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LOCALLY MINIMAL PROJECTIONS IN $B(H)$

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ABSTRACT. Given an n-tuple $\{a_1, \dots, a_n\}$ of self-adjoint operators on an infinite dimensional Hilbert space H, we say that a projection p in $B(H)$ is *locally minimal* for $\{a_1, \dots, a_n\}$ if each pa_jp, for $j = 1, \dots, n$, is a scalar multiple of p. In Theorem 1.8 we show that for any such $\{a_1, \dots, a_n\}$ and any positive integer k there exists a projection p of rank k that is locally minimal for $\{a_1, \dots, a_n\}$. If we further assume that $\{a_1, \dots, a_n, 1\}$ is a linearly independent set in the Calkin algebra, then in Theorem 2.10 we prove that p can be chosen of infinite rank.

KEYWORDS: *Projections, Hilbert space operators, locally minimal projections.*

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0. INTRODUCTION

Given an n-tuple $\{a_1, \dots, a_n\}$ of bounded self-adjoint operators on an infinite dimensional Hilbert space H, we say that a projection p in $B(H)$ (the algebra of all bounded linear operators on H) is *locally minimal* for $\{a_1, \dots, a_n\}$ if each pa_jp, for $j = 1, \dots, n$, is a scalar multiple of p. The name is derived from the fact that a projection p in a C^*-algebra A is a minimal projection if pAp consists of multiples of p. In Theorem 1.8 we show that for any such $\{a_1, \dots, a_n\}$ and any positive integer k there exists a projection p of rank k that is locally minimal for $\{a_1, \dots, a_n\}$. If we further assume that $\{a_1, \dots, a_n, 1\}$ is a linearly independent set in the Calkin algebra (i.e. the quotient of $B(H)$ by the ideal of compact operators on H), then in Theorem 2.10 we prove that p can be chosen of infinite rank. In Example 2.11 we show why our theorems cannot be improved, in general, although better results are often possible in special cases, using our methods.

When only a finite set of operators is under study, we are not interested in the algebra generated by these operators. In this situation local minimality appears to be the right concept to study. For instance, suppose bounded self-adjoint operators $\{a_1, \dots, a_n\}$ represent physical observables in a quantum system, states are represented by vectors in H, and the expectation value of an observable a_j in a state η is given by $\langle a_j\eta, \eta\rangle$. If p is locally minimal for $\{a_1, \dots, a_n\}$, as η moves among the states under p (i.e. such that $p\eta = \eta$), then the expectation value of each of the observables $\{a_1, \dots, a_n\}$ does not change. In a later paper we plan to study some physically interesting examples and see if a physical interpretation can be imputed to the phenomenon of local minimality. Some of the more promising examples will require the methods of this paper to be extended to unbounded operators, where the technical difficulties are non-trivial. For other physical applications it is natural to assume that the operators $\{a_1, \dots, a_n\}$ lie in a type II or type III factor and then ask that the projection p lie in that same von Neumann algebra. Our present methods do not generalize to that case.

When we began this study we wanted to carry over Samet's extension of Lyapunov's theorem [3], p. 471, to non-commutative situations in the same way that we generalized Lyapunov's theorem itself to non-commutative situations in our Memoir ([1]). A key idea in Samet's proof was the notion of complete linear independence. A set of functions on a measure space is *completely linearly independent* if, for any set E of positive measure, the restriction of the set of functions to E is linearly independent. Samet shows that complete linear independence is the generic case. As we show in our Theorems 1.8 and 2.10, the natural non-commutative version of complete linear independence fails in $B(H)$. Therefore a generalization of Samet's theorem will have to come from other methods.

1. THE FINITE RANK THEOREM

LEMMA 1.1. *If* $a = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$ *and* $b = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}$ *are* 2×2 *diagonal self-adjoint matrices such that* $\alpha_1 \geqslant \beta_1$, *and* $\beta_2 \geqslant \alpha_2$, *then there is a vector* η *in* $\mathbb{C}^2$ *such that*

$$\langle a\eta, \eta\rangle = \langle b\eta, \eta\rangle.$$

Proof. We may assume that $\alpha_1 > \beta_1$ and $\beta_2 > \alpha_2$ since otherwise, the assertion is trivial. Write

$$t = \frac{\beta_2 - \alpha_2}{\alpha_1 - \beta_1 + \beta_2 - \alpha_2}$$

and set

$$\eta_t = \begin{bmatrix} \sqrt{t} \\ \sqrt{1-t} \end{bmatrix}.$$

Note that since $\alpha_1 > \beta_1$ and $\alpha_2 < \beta_2$, we have $0 < t < 1$ and so $\|\eta_t\| = 1$. It is easy to check that, with this choice, we have

$$\langle a\eta_t, \eta_t\rangle = \langle b\eta_t, \eta_t\rangle. \quad \blacksquare$$

