J. OPERATOR THEORY
48(2002), 621–632

HILBERT C^*-MODULES WITH A PREDUAL

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Communicated by William B. Arweson

Abstract. We extend Sakai's characterization of von Neumann algebras to the context of Hilbert C^*-modules. If A, B are C^*-algebras and X is a full Hilbert A-B-bimodule possessing a predual such that left, respectively right, multiplications are weak*-continuous, then $\mathrm{M}(A)$ and $\mathrm{M}(B)$ are W^*-algebras, the predual is unique, and X is selfdual in the sense of Paschke. For unital A, B the above continuity requirement is automatic.

We determine the dual Banach space X^* of a Hilbert A-B-bimodule X and show that Paschke's selfdual completion of X is isomorphic to the bidual X^{**}, which is a Hilbert C^*-module in a natural way. We conclude with a new approach to multipliers of Hilbert C^*-bimodules.

Keywords: *Hilbert W^*-module, Hilbert C^*-module, correspondence.*

MSC (2000): 46L08.

In [8] Paschke observed that the C^*-algebra of adjointable operators $\mathcal{L}(E)$ on a Hilbert C^*-module E over a von Neumann algebra need not be a von Neumann algebra itself. However, he showed that, if one asks E to be selfdual in a sense analogous to the selfduality of Hilbert spaces, then $\mathcal{L}(E)$ is a von Neumann algebra. Selfdual Hilbert C^*-module have henceforth been used in the theory of Morita equivalence ([10]) and in index theory ([1]). In the present paper, we shall give a new approach to these modules in the spirit of Sakai's abstract characterization of von Neumann algebras ([11], [12]).

In Section 1 we shall recall some basic concepts and introduce our definition of Hilbert W^*-modules along with some examples. In particular, we shall see that the modules appearing in Rieffel's theory of Morita equivalence of von Neumann algebras are Hilbert W^*-modules, yielding a new proof of their selfduality which does not rely on an orthonormalization procedure. We shall also see that every correspondence in the sense of Connes ([3]) gives rise to a Hilbert W^*-module.

Section 2 contains our main result, Theorem 2.6, which states that every Hilbert W^*-module X over C^*-algebras A and B is selfdual, the multiplier algebras $\mathrm{M}(A)$, $\mathrm{M}(B)$ are von Neumann algebras, the predual is unique, and the module operations are separately weak*-continuous. To achieve this, we characterize in Lemma 2.2 those bounded B-linear mappings $\rho : X \to B$ which can be obtained as $\rho = (\xi|\cdot)_B$ for an appropriate element $\xi \in X$. This leads to an elementary proof of the selfduality of finitely generated Hilbert C^*-modules (Proposition 2.3 and Remark 2.4). A converse of Theorem 2.6 is provided by Proposition 2.9, which gives various equivalent conditions for a Hilbert C^*-module over a von Neumann algebra to be a Hilbert W^*-module, simplifying some arguments in [5] and [1].

In Section 3, we shall show that the topological dual X^* of a Hilbert B-module X is isomorphic to the projective tensor product of the dual B^* with X (or rather the adjoint module X^o), which implies that Paschke's selfdual completion of X is isomorphic to the enveloping Hilbert W^*-module of Section 1. Moreover, we shall see that the bounded module mappings into the coefficient C^*-algebras, extensively used in Section 2, can be thought of as one-sided multipliers of X and realized in X^{**}, which permits to define the multiplier bimodule of X as the intersection of left and right multipliers, providing a new approach to the multiplier bimodule of Echterhoff and Raeburn ([4]).

CONVENTIONS. We shall denote the topological dual of a Banach space V by V^*. Whenever we apply a bilinear operation to sets, we shall mean the linear span of all possible products.

1. DEFINITIONS AND EXAMPLES

A *Hilbert A-B-bimodule* ([2]), or simply a *Hilbert C^*-bimodule*, is an A-B-bimodule X together with inner products ${}_A(\cdot|\cdot) : X \times X \to A$, and $(\cdot|\cdot)_B : X \times X \to B$ such that

$$ {}_A(\xi|\eta)\zeta = \xi(\eta|\zeta)_B, \quad \text{for all } \xi, \eta, \zeta \in X. $$

We also require that X is complete with respect to the norm $\|\xi\| := \|(\xi|\xi)_B\|$, $\xi \in X$, and note that $\|(\xi|\xi)_B\| = \|{}_A(\xi|\xi)\|$ for all $\xi \in X$ ([2]). By convention ${}_A(\cdot|\cdot)$ is assumed to be A-linear in its first argument, whereas $(\cdot|\cdot)_B$ is B-linear in its second argument. We will often assume that X is *full*, i.e., the closed linear span of ${}_A(X|X)$, and $(X|X)_B$ is all of A, and B, respectively. Full Hilbert A-B-bimodules will be referred to as *equivalence bimodules*.

A right B-module E is a *Hilbert B-module*, or *Hilbert C^*-module*, if it possesses a B-valued inner product and is complete with respect to the corresponding norm ([6], [13]). Every such module gives rise to a *Hilbert A-B-bimodule* with $A = \mathcal{K}(E)$, the C^*-algebra of *generalized compact operators*, which is generated by the "rank one" operators $\theta_{\xi,\eta}$, $\xi, \eta \in E$ where $\theta_{\xi,\eta}(\zeta) = \xi(\eta|\zeta)_B$, $\zeta \in E$. The left handed inner product is then defined by ${}_A(\xi|\eta) := \theta_{\xi,\eta}$, $\xi, \eta \in E$. Replacing B by the closed ideal generated by $(E|E)_B$, we obtain an equivalence bimodule. In the present article, we shall thus consider Hilbert C^*-modules as special cases of equivalence bimodules.

