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REGULARITY OF PROJECTIONS REVISITED

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Abstract. The concept of *regularity* in the meta-topological setting of projections in the double dual of a C^*-algebra addresses the interrelations of a projection p with its closure $\overline{p}$, for instance in the form that such projections act identically, in norm, on elements of the C^*-algebra. This concept has been given new actuality with the recent plan of Peligrad and Zsido to find a meaningful notion of Murray-von Neumann type equivalence among open projections.

Although automatic in the commutative case, it has been known since the late sixties that regularity fails for many projections. The original investigations, however, did not answer a question such as: *Are all open and dense projections regular in $\mathfrak{A}$, when $\mathfrak{A}$ is simple?* We report here that this and related questions have negative answers. In the other direction, we supply positive results on regularity of large open projections.

Keywords: *Regular projection, open projection, dense projection, enveloping von Neumann algebra, simple C^*-algebra, AF algebra.*

MSC (2000): Primary 46L05; Secondary 46L85.

1. INTRODUCTION

For a pair of elements x, b in a C^*-algebra $\mathfrak{A}$ with $b \geqslant 0$ and $\|b\| = 1$, consider the quantity

$$\|x\|_b = \sup_{n \geqslant 0} \|xb^{1/n}x^*\|^{1/2}.$$

It is easy to see (Lemma 2.4 below) that $\|\cdot\|_b$ is a seminorm, and that $\|\cdot\|_b \leqslant \|\cdot\|$, but clearly $\|\cdot\|_b$ will fail to be a norm if b is a zero-divisor. In fact, for many C^*-algebras $\mathfrak{A}$, $\|\cdot\|_b$ coincides with $\|\cdot\|$ for all positive elements b of $\mathfrak{A}$ that have norm one and are not zero-divisors. This is easily seen when $\mathfrak{A}$ is Abelian, and we observe below (Proposition 2.5) that it also is the case when $\mathfrak{A}$ is of the form $C(X, \mathbf{M}_n)$.

Prompted by a question of Peligrad and Zsido ([19]) we answer here, in the negative, the question of whether $\|\cdot\|_b$ is a norm for a positive, norm one non-zero divisor b in a C^*-algebra $\mathfrak{A}$. Intriguingly, at the present stage of our investigations, we can not provide a direct, self-contained example of a positive, norm one element b in a C^*-algebra which is not a zero-divisor, and for which $\|\cdot\|_b \neq \|\cdot\|$. Nevertheless, the theory of *open dense nonregular projections*, to which we contribute in this paper, shows that such elements do exist in, e.g., the stabilized 2^∞ UHF algebra.

The concept of *regularity* for a projection in the double dual of a C^*-algebra $\mathfrak{A}^{**}$ was introduced by Tomita in [20] and subsequently studied by Effros and Akemann in [13], [1] and [2]. We are going to collect several known characterizations of this notion in Section 2.2 below. If we restrict our attention to the classes of *open* and *dense* projections in $\mathfrak{A}^{**}$ (see Section 2.1), regularity of the dense and open projection p reduces to the simple property that

$$\|a\| = \|ap\|,$$

for all $a \in \mathfrak{A}$. The relevance of this concept to our discussion of $\|\cdot\|_b$ is perhaps already clear to the reader, but will be explained in depth in Section 2.3.

Every projection in the double dual of a separable C^*-algebra dominates enough regular projections to be a strong limit of a sequence of such by Corollary 6 from [20]. It is clear that every closed projection is regular, and not hard to prove (a result attributed to Kaplansky) that every central projection is regular. An example of an open and nonregular projection was found in Example I.2 from [3], and it was proved in [1] that every open projection of a von Neumann algebra is regular, but the initial work of the first author left open the following questions for a general C^*-algebra $\mathfrak{A}$:

(I) Are all open and dense projections regular in $\mathfrak{A}$?

(II) Are all open projections regular in $\mathfrak{A}$, when $\mathfrak{A}$ is simple?

(III) Are all open and dense projections regular in $\mathfrak{A}$, when $\mathfrak{A}$ is simple?

We must report here that there exist AF algebras in which (I)–(III) fail; in fact, problem (I) is equivalent to the problem of finding an element b as described above. The existence of such projections is expected to cause complications in the program of [19] to find a meaningful notion of Murray-von Neumann type equivalence among open projections, which sparked a renewed interest in regularity. However, our investigations turned up a collection of positive results, proving that many dense and open projections are in fact regular. It is our hope that they can be used to work around such complications.

2. PRELIMINARIES

2.1. Notation. We are going to work a great deal with the C^*-algebra $\mathbb{K}$ of compact operators on a separable Hilbert space, and with the 2^∞ UHF algebra which shall be denoted by $\mathfrak{M}$. We denote generic C^*-algebras by $\mathfrak{A}$, and always consider them to be subalgebras of their double dual $\mathfrak{A}^{**}$. We denote by $\mathbf{z}$ the sum of all minimal central projections of $\mathfrak{A}^{**}$. This is exactly the central cover of the atomic representation of $\mathfrak{A}$, i.e. the cover of the sum of all irreducible representations.

In $\mathfrak{A}^*$ we will be working with the sets $Q(\mathfrak{A})$, $S(\mathfrak{A})$, $P(\mathfrak{A})$ consisting of quasi-states, states and pure states, respectively. Given a projection $p \in \mathfrak{A}^{**}$, we also need:

$$\begin{aligned}
L(p) &= {}^{\perp}(\mathfrak{A}^{**}(1-p)) = \{\varphi \in \mathfrak{A}^* \mid \forall a \in \mathfrak{A}^{**},\ \varphi(a(1-p)) = 0\} \\
F(p) &= \{\varphi \in Q(\mathfrak{A}) \mid \varphi(1-p) = 0\} \\
P(p) &= F(p) \cap P(\mathfrak{A}) = \{p \in P(\mathfrak{A}) \mid \varphi(p) = 1\}
\end{aligned}$$

and call these sets the left invariant subspace, the face and the set of pure states *supported* by p, respectively. If one of the sets $L(p), F(p), C(p)$ is weak* closed, so are the other two. This property defines a *closed* projection ([17], 3.11.9). The *closure* of a general projection p is the smallest closed projection $\overline{p}$ which dominates p; by [13]it is uniquely determined by

$$L(\overline{p}) = \overline{L(p)}.$$

As we shall see below, it is not true in general that $F(\overline{p}) = \overline{F(p)}$, leading to the definition of regular projections.

With definitions of closedness and closure in place, we can derive the following notions in a straightforward way: An *open* projection is one whose complement is closed, and a *dense* projection is one whose closure is the identity. We also define *compact* projections as those closed projections which are dominated by an element in $\mathfrak{A}$.

2.2. Regularity. Let us collect several known characterizations of regularity as follows:

Theorem 2.1. *Let $p \in \mathfrak{A}^{**}$ be a projection. Then the following conditions are equivalent:*

(i) $\forall a \in \mathfrak{A}$, $\|a\overline{p}\| = \|ap\|$;
(ii) $\forall a \in M(\mathfrak{A})$, $p \leqslant a \Rightarrow \overline{p} \leqslant a$;
(iii) $(L(p)_1)^- = L(\overline{p})_1$;
(iv) $F(p)^- = F(\overline{p})$.

The conditions (iii)–(iv) were used in the original work of Tomita and Effros on regularity, and their equivalence in presence of unitality was proved in 6.1. from [13]. The proof works in the general case with minor modifications, see the remark below. The condition (i), and its equivalence with (iii), is found in II.12 of [1]. Condition (ii) is from Theorem 18 of [18].

