that $\rho'=\mathrm{d}U$.

Conversely, for any unitary representation U there exists a unique non-degenerate $*$-representation ρ of $L^1(G)$ and so of $\mathcal{B}=C_0^\infty(G)$ such that (3.8) holds. By the above reasoning, we then have $\rho'=\mathrm{d}U$.

Summarizing, we have shown that the $*$-representations of the $*$-algebra $\mathcal{X}$ derived from the pair $(\mathcal{X},\mathcal{B})$ are precisely the G-integrable representations of the $*$-algebra $\mathcal{E}(\mathfrak{g})$ (in the sense of [14], Chapter 10). That is, the $*$-representations ρ' are the $*$-representations $\mathrm{d}U$ for unitary representations U of the Lie group G.

4. EXAMPLES RELATED TO THE WEYL CALCULUS

The Weyl calculus of pseudodifferential operators on $\mathbb{R}^n$ can be used to construct further examples of compatible pairs. We restrict ourselves to the case $n=1$ and refer to the books [3] and [17] (see also [4]) for the notation and the facts on the Weyl calculus needed in what follows.

Let $\mathcal{P}$ and $\mathcal{Q}$ be the self-adjoint operators and let $W(s,t)$ be the unitary operator on the Hilbert space $L^2(\mathbb{R})$ defined by

$$(\mathcal{P}f)(x)=\frac{1}{2\pi\mathrm{i}}f'(x),\quad (\mathcal{Q}f)(x)=xf(x),\quad W(s,t)=\mathrm{e}^{2\pi\mathrm{i}(s\mathcal{Q}+t\mathcal{P})},\quad s,t\in\mathbb{R}.$$

To any measurable function a on $\mathbb{R}^2$ such that its Fourier transform

$$\widehat{a}(x,y)=\iint \mathrm{e}^{-2\pi\mathrm{i}(xs+yt)}a(s,t)\,\mathrm{d}s\,\mathrm{d}t \tag{4.1}$$

is in $L^1(\mathbb{R}^2)$, the Weyl calculus assigns an operator $\mathrm{Op}(a)$ on the Hilbert space $L^2(\mathbb{R})$ by

$$\mathrm{Op}(a) = \iint \widehat{a}(s,t) W(s,t)\, \mathrm{d}s\, \mathrm{d}t. \tag{4.2}$$

The integral is defined as a Bochner integral because $\widehat{a} \in L^1(\mathbb{R}^2)$. The adjoint operator $\mathrm{Op}(a)^*$ and the operator product $\mathrm{Op}(a)\mathrm{Op}(b)$ (at least for "nice" symbols a and b) are given by

$$\mathrm{Op}(a)^* = \mathrm{Op}(a^+) \quad \text{and} \quad \mathrm{Op}(a)\mathrm{Op}(b) = \mathrm{Op}(a\#b), \tag{4.3}$$

where

$$a^+(x,y) := \overline{a(x,y)}, \tag{4.4}$$

$$\begin{aligned}(a\#b)&(x_1,x_2)\\ &:= \iiiint a(u_1,u_2) b(v_1,v_2) \mathrm{e}^{4\pi \mathrm{i}[(x_1-u_1)(x_2-v_2)-(x_1-v_1)(x_2-u_2)]}\, \mathrm{d}u_1\, \mathrm{d}u_2\, \mathrm{d}v_1\, \mathrm{d}v_2.\end{aligned} \tag{4.5}$$

Because of the formulas (4.3) it is natural to expect that a vector space $\mathcal{B}$ of sufficiently nice functions becomes a $*$-algebra with involution (4.4) and product (4.5) provided that $a^+ \in \mathcal{B}$ and $a\#b \in \mathcal{B}$ when $a,b \in \mathcal{B}$. An example of such a $*$-algebra is the Schwartz space $\mathcal{S}(\mathbb{R}^2)$. Another example is obtained as follows: Let $\mathcal{A}(\mathbb{R}^2)$ denote the vector space of all holomorphic functions f on $\mathbb{C}^2$ such that for all $s_j, c_j, d_j \in \mathbb{R}$, $c_j < d_j$, $j = 1,2$, we have

$$\sup \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| f(x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2) \right|^2 \mathrm{e}^{s_1 x_1 + s_2 x_2}\, \mathrm{d}x_1\, \mathrm{d}x_2 < \infty,$$

where the supremum is taken over the set $\{(y_1,y_2) \in \mathbb{R}^2 : c_j < y_j < d_j, j = 1,2\}$. Then $\mathcal{B}_1 := \mathcal{S}(\mathbb{R}^2)$ and $\mathcal{B}_2 := \mathcal{A}(\mathbb{R}^2)$ are both $*$-algebras with product and involution defined by (4.5) and (4.4), respectively. This was shown in [4], Proposition 1, for $\mathcal{B}_1 := \mathcal{S}(\mathbb{R}^2)$ and in [16], Lemma 1.11, for $B_2 = \mathcal{A}(\mathbb{R}^2)$.