1.1. Definition. A *Hilbert W^*-module over A and B* is an equivalence bimodule X possessing a predual, i.e., $X = V^*$, where V^* is the topological dual of a Banach space V, such that the mappings $\xi \mapsto a\xi$ and $\xi \mapsto \xi b$ are weak*-continuous for all $a \in A$, $b \in B$.

1.2. Remark. It will be useful to have the following reformulation of the continuity requirements of Definition 1.1. The topological dual X^* becomes a Banach B-A-module by setting $bfa := f(a \cdot b)$, $a \in A$, $b \in B$. Identifying V with its canonical image in X^*, the above continuity requirements are equivalent to asking V to be a B-A-submodule of X^*.

We shall tacitly use the identification $V \subseteq X^*$, as well as the bimodule structure of V.

In the following examples the C^*-algebras A and B come naturally embedded in von Neumann algebras $\mathcal{M}, \mathcal{N}$, respectively. We shall thus also speak of *Hilbert W^*-modules over $\mathcal{M}$ and $\mathcal{N}$*. Of course, the C^*-algebras are easily recovered as the closed linear span of the inner products. In general, A and B need not coincide with $\mathcal{M}$ and $\mathcal{N}$. This is easily seen by considering an infinite dimensional Hilbert space $\mathcal{H}$, which is a Hilbert W^*-module over $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})$ and $\mathbb{C}$.

1.3. Concrete Hilbert W^*-modules. We let $\mathcal{H}$ be a Hilbert space and $X \subseteq \mathcal{L}(\mathcal{H})$ a weakly closed subspace such that $XX^*X \subseteq X$. Then X becomes a *concrete Hilbert W^*-module over* $A := \overline{XX^*}$, $B := \overline{X^*X}$, putting $a \cdot \xi \cdot b := a\xi b$, and ${}_A(\xi|\eta) := \xi\eta^*$, $(\xi|\eta)_B := \xi^*\eta$ for all $a \in A$, $b \in B$, $\xi, \eta \in X$. We shall see below (Remark 2.7) that every Hilbert W^*-module is isomorphic to a concrete one.

For a given Hilbert A-B-bimodule X, we shall need its *adjoint* $X^o := \{\xi^o : \xi \in X\}$ which becomes a Hilbert B-A-module upon setting

$$(\xi^o|\eta^o)_A := {}_A(\xi|\eta), \quad {}_B(\xi^o|\eta^o) := (\xi|\eta)_B, \quad \text{and} \quad b\xi^o a := (a^*\xi b^*)^o,$$

as well as its *linking algebra* L ([2]), which is defined by

$$L = \left\{ \begin{pmatrix} a & \xi \\ \eta^o & b \end{pmatrix} : a \in A,\, b \in B,\, \xi \in X,\, \eta^o \in X^o \right\},$$

$$\begin{pmatrix} a_1 & \xi_1 \\ \eta_1^o & b_1 \end{pmatrix} \begin{pmatrix} a_2 & \xi_2 \\ \eta_2^o & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + {}_A(\xi_1|\eta_2) & a_1\xi_2 + \xi_1 b_2 \\ \eta_1^o a_2 + b_1\eta_2^o & (\eta_1|\xi_2)_B + b_1b_2 \end{pmatrix},$$

$$\begin{pmatrix} a & \xi \\ \eta^o & b \end{pmatrix}^* = \begin{pmatrix} a^* & \eta \\ \xi^o & b^* \end{pmatrix}.$$

There is a unique norm turning L into a C^*-algebra ([2]). Setting $p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}(L)$, and $q := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{M}(L)$, we obtain $A \cong pLp$, $B \cong qLq$, and $L \cong pLq$.

If $\mathcal{M}$ is a von Neumann algebra and $p, q \in \mathcal{M}$ are projections, then $p\mathcal{M}q$ is a Hilbert W^*-module over $p\mathcal{M}p$, $q\mathcal{M}q$, using the product of $\mathcal{M}$ and putting ${}_{p\mathcal{M}p}(\xi|\eta) := \xi\eta^*$, $(\xi|\eta)_{q\mathcal{M}q} := \xi^*\eta$ for $\xi, \eta \in p\mathcal{M}q$. This observation is used in the following examples.

1.4. THE ENVELOPING HILBERT W^*-MODULE. Let X be a Hilbert A-B-bimodule. Then its Banach space bidual X^{**} is a Hilbert W^*-module over the enveloping von Neumann algebras A^{**}, B^{**} in a natural way. Indeed, embedding X in its linking algebra L, it is easy to see that $A^{**} \cong pL^{**}p$, $B^{**} \cong qL^{**}q$, and $X^{**} \cong pL^{**}q$, where $p, q \in \mathrm{M}(L)$ are the canonical projections. So that X^{**} inherits its structure of a Hilbert W^*-module from L^{**}.