REMARK 2.2. If $\varphi_\lambda \to \varphi$ weak* in $Q(\mathfrak{A})$, and $\mathfrak{A}$ is unital, then $\|\varphi_\lambda\| = \varphi_\lambda(1) \to \varphi(1) = \|\varphi\|$. This is not true in the non-unital case; for instance a net of states can converge to weak* to zero. But if $\|\varphi\| = 1$, then in all cases $\|\varphi_\lambda\| \to \|\varphi\|$. One needs to use this observation to amend Effros' proof of 6.1 from [13] to the non-unital case. But it will not carry through in establishing equivalence between (i)–(iv) and the property

(v) $\overline{\{\varphi \in (\mathfrak{A}^*)_+ \mid \varphi(1-p) = 0\}} = (\overline{L(p)})_+$;

which is also considered in that result. In fact, it is shown in [11] that this condition is strictly weaker than regularity in the non-unital case. The reader is referred to [14] for details about this complication.

2.3. Preamble on non-zero divisors.

DEFINITION 2.3. Let a C^*-algebra $\mathfrak{A}$ be given, and fix $b \in \mathfrak{A}$ with $b \geqslant 0$ and $\|b\| = 1$. We define

(i) $\|x\|_b = \sup\limits_{n \geqslant 0} \|xb^{1/n}x^*\|^{1/2}$;

(ii) $\gamma(b) = \inf\limits_{x \neq 0} \frac{\|x\|_b}{\|x\|}$.

When $b \in \mathfrak{A}$ is positive and has norm one, we write $[b]$ for its range projection in $\mathfrak{A}^{**}$. We then get:

LEMMA 2.4. *We have $\|x\|_b = \|x[b]\|$, so $\|\cdot\|_b$ is a seminorm. The equality*

$$\|\cdot\| = \|\cdot\|_b$$

holds precisely when $[b]$ is a dense regular projection.

Proof. The norm equality holds as $b^{1/n} \nearrow [b]$ in $\mathfrak{A}^{**}$, and the second claim is immediate from Theorem 2.1. ∎

If b is a zero divisor $\gamma(b) = 0$, and if b is strictly positive, then $[b] = 1$ and $\gamma(b) = 1$. In the case of a commutative $\mathfrak{A}$, one easily sees that 0 and 1 are the only possible values of $\gamma(b)$. This generalizes further:

PROPOSITION 2.5. *Let Ω be a locally compact Hausdorff space. If $b \in C_0(\Omega, \mathbf{M}_n)$ is positive, has norm one, and is not a zero-divisor, then $\gamma(b) = 1$.*

Proof. Suppose that $\|x[b]\| < 1$ and $\|x\| = 1$. Let U be an open set such that for ω in U, $\|x(\omega)\| > \|x[b]\|$. Since $\|x(\omega)[b](\omega)\| < 1$ for each ω in V, therefore $[b](\omega)$ is not equal to 1 for ω in V, hence $(1 - f(b))(\omega)$ is not zero for such ω. Let then ω_0 be chosen so that the number k of nonzero eigenvalues of $b(\omega)$ is maximal at $\omega = \omega_0$ for $\omega \in U$. We are going to find $r > 0$ and a neighborhood $V \subseteq U$ of ω_0 such that

$$\mathrm{sp}(b(\omega)) \cap \left(0, \frac{r}{2}\right) = \emptyset \quad \text{for all } \omega \in V.$$

If $k = 0$, we achieve this by $r = 1$ and $V = U$. When $k > 0$, label the distinct nonzero eigenvalues of $b(\omega_0)$ as $s_1 < s_2 < \cdots < s_k$. Let r be the minimum distance between any two eigenvalues of $b(\omega_0)$ (including the 0 eigenvalue). Using functional calculus with spike functions supported around each nonzero eigenvalue of $b(\omega_0)$, we find for each j an open neighborhood V_j of ω_0 such that for any $\omega \in V_j$, $b(\omega)$

has an eigenvalue in the interval $(s_j - r/4, s_j + r/4)$. With $V = U \cap V_1 \cap \cdots \cap V_k$ we have then that for each $\omega \in V$, $b(\omega)$ has an eigenvalue in each $(s_j - r/4, s_j + r/4)$. By maximality of k, $b(\omega)$ can't have an eigenvalue in the interval $(0, r/2)$.

Let f be a continuous function on $[0,1]$ that is 0 at 0 and equal to 1 on $[r/2, 1]$. Then

$$(1 - f(b))(\omega)b(\omega) = 0$$

for each ω in V. Now find a Urysohn function g on Ω that is 1 at ω_0 and 0 outside V. The operator $\omega \mapsto g(\omega)(1 - f(b))(\omega)$ is thus a nonzero element of $C_0(\Omega, \mathbf{M}_n)$ that is orthogonal to b. ∎

Our Theorems 3.5 and 3.9 below will show that there exists a positive, norm one, element of $C([0,1], \mathbb{K})$ or $\mathfrak{M} \otimes \mathbb{K}$, not a zero divisor, but with $\gamma(b) < 1$.

Remark 2.6. For $t \in [0,1]$, let

$$\mathcal{R}_t = \{b \in \mathfrak{A}^+ \mid \|b\| = 1,\, \gamma(b) = t\}.$$

Define

$$\mathcal{N} = \{b \in A^+ \mid \|b\| = 1,\, b \text{ is not a zero divisor}\}.$$

Our results above open the discussion of when $\mathcal{N} = \mathcal{R}_1$. Related questions, which we shall not attempt to answer here, are what properties $\mathcal{R}_t$ have (other than automorphism invariance), whether all $\mathcal{R}_t$ are nonempty when $\mathcal{R}_1 \neq \mathcal{N}$, and whether, when $b \in \mathcal{R}_t$ for $t < 1$, we can find $x \in \mathfrak{A}_+ \setminus \{0\}$ such that $\sup_n ||b^{1/n}x|| = t||x||$.

3. Open and nonregular projections

The present section contains all our examples establishing existence of open and nonregular projections under additional assumptions. The first technical results, collected in Section 3.1 below, provide the foundation for all the results in Sections 3.2 and 3.3. Section 3.4 does not depend upon 3.1.

3.1. A vector-valued function. The following notation will be used in all of this section. H is an infinite dimensional, separable Hilbert space. In H, we choose a distinguished unit vector y. The space Ω is some perfect and compact metric space. Recall that Ω is separable and let S be a countably infinite dense set in Ω, enumerated without repetitions as $S = \{s_n\}$.

Lemma 3.1. *For any $s \in S$ there is a norm continuous function $z : \Omega \setminus \{s\} \to H$ with the properties*

$$\|z(\omega)\| = 1, \quad \textit{for all } \omega \in \Omega \setminus \{s\} \quad \textit{and} \quad \text{weak} \lim_{\omega \to s} z(\omega) = 0.$$

Proof. Choose a basis $\{e_n\}$ for H. Recall that Ω is metric; we may assume that $\operatorname{diam}(\Omega) < 1$. We consider pairwise overlapping annuli centered at s defined as

$$A_n = \left\{ \omega \in \Omega \;\middle|\; \frac{1}{n+2} < d(\omega, s) < \frac{1}{n} \right\};$$

clearly this covers the paracompact space $\Omega\backslash\{s\}$. Choosing a locally finite partition of unity for $\Omega\backslash\{s\}$ subordinate to the A_n we get a family $\{\psi_n\}$ with the properties:

(i) for each $\omega \in \Omega \setminus \{s\}$, $\psi_n(\omega) \neq 0$ for no more than two values of $n \in \mathbb{N}$;

(ii) for each $\omega \in \Omega \setminus \{s\}$, $\sum_{n=1}^{\infty} \psi_n(\omega) = 1$;

(iii) for each N, there exists a neighborhood U of s with the property that $\psi_1|U = \cdots = \psi_N|U = 0$.