For $a \in \mathcal{B}_j$, $j = 1,2$, the operator $\mathrm{Op}(a)$ is bounded. From the formulas (4.3) it follows that the norm $\|\cdot\|$ on $\mathcal{B}_j$ defined by

$$\|a\| := \|\mathrm{Op}(a)\|$$

is a submultiplicative and $*$-invariant. Therefore, $\mathcal{B}_1$ and $\mathcal{B}_2$ are normed $*$-algebras.

Let $\mathcal{X}_1$ be the unital algebra with two generators p and x and defining relation

$$\mathsf{p}\mathrm{x} - \mathrm{x}\mathsf{p} = -\mathrm{i} \cdot 1.$$

Clearly, $\mathcal{X}_1$ is a $*$-algebra with involution determined by $\mathsf{p}^+ = \mathsf{p}$ and $\mathrm{x}^+ = \mathrm{x}$. One easily checks that there is a left action of the algebra $\mathcal{X}_1$ on $\mathcal{B}_1 = \mathcal{S}(\mathbb{R}^2)$ such that

$$\mathsf{p} \triangleright a := \left(\frac{1}{2\mathrm{i}} \frac{\partial}{\partial x_1} + 2\pi x_2 \right) a, \quad \mathrm{x} \triangleright a := \left(x_1 - \frac{1}{4\pi \mathrm{i}} \frac{\partial}{\partial x_2} \right) a. \tag{4.6}$$

In terms of the operators $\mathcal{P}_j = \frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial x_j}$ and $\mathcal{Q}_j = x_j$ on the Hilbert space $L^2(\mathbb{R}^2)$ the latter can be rewritten as

$$\mathsf{p} = \pi\mathcal{P}_1 + 2\pi\mathcal{Q}_2, \quad \mathrm{x} = \mathcal{Q}_1 - \frac{1}{2}\mathcal{P}_2. \tag{4.7}$$

Suppose that q is a complex number of modulus one. Let $\mathcal{X}_2$ be the coordinate $*$-algebra $\mathcal{O}(\mathbb{R}^2_q)$ of the real quantum plane. It is defined as follows (see, for instance, [2] or [6]). As an algebra, $\mathcal{O}(\mathbb{R}^2_q)$ has two generators x and y with defining relation

$$\mathrm{xy} = q\mathrm{yx}.$$

The involution is defined by the requirements $\mathrm{x}^+ = \mathrm{x}$ and $\mathrm{y}^+ = \mathrm{y}$.

We write $q = \mathrm{e}^{2\pi\mathrm{i}\gamma}$ with a *fixed* real number γ and we take two real numbers α and β such that $\alpha\beta = \gamma$. Then we define a left action of the algebra $\mathcal{X}_2$ on $\mathcal{B}_2 = \mathcal{A}(\mathbb{R}^2)$ by

$$(\mathrm{x}\triangleright a)(x_1,x_2) = \mathrm{e}^{2\pi\alpha x_1} a\Big(x_1, x_2 + \mathrm{i}\frac{\alpha}{2}\Big), \quad (\mathrm{y}\triangleright a)(x_1,x_2) = \mathrm{e}^{2\pi\beta x_2} a\Big(x_1 - \mathrm{i}\frac{\beta}{2}, x_2\Big). \tag{4.8}$$

Since $x \triangleright (y \triangleright a) = qy \triangleright (x \triangleright a)$ as easily verified, the latter gives a well-defined left action of $\mathcal{X}_2$ on $\mathcal{B}_2$. The operator $\mathrm{e}^{2\pi c\mathcal{P}}$, $c \in \mathbb{R}$, acts on functions of its domain as

$$\left(\mathrm{e}^{2\pi c\mathcal{P}} f\right)(x) = f(x - c\mathrm{i});$$

see Lemma 1.1 in [15] for a precise statement. Using this fact, formula (4.8) means that

$$\mathrm{x} = \mathrm{e}^{2\pi\alpha\mathcal{Q}_1} \otimes \mathrm{e}^{-\pi\alpha\mathcal{P}_2}, \quad \mathrm{y} = \mathrm{e}^{\pi\beta\mathcal{P}_1} \otimes \mathrm{e}^{2\pi\beta\mathcal{Q}_2}. \tag{4.9}$$

LEMMA 4.1. *$(\mathcal{X}_1, \mathcal{B}_1)$ and $(\mathcal{X}_2, \mathcal{B}_2)$ are compatible pairs.*

Proof. It suffices to check (1.1) for the generators p, x and x, y, respectively. By (4.7), for the generators p and x condition (1.1) means that

$$(\mathcal{P}_1 a + 2\mathcal{Q}_2 a)^+ \# b = a^+ \# (\mathcal{P}_1 b + 2\mathcal{Q}_2 b), \quad (2\mathcal{Q}_1 a - \mathcal{P}_2 a)^+ \# b = a^+ \# (2\mathcal{Q}_1 b - \mathcal{P}_2 b).$$