In the following, we will consider X as a subset of X^{**}. We note that the module operations of X^{**} are the unique separately weak*-continuous extensions of those of X.

1.5. INTERTWINERS AND MORITA EQUIVALENCE. For two representations $\pi_i : A \to \mathcal{L}(\mathcal{H}_i)$, $i \in \{1, 2\}$ of a C^*-algebra A the space of intertwiners

$$X = \{T \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1) : T\pi_2(a) = \pi_1(a)T \text{ for all } a \in A\}$$

is a Hilbert W^*-module over the commutants $\pi_1(A)'$ and $\pi_2(A)'$. Indeed, considering the direct sum representation $\pi_1 \oplus \pi_2 : A \to \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ with canonical projections $p_1, p_2 \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, and setting $L := (\pi_1 \oplus \pi_2)(A)'$, we obtain $\pi_1(A)' \cong p_1Lp_1$, $\pi_2(A)' \cong p_2Lp_2$, and $X \cong p_1Lp_2$. The so obtained Hilbert W^*-module is a fundamental device in the theory of Morita equivalence of von Neumann algebras ([10]).

1.6. CORRESPONDENCES. We let $\mathcal{M}, \mathcal{N}$ be von Neumann algebras. A correspondence in the sense of Connes ([3]) is a Hilbert space $\mathcal{H}$ which is a normal $\mathcal{M}$-$\mathcal{N}$-bimodule, or equivalently, which possesses a normal representation of $\mathcal{M} \otimes \mathcal{N}^{\mathrm{opp}}$, where $\mathcal{N}^{\mathrm{opp}}$ is the opposite algebra of $\mathcal{N}$. Every correspondence $\mathcal{H}$ gives rise to a Hilbert W^*-module by comparing it to the identity correspondence over $\mathcal{N}$, which is simply given by the standard Hilbert space $L^2(\mathcal{N})$ for $\mathcal{N}$ and the module multiplications $a\xi b := aJb^*J\xi$, $a, b \in \mathcal{N}$, $\xi \in L^2(\mathcal{N})$, where J is Tomita's involution. Indeed, the space of bounded right module operators

$$X := \{T \in \mathcal{L}(L^2(\mathcal{N}), \mathcal{H}) : T(\xi a) = T(\xi)a,\ a \in \mathcal{N},\ \xi \in L^2(\mathcal{N})\}$$

is a Hilbert W^*-module, which is most easily seen by letting π_1, π_2 denote the right representations of $\mathcal{N}^{\mathrm{opp}}$ on $\mathcal{H}$ and $L^2(\mathcal{N})$, respectively, and observing that X is a special case of the previous example with $A = \mathcal{N}^{\mathrm{opp}}$. Hence, X is a Hilbert W^*-module over $\pi_1(\mathcal{N}^{\mathrm{opp}})' \supseteq \mathcal{M}$ and $\pi_2(\mathcal{N}^{\mathrm{opp}})' \cong \mathcal{N}$.

2. MAIN RESULTS

Paschke calls a Hilbert B-module E *selfdual* ([8]) if for every bounded B-linear mapping $\rho : E \to B$ there is an element $\eta \in E$ such that $\rho = (\eta|\cdot)_B$. This terminology is somewhat misleading since it suggests that the space of all bounded B-linear mappings $\mathcal{B}_B(X, B)$ is a dual of E. In general, however, there is no natural way of constructing a B-valued inner product on $\mathcal{B}_B(X, B)$. So that, at least in the category of Hilbert C^*-modules, $\mathcal{B}_B(X, B)$ is not a dual of E. However, we shall stick to Paschke's terminology.

A Hilbert A-B-module X is said to be *B-selfdual* if it is selfdual as a Hilbert B-module. Correspondingly, we shall say that X is *A-selfdual* if, for every bounded A-linear mapping $\lambda : X \to A$, there is an element $\eta \in X$ such that $\lambda = {}_A(\cdot|\eta)$.

2.1. DEFINITION. A bounded B-linear mapping $\rho : X \to B$ is said to *vanish at infinity* if, for every approximate identity $\{u_\alpha\} \subseteq A$,

$$\|\rho(u_\alpha \cdot) - \rho\| \to 0$$

with respect to the operator norm.

If $\{u_\alpha\} \subseteq A$ is an approximate unit, then $\|u_\alpha\xi - \xi\| \to 0$ for every $\xi \in X$. In the proof of the following lemma, we shall use a particular approximate unit $\Big\{u_\alpha := \sum\limits_{i=1}^{n_\alpha} {}_A(\eta_i^\alpha|\eta_i^\alpha)\Big\}$ where $\{\eta_i^\alpha\} \subseteq X$ (cf. [2]).

2.2. LEMMA. *For a bounded B-linear mapping $\rho : X \to B$, the following conditions are equivalent:*

(i) *ρ vanishes at infinity;*

(ii) *there is $\eta \in X$ such that $\rho = (\eta|\cdot)_B$.*

Proof. We assume that $\rho : X \to B$ vanishes at infinity. Putting $\eta_\alpha = \sum\limits_i \eta_i^\alpha \rho(\eta_i^\alpha)^*$, where $\{\eta_i^\alpha\} \subseteq X$ determines an approximate identity for A, we find that, for every $\xi \in X$,

$$(\eta_\alpha|\xi)_B = \sum \rho(\eta_i^\alpha)(\eta_i^\alpha|\xi)_B = \rho\Big(\sum \eta_i^\alpha(\eta_i^\alpha|\xi)_B\Big) = \rho\Big(\sum {}_A(\eta_i^\alpha|\eta_i^\alpha)\xi\Big).$$

Thus, by assumption, $(\eta_\alpha|\cdot)_B \to \rho$ in norm. But as $\|(\eta_\alpha|\cdot)_B\| = \|\eta_\alpha\|$, we see that $\{\eta_\alpha\}$ is a Cauchy-net. Letting $\eta = \lim \eta_\alpha$, we thus obtain $\rho = (\eta|\cdot)_B$.