LEMMA 1.2. *Suppose* k *is a positive integer,* a, b, *and* $c_1, \dots, c_m$ *are diagonal self-adjoint matrices of dimension at least* $2k$ *and* $\alpha_1, \alpha_2, \dots, \alpha_{2k}$ *and* $\beta_1, \beta_2, \dots$ $\dots, \beta_{2k}$, *are the eigenvalues lists for* a *and* b, *respectively. If* $\alpha_i \geqslant \beta_i$ *for* $i = 1, 2, \dots, k$ *and* $\alpha_i \leqslant \beta_i$ *for* $i = k+1, k+2, \dots, 2k$, *then there is a projection* p *of* *rank k whose range is contained in the span of the first $2k$ eigenvectors of a (or b) such that:*

(i) *$pap = pbp$;*

(ii) *pap, pbp and $pc_1p, \dots, pc_mp$ can be simultaneously diagonalized.*

Proof. Write $\eta_1, \eta_2, \dots, \eta_k, \eta_{k+1}, \dots, \eta_{2k}$ for the eigenvectors corresponding to the eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_{2k}$ and consider the 2×2 matrices a_i and b_i formed by restricting a and b to the span of $\{\eta_i, \eta_{k+i}\}$, $i = 1, 2, \dots, k$. These matrices have the form

$$a_i = \begin{bmatrix} \alpha_i & 0 \\ 0 & \alpha_{k+i} \end{bmatrix} \quad \text{and} \quad b_i = \begin{bmatrix} \beta_i & 0 \\ 0 & \beta_{k+i} \end{bmatrix}.$$

Since $\alpha_i \geqslant \beta_i$ and $\alpha_{k+i} \leqslant \beta_{k+i}$, we may apply Lemma 1.1 to get a unit vector ξ_i in the span of $\{\eta_i, \eta_{k+i}\}$ such that $(a\xi_i, \xi_i) = (b\xi_i, \xi_i)$. Let p denote the projection onto the span of $\{\xi_1, \dots, \xi_k\}$. Since distinct ξ_i's are formed from orthogonal eigenvectors we get that $\{\xi_1, \dots, \xi_k\}$ is orthonormal and if $i \neq j$, then $(a\xi_i, \xi_j) = (b\xi_i, \xi_j) = (c_1\xi_i, \xi_j) = \cdots = (c_m\xi_i, \xi_j) = 0$. Since $(a\xi_i, \xi_i) = (b\xi_i, \xi_i)$ for each i, we get $pap = pbp$ and $pap, pbp, pc_1p, \dots, pc_mp$ are simultaneously diagonalizable, as desired. ∎

LEMMA 1.3. *If k is a positive integer and $\alpha_1, \dots, \alpha_{3k}$ and $\beta_1 \dots, \beta_{3k}$ are sequences of positive numbers, then either*

(i) *$\{i : \alpha_i = \beta_i\}$ has at least k elements or,*

(ii) *there is an index set $\{i_1, \dots, i_{2k}\} \subset \{1, 2, \dots, 3k\}$ and a positive real number t such that*

$$\alpha_{i_1} \geqslant t\beta_{i_1}, \dots, \alpha_{i_k} \geqslant t\beta_{i_k} \quad \textit{and} \quad \alpha_{i_{k+1}} \leqslant t\beta_{i_{k+1}}, \dots, \alpha_{i_{2k}} \leqslant t\beta_{i_{2k}}.$$

Proof. If $\{i : \alpha_i = \beta_i\}$ has at least k members, we are done. So suppose that this set has fewer than k elements. In this case, relabeling if necessary, we may assume that $\alpha_i \neq \beta_i$ for $i = 1, 2, \dots, 2k$. Now write $\gamma_i = \alpha_i/\beta_i$ for $i = 1, 2, \dots, 2k$ and relabel so that $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_{2k}$. Observe that each $\gamma_i > 0$ since the α's and β's are positive. Now take $t = \gamma_k$ With this we have

$$\frac{\gamma_1}{t} \geqslant \frac{\gamma_2}{t} \geqslant \cdots \geqslant \frac{\gamma_k}{t} = 1 \geqslant \frac{\gamma_{k+1}}{t} \geqslant \frac{\gamma_{k+2}}{t} \geqslant \cdots \geqslant \frac{\gamma_{2k}}{t}.$$