Both relations are easily derived from the definition (4.5) of the twisted product # using partial integration. Note that the corresponding boundary terms vanish because the functions a and b are in $\mathcal{S}(\mathbb{R}^2)$. We omit the details. For the generators x and y of $\mathcal{X}_2$ formula (1.1) says that

$$(\mathrm{e}^{2\pi\alpha\mathcal{Q}_1}\mathrm{e}^{-\pi\alpha\mathcal{P}_2} a)^+ \# b = a^+ \# (\mathrm{e}^{2\pi\alpha\mathcal{Q}_1}\mathrm{e}^{-\pi\alpha\mathcal{P}_2} b),$$
$$(\mathrm{e}^{\pi\beta/2\mathcal{P}_1}\mathrm{e}^{2\pi\beta\mathcal{Q}_2} a)^+ \# b = a^+ \# (\mathrm{e}^{\mathsf{p}\beta\mathcal{P}_1}\mathrm{e}^{2\pi\beta\mathcal{Q}_2} b).$$

Both identities follow at once from the formulas stated in [16], Lemma 11, combined with the fact that $(\mathrm{e}^{c\mathcal{P}_j} a)^+ = \mathrm{e}^{-c\mathcal{P}_j} a^+$ for $c \in \mathbb{R}$ and $a \in \mathcal{A}(\mathbb{R}^2)$. ∎

Now we turn to $*$-representations. If $\mathcal{K}$ is a Hilbert space, then it obvious from (4.3) that the formula

$$\rho_0(a) = \mathrm{Op}(a) \otimes I, \quad a \in \mathcal{B}_j, \tag{4.10}$$

defines a continuous $*$-representation of the normed $*$-algebra $\mathcal{B}_j$ on the Hilbert space $L^2(\mathbb{R}) \otimes \mathcal{K}$.

Lemma 4.2. *Any continuous $*$-representation of the normed $*$-algebra $\mathcal{B}_j$, $j=1,2$, is unitarily equivalent to a $*$-representation ρ_0.*

This assertion is probably known, but we could not find it explicitely in the literature. Thus we include a sketch of proof.

Sketch of proof. Suppose that ρ is a continuous $*$-representation of the normed $*$-algebra $\mathcal{B}_j$ on a Hilbert space $\mathcal{G}$. Since ρ is a direct sum of cyclic $*$-representation, we can assume without loss of generality that ρ is cyclic. Let $\varphi \in \mathcal{G}$ be a cyclic vector for ρ. For $a \in \mathcal{B}_j$ and $s,t \in \mathbb{R}$, we set

$$a_{s,0}(x_1,x_2) := \mathrm{e}^{2\pi \mathrm{i} s x_1} a\Big(x_1, x_2 - \frac{s}{2}\Big), \quad a_{0,t}(x_1,x_2) := \mathrm{e}^{2\pi t x_2} a\Big(x_1 + \frac{t}{2}, x_2\Big). \tag{4.11}$$

Since $a^+_{s,0} \# a_{s,0} = a^+ \# a$ as easily computed, we have

$$\begin{aligned}\|\rho(a_{s,0})\varphi\|^2 &= \langle \rho(a_{s,0})\varphi, \rho(a_{s,0})\varphi\rangle = \langle \rho(a^+_{s,0}\# a_{s,0})\varphi, \varphi\rangle\\ &= \langle \rho(a^+\# a)\varphi,\varphi\rangle = \langle \rho(a)\varphi, \rho(a)\varphi\rangle = \|\rho(a)\varphi\|^2.\end{aligned}$$

Hence there is an isometric map $U(s)$ of $\rho(\mathcal{B}_j)\varphi$ onto itself such that $U(s)\rho(a)\varphi = \rho(a_{s,0})\varphi, a \in \mathcal{B}_j$. Obviously, $U(s_1+s_2) = U(s_1)U(s_2)$ for $s_1, s_2 \in \mathbb{R}$ and $U(0) = I$. Moreover, $\|a_{s,0} - a\|_0 \to 0$ as $s \to 0$ and hence $U(s)\psi \to \psi$ in $\mathcal{G}$ for $\psi \in \rho(\mathcal{B}_j)\varphi$ as $s \to 0$ by the continuity of ρ. Since $\rho(\mathcal{B}_j)\varphi$ is dense in $\mathcal{G}$, $U(s)$ extends to a unitary operator on $\mathcal{G}$ and $s \to U(s)$ is a strongly continuous one-parameter unitary group on $\mathcal{G}$. Similarly, there is another strongly continuous one-parameter unitary group $t \to V(t)$ on $\mathcal{G}$ such that $V(t)\rho(a)\varphi = \rho(a_{0,t})\varphi$. From their definitions we immediately derive that the unitary groups U and V satisfy the Weyl relation

$$V(t)U(s) = \mathrm{e}^{2\pi \mathrm{i} st} U(s)V(t), \quad s,t \in \mathbb{R}.$$