For the reverse implication, we observe that $(\eta|u_\alpha\cdot)_B = (u_\alpha\eta|\cdot)_B$ converges uniformly to $(\eta|\cdot)_B$ since $\|u_\alpha\eta - \eta\| \to 0$. ∎

A Hilbert A-B-bimodule X is said to be *finitely generated* as a right B-module if there is a finite set $\{\xi_i\}_{1\leqslant i\leqslant n} \subseteq X$ such that $X = \sum\limits_{i=1}^{n} \xi_i B$.

2.3. Proposition. *Every Hilbert A-B-bimodule which is finitely generated as a right B-module is B-selfdual.*

Proof. We have to show, by Lemma 2.2, that every bounded B-linear mapping $\rho : X \to B$ vanishes at infinity. By assumption, there is $\{\xi_i\}_{1\leqslant i\leqslant n} \subseteq X$ such that the bounded B-linear mapping

$$P : B^n \to X, \quad (b_i) \to \sum_{i=1}^{n} \xi_i b_i$$

is surjective, hence, open by the open mapping theorem. There is thus a real constant $\gamma > 0$ such that, for every $\xi \in X$, there is $(b_i) \in B^n$ with $P((b_i)) = \xi$ and $\|(b_i)\| \leqslant \gamma\|\xi\|$. The latter condition implies that $\|b_i\| \leqslant \gamma\|\xi\|$, $1 \leqslant i \leqslant n$. For an approximate identity $\{u_\alpha\} \subseteq A$ and $\xi = P((b_i)) = \sum \xi_i b_i \in X$ we have

$$\begin{aligned}\|\rho(\xi) - \rho(u_\alpha \xi)\| &= \Big\|\rho\Big(\sum \xi_i b_i - \sum u_\alpha \xi_i b_i\Big)\Big\| = \Big\|\sum \rho(\xi_i - u_\alpha \xi_i) b_i\Big\| \\ &\leqslant \sum \|\rho\|\, \|\xi_i - u_\alpha \xi_i\|\, \|b_i\| \leqslant \gamma\|\xi\|\, \|\rho\| \sum \|\xi_i - u_\alpha \xi_i\|.\end{aligned}$$

Hence, $\|\rho - \rho(\,\cdot\, u_\alpha)\| \to 0$ as desired. ∎

2.4. Remark Using the adjoint X^o of the Hilbert A-B-bimodule X, it is clear that there are corresponding results for the left module structure of X. In particular, if X is finitely generated as an A-module, then it is A-selfdual.

For Theorem 2.6 below, we need to recall the relationship between multipliers of A, B and operators on an equivalence bimodule X over A and B.

We let $\mathrm{LM}(A)$, and $\mathrm{RM}(B)$, denote the Banach algebras of left, respectively right, multipliers of A, and B ([9]). For every $a \in \mathrm{LM}(A)$ and approximate unit $\{u_\alpha\} \subseteq A$ it is easy to see that, for $\xi \in X$, $((a u_\alpha)\xi)_\alpha$ is a Cauchy-net. We may thus define $\Lambda(a)\xi := \lim(a u_\alpha)\xi$, which is independent of the choice of the approximate unit. This yields an isometric isomorphism $\Lambda : \mathrm{LM}(A) \to \mathcal{B}_B(X)$, where $\mathcal{B}_B(X) = \mathcal{B}_B(X, X)$ denotes the Banach algebra of bounded B-linear operators on X. The inverse of Λ is obtained by observing that for $T \in \mathcal{B}_B(X)$, we have $T{}_A(\xi|\eta)\zeta = T\xi(\eta|\zeta)_B = {}_A(T\xi|\eta)\zeta$. So that T gives rise to a right centralizer of A.

The isomorphism Λ takes the multiplier algebra $\mathrm{M}(A)$ onto the C^*-algebra of adjointable (with respect to $(\,\cdot\,|\,\cdot\,)_B$) operators $\mathcal{L}_B(X)$ (cf. [7]). Similarly, there is an isomorphism from $\mathrm{RM}(B)$ onto the Banach algebra of bounded A-linear mappings $\mathcal{B}_A(X)$, such that $\mathrm{M}(B)$ is mapped onto the C^*-algebra of adjointable (with respect to ${}_A(\,\cdot\,|\,\cdot\,)$) operators $\mathcal{L}_A(X)$.