Let

$$z(\omega) = \sum_{n\in\mathbb{N}} \sqrt{\psi_n(\omega)} e_n.$$

This is locally a finite sum yielding unit vectors by (i) and (iii). Let $v \in H$ and $\varepsilon > 0$ be given. Choose N such that $\langle v, e_n\rangle \leqslant \varepsilon/2$ when $n \geqslant N$ and U a neighborhood of s as in (ii). We then have for any $\omega \in U$, by (i) again, that

$$|\langle z(\omega), v\rangle| = \left| \sum_{n=N+1}^{\infty} \sqrt{\psi_n(\omega)}\, \langle e_n, v\rangle \right| \leqslant |\langle e_{n_1}, v\rangle| + |\langle e_{n_2}, v\rangle| \leqslant \varepsilon.$$

Here n_1, n_2 are the two values of condition (i). ∎

LEMMA 3.2. *Let H, y, Ω and S be as above. For any $\delta > 0$ there are functions*

$$x : \Omega \to H, \quad \mu : S \to (0,1)$$

with the properties:

(i) $\|x(\omega)\| = 1$ *for all* $\omega \in \Omega$;

(ii) $\|y - x(\omega)\| < \delta$ *for all* $\omega \in \Omega$;

(iii) x *is norm continuous at any* $\omega \in \Omega \setminus S$;

(iv) *for any* $s \in S$,

$$\text{weak} \lim_{\omega\to s} x(\omega) = \mu(s)x(s).$$

Proof. Fix $\eta > 0$ such that $1 - 1/\sqrt{1+\eta} + \sqrt{\eta} < \delta$. Choose a basis for H indexed over $\{0\} \cup \mathbb{N} \times \mathbb{N}$, say $\{e_0, e_{ij}\}$, with $e_0 = y$. Set $H_n = \overline{\operatorname{span}\{e_{ni} \mid i \in \mathbb{N}\}}$. Choose functions $z_n : \Omega \setminus \{s_n\} \to H_n$ according to Lemma 3.1, and let

$$z_n(\omega) = \begin{cases} \sqrt{\eta 2^{-n}} z_n(\omega), & \omega \neq s_n, \\ 0, & \omega = s_n. \end{cases}$$

By orthogonality, for all $\omega \in \Omega \setminus S$, we have

$$\left\| y + \sum_{m\in\mathbb{N}} z_m(\omega) \right\|^2 = 1 + \eta$$

and

$$\left\| \sum_{m=N}^{\infty} z_m(\omega) \right\| = \sqrt{\eta 2^{-N+1}}. \tag{3.1}$$

Since

$$\text{weak} \lim_{\omega\to s_n} \left[y + \sum_{m\in\mathbb{N}} z_m(\omega) \right] = y + \sum_{m\neq n} z_m(s_n)$$

we get again by orthogonality that the limit vector has length $\sqrt{1+\eta(1-2^{-n})}$. We are then ready to define

$$x(\omega)=\begin{cases}\frac{1}{\sqrt{1+\eta}}\Big[y+\sum_{m\in\mathbb{N}}z_m(\omega)\Big], & \omega\in\Omega\setminus S,\\[2ex] \frac{1}{\sqrt{1+\eta(1-2^{-n})}}\Big[y+\sum_{m\neq n}z_m(\omega))\Big], & \omega=s_n;\end{cases}$$

and

$$\mu(s_n)=\frac{\sqrt{1+\eta}}{\sqrt{1+\eta(1-2^{-n})}},$$

where we note that $\mu(s_n)\nearrow 1$ as $n\to\infty$. The condition (i) is now clearly met, and we have from (3.1) that

$$\|y-x(\omega)\|\leqslant\left(1-\frac{1}{\sqrt{1+\eta}}\right)+\left\|\sum_{m\in\mathbb{N}}z_m(\omega)\right\|=1-\frac{1}{\sqrt{1+\eta}}+\sqrt{\eta}<\delta$$

for any $\omega\in\Omega\setminus S$. In fact, this inequality readily extends to all of Ω, proving that $x(\omega)$ satisfies (ii). To prove that it satisfies (iii), fix $\omega_0\in\Omega\setminus S$ and $\delta>0$. For an N to be determined later, choose a neighborhood U_N of ω_0 such that $U_N\cap S\subseteq\{s_n\mid n\geqslant N\}$. For $\omega\in U_N\setminus S$ we get

$$\|x(\omega_0)-x(\omega)\|\leqslant 2\frac{\sqrt{\eta 2^{-N+1}}}{\sqrt{1+\eta}}+\frac{1}{\sqrt{1+\eta}}\sum_{m=1}^{N-1}\|z(\omega_0)-z(\omega)\|$$

by (3.1) again. A similar, slightly more complicated computation gives that for $\omega\in U_N\cap S$,

$$\|x(\omega_0)-x(\omega)\|\leqslant 2\frac{1-\mu(s_N)}{\sqrt{1+\eta}}+2\frac{\sqrt{\eta 2^{-N+1}}}{\sqrt{1+\eta}}+\frac{1}{\sqrt{1+\eta}}\sum_{m=1}^{N-1}\|z(\omega_0)-z(\omega)\|, \tag{3.2}$$

where we have used that the missing term in $x(\omega)$ in this case is not among the first $N-1$ and that $s(\mu_n)\geqslant s(\mu_N)$ when $n\geqslant N$. Thus the estimate in (3.2) holds throughout U_N. We can choose N such that the two first terms above sum to no more than $\varepsilon/2$, and then the norm continuity of each $z_1,\dots,z_{N-1}$ on U_N to find a neighborhood $\omega\in V\subseteq U_N$ on which the last term is also bounded by $\varepsilon/2$.

To prove (iv) it suffices to consider the case of sequences $\omega_k\to s_{n_0}$. We may even reduce to the two cases of $\{\omega_k\}\cap S=\emptyset$ and $\{\omega_k\}\subseteq S$. In the first case, we note that $\omega\mapsto\langle z_m(\omega),z\rangle$ can be extended with zero to a continuous function on Ω for each m, uniformly bounded by $\sqrt{\eta 2^{-m}}\|z\|$. On $\Omega\setminus S$ we have

$$\langle x(\omega),z\rangle=\frac{1}{\sqrt{1+\eta}}\left(\langle y,z\rangle+\sum_{m\in\mathbb{N}}\langle z_m(\omega),z\rangle\right)$$

and the expression on the right is continuous on all of Ω by the above. Hence

$$\operatorname{weak}\lim_{k\to\infty}x(\omega_k)=\frac{1}{\sqrt{1+\eta}}\left(y+\sum_{\substack{m\in\mathbb{N}\\ m\neq n_0}}z_m(s_{n_0})\right)=\mu(s_{n_0})x(s_{n_0}).$$

In the second case, we write $\omega_k = s_{n_k}$ and let $\varepsilon > 0$ be given. Because all values s_n are distinct, we can find, for some $N > n_0$ to be determined, a neighborhood U'_N of s_{n_0} with the property

$$U'_N \cap S \subseteq \{s_{n_0}\} \cup \{s_n \mid n \geqslant N\}. \tag{3.3}$$

Choose K with $s_{n_k} \in U'_N$ for all $k \geqslant K$. As in the proof of (iii), we get that

$$\begin{aligned}\langle \mu(s_{n_0})x(s_{n_0}) - x(s_{n_k}), z\rangle &\leqslant 2\left(\frac{1}{\sqrt{1+\eta}} - \frac{1}{\sqrt{1+\eta(1-2^{-N})}}\right)\|z\| \\ &+ 2\frac{\sqrt{\eta 2^{-N+1}}}{\sqrt{1+\eta}}\|z\| + \frac{1}{\sqrt{1+\eta}}\sum_{m=1}^{n_0-1}\|z_m(s_{n_0}) - z_m(s_{n_k})\|\,\|z\| \\ &+ \frac{\langle z_{n_0}(s_{n_k}), z\rangle}{\sqrt{1+\eta}} + \frac{1}{\sqrt{1+\eta}}\sum_{m=n_0+1}^{N-1}\|z_m(s_{n_0}) - z_m(s_{n_k})\|\,\|z\|.\end{aligned}$$

Choose first N to make the two first terms less than $\varepsilon/5$, a corresponding K, and then $K_1 \geqslant K$ so that the last three terms each are less than $\varepsilon/5$ for $k \geqslant K_1$. ∎

We now consider $\mathfrak{A} = C(\Omega, \mathbb{K})$, where $\mathbb{K}$ is considered as the compact operators on our Hilbert space H. By standard identifications, $\mathfrak{A}^{**}\mathbf{z} = \{f : \Omega \to \mathbb{B}(H) \mid f \text{ is bounded}\}$. We define r_x in $\mathfrak{A}^{**}\mathbf{z}$ by defining it at each ω in Ω to be the projection on the span of $x(\omega)$.