Therefore, by the Stone-von Neumann uniqueness theorem for the canonical commutation relation (see e.g. [10]), there exists a Hilbert space $\mathcal{K}$ and a unitary map T of $\mathcal{G}$ onto $L^2(\mathbb{R}) \otimes \mathcal{K}$ such that

$$T^{-1}U(s)T = W(s,0) \otimes I, \quad T^{-1}V(t)T = W(0,t) \otimes I \quad \text{for } s,t \in \mathbb{R}. \tag{4.12}$$

Let us abbreviate $\widetilde{W}(s,t) := \mathrm{e}^{-\pi \mathrm{i} st} U(s)V(t)$ and $\widetilde{\rho}(a) := \iint \widehat{a}(s,t)\widetilde{W}(s,t)\,\mathrm{d}s\,\mathrm{d}t$, where $a \in \mathcal{B}_j$. Since

$$W(s,t) = \mathrm{e}^{\pi \mathrm{i} st} W(s,0)W(0,t), \quad s,t \in \mathbb{R}, \tag{4.13}$$

it follows from (4.12) that $T^{-1}\widetilde{W}(s,t)T = W(s,t) \otimes I$ and hence

$$T^{-1}\widetilde{\rho}(a)T = \mathrm{Op}(a) \otimes I = \rho_0(a), \quad a \in \mathcal{B}_j. \tag{4.14}$$

On the other hand, from the definition of $\mathrm{Op}(a)$ one derives that

$$W(s,0)\mathrm{Op}(a) = \mathrm{Op}(a_{s,0}) \quad \text{and} \quad W(0,t)\mathrm{Op}(a) = \mathrm{Op}(a_{0,t}).$$

By the defnition of $\widetilde{W}(s,0) = U(s)$ and $\widetilde{W}(0,t) = V(t)$, the latter implies that

$$\widetilde{W}(s,t)\rho(b)\varphi = \rho(\mathrm{Op}^{-1}(W(s,t)\mathrm{Op}(b)))\varphi$$

and so

$$\begin{aligned}\widetilde{\rho}(a)\rho(b)\varphi &= \rho\bigg(\mathrm{Op}^{-1}\bigg(\bigg(\iint \widehat{a}(s,t)W(s,t)\,\mathrm{d}s\,\mathrm{d}t\bigg)\mathrm{Op}(b)\bigg)\bigg)\varphi\\ &= \rho(\mathrm{Op}^{-1}(\mathrm{Op}(a)\mathrm{Op}(b)))\varphi = \rho(\mathrm{Op}^{-1}(\mathrm{Op}(a\# b)))\varphi = \rho(a)\rho(b)\varphi\end{aligned}$$

for $a,b \in \mathcal{B}_j$. Since $\rho(\mathcal{B}_j)\varphi$ is dense in $\mathcal{G}$, we obtain $\widetilde{\rho}(a) = \rho(a)$. Thus, by (4.14) we have $T^{-1}\rho(a)T = \rho_0(a)$ which completes the proof of Lemma 4.2. ∎

Finally, let us describe the $*$-representation ρ_0' of the $*$-algebra $\mathcal{X}_j$ derived from the $*$-representation ρ_0 of $\mathcal{B}_j$. First let $j=1$. Since $W(s,0)=\mathrm{e}^{2\pi \mathrm{i} s\mathcal{Q}}$ and $W(0,t)=\mathrm{e}^{2\pi \mathrm{i} t\mathcal{P}}$, it follows from (4.11) and (4.13) by differentation at $s=0$ and $t=0$, respectively, that

$$\mathcal{Q}\mathrm{Op}(a)=\mathrm{Op}\Big(x_1a-\frac{1}{4\pi\mathrm{i}}\frac{\partial a}{\partial x_2}\Big),\quad \mathcal{P}\mathrm{Op}(a)=\mathrm{Op}\Big(\frac{1}{2\mathrm{i}}\frac{\partial a}{\partial x_1}+2\pi x_2 a\Big).$$

Combining the latter with (4.6) we conclude that

$$\rho_0'(\mathrm{x})\rho_0(a)=\rho_0(\mathrm{x}\triangleright a)=\mathrm{Op}(\mathrm{x}\triangleright a)\otimes I=\mathcal{Q}\mathrm{Op}(a)\otimes I=(\mathcal{Q}\otimes I)\rho_0(a)$$

and similarly $\rho_0'(\mathsf{p})\rho_0(a)=(\mathcal{P}\otimes I)\rho_0(a)$ for $a\in\mathcal{B}_1$, so that

$$\overline{\rho_0'(\mathrm{x})}=\mathcal{Q}\otimes I\quad\text{and}\quad \overline{\rho_0'(\mathsf{p})}=\mathcal{P}\otimes I.$$

That is, up to unitary equivalence the $*$-representations ρ_0' of $\mathcal{X}_1$ are precisely the orthogonal direct sums of the Schrödinger representation of the $*$-algebra $\mathcal{X}_1$ with domain $\mathcal{S}(\mathbb{R})$ on the Hilbert space $L^2(\mathbb{R})$.