In other words, in this case we get

$$\alpha_1 \geqslant t\beta_1, \alpha_2 \geqslant t\beta_2, \dots, \alpha_k = t\beta_k$$

and

$$\alpha_{k+1} \leqslant t\beta_{k+1}, \alpha_{k+2} \leqslant t\beta_{k+2}, \dots, \alpha_{2k} \leqslant t\beta_{2k},$$

as desired. ∎

LEMMA 1.4. *If $a_1, a_2, \ldots, a_n$ are self-adjoint operators acting on an infinite dimensional Hilbert space H, then there is a projection p of infinite rank such that $pa_1p, pa_2p, \ldots, pa_np$ are each diagonal with respect to some fixed orthonormal basis for the range of p.*

Proof. Fix a unit vector ξ_1 in H and select a unit vector, ξ_2 such that

$$\xi_2 \in \{\xi_1, a_1\xi_1, \ldots, a_n\xi_1\}^\perp.$$

Let us now continue by induction. Suppose that orthonormal vectors $\xi_1, \ldots, \xi_k$ have been selected such that if $i < j \leqslant k$, then

$$\xi_j \in \{\xi_i, a_1\xi_i, \ldots, a_n\xi_i : 1 \leqslant i < j\}^\perp.$$

As H is infinite dimensional, we may select a unit vector ξ_{k+1} such that

$$\xi_{n+1} \in \{\xi_i, a_1\xi_i, \ldots, a_n\xi_i : 1 \leqslant i \leqslant n\}^\perp.$$

This produces an infinite orthonormal sequence $\xi_1, \xi_2, \ldots$ such that $\langle a_i\xi_j, \xi_k\rangle = 0$ if $j < k$ and $1 \leqslant i \leqslant n$. Since each a_i is self-adjoint we also get $\langle a_i\xi_j, \xi_k\rangle = 0$ if $j > k$. Hence, if we let p denote the projection onto the span of ξ_n's, then each pa_ip is diagonal. ∎

LEMMA 1.5. *If $a_1, \ldots, a_n$ are injective operators acting on an infinite dimensional Hilbert space such that each a_i is either positive or negative and k is a positive integer, then there is a projection p of rank k and nonzero real numbers $t_2, \ldots, t_n$ such that*

$$pa_1p = t_2pa_2p = \cdots = t_npa_np.$$

Proof. First assume that each a_i is positive and injective. By Lemma 1.4 we may find a projection p_1 of infinite rank such that $p_1a_ip_1$ is diagonal for each i.

Now write $N = 3^{n-1}k$. If $N/3$ of the eigenvalues of the first N eigenvalues of $p_1a_1p_1$ agree with the corresponding eigenvalues of $p_1a_2p_1$, then let $p_2 < p_1$ denote the projection onto the span of the associated eigenvectors. Otherwise, by Lemma 1.3 we may find a number $t_2 > 0$ such that $N/3$ of the first N eigenvalues of $p_1a_1p_1$ are greater than or equal to the corresponding eigenvalues of $t_2p_1a_2p_1$ and also find $N/3$ disjoint indices among the first N eigenvalues where this inequality is reversed. Next, applying Lemma 1.2 we may find a projection $p_2 < p_1$ of rank $N/3$ such that

$$p_2a_1p_2 = t_2p_2a_2p_2$$

and such that $p_2a_1p_2, \ldots, p_2a_np_2$ are all diagonal.

Now proceed by induction. At the next stage we apply the argument of the previous paragraph to get a projection $p_3 \leqslant p_2$ of rank $N/3^2$ and a positive real number t_3 such that

$$p_3a_1p_3 = t_3p_3a_3p_3$$

and $p_3a_1p_3, \ldots, p_3a_np_3$ are all diagonal.

After $n-1$ steps we get a projection $p = p_{n-1}$ of rank $N/3^{n-1} = k$ and positive numbers $t_2, \ldots, t_n$ such that

$$pap = t_2pa_2p = \cdots = t_npa_np.$$

Thus the lemma has been established when all of the operators are positive. Now suppose that some of the operators are negative. Relabeling if necessary, we may assume that for some $1 \leqslant k < n$, $a_1, \ldots, a_k$ are positive and $a_{k+1}, \ldots, a_n$ are negative. In this case, we may apply the first part of the proof to $a_1, \ldots, a_k, -a_{k+1}, \ldots, -a_n$ to get a projection p of rank k and scalars $t_2, \ldots, t_n$ such that

$$pa_1p = t_2pa_2p = \cdots = t_kpa_kp = -t_{k+1}pa_{k+1} = \cdots = -t_npa_np.$$

The proof may now be completed by replacing $t_{k+1}, \ldots, t_n$ with $-t_{k+1}, \ldots, -t_n$. ∎

LEMMA 1.6. *If a is a self-adjoint $2k \times 2k$ matrix, then there are orthonormal vectors $\xi_1, \ldots, \xi_k$ in $\mathbb{C}^{2k}$ such that if p denotes the projection onto the span of the ξ_i's, then $pap = \lambda p$ for some $\lambda \in \mathbb{R}$.*