2.5. Lemma. *Let X be a Hilbert C^*-module over the von Neumann algebra $\mathcal{M}$. If X possesses a predual such that all $(\,\cdot\,|\xi)_{\mathcal{M}}$, $\xi \in X$ are continuous on bounded subsets for the corresponding weak*-topologies, then X is $\mathcal{M}$-selfdual.*

Proof. We let $\rho : X \to \mathcal{M}$ be a bounded $\mathcal{M}$-linear mapping and choose an approximate identity $\{u_\alpha\} \subseteq A := \mathcal{K}(X)$. Since, for fixed α, $\rho(u_\alpha\,\cdot\,)$ vanishes at infinity, there is $\eta_\alpha \in X$ such that $\rho(u_\alpha\,\cdot\,) = (\eta_\alpha|\,\cdot\,)_B$. Moreover, $\|\eta_\alpha\| = \|\rho(u_\alpha\,\cdot\,)\| \leqslant \|\rho\|$. Since the unit ball of X is weak*-compact we can choose a subnet $\{\eta_{\alpha_i}\}$ with weak*-limit point $\eta \in X$. So that, for all $\xi \in X$,

$$(\eta|\xi)_B = \lim_i (\eta_{\alpha_i}|\xi)_B = \lim_i \rho(u_{\alpha_i}\xi) = \rho(\xi). \quad ∎$$

In what follows, we shall denote the projective tensor product of Banach spaces by $\widehat{\otimes}$. Its module version is obtained by an obvious quotient construction. We use the canonical identification $(X\widehat{\otimes}Y)^* \cong \mathcal{B}(X, Y^*)$, for Banach spaces X, Y.

2.6. THEOREM. *Let X be a Hilbert W^*-module, and let V denote a predual of X. Then X is A-selfdual and B-selfdual. Moreover, $M(A)$ and $M(B)$ are W^*-algebras with preduals $X\widehat{\otimes}_B V$ and $V\widehat{\otimes}_A X$, respectively. The predual V is unique, and the module multiplications and inner products are separately weak*-continuous.*

Proof. As $X\widehat{\otimes}_B V$ is a quotient of $X\widehat{\otimes}V$, there is an isometry $\Theta : (X\widehat{\otimes}_B V)^* \to \mathcal{B}(X, V^*) = \mathcal{B}(X)$ such that, for all $F \in (X\widehat{\otimes}_B V)^*$,

$$f(\Theta(F)\xi) = F(\xi \otimes_B f), \quad f \in V, \xi \in X.$$

The range of Θ actually coincides with the subset $\mathcal{B}_B(X)$ of bounded B-linear mappings. Indeed, it is clear that $\operatorname{im}\Theta \subseteq \mathcal{B}_B(X)$. On the other hand, every $T \in \mathcal{B}_B(X)$ defines a functional $F : \xi \otimes_B f \mapsto f(T\xi)$ such that $\Theta(F) = T$.

Similarly, we obtain $(V\widehat{\otimes}_A X)^* \cong \mathcal{B}_A(X) \cong \mathrm{RM}(B)$ with duality given by

$$\langle f \otimes_A \xi, b\rangle = f(\xi b), \quad f \in V, \xi \in X, b \in \mathrm{RM}(B).$$

Hence, $\mathrm{RM}(B)$ is a dual Banach space and carries thus a weak*-topology. Now observe that $(\eta|\cdot)_B$, $\eta \in X$, is continuous on bounded subsets with respect to the corresponding weak*-topologies. In fact, for $\sum f_i \otimes_A \xi_i \in V\widehat{\otimes}_A X$, we have

$$\left\langle \sum f_i \otimes_A \xi_i, (\eta|\cdot)_B \right\rangle = \sum f_i(\xi_i(\eta|\cdot)_B) = \sum f_i({}_A(\xi_i|\eta)\cdot) \in V.$$

Hence, by the same arguments as in the proof of Lemma 2.5, X is B-selfdual. It follows as in Hilbert space theory, that every bounded B-linear mapping is adjointable. Consequently, $\mathrm{M}(A) \cong \mathcal{L}_B(X) = \mathcal{B}_B(X)$ is a W^*-algebra.

Analogously, one obtains that X is A-selfdual and $(V\widehat{\otimes}_A X)^* \cong \mathrm{M}(B)$.

For the last assertion, we observe that the multiplier algebra $\mathrm{M}(L)$ of the linking algebra L is isomorphic to the Banach space dual of

$$\begin{pmatrix} X\widehat{\otimes}_B V & V \\ V & V\widehat{\otimes}_B X \end{pmatrix},$$

and hence a W^*-algebra. So that the remaining assertions follow from the well-known theorems of Sakai. ∎

2.7. REMARK. Let X be a Hilbert W^*-module and L its linking algebra. We have seen in the proof of Theorem 2.6 that $\mathcal{M} = \mathrm{M}(L)$ is a W^*-algebra. Hence, every Hilbert W^*-module is a corner of a W^*-algebra.

Every faithful normal representation π of $\mathrm{M}(L)$ on a Hilbert space $\mathcal{H}$ yields a realization of X, $\pi(X)$, which obviously satisfies $\pi(X)\pi(X)^*\pi(X) \subseteq \pi(X)$. So that X is isomorphic to a concrete Hilbert W^*-module.

2.8. THEOREM. *Let X be an equivalence bimodule over unital C^*-algebras A and B. If X possesses a predual, then it is a Hilbert W^*-module and A, B are von Neumann algebras.*

Proof. By our assumtions, we have that $\mathrm{LM}(A) = A$ and $\mathrm{RM}(B) = B$. But we saw in the proof of Theorem 2.6 without using the continuity requirements of

Definition 1.1, that $\mathrm{LM}(A)$, and $\mathrm{RM}(B)$, are dual Banach spaces. Hence, A and B are von Neumann algebras.