Now let $j=2$. From the formulas in [16], Lemma 1.9, we then have

$$\mathrm{e}^{2\pi\alpha\mathcal{Q}}\mathrm{Op}(a)=\mathrm{Op}(\mathrm{x}\triangleright a),\quad \mathrm{e}^{2\pi\beta\mathcal{P}}\mathrm{Op}(a)=\mathrm{Op}(\mathrm{y}\triangleright a), \tag{4.15}$$

where $x\triangleright a$ and $y\triangleright a$ are defined by (4.8). From (4.15) and (1.2) we obtain

$$\overline{\rho_0'(\mathrm{x})}=\mathrm{e}^{2\pi\alpha\mathcal{Q}}\otimes I,\quad \overline{\rho_0'(\mathrm{y})}=\mathrm{e}^{2\pi\beta\mathcal{P}}\otimes I.$$

This $*$-representation ρ_0 appears (in a slighty different notation) in the work by M. Rieffel ([12]) on the quantum plane.

We illustrate the preceding considerations by three other closely related examples. Let $\mathcal{B}_3$ be the normed $*$-algebra $\mathcal{B}_2\oplus\mathcal{B}_2\oplus\mathcal{B}_2\oplus\mathcal{B}_2$. There is a left action of $\mathcal{X}_2$ on $\mathcal{B}_3$ such that

$$\begin{aligned}\mathrm{x}\triangleright(a_1,a_2,a_3,a_4)&=(\mathrm{x}\triangleright a_1,\mathrm{x}\triangleright a_2,-\mathrm{x}\triangleright a_3,-\mathrm{x}\triangleright a_4),\\ \mathrm{y}\triangleright(a_1,a_2,a_3,a_4)&=(\mathrm{y}\triangleright a_1,-\mathrm{y}\triangleright a_2,\mathrm{y}\triangleright a_3,-\mathrm{y}\triangleright a_4)\end{aligned}$$

for $a_1,a_2,a_3,a_4\in\mathcal{B}_2$, where $\mathrm{y}\triangleright a$ and $\mathrm{y}\triangleright a$ are defined by (4.9). Obviously, $(\mathcal{X}_2,\mathcal{B}_3)$ is a compatible pair.

For arbitrary $\varepsilon_1,\varepsilon_2\in\{+,-\}$, there is a $*$-representation $\rho'_{\varepsilon_1\varepsilon_2}$ of $\mathcal{X}_2$ derived from a continuous $*$-representation $\rho_{\varepsilon_1\varepsilon_2}$ of $\mathcal{B}_3$ such that

$$\overline{\rho'_{\varepsilon_1\varepsilon_2}(\mathrm{x})}=\varepsilon_1\mathrm{e}^{2\pi\alpha\mathcal{Q}}\otimes I,\quad \overline{\rho'_{\varepsilon_1\varepsilon_2}(\mathrm{y})}=\varepsilon_2\mathrm{e}^{2\pi\beta\mathcal{P}}\otimes I.$$

It is easily seen that the $*$-representation ρ' of $\mathcal{X}_2$ associated with the pair $(\mathcal{X}_2,\mathcal{B}_3)$ are precisely the orthogonal direct sums of some $*$-representations ρ'_{++},ρ'_{+-}, ρ'_{-+},ρ'_{--}.

Finally, let $\mathcal{B}_4$ the $*$-algebra $\mathcal{B}_2\otimes M_2(\mathbb{C})\cong M_2(\mathcal{B}_2)$ equipped with the C^*-matrix norm derived from the C^*-norm of $\mathcal{B}_2$. We suppose now that α and β are real numbers such that

$$\alpha\beta=\gamma+\frac{1}{2}. \tag{4.16}$$

Then there exists a left action of the algebra $\mathcal{X}_2$ on $\mathcal{B}_4$ given by

$$\mathrm{x}\triangleright\begin{pmatrix}a_1&a_2\\a_3&a_4\end{pmatrix}=\begin{pmatrix}\mathrm{x}\triangleright a_1&\mathrm{x}\triangleright a_2\\-\mathrm{x}\triangleright a_3&-\mathrm{x}\triangleright a_4\end{pmatrix},\quad \mathrm{y}\triangleright\begin{pmatrix}a_1&a_2\\a_3&a_4\end{pmatrix}=\begin{pmatrix}\mathrm{y}\triangleright a_3&\mathrm{y}\triangleright a_4\\\mathrm{y}\triangleright a_1&\mathrm{y}\triangleright a_2\end{pmatrix},$$

where

$$\mathrm{x} \triangleright a := \mathrm{e}^{2\pi\alpha\mathcal{Q}_1} \otimes \mathrm{e}^{-\pi\alpha\mathcal{P}_2} a, \quad \mathrm{y} \triangleright \mathrm{e}^{\pi\beta\mathcal{P}_1} \otimes \mathrm{e}^{2\pi\beta\mathcal{Q}_2} a.$$