LEMMA 3.3. *With x as in Lemma 3.2, for $\delta < 1$, we have $\overline{r}_x\mathbf{z} = r_x$.*

Proof. Clearly $r_x = r_x\mathbf{z} \leqslant \overline{r}_x\mathbf{z}$, to reach a contradiction assume $r_x < \overline{r}_x\mathbf{z}$. This implies that $(\overline{r}_x - r_x)\mathbf{z}$ dominates a minimal projection, so some pure state of $C(\Omega, \mathbb{K})$ will evaluate to one on $\overline{r}_x - r_x$. Such a pure state will be given by

$$f \mapsto \langle f(\omega_0)z_0, z_0\rangle$$

for some choice of $\omega_0 \in \Omega$ and unit vector $z_0 \in H$, so we have, using the identification of $\mathfrak{A}^{**}\mathbf{z}$ as operator-valued functions on Ω, that

$$\overline{r}_x(\omega_0)z_0 = z_0, \quad r_x(\omega_0)z_0 = 0.$$

By the definition of r_x, the second property implies orthogonality of z_0 and $x(\omega_0)$. Let $p, q \in \{f : \Omega \to \mathbb{B}(H) \mid f \text{ is bounded}\}$ be given as projections onto $\operatorname{span}\{z_0, y\}$, $\operatorname{span}\{px(\omega)\}$, respectively (recall that y is the distinguished unit vector in H). Since p is a constant projection, $p = p'\mathbf{z}$ where $p' \in \mathfrak{A}$. Note also that q has constant rank one because

$$\|px(\omega)\| \geqslant \|y\| - \|p(y - x(\omega))\| \geqslant 1 - \delta.$$

In fact, we are going to prove that q varies norm continuously with ω. Before we do so, let us show how this leads to the desired contradiction. As above, $q = q'\mathbf{z}$ where $q' \in \mathfrak{A}$. Since we have arranged that $p - q(\omega)$ annihilates $x(\omega)$ for all ω, we get $(p - q)r_x = 0$. This entails that $r_x \leqslant 1 - (p' - q')$, and since the larger projection here is closed, even that $\overline{r}_x \leqslant 1 - (p' - q')$. Thus $(p - q)\overline{r}_x = 0$, leading to the contradiction

$$0 = (p - q(\omega_0))\overline{r}_x(\omega_0)z_0 = (p - q(\omega_0))z_0 = z_0.$$

To see that $\omega \mapsto q(\omega)$ is norm continuous, note that by what has already been said, $z(\omega) = (1/\|p(x(\omega))\|)px(\omega)$ is always defined. We have that p, being of finite rank, is weak-norm continuous, so we conclude from Lemma 3.2 (iii)–(iv) that

$$\lim_{\omega \to \omega_0} px(\omega) = \begin{cases} \mu(\omega_0)px(\omega_0), & \text{if } \omega_0 \in S, \\ px(\omega_0), & \text{if } \omega_0 \notin S; \end{cases}$$

showing that $\omega \mapsto z(\omega)$ is a norm continuous function in H. When now v is any unit vector, we have

$$\begin{aligned} \|(q(\omega) - q(\omega_0))(v)\| &= \| \langle v, z(\omega) \rangle z(\omega) - \langle v, z(\omega_0) \rangle z(\omega_0) \| \\ &\leqslant \| \langle v, z(\omega) - z(\omega_0) \rangle z(\omega) \| + \| \langle v, z(\omega_0) \rangle (z(\omega) - z(\omega_0)) \| \\ &\leqslant \|z(\omega) - z(\omega_0)\|^{1/2} \|v\|^{1/2} \|z\| + \|v\|^{1/2} \|z(\omega_0)\|^{1/2} \|z(\omega) - z(\omega_0)\| \\ &\leqslant \|z(\omega) - z(\omega_0)\|^{1/2} + \|z(\omega) - z(\omega_0)\|, \end{aligned}$$

and q is norm continuous. ∎

3.2. Dense nonregularity.

We are going to prove that $1 - \overline{r}_x$ is dense and nonregular.

PROPOSITION 3.4. *With x as in Lemma 3.2, for any δ, we have:*
(i) *r_x is discontinuous in norm at any $s \in S$;*
(ii) *$1 - \overline{r}_x$ is dense;*
If furthermore $\delta < 1/5$, we have
(iii) *$1 - \overline{r}_x$ is not regular.*

Proof. For (i), apply r_x to $x(s)$ and note that, as ω approaches s,

$$\omega \mapsto r_x(\omega)x(s) = \langle x(s), x(\omega) \rangle x(\omega)$$

cannot converge in norm to a unit vector because $\mu(s) < 1$.

For (ii), it suffices to show that $\overline{r}_x$ could not dominate a nonzero positive element of $\mathfrak{A}$. If it did, then by functional calculus, it would also dominate an element b with the property that $b(\omega)$ was a projection on a nonempty open set U of Ω. By Lemma 3.3, r_x would dominate the canonical image of b in $\mathbf{z}\mathfrak{A}^{**}$, and since $r_x(\omega)$ is rank 1 then $b(\omega) = r_x(\omega)$ on U. This contradicts (i) as b is norm continuous at any $s_n \in U$.

For (iii), let $s \in S$, and define a constant element $c \in \mathfrak{A}$ by $c(\omega) = r_x(s)$ for all ω. Since $\overline{r}_x\mathbf{z} = r_x$ by Lemma 3.3,

$$(1 - \overline{r}_x(\omega))c(\omega)\mathbf{z} = (1 - r_x(\omega))r_x(s) = (r_x(s) - r_x(\omega))r_x(s),$$

so for any $\omega \in \Omega$ and any unit vector $v \in H$,

$$\begin{aligned} \|(r_x(s) - r_x(\omega))v\| &= \| \langle v, x(s) \rangle x(s) - \langle v, x(\omega) \rangle x(\omega) \| \\ &\leqslant 2\delta + \| \langle v, y \rangle y - \langle v, y \rangle y \| + 2\delta \leqslant \frac{4}{5}. \end{aligned}$$

This gives, using 4.3.15 from [17]

$$\begin{aligned}\|(1-\overline{r}_x)c\| = \|(1-\overline{r}_x)c\mathbf{z}\| &= \sup_{\omega,\|z\|\leqslant 1} \|(r_x(s)-r_x(\omega))r_x(s)z\| \\ &\leqslant \sup_{\omega,\|v\|\leqslant 1} \|(r_x(s)-r_x(\omega))v\| \leqslant \frac{4}{5} \\ &< 1 = \|c\| = \|\overline{1-\overline{r}_x c}\|,\end{aligned}$$

and $1-\overline{r}_x$ is not regular, by Theorem 2.1 (i). ∎

Theorem 3.5. *Let M be the Cantor set. The C^*-algebras $C([0,1],\mathbb{K})$ and $C(M,\mathbb{K})$ both have open, dense and nonregular projections. The latter C^*-algebra is AF.*

Remark 3.6. The reader may wonder whether $1-r_x$ is regular. In fact, since it dominates $1-\mathbf{z}$ by construction, and since every ideal of $C(\Omega,\mathbb{K})$ has diffuse states, our Proposition 4.3 below will show that it is indeed regular.