Continuing as in the final paragraph of the proof of Theorem 2.6, we see that the linking algebra L of X is a W^*-algebra and A, B, X are corners in L. The weak*-continuity of $a \in A$, $b \in B$ as operators on X now follows from the corresponding theorems of Sakai. ∎

Suppose we are given a Hilbert C^*-module over a von Neumann algebra. We may ask for further conditions which imply that it is a Hilbert W^*-module. The following proposition gives three such conditions. Its proof simplifies some arguments in [1] and [5], where the equivalence of conditions two and three is also proved.

2.9. PROPOSITION. *For a Hilbert C^*-module X over the von Neumann algebra $\mathcal{M}$ the following conditions are equivalent:*

(i) *X possesses a predual V such that all $(\,\cdot\,|\xi)_{\mathcal{M}}$, $\xi \in X$ are continuous on bounded subsets for the corresponding weak*-topologies;*

(ii) *X is $\mathcal{M}$-selfdual;*

(iii) *the unit ball of X is complete with respect to the locally convex topology induced by the family of seminorms*

$$N = \{|\varphi((\,\cdot\,|\xi)_{\mathcal{M}})| : \varphi \in \mathcal{M}_*,\, \xi \in X\};$$

(iv) *X is a Hilbert W^*-module.*

Proof. The implication (i) $\Rightarrow$ (ii) is Lemma 2.5.

(ii) $\Rightarrow$ (iii) Let $\{\eta_\alpha\}$ be a N-Cauchy net in the unit ball of X. For every $\varphi \in \mathcal{M}_*$ and $\xi \in X$, the net $\{\varphi((\eta_\alpha|\xi)_{\mathcal{M}})\}$ is Cauchy and bounded by $\|\xi\|\,\|\varphi\|$. Hence $\varphi \mapsto \lim \varphi((\eta_\alpha|\xi)_{\mathcal{M}})$ defines a bounded functional on $\mathcal{M}_*$, and there is $\rho(\xi) \in \mathcal{M}$ such that $\text{w*-}\lim\,(\eta_\alpha|\xi)_{\mathcal{M}} = \rho(\xi)$ and $\|\rho(\xi)\| \leqslant \|\xi\|$. Clearly, ρ is $\mathcal{M}$-linear and by assumption there is $\eta \in X$ with $\rho(\xi) = (\eta|\xi)_{\mathcal{M}}$. It follows that $N\text{-}\lim \eta_\alpha = \eta$.

(iii) $\Rightarrow$ (iv) Let V denote the norm closure of the linear span of $\{\overline{\varphi((\,\cdot\,|\xi)_{\mathcal{M}})} : \varphi \in \mathcal{M}_*,\, \xi \in X\}$ in X^*. The unit ball of X is still complete with respect to the weak topology induced by V. Note that V is a separating Banach space of functionals for X. We may thus consider the canonical embedding of X into V^*. This embedding is a homeomorphism with respect to the topologies induced by V, and the image of X is weak*-dense by the Hahn-Banach theorem. Consequently, the unit ball of X is weak*-complete in V^* and therefore weak*-closed. So is all of X by the Krein-Smulian theorem. Hence, $X = V^*$.

The continuity assertions follow immediately, taking $f = \sum \overline{\varphi_i((\,\cdot\,|\xi_i)_{\mathcal{M}})}$ and $a \in \mathcal{L}(X)$, $b \in \mathcal{M}$, from

$$bfa = \sum \overline{\varphi_i((a\cdot b)|\eta_i)_{\mathcal{M}})} = \sum \overline{\varphi_i((b^*(\,\cdot\,|a^*\eta)_{\mathcal{M}})} \in V \quad \text{and}$$
$$\|bfa\| = \|f(a\cdot b)\| \leqslant \|a\|\,\|b\|\,\|f\|.$$

The remaining implication (iv) $\Rightarrow$ (i) is part of Theorem 2.6. ∎

3. THE ENVELOPING HILBERT C^*-MODULE AND MULTIPLIERS

For every $\varphi \in B^*$ and $\xi \in X$, $\varphi((\xi|\cdot)_B)$ yields a continuous functional on X, whose norm is bounded by $\|\varphi\|\,\|\xi\|$. Considering B^* as a B-bimodule, we thus obtain a bounded homomorphism of B-A-modules $\Theta : B^*\widehat{\otimes}_B X^o \to X^o$, $\varphi \otimes \xi^o \mapsto \varphi((\xi|\cdot)_B)$, where the bimodule structure of X^o is given by $b\Phi a := \Phi(a \cdot b)$, $\Phi \in X^o$, $a \in A$, $b \in B$, and similarly for B^*.

The following proposition shows that Θ is isometric and onto.

3.1. PROPOSITION. *For a Hilbert A-B-bimodule X the mapping $\varphi \otimes \xi^o \mapsto \varphi((\xi|\cdot)_B)$ yields an isometric isomorphism of Banach B-A-modules, i.e.,*

$$B^*\widehat{\otimes}_B X^o \cong X^*.$$

Similarly, $\xi^o \otimes \varphi \mapsto \varphi({}_A(\cdot|\xi))$ yields an isometric isomorphism $X^o\widehat{\otimes}_A A^* \cong X^*$.