Note that the latter formula coincides with (4.9), but now we have assumed $\alpha\beta = \gamma + \frac{1}{2}$ rather than $\alpha\beta = \gamma$. Using the fact that $(\mathcal{X}_2, \mathcal{B}_2)$ is a compatible pair, a straightforward computation shows that $(\mathcal{X}_2, \mathcal{B}_4)$ is also a compatible pair.

For a Hilbert space $\mathcal{G}$, there is a continuous $*$-representation ρ_0 of $\mathcal{B}_4$ on the Hilbert space $L^2(\mathbb{R}) \otimes \mathbb{C}^2 \otimes \mathcal{G}$ such that

$$\rho_0(a \otimes m) = \mathrm{Op}(a) \otimes m \otimes I, \quad B_2, m \in M_2(\mathbb{C}).$$

From Lemma 4.2 it follows immediately that any continuous $*$-representation of $\mathcal{B}_4$ is unitarily equivalent to such a $*$-representation ρ_0. Using the facts that $\mathrm{Op}(\mathrm{x} \triangleright a) = \mathrm{e}^{2\pi\alpha\mathcal{Q}}\mathrm{Op}(a)$, $\mathrm{Op}(\mathrm{y} \triangleright a) = \mathrm{e}^{2\pi\beta\mathcal{P}}\mathrm{Op}(a)$, we derive that the closures of the operators $\rho_0'(\mathrm{x})$ and $\rho_0'(\mathrm{y})$ are the self-adjoint operator matrices

$$\overline{\rho_0'(\mathrm{x})} = \begin{pmatrix} \mathrm{e}^{2\pi\alpha\mathcal{Q}} \otimes I & 0 \\ 0 & -\mathrm{e}^{2\pi\alpha\mathcal{Q}} \otimes I \end{pmatrix}, \quad \overline{\rho_0'(\mathrm{y})} = \begin{pmatrix} 0 & \mathrm{e}^{2\pi\beta\mathcal{P}} \otimes I \\ \mathrm{e}^{2\pi\beta\mathcal{P}} \otimes I & 0 \end{pmatrix}.$$

Finally, we consider a $*$-algebra and their representations in the above context which was used by S.L. Woronowicz in his approach to the quantum $ax+b$-group ([20]). Let $\mathcal{X}_3$ denote the $*$-algebra generated by three hermitean elements x, y, χ and defining relations

$$xy = qyx, \quad x\chi = \chi x, \quad y\chi = -\chi y, \quad \chi^2 = 1.$$

Then there is a left action of $\mathcal{X}_3$ on the $*$-algebra $\mathcal{B}_4 = \mathcal{B}_2 \otimes M_2(\mathbb{C})$ defined by

$$\begin{aligned} x \triangleright \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} &= \begin{pmatrix} x \triangleright a_1 & x \triangleright a_2 \\ x \triangleright a_3 & x \triangleright a_4 \end{pmatrix}, \\ y \triangleright \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} &= \begin{pmatrix} y \triangleright a_1 & y \triangleright a_2 \\ -y \triangleright a_3 & -y \triangleright a_4 \end{pmatrix}, \\ \chi \triangleright \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} &= \begin{pmatrix} a_3 & a_4 \\ a_1 & a_2 \end{pmatrix}, \end{aligned}$$

where $x \triangleright a$ and $y \triangleright a$ are given by (4.9). It is not difficult to check that $(\mathcal{X}_3, \mathcal{B}_4)$ is a compatible pair and that for the corresponding $*$-representation ρ_0' of $\mathcal{X}_3$ we have

$$\overline{\rho_0'(x)} = \begin{pmatrix} \mathrm{e}^{2\pi\alpha\mathcal{Q}} \otimes I & 0 \\ 0 & \mathrm{e}^{2\pi\alpha\mathcal{Q}} \otimes I \end{pmatrix}, \quad \overline{\rho_0'(y)} = \begin{pmatrix} \mathrm{e}^{2\pi\beta\mathcal{P}} \otimes I & 0 \\ 0 & -\mathrm{e}^{2\pi\beta\mathcal{P}} \otimes I \end{pmatrix},$$

where $\alpha\beta = \gamma$.