3.3. Dense nonregularity in simple C^*-algebras.

We now proceed to show the existence of a dense open nonregular projection in $\mathfrak{M}\otimes\mathbb{K}$. Our proof is based on pushing the projection constructed in the previous section in $C(M)\otimes\mathbb{K}$, with M the Cantor set, into $\mathfrak{M}\otimes\mathbb{K}$. This will prove to be possible because, as noted in [8], $C(M)$ sits inside $\mathfrak{M}$ as a diagonal MASA on which the "diagonal compression" map $E:\mathfrak{M}\to C(M)$ is a faithful conditional expectation (a projection of norm one).

One should note right away that since, as we shall see in Remark 3.10, regularity is not a hereditary property, one can not in general conclude the existence of a nonregular projection in a C^*-algebra $\mathfrak{A}$ from the existence of such a projection in a subalgebra of $\mathfrak{A}$. The proposition below, a collection of well-known results, compiles the information we shall need to make such an argument work.

Proposition 3.7. *There is a faithful conditional expectation E from the 2^∞* UHF *algebra $\mathfrak{M}$ onto its diagonal MASA $C(M)$. Then E induces a map*

$$E\otimes\mathrm{id}:\mathfrak{M}\otimes\mathbb{K}\to C(M)\otimes\mathbb{K}$$

which is again a faithful conditional expectation. Furthermore, the normal extension

$$(E\otimes\mathrm{id})^{**}:(\mathfrak{M}\otimes\mathbb{K})^{**}\to(C(M)\otimes\mathbb{K})^{**}$$

is also a conditional expectation.

Proof. The existence of E, which is faithful as it is trace-preserving, is noted in [8]. The tensor product then exists as E is a completely positive map, and by [21], it will be a faithful conditional expectation. That also $(E\otimes\mathrm{id})^{**}$ is a conditional expectation follows by routine duality arguments. ∎

Denote by ι the inclusion map of $C(M)$ into $\mathfrak{M}$. Since ι is a $*$-monomorphism, so is the canonically induced map

$$(\iota \otimes \mathrm{id})^{**} : (C(M) \otimes \mathbb{K})^{**} \to (\mathfrak{M} \otimes \mathbb{K})^{**}.$$

PROPOSITION 3.8. *Let p be an open dense nonregular projection for $C(M) \otimes \mathbb{K}$. Then the projection $(\iota \otimes \mathrm{id})^{**}(p)$ is an open, dense, nonregular projection for $\mathfrak{M} \otimes \mathbb{K}$.*

Proof. Since p is open, dense and nonregular, there is a nonzero element $d_0 \in C(M) \otimes \mathbb{K}$ such that $\|d_0\| = 1 > \|pd_0\|$. Since $(\iota \otimes \mathrm{id})^{**}$ is a $*$-monomorphism, $(\iota \otimes \mathrm{id})^{**}(p)$ is an open projection. Suppose we can prove that it is dense for $\mathfrak{M} \otimes \mathbb{K}$. Then it must be nonregular since $\|d_0\| = 1 > \|pd_0\|$ and hence $1 = \|(\iota \otimes \mathrm{id})(d_0)\| > \|(\iota \otimes \mathrm{id})^{**}(p)(\iota \otimes \mathrm{id})(d_0)\|$.

It remains to show that $(\iota \otimes \mathrm{id})^{**}(p)$ is dense for $\mathfrak{M} \otimes \mathbb{K}$. Let b denote a positive element of $\mathfrak{M} \otimes \mathbb{K}$ such that $(\iota \otimes \mathrm{id})^{**}(p)b = 0$. Using the properties of conditional expectations,

$$0 = (E \otimes \mathrm{id})^{**}((\iota \otimes \mathrm{id})^{**}(p)b) = (\iota \otimes \mathrm{id})^{**}(p)(E \otimes \mathrm{id})^{**}(b) = (\iota \otimes \mathrm{id})^{**}(p(E \otimes \mathrm{id})(b)).$$

But $(\iota \otimes \mathrm{id})^{**}$ is injective, so this means that $p(E \otimes \mathrm{id})(b) = 0$. Since p is dense for $C(M) \otimes \mathbb{K}$ and $E \otimes \mathrm{id}(b)$ is in $C(M) \otimes \mathbb{K}$, $E \otimes \mathrm{id}(b) = 0$. By Proposition 3.7 above, $E \otimes \mathrm{id}$ is faithful, so $b \geqslant 0$ and $E \otimes \mathrm{id}(b) = 0$ imply that $b = 0$. Thus $(\iota \otimes \mathrm{id})^{**}(p)$ is dense, and we are done. ∎

THEOREM 3.9. *There is a non-regular dense open projection for the stabilized* 2^{∞} UHF *algebra $\mathfrak{M} \otimes \mathbb{K}$.*

Proof. Combine Proposition 3.8 and Theorem 3.5. ∎

REMARK 3.10. Consider

$$\mathfrak{A} = \left\{ (A_n)_{n \in \mathbb{Z}} \in (\mathbf{M}_2)^{\mathbb{Z}} \;\middle|\; \lim_{n \to \pm\infty} A_n \text{ exists} \right\}$$

and its subalgebra

$$\mathfrak{B} = \left\{ (A_n)_{n \in \mathbb{Z}} \in \mathfrak{A} \;\middle|\; \lim_{n \to +\infty} A_n = \lim_{n \to -\infty} A_n \right\}.$$

As the irreducible representations of $\mathfrak{B}$ (all 2-dimensional) can be naturally parametrized over $\mathbb{Z} \cup \{\infty\}$, we can specify an open projection p in $\mathfrak{B}^{**}$ by

$$p\mathbf{z}(n) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & n \in \mathbb{N}_0, \\ \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}, & n \in -\mathbb{N}, \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & n = \infty. \end{cases}$$

Since one finds that $\overline{p}\mathbf{z}$ differs only from $p\mathbf{z}$ by having $\overline{p}\mathbf{z}(\infty) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, one can get by considering the element $b \in \mathfrak{B}$ which is constantly $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ that p is not regular. However, p considered as an element of $\mathfrak{A}$ has a closure which lies in $\mathfrak{B}$ itself, and thus must be regular.

3.4. NONREGULARITY IN $\mathfrak{M}$. At present, we do not see a way to amend our construction to prove that there is a dense open nonregular projection for the 2^∞ UHF algebra $\mathfrak{M}$ itself, but only for its stabilized version. The difference in this respect between $C(\Omega, \mathbf{M}_n)$ and $C(\Omega, \mathbb{K})$ may seem to indicate that this is due to some deeper phenomenon.

However, we can supply an example of a nonregular open projection associated to $\mathfrak{M}$, and we supply the details here as we feel the methods may be of independent value.

LEMMA 3.11. *If $p \in \mathfrak{A}^{**}$ is regular and dominated by $\mathbf{z}$, then*

$$\overline{P(p)} = P(\overline{p}),$$

where the closure on the left hand side is relative to $P(\mathfrak{A})$.