Proof. We shall show that the adjoint of Θ, $\Theta^* : X^{**} \to (B^*\widehat{\otimes}_B X^o)^*$ is a surjective isometry. Clearly, Θ itself will then be a surjective isometry.

Let us observe that for all $\eta \in X^{**}$, $\Theta^*(\eta)(\varphi \otimes_B \xi) = \varphi((\xi|\eta)_{B^{**}})$, which is obvious for elements in X and follows for general elements by continuity. It will be convenient to identify $(B^*\widehat{\otimes}_B X^o)^*$ and $\mathcal{B}_B(X^o, B^{**})$ via

$$\mathcal{B}_B(X^o, B^{**}) \to (B^*\widehat{\otimes}_B X^o)^*, \quad f \mapsto (\varphi \otimes_B \xi \mapsto \varphi(f(\xi))).$$

Hence, $\Theta^*(\eta) = (\cdot|\eta)_{B^{**}}$ for all $\eta \in X^{**}$. To see the surjectivity of Θ^*, let $f \in \mathcal{B}_B(X^o, B^{**})$. Its weak*-continuous extension $\widetilde{f} : (X^o)^{**} \to B^{**}$ is B^{**}-linear, so that, by Theorem 2.6, there is $\eta \in X^{**}$ such that, for all $\xi^o \in X^o$, $f(\xi^o) = \widetilde{f}(\xi^o) = (\xi|\eta)_{B^{**}} = \Theta^*(\eta)(\xi^o)$.

To establish that Θ^* is isometric, we need to show that $(\cdot|\eta)_{B^{**}}$ attains its norm $\|\eta\|$ on the subspace $X \subseteq X^{**}$. To this end we shall work in the linking algebra L of X. We identify X and X^{**} with the subspaces $pLq \subseteq L$ and $pL^{**}q \subseteq L^{**}$, respectively. It is a consequence of Kaplansky's density theorem that pL_1q is weak*-dense in $pL_1^{**}q$. Choosing a state $\psi \in L^*$ such that $\psi(\|\eta\|^{-1}\eta^*\eta) = \|\eta\|$, there is thus, for every $\varepsilon > 0$, an element $\xi \in X_1$ with

$$\|\eta\| - \varepsilon \leqslant |\psi(\xi^*\eta)| \leqslant \|(\xi|\eta)_{B^{**}}\| \leqslant \|\Theta^*(\eta)\|. \quad \blacksquare$$

In [8] Paschke showed that, for a Hilbert B-module X, $\mathcal{B}_B(B^{**} \otimes_B X^o, B^{**})$ can be equipped with a B^{**}-valued inner product turning it into a B^{**}-selfdual Hilbert C^*-module, which, as X can be isometrically embedded in $\mathcal{B}_B(B^{**} \otimes_B X^o, B^{**})$, was called the *selfdual completion* by him. Using that $\mathcal{B}_B(B^{**} \otimes_B X^o, B^{**})$ is canonically isomorphic to $\mathcal{B}_B(X^o, B^{**})$, we obtain the following corollary.

3.2. COROLLARY. *The selfdual completion $\mathcal{B}_B(X^o, B^{**})$ of a Hilbert A-B-bimodule X is isomorphic to the enveloping Hilbert W^*-module X^{**}. Similarly, we have $\mathcal{B}_A(X^o, A^{**}) \cong X^{**}$.*

3.3. DEFINITION. For a Hilbert A-B-bimodule X we put

$$\mathrm{RM}(X) = \{\eta \in X^{**} : (X|\eta)_{B^{**}} \subseteq B\},$$
$$\mathrm{LM}(X) = \{\eta \in X^{**} : {}_{A^{**}}(\eta|X) \subseteq A\},$$
$$\mathrm{M}(X) = \mathrm{LM}(X) \cap \mathrm{RM}(X).$$

Clearly, the isomorphism $\Theta^* : X^{**} \to \mathcal{B}_B(X^o, B^{**})$, $\eta \mapsto (\,\cdot\,|\eta)_{B^{**}}$ from the proof of Proposition 3.1 maps $\mathrm{RM}(X)$ onto $\mathcal{B}_B(X^o, B)$, and the corresponding map $X^{**} \to \mathcal{B}_A(X^o, A^{**})$ takes $\mathrm{LM}(X)$ onto $\mathcal{B}_A(X^o, A)$. So that $\mathrm{RM}(X) = X$, respectively $\mathrm{LM}(X) = X$, if and only if X is B-selfdual, respectively, A-selfdual.