5. ANOTHER EXAMPLE: QUANTUM $SU_q(1,1)$ GROUP

Suppose that q is a real number such that $q > 0$, $q \neq 1$. Let $\mathcal{X}$ denote the coordinate $*$-algebra $\mathcal{O}(SU_q(1,1))$ of the quantum group $SU_q(1,1)$ (see, for instance, [2], [6] or [19]). That is, $\mathcal{X}$ is the $*$-algebra with two generators a and c and defining relations

$$ac = qca, \quad ac^+ = qc^+a, \quad cc^+ = c^+c, \quad a^+a - c^+c = 1, \quad aa^+ - q^2c^+c = 1.$$

Let $\mathcal{B}$ be the $*$-algebra generated by the $*$-algebra $C_0(\mathbb{C})$ of compactly supported continuous functions on the complex plane $\mathbb{C}$ with usual algebraic structure and a single generator v satisfying the relations $vf(z) = f(qz)v$, $f \in C_0(\mathbb{C}), z \in \mathbb{C}$, and $vv^+ = v^+v = 1$. We consider $\mathcal{B}$ as $*$-subalgebra of bounded operators acting on the Hilbert space $L^2(\mathbb{C})$ equipped with the operator norm.

Then there is a left action of $\mathcal{X}$ on $\mathcal{B}$ defined by

$$\begin{aligned} &a \triangleright v^k f(z) = v^{k+1}\sqrt{1 + q^{-2k}|z|^2}\, f(z), \\ &c \triangleright v^k f(z) = q^{-2k} v^k z f(z), \qquad\qquad f \in C_0(\mathbb{C}),\ k \in \mathbb{Z}. \end{aligned}$$

One easily checks that $(\mathcal{X}, \mathcal{B})$ is a compatible pair.

Let ρ be a non-degenerate continuous $*$-representation of $\mathcal{B}$ on a Hilbert space $\mathcal{H}$. It is not difficult to show that there exists a spectral measure $E(z), z \in \mathbb{C}$, and a unitary operator u on $\mathcal{H}$ satisfying $uE(z)u^{-1} = E(qz)$, $z \in \mathbb{C}$, such that

$$\rho(v^k f) = u^k \int f(z)\, \mathrm{d}E(z), \quad f \in C_0(\mathbb{C}),\ k \in \mathbb{Z}.$$

The operators $\overline{\rho'(a)}$ and $\overline{\rho'(c)}$ of the associated $*$-representation ρ' of $\mathcal{X}$ act as

$$\rho'(a) = u\sqrt{1 + (z)^2} \quad \text{and} \quad \rho'(c) = z.$$

These are precisely the $*$-representations of the $*$-algebra $\mathcal{X} = \mathcal{O}(SU_q(1,1))$ considered in [19], p. 79.

6. OUTLOOK

In the preceding sections we have investigated a variety of examples of compatible pairs $(\mathcal{X}, \mathcal{B})$ and the corresponding well-behaved $*$-representations. Here we add some remarks on these concepts and on possible modifications.

1. First, let us emphasize that the above approach does not solve the problem of *selecting* the well-behaved $*$-representations of a given $*$-algebra. As mentioned in the introduction, this essentially depends on the specific structure of the $*$-algebra and on the aim of the considerations. The notion of a compatible pair is only a proposal of a general concept in order to treat appearently different examples in the same general context.

2. For a compatible pair $(\mathcal{X}, \mathcal{B})$, let τ denote the locally convex topology on $\mathcal{B}$ defined by the family of seminous $p_x(b) = \|x \triangleright b\|$, $x \in \mathcal{X}, b \in \mathcal{B}$. In all above examples, $\mathcal{B}[\tau]$ is a metrizable $*$-algebra with jointly continuous multiplication. Therefore, the completion $\widetilde{\mathcal{B}}$ of $\mathcal{B}[\tau]$ is a Frechet $*$-algebra and the action of $\mathcal{X}$

on $\mathcal{B}$ extends by continuity to an action on $\widetilde{\mathcal{B}}$ which also satisfies the compatibility condition (1.1). Thus, another possible approach might be to replace the (uncomplete) normed $*$-algebra $(\mathcal{B}, \|\cdot\|)$ by the Frechet $*$-algebra $\widetilde{\mathcal{B}}[\tau]$.

3. The reason why we have defined compatible pairs by means of condition (1.1) is that it seems to be the *weakest* requirement on the left action $\triangleright$ which ensures that a non-degenerate $*$-representation of $\mathcal{B}$ yields a well-defined $*$-representation of $\mathcal{X}$ by means of formula (1.2). However, for compatible O^*-pairs (Example 1.3) and for all compatible pairs occuring in this paper we have much more structure: In all theses cases the $*$-algebras $\mathcal{X}$ and $\mathcal{B}$ are $*$-subalgebras of a larger $*$-algebra $\mathcal{A}$ and the left action $x \triangleright b$ is just the product $x \cdot b$ of $x \in \mathcal{X}$ and $b \in \mathcal{B}$ in the algebra $\mathcal{A}$. It is obvious that the $*$-algebra axioms imply that condition (1.1) is fulfilled.