Proof. We first apply the assumption that $p \leqslant \mathbf{z}$ to prove that $\overline{\mathrm{co}}(P(p) \cup \{0\}) = (p)^-$. Since $F(p)^-$ is a weak* closed convex set containing $P(p)$, the inclusion from left to right is clear. By the double polar theorem (as it is found in, e.g., [10], IV.1, Proposition 3) applied to the real spaces $(\mathfrak{A}_{\mathrm{sa}}, \|\cdot\|)$ and $(\mathfrak{A}^*_{\mathrm{sa}}, \mathrm{weak}^*)$ in duality (cf. [17], 3.1.1) and the fact that $0 \in F(p)$, the other inclusion is equivalent to $P(p)^{\circ\circ} \supseteq F(p)^{\circ\circ}$, which follows from

$$\begin{aligned} P(p)^\circ &= \{a \in \mathfrak{A}_{\mathrm{sa}} \mid \forall \varphi \in P(p),\ \varphi(a) \geqslant -1\} = \{a \in \mathfrak{A}_{\mathrm{sa}} \mid \forall \varphi \in P(\mathfrak{A}),\ \varphi^{**}(pap) \geqslant -1\} \\ &= \{a \in \mathfrak{A}_{\mathrm{sa}} \mid \forall \pi \in \widehat{\mathfrak{A}},\ \pi^{**}(pap) \geqslant -\mathbf{1}\} = \{a \in \mathfrak{A}_{\mathrm{sa}} \mid pap\mathbf{z} \geqslant -\mathbf{z}\} \\ &= \{a \in \mathfrak{A}_{\mathrm{sa}} \mid pap \geqslant -\mathbf{1}\} \subseteq F(p)^\circ. \end{aligned}$$

For the inclusion from left to right in the second equality, let $a \in \mathfrak{A}_{\mathrm{sa}}$ with $\varphi(a) \geqslant -1$ for all $\varphi \in P(p)$ be given and set $\alpha = \psi(p)$ for a given $\psi \in P(\mathfrak{A})$. If $\alpha = 0$, clearly $\psi^{**}(pap) \geqslant -1$. If $\alpha > 0$, one uses that $\psi' = \alpha^{-1}\psi(p \cdot p)$ is an element of $P(p)$. The assumption on p is used in the inclusion from left to right of the last equation; if $pap\mathbf{z} \geqslant -\mathbf{z}$, we have

$$pap = pap\mathbf{z} \geqslant -\mathbf{z} \geqslant -\mathbf{1}.$$

When $P(p)^-$ denotes the closure of $P(p)$ relative to $P(\mathfrak{A})$, it is clear that $P(p)^- \subseteq P(\overline{p})$. Using the fact that p is regular along with the above observation, we get

$$\overline{\mathrm{co}}(P(p) \cup \{0\}) = F(p)^- = F(\overline{p}).$$

By Appendice B 14 of [12] the extremal points of $F(\overline{p})$ are contained in $(P(p) \cup \{0\})^-$, and as $F(\overline{p})$ is a face, these are exactly $P(\overline{p}) \cup \{0\}$. We conclude that $P(\overline{p}) \subseteq P(p)^-$, as required. ∎

Lemma 3.12. *Assume that q is an open projection of $\mathfrak{A}$. Then q is regular if and only iff $q\mathbf{z}$ is. In this case, $\overline{q\mathbf{z}} = \overline{q}$.*

Proof. Assume that q is regular. Since q is open, q is a limit of a net of compact projections q_k dominated by q. Since $F(\mathbf{z}q_k)^- = F(q_k)$, then $F(q)$ is contained in the closure of the union of the $F(\mathbf{z}q_k)$. Thus $F(q)$ is contained in $F(\mathbf{z}q)^-$, and

$$F(\overline{q\mathbf{z}}) \subseteq F(\overline{q}) = \overline{F(q)} = \overline{F(q\mathbf{z})} \subseteq F(\overline{q\mathbf{z}}).$$

In the other direction, suppose that $\mathbf{z}q$ is regular, and let r be the closure of $\mathbf{z}q$. Then $\mathbf{z}r$ dominates $\mathbf{z}q$. Since r is closed and q is open, 4.3.15 of [17] shows that r dominates q. Thus the closure of q is dominated by r, hence it equals r. We conclude that $\overline{q\mathbf{z}} = \overline{q}$. But $\mathbf{z}q$ regular means that for any a in $\mathfrak{A}$, $\|\mathbf{z}qa\| = \|ra\|$, but also $\|\mathbf{z}qa\| \leqslant \|qa\| \leqslant \|ra\|$ because of the ordering of $\mathbf{z}q, q$ and r. Thus $\|qa\| = \|ra\|$, so q is regular. ∎

Now for an open projection q, we can look at $q\mathbf{z}$ and use Lemma 3.12 to check for regularity.

Proposition 3.13. *If q is open and regular, and if $\overline{P(\mathbf{z}q)} = P(\mathbf{z}q)$, then $q \in M(\mathfrak{A})$.*

Proof. Since q is regular, $\mathbf{z}q$ is regular by Lemma 3.12 above. By Lemma 3.11, regularity of $\mathbf{z}q$ means that $P(\mathbf{z}q)^- = P(\overline{q})$, implying that $q\mathbf{z} = \overline{q}\mathbf{z}$. As in 4.3.15 of [17] because q and q^- are both semicontinuous, we conclude that $q = \overline{q}$. Since q is already open, it lies in $M(\mathfrak{A})$ by 2.2 from [6]. ∎

We now apply the results above to the "perfection" $\mathfrak{A}_c$ of a C^*-algebra $\mathfrak{A}$ introduced in [7]. This C^*-algebra is defined as

$$\{a \in \mathfrak{A}^{**}\mathbf{z} \mid a, a^*, aa^* \text{ are weak*-continuous on } P(\mathfrak{A}) \cup \{0\}\},$$

and has subsequently been studied in [4] and [9]. This notion ties in with regularity as follows:

Corollary 3.14. *If p is an open and regular projection of $\mathfrak{A}$, and if $p\mathbf{z} \in \mathfrak{A}_c$, then in fact $p \in M(\mathfrak{A})$.*

Proof. By definition of $\mathfrak{A}_c$, elements of $\mathfrak{A}_c$ are continuous on the pure states of $\mathfrak{A}$. Hence $P(p\mathbf{z})^- = P(p\mathbf{z})$, and Proposition 3.13 applies. ∎

We are now ready to prove the existence of a nonregular projection in $\mathfrak{M}$. Before we give the proof, let us review the following key notions from [5] and [4]:

DEFINITION 3.15. A net x_λ of elements in $\mathfrak{A}^+$ with $\|x_\lambda\| = 1$ *excises* the state $\varphi \in S(\mathfrak{A})$ if

$$\|x_\lambda(a - \varphi(a))x_\lambda\| \to 0, \quad \forall a \in \mathfrak{A}.$$

REMARK 3.16. To check that a net x_λ in A^+ with $\|x_\lambda\| = 1$ excises a state φ it suffices to check the convergence for each a in a dense subset of $\mathfrak{A}$.

DEFINITION 3.17. Let $\mathfrak{A}$ be a C^*-algebra.

(i) ([4], 2.1) A sequence a_n of $\mathfrak{A}$ is *diffuse* if for every net φ_λ in $P(\mathfrak{A}) \cup \{0\}$, converging weak* in $P(\mathfrak{A}) \cup \{0\}$, we have

$$\lim_{\lambda,n} \varphi_\lambda(a_n^* a_n + a_n a_n^*) = 0.$$

(ii) ([4], 3.1) An orthogonal, positive, norm one sequence a_n in $\mathfrak{A}$ is *truly diffuse* if for any increasing sequence n_k in $\mathbb{N}$, the sequence

$$\sum_{j=n_k}^{n_{k+1}-1} a_j$$

is diffuse.

(iii) ([5]) A quasi-state $\varphi \in Q(\mathfrak{A})$ is *diffuse* when $\varphi(\mathbf{z}) = 0$.

THEOREM 3.18. *The* 2^∞ UHF *algebra* $\mathfrak{M}$ *contains a nonregular open projection.*

Proof. By 3.7 from [4] there is a factor state φ of type II_∞, a sequence $\{p_n\}$ of orthogonal projections in $\mathfrak{M}$ and a dense sequence $\{a_n\}$ in $\mathfrak{M}$ such that for $1 \leqslant k \leqslant n$ and $m > n$,

(i) $p_n(a_k - \varphi(a_k))p_n = 0$;

(ii) $p_m a_k p_n = 0 = p_n a_k p_m$.

We first show that $\{p_n\}$ is a truly diffuse sequence. Let $\{n_k\}$ be an increasing sequence of natural numbers and define $c_k = \sum_{j=n_k}^{n_{k+1}-1} p_j$; by Definition 3.17 (ii) we need to show that $\{c_k\}$ is a diffuse sequence. Using 2.14 from [4] we can conclude that $\{c_k\}$ is a diffuse sequence if it excises a diffuse state. Since φ is of type II_∞, it is diffuse. Since $\|c_k\| = 1$ for all k, by Remark 3.16 we only need to check the excising condition on the dense set $\{a_k\}$, and here it is immediately verified for each a_k by (i) and (ii) above. Thus $\{p_n\}$ is a truly diffuse sequence.