3.4. PROPOSITION. *For a Hilbert A-B-bimodule X, we have*

$$M(X) = \{\eta \in X^{**} : A\eta + \eta B \subseteq X\} \cong \mathcal{L}_A(X^o, A) \cong \mathcal{L}_B(X^o, B).$$

Proof. Let $\eta \in X^{**}$ be such that $A\eta + \eta B \subseteq X$. For every $\xi \in X$ and each approximate identity $\{u_\alpha\} \subseteq A$, we have

$$(\xi|\eta)_{B^{**}} = \lim_\alpha (u_\alpha \xi|\eta)_{B^{**}} = \lim_\alpha (\xi|u_\alpha \eta)_{B^{**}} \subseteq B.$$

Similarly, ${}_{A^{**}}(\eta|X) \subseteq A$. Hence, $\eta \in \mathrm{M}(X)$. For the reverse inclusion, we note that every $\eta \in \mathrm{M}(X)$ and $a \in A$ defines a bounded B-linear mapping $\rho = (\,\cdot\,|a\eta)_{B^{**}} = (a^* \cdot |\eta)_{B^{**}} : X^o \to B$, which obviously vanishes at infinity. By Lemma 2.2 there is $\zeta^o \in X^o$ such that $(\,\cdot\,|a\eta)_{B^{**}} = (\,\cdot\,|\zeta)_B = (\,\cdot\,|\zeta)_{B^{**}}$, whence $a\eta = \zeta \in X$. Similarly, we have $\eta B \subseteq X$ using Remark 2.4.

If $\eta \in \mathrm{M}(X)$, then an adjoint of ${}_{A^{**}}(\eta|\,\cdot\,) : X^o \to A$, $\xi^o \mapsto {}_{A^{**}}(\eta|\xi)$ is given by $A \to X^o$, $a \mapsto \eta^o a$. Hence, $\Lambda(\eta) := {}_{A^{**}}(\eta|\,\cdot\,) \in \mathcal{L}_A(X^o, A)$, and Λ is an A-B-linear isometry. Surjectivity of Λ is clear from Definition 3.3 and the remark thereafter.

Similarly, $\mathrm{M}(X) \cong \mathcal{L}_B(X^o, B)$. ∎

3.5. REMARK. In [4] Echterhoff and Raeburn introduced multiplier bimodules in the following way. A *double centralizer* (m_A, m_B) is a pair consisting of A-linear and B-linear mappings $m_A : A \to X$, $m_B : B \to X$, respectively, such that the compatibility relation

$$m_A(a)b = am_B(b), \quad a \in A,\, b \in B$$

is satisfied. Boundedness of such mappings follows as in the C^*-algebra case. The multiplier bimodule is defined as the set of all double centralizers of X.

Let us briefly show that this definition coincides with ours. Of course, each $\xi \in \mathrm{M}(X)$ defines a double centralizer via

$$(a \mapsto a\xi, b \mapsto \xi b).$$

If, on the other hand, (m_A, m_B) is a double centralizer, we put $\eta = m_A^{**}(1)$, $\eta' = m_B^{**}(1)$ and observe that the compatibility of m_A and m_B implies that

$$a\eta b = am_A^{**}(1)b = m_A(a)b = am_B(b) = a\eta' b, \quad a \in A,\, b \in B.$$

Using weak*-continuity we find that $\eta = \eta'$. Consequently, $A\eta + \eta B = m_A(A) + m_B(B) \subseteq X$, whence $\eta \in \mathrm{M}(X)$.

The following proposition is contained in [4]. We give an alternative proof.

3.6. PROPOSITION. *The multiplier bimodule* $M(X)$ *is the unique Hilbert* $M(A)$*-*$M(B)$*-bimodule satisfying the following properties:*

(i) $AM(X) + M(X)B \subseteq X$;

(ii) *if* M *is another* A*-*B*-bimodule which contains* X *and satisfies* (i), *then there exists a unique* A*-*B*-linear mapping* $M \to M(X)$ *which is the identity on* X.

Proof. We shall first show that $\mathrm{M}(X)$ is a Hilbert $\mathrm{M}(A)$-$\mathrm{M}(B)$-bimodule. Let $\xi, \eta \in \mathrm{M}(X)$. Then $B(\xi|\eta)_{B^{**}} + (\xi|\eta)_{B^{**}}B = (\xi B|\eta)_{B^{**}} + (\xi|\eta B)_{B^{**}} \subseteq B$. Hence, $(\mathrm{M}(X)|\mathrm{M}(X))_{B^{**}} \subseteq \mathrm{M}(B)$, and similarly, ${}_{A^{**}}(\mathrm{M}(X)|\mathrm{M}(X)) \subseteq \mathrm{M}(A)$. The inclusion $\mathrm{M}(A)\mathrm{M}(X) + \mathrm{M}(X)\mathrm{M}(B) \subseteq \mathrm{M}(X)$ follows immediately from the characterization of multipliers given in Proposition 3.4.

Statement (i) is part of Proposition 3.4. Suppose that M is another A-B-bimodule with $X \subseteq M$ and $AM + MB \subseteq X$. Every $m \in M$ defines a double centralizer $(a \mapsto am, b \mapsto mb)$, hence, using the preceding remark, an element in $\mathrm{M}(X)$. It is straightforward to check that this mapping has the required properties.

Suppose now that $\Phi, \Psi : M \to \mathrm{M}(X)$ are two module mappings fixing X. Then $(\Phi(m) - \Psi(m))b = \Phi(mb) - \Psi(mb) = mb - mb = 0$ with $m \in M$, $b \in B$. Consequently, $\Phi(m) = \Psi(m)$, $m \in M$.

The uniqueness of $\mathrm{M}(X)$ follows from the usual universality considerations. ∎

Acknowledgements. The results in this paper are part of the author's doctoral thesis "Interplay between Noncommutative Topology and Operators on C^*-algebras" written at the University of Tübingen under the supervision of M. Mathieu.

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Received July 15, 2000.