4. Let $\mathcal{X}$ be a $*$-algebra with unit and let $\triangleright$ be a left action of $\mathcal{X}$ on another $*$-algebra $\mathcal{B}$. On the direct sum $\mathcal{A} := \mathcal{X} \oplus \mathcal{B}$ of vector spaces $\mathcal{X}$ and $\mathcal{B}$ we define a product

$$(x + a)(y + b) := xy + (y^* \triangleright a^*)^* + x \triangleright b + ab, \quad x, y \in \mathcal{X},\ a, b \in \mathcal{B},$$

and an involution $(x + a)^* := x^* + a^*$. With these structures, $\mathcal{A}$ is a $*$-algebra if and only if conditions (1.1) and

$$(x \triangleright a)b = x \triangleright (ab) \quad \text{and} \quad (x \triangleright (y \triangleright a)^*)^* = y \triangleright (x \triangleright a^*)^*, \quad x, y \in \mathcal{X},\ a, b \in \mathcal{B},$$

are fulfilled. If this is true, then $\mathcal{X}$ and $\mathcal{B}$ are $*$-subalgebras of $\mathcal{A}$ and the left action $\triangleright$ of $\mathcal{X}$ on $\mathcal{B}$ is the product in the algebra $\mathcal{A}$. Such a $*$-algebra $\mathcal{A}$ has been used as "function algebra" on the quantum quarter plane in [16].

5. Let $(\mathcal{X}, \mathcal{B})$ be a compatible pair. The elements $x \in \mathcal{B}$ can be interpreted as multipliers of the algebra $\mathcal{B}$. The representation ρ' of $\mathcal{X}$ derived from the representation ρ of $\mathcal{B}$ (see Proposition 1.1) can be considered as the induced representation of ρ in the sense of M. Rieffel ([11]). We shall discuss these aspects elsewhere.

REFERENCES

1. J. Dixmier, P. Malliavan, Factorisations de functions et de vecteurs indefiniment differentiables, *Bull. Sci. Math.* **5**(1978), 305–330.
2. L.D. Faddeev, N.Yu. Reshetikhin, L.A. Takhtajan, Quantization of Lie groups and Lie algebras, *Leningrad Math. J.* **1**(1990), 193–225.
3. G.B. Folland, *Harmonic Analysis in Phase Space*, Princeton Univ. Press, Princeton 1989.
4. J.M. Gracia-Bondia, J.C. Varilly, Algebras of distributions for phase-space quantum mechanics. I, *J. Math. Phys.* **29**(1988), 869–879.
5. A. Inoue, *Tomita-Takesaki Theory in Algebras of Unbounded Operators*, Lecture Notes Math., vol. 1699, Springer-Verlag, 1998.
6. A.U. Klimyk, K. Schmüdgen, *Quantum Groups and their Representations*, Texts Monogr. Physics, Springer-Verlag, Berlin 1997.
7. M.A. Naimark, *Normed Rings*, P. Noordhoff, Groningen 1964.
8. V.L. Ostrowski, Yu.S. Samoilenko, Unbounded operators satisfying non-Lie commutation relations, *Rep. Math. Phys.* **28**(1989), 91–104.
9. R.T. Powers, Self-adjoint algebras of unbounded operators, *Comm. Math. Phys.* **21**(1971), 85–124.

10. C.R. PUTNAM, *Commutation Properties of Hilbert Space Operators*, Springer Verlag, Heidelberg 1967.
11. M. RIEFFEL, Induced representations of C^*-algebras, *Adv. Math.* **13**(1974), 176–257.
12. M. RIEFFEL, *Quantum Deformations for Actions of* $\mathbb{R}^d$, Mem. Amer. Math. Soc., vol. 506, Amer. Math. Soc., Providence, RI, 1993.
13. K. SCHMÜDGEN, On the Heisenberg commutation relation. II, *Publ. Res. Inst. Math. Sci.* **19**(1983), 601–671.
14. K. SCHMÜDGEN, *Unbounded Operator Algebras and Representation Theory*, Birkhäuser Verlag, Basel 1990.
15. K. SCHMÜDGEN, Integrable operator representations of $\mathbb{R}_q^2, X_{q,\gamma}$ and $SL_q(2,\mathbb{R})$, *Comm. Math. Phys.* **159**(1994), 217–237.
16. K. SCHMÜDGEN, On the quantum quarter plane and the real quantum plane, *Internat. J. Math.* **13**(2002), 279–321; see also mathQA/0005225.
17. E.M. STEIN, *Harmonic Analysis*, Princeton Univ. Press, Princeton 1993.
18. G. WARNER, *Harmonic Analysis on Semi-Simple Lie Groups*, Springer Verlag, Berlin 1972.
19. S.L. WORONOWICZ, Unbounded elements affiliated with C^*-algebras and noncompact quantum groups, *Comm. Math. Phys.* **136**(1991), 399–432.
20. S.L. WORONOWICZ, Quantum $ax + b$-group, preprint, Warsaw 1999.

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