Set $p = \sum_{n=1}^{\infty} p_n$, where the sum is taken in $\mathfrak{M}^{**}$. If p were in $\mathfrak{M}$, then by Dini's theorem the sequence of partial sums $\left\{\sum_{n=1}^{k} p_n\right\}$, would have to converge to p in norm, and that is impossible. Thus p does not lie in $\mathfrak{M}$.

We next show that $p\mathbf{z}$ lies in $\mathfrak{M}_c$. Since $p\mathbf{z}$ is a projection and $\mathfrak{M}$ is separable, it sufffices to assume that $\{\psi_n\}$ is a sequence of pure states of $\mathfrak{M}$ that converges to a pure state ψ of $\mathfrak{M}$ and to show that $(\psi_n - \psi)(p) \to 0$. Let $\varepsilon > 0$ be given. Choose n_0 such that $\psi\Big(\sum_{j=n_0}^{\infty} p_j\Big) < \varepsilon/3$. This is possible since $\sum_{n=1}^{\infty} p_n$ is weak* convergent in $\mathfrak{M}^{**}$ and ψ is weak* continuous on A^{**}. Since $\{p_n\}$ is a truly diffuse sequence, 3.2 of [4] allows us to find $n_1 > n_0$ such that for $j \geqslant n_1$, $\psi_j\Big(\sum\limits_{n=n_1}^{\infty} p_n\Big) < \varepsilon/3$. Choose $n_2 > n_1$ such that for $j > n_2$, $|(\psi_j - \psi)\Big(\sum\limits_{i=1}^{n_1-1} p_i\Big) < \varepsilon/3$. Now for $j > n_2$,

$$\left|\Big(\psi_j-\psi\Big)\Big(\sum_{i=1}^{\infty}p_i\Big)\right| < \left|\Big(\psi_j-\psi\Big)\Big(\sum_{i=1}^{n_1-1}p_i\Big)\right|+\left|\psi_j\Big(\sum_{n=n_1}^{\infty}p_n\Big)\right|+\left|\psi\Big(\sum_{n=n_1}^{\infty}p_n\Big)\right| < \varepsilon.$$

This shows that $p\mathbf{z}$ is in $\mathfrak{M}_c$. Since p was not in $M(\mathfrak{M}) = \mathfrak{M}$, then p is not regular by Corollary 3.14. ∎

4. AUTOMATIC REGULARITY OF LARGE PROJECTIONS

In this section, we present a few positive results on regularity that we found while trying to settle the general questions described in the introduction. It is our hope that they can be used to work around some of the complications that the existence of non-regular dense open projections lead to.

4.1. "BOTTOM UP" REGULARITY. The following lemma follows directly from the definitions and the fact that central projections are regular. Nevertheless, it plays a key role in establishing our more profound results at the end of the section.

LEMMA 4.1. *Let $p \in \mathfrak{A}^{**}$ be a projection.*

(i) *p is regular and dense if and only if*

$$\forall a \in \mathfrak{A}, \quad \|ap\| = \|a\|.$$

(ii) *If p dominates a regular and dense projection, then p is also regular and dense.*

(iii) *If p dominates a central and dense projection, then p is regular and dense.*

As an example of the strength of this form of reasoning, note that it gives a short alternative proof of 3.4 from [19], since when $\mathbb{K}$ is an essential ideal in $\mathfrak{A}$ and p is dense and open in $\mathfrak{A}^{**}$, then p must dominate the cover of $\mathbb{K}$, which is dense and central.

Before we move on to further consequences, we need a few preliminaries:

REMARK 4.2. If $\mathfrak{I}$ is a closed ideal of $\mathfrak{A}$, then there is a central open projection x in $\mathfrak{A}^{**}$ such that $\mathfrak{I} = \mathfrak{A} \cap x\mathfrak{A}^{**}$. In this setting, and $\mathfrak{I}^{**}$ is isometrically isomorphic to $x\mathfrak{A}^{**}$ (see [17], 3.10.7). Further, $(\mathfrak{A}/\mathfrak{I})^{**}$ is isometrically isomorphic to $(1-x)\mathfrak{A}^{**}$. The first isomorphism respects $\mathbf{z}$ in the sense that

$$x\mathbf{z}_{\mathfrak{A}} = \mathbf{z}_{\mathfrak{I}}$$

when $\mathbf{z}_{\mathfrak{I}}$ denotes the sum of the minimal projections in $\mathfrak{I}^{**}$ and $\mathbf{z}_{\mathfrak{A}}$ denotes the sum of the minimal projections in $\mathfrak{A}^{**}$ (see [17], 3.13.6 (iii)).

The notion of a *scattered* C^*-algebra from [15] will also be useful. Here, a C^*-algebra is defined to be scattered if no state of $\mathfrak{A}$ is diffuse, cf. Definition 3.17 (iii). By 2.2 from [15] $\mathfrak{A}$ is scattered precisely if $\mathbf{z} = 1$. We use these facts in the next proof.

PROPOSITION 4.3. *Let $\mathfrak{A}$ be a C^*-algebra that has no nonzero scattered ideal. Then any projection dominating $1 - \mathbf{z}$ is a dense regular projection.*

Proof. By Lemma 4.1 (iii) it suffices to prove that $1 - \mathbf{z}$ is dense. Let $x = 1-(1-\mathbf{z})^-$. If $x = 0$, then $1-\mathbf{z}$ is dense, so, to reach a contradiction, assume that $x \neq 0$. Since x is an open central projection, $\mathfrak{I} = \mathfrak{A} \cap x\mathfrak{A}^{**}$ is a nonzero ideal. By hypothesis $\mathfrak{I}$ is not scattered, so after identification as explained above, $\mathbf{z}_{\mathfrak{I}} < x$. Thus, by Remark 4.2 again,

$$0 < (x - \mathbf{z}_{\mathfrak{I}}) \leqslant 1 - \mathbf{z}_{\mathfrak{A}},$$

contradicting the definition of x. ∎

COROLLARY 4.4. *If $\mathfrak{A}$ is antiliminary, then any projection dominating $1-\mathbf{z}$ is dense and regular.*

Proof. By 3.2 from [15] any scattered C^*-algebra is type I. Since $\mathfrak{A}$ is antiliminary, it has no nonzero type I ideals, so the conclusion follows by Proposition 4.3. ∎

PROPOSITION 4.5. *Assume that $\mathfrak{A}$ is a C^*-algebra with a faithful tracial state τ. If p is an projection in $\mathfrak{A}^{**}$ such that $\tau(p) = 1$, then p is regular and dense.*

Proof. Let x be the support projection of τ in $\mathfrak{A}^{**}$. Since τ is a trace, it is unitarily invariant, and that immediately implies that x is a central projection. Since $\tau(p) = 1$, p must dominate x by the definition of x as the support projection of τ. Since τ is faithful, x is dense. Lemma 4.1 (iii) implies that p is regular and dense. ∎

REMARK 4.6. The situation in Proposition 4.5 can arise in many ways. For example, let $\mathfrak{M}$ be the 2^∞ UHF algebra with trace τ. Recursively choose an orthogonal family $\{p_n\}$ of projections in $\mathfrak{M}$ such that $\tau(p_n) = 2^{-n}$. (There are uncountably many distinct ways to do this.) Then $\sum\{p_n\}$, taken in $\mathfrak{M}^{**}$, is an open projection with trace 1. Even in an algebra with no nontrivial projections, e.g. the reduced C^*-algebra of the free group on two generators, this same recursive construction can take place, except that the projections p_n will be open in $\mathfrak{M}^{**}$, not lying in $\mathfrak{M}$ itself. We view this as a “bottom up” method of constructing dense, regular open projections. By contrast, the “top down” method of Corollary 4.9 below shows that certain projections that are constructed by subtracting closed projections from the identity are also regular and dense.

We end this section with the following useful lemma, in which we tacitly invoke the isomorphisms from Remark 4.2. We point out that one direction of Lemma 4.7 (ii) below was already noted in 3.5 from [19].

LEMMA 4.7. *Let $\mathfrak{A}$ be a C^*-algebra with an ideal $\mathfrak{I}$ whose central cover in $\mathfrak{A}^{**}$ is x.*

(i) *A projection q in $\mathfrak{A}^{**}$ is regular and dense for $\mathfrak{A}$ if and only if xq is regular and dense for $\mathfrak{I}$ and $(1-x)q$ is regular and dense for $\mathfrak{A}/\mathfrak{I}$.*

(ii) *If $\mathfrak{I}$ is essential, then q is regular and dense for $\mathfrak{A}$ if and only if xq is regular and dense for $\mathfrak{I}$.*

Proof. The forward direction is trivial from Lemma 4.1 (i). Now assume that xq is regular and dense for $\mathfrak{I}$ and $(1-x)q$ is regular and dense for $\mathfrak{A}/\mathfrak{I}$. Let b be a norm 1 element of $\mathfrak{A}$. Suppose that $\|bq\| < 1$. Then $\|b\| = 1$ implies that either $\|xb\| = 1$ or $\|(1-x)b\| = 1$. If $\|(1-x)b\| = 1$, then $\|(1-x)bq\| = 1$ by the regularity and density of $(1-x)q$, contradicting $\|bq\| < 1$. Thus we can assume that $\|xb\| = 1$. Choose a positive norm one element a of $\mathfrak{I}$ such that $\|ab\| > \|bq\|$. This is possible since x is the weak* limit of elements of $\mathfrak{I}$ of norm less than one, hence xb is the weak* limit of elements of the form ab. Since norm closed balls are also weak* closed (by the definition of the dual space norm), the set of all elements of the form ab can't all be contained in a ball about 0 with radius strictly less than $\|xb\| = 1$. However, ab lies in $\mathfrak{I}$, so by regularity and density of xq,

$$\|bq\| = \|a\|\,\|bq\| \geqslant \|abxq\| = \|ab\| > \|bq\|,$$

a contradiction, proving (i)

By Lemma 4.1, it suffices to show that if xq is regular and dense for $\mathfrak{I}$, then q is regular and dense for $\mathfrak{A}$. Let b lie in $\mathfrak{A}$ with $\|b\| = 1$. It suffices to show that $\|qb\| = 1$. Since $\mathfrak{I}$ is essential, the map $A \mapsto xA$ has no kernel, hence is isometric. Thus $\|xb\| = 1$, and the proof of (ii) now proceeds as the proof of (i). ∎

4.2. "TOP DOWN" REGULARITY. We end the paper by a collection of results which generalize 3.6 from [19]. Many ingredients in the proof below were indeed borrowed from that source.

THEOREM 4.8. *Let $\{p_n\}_{n\in\mathcal{N}}$ be a countable family of minimal projections in $\mathfrak{A}^{**}$, with the property that with*

$$\mathcal{F} = \{n \in \mathcal{N} \mid p_n \in \mathfrak{A}\},$$

then $\mathcal{F}$ is a finite (possibly void) set. Set

$$p = \bigvee_{n\in\mathcal{F}} p_n, \quad q = 1 - \bigvee_{n\in\mathcal{N}} p_n.$$

Then q is regular and $q^- = 1 - p$.

Proof. The proof will be in steps and will keep the notation above. Other symbols may be reused from one step to the next.

STEP 1. REDUCTION TO $\mathcal{F} = \emptyset$. Since each p_n for n in F is both open and closed, so is the supremum by 2.5 and 2.9 from [1], hence $1-p$ is both open and closed, and consequently a multiplier of $\mathfrak{A}$ ([16], 2.5). Thus q^- can be no larger than $1-p$. Both conclusions of the theorem will therefore follow if we can show that for any positive, norm 1 element b of $(1-p)\mathfrak{A}(1-p)$, $\|pb\| = 1$. We may thus, without loss of generality, assume that $p = 0$, i.e. that $\mathcal{F}$ is void.

STEP 2. REDUCTION TO CASES. By 6.2.7 from [17] there is a largest type I ideal $\mathfrak{I}$ of $\mathfrak{A}$ and $\mathfrak{A}/\mathfrak{I}$ is antiliminary. Let x be the central cover of $\mathfrak{I}$ in $\mathfrak{A}^{**}$. By Lemma 4.7 (i) it suffices to show that xq is regular and dense for $\mathfrak{I}$ and $(1-x)q$ is regular and dense for $\mathfrak{A}/\mathfrak{I}$. We shall show in the next paragraph that both xq and $(1-x)q$ can be expressed as required in the hypothesis of the present theorem. Thereafter it will suffice to demonstrate the theorem separately in the cases below.

Note that since x is central, a rank 1 projection in $\mathfrak{A}^{**}$ lies under x or under $1-x$. Thus the projections $\{p_n \mid n \in \mathbb{N}\}$ are partitioned into two subsets, those lying under x and those lying under $1-x$. Obviously a projection that lies in $\mathfrak{I}$ must lie in $\mathfrak{A}$. Therefore xq is the complement of the supremum of a countable family of rank 1 projections in $\mathfrak{I}^{**}$, none of which lies in $\mathfrak{I}$. As for $(1-x)q$, clearly it is the complement of the supremum of a countable family of rank 1 projections in $(\mathfrak{A}/\mathfrak{I})^{**}$. Since $\mathfrak{A}/\mathfrak{I}$ is antiliminary, it can't contain any rank one projections by 6.1.7 from [17]. Therefore both xq and $(1-x)q$ can be expressed as hypothesized.

STEP 3. THE ANTILIMINARY CASE. Assume that $\mathfrak{A}$ is antiliminary. Since q dominates $1-\mathbf{z}$, we are done by Corollary 4.4.

STEP 4. THE TYPE I CASE. Assume that $\mathfrak{A}$ is type I. By 6.2.11 from [17] $\mathfrak{A}$ contains an essential ideal $\mathfrak{J}$ that has continuous trace. Arguing as above with Lemma 4.7 (ii) we can pass to the case below.

STEP 5. THE CONTINUOUS TRACE CASE. Assume that $\mathfrak{A}$ has continuous trace, and recall that the spectrum of $\mathfrak{A}$ is a locally compact Hausdorff space by 6.1.11 from [17]. Since q dominates the complement of the supremum r of the central covers of the $\{p_n\}$, it suffices to prove that r is a dense central projection. This follows by a category argument, since r is represented in the spectrum of $\mathfrak{A}$ as the complement of a countable set (namely the central covers of the projections $\{p_n\}$), which is still a dense set. Of course countability of $\{p_n\}$ is crucial here. ∎

COROLLARY 4.9. *Let $\mathfrak{A}$ be any C^*-algebra, and suppose $p \in \mathfrak{A}^{**}$ has finite codimension. Then p is regular.*

COROLLARY 4.10. *If $\mathfrak{A}$ contains no minimal projections, and $\{\varphi_n\}_{n\in\mathbb{N}}$ are pure states of $\mathfrak{A}$ with $\{p_n\}_{n\in\mathbb{N}}$ their corresponding support projections in $\mathfrak{A}^{**}$, then*

$$1 - \bigvee_{n\in\mathbb{N}} p_n$$

is regular and dense.

REMARK 4.11. Note that when every ideal of $\mathfrak{A}$ has a diffuse state, the countability condition in the corollary above is unnecessary as any projection of the form

$$1 - \bigvee_{i\in I} p_i,$$

with p_i minimal, dominates $1-\mathbf{z}$ which is regular and dense by Proposition 4.3.

Indeed, it would be possible to strengthen the results above further by combining these ideas. We refrain from this for the moment.

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