Proof. We may assume that a is diagonal with diagonal entries $\alpha_1 \leqslant \cdots \leqslant \alpha_{2k}$. Let $\eta_1, \ldots, \eta_{2k}$ denote the standard basis vectors and fix a real number λ such that $\alpha_k \leqslant \lambda \leqslant \alpha_{k+1}$. Next if $i \leqslant k$, then write

$$t_i = \begin{cases} 0 & \text{if } \alpha_i = \lambda = \alpha_{k+i}, \\ \frac{\alpha_{k+i} - \lambda}{\alpha_{k+i} - \alpha_i} & \text{otherwise.} \end{cases}$$

If we set

$$\xi_i = \sqrt{t_i}\eta_i + \sqrt{1 - t_i}\eta_{k+i}, \quad 1 \leqslant i \leqslant k,$$

then it is straightforward to check that the ξ_i's are orthonormal and $\langle a\xi_i, \xi_i \rangle = \lambda$ for each index i. Further, since a is diagonal, we get that $\langle a\xi_i, \xi_j \rangle = 0$ if $i \neq j$.

Hence, if p denotes the projection onto the span of the ξ_i's, then $pap = \lambda p$. ∎

LEMMA 1.7. *If n is a positive integer and for each $1 \leqslant i \leqslant n$, $\{\alpha_{ik}\}_{k=1}^{\infty}$ is an infinite real sequence, then there is an infinite index set $\{j_1, j_2, \ldots\}$ such that for each i the subsequence $\{\alpha_{1,j_1}, \alpha_{i,j_2}, \ldots\}$ is either strictly positive, strictly negative or 0.*

Proof. Write

$$\sigma_{1,+} = \{\alpha_{1j} : \alpha_{ij} > 0\}, \quad \sigma_{1,-} = \{\alpha_{1j} : \alpha_{1j} < 0\}, \quad \sigma_{1,0} = \{\alpha_{1j} : \alpha_{1j} = 0\}$$

and let σ_1 denote the first of these sets that is infinite. Now continue by induction. Suppose that for some index $1 \leqslant i < n$, infinite subsets $\sigma_1 \supset \cdots \supset \sigma_i$ have been selected such that if $1 \leqslant j \leqslant i$, then $\{\alpha_{jk} : k \in \sigma_j\}$ is either strictly positive, strictly negative or 0. Arguing as above, we may find an infinite subset σ_{i+1} of σ_i such that $\{\alpha_{i+1,k} : k \in \sigma_{i+1}\}$ is either strictly positive, strictly negative or 0. The proof is now completed by taking $\{j_1, j_2, \ldots\} = \sigma_n$. ∎

THEOREM 1.8. (The Finite Rank Theorem) *If $a_1, \dots, a_n$ are bounded, self-adjoint operators acting on an infinite dimensional Hilbert space H and k is a positive integer, then there are real numbers $t_1, \dots, t_n$ and a projection p of rank k such that*

$$pa_ip = t_ip, \quad 1 \leqslant i \leqslant n.$$

Proof. Applying Lemma 1.4, we may find a projection q_1 of infinite rank such that each $q_1a_iq_1$ is diagonal with respect to some fixed orthonormal basis for q_1H. Now apply Lemma 1.7 to the eigenvalue sequences of the $q_1a_iq_1$'s to get a diagonal projection $q_2 \leqslant q_1$ of infinite rank and such that for each index i, $q_2a_iq_2$ is either strictly positive, strictly negative or 0 on q_2H.

Let $t_i = 0$ for each index i such that $q_2a_iq_2 = 0$. If $q_2a_iq_2 = 0$ for each i, then we may take $p = q_2$ and the proof is complete. Otherwise, restricting to the nonzero $q_2a_iq_2$'s, we may assume that $q_2a_iq_2$ is injective for each i. In this case we may apply Lemma 1.5 and get a projection $p_1 \leqslant q_2$ of rank at least $2k$ and $s_2, \dots, s_n \neq 0$ such that $p_1a_1p_1 = s_2p_1a_2p_1 = \cdots = s_np_1a_np_1$. The proof is completed by applying Lemma 1.6 to $p_1a_1p_1$ to get a projection p of rank k with the desired properties and setting $t_1 = \lambda_1$ and $t_j = s_j\lambda_j$ for $j > 1$. ∎

2. THE INFINITE RANK THEOREM

DEFINITION 2.1. If $x_1, x_2, \dots, x_n$ are bounded linear operators on an infinite dimensional separable Hilbert space H, their *joint essential numerical range* is by definition

$$W_{\mathrm{e}}(x_1, x_2, \dots, x_n) = \{(f(x_1), f(x_2), \dots, f(x_n)) : f \text{ is a singular state on } B(H)\}.$$

REMARK 2.2. Recall that a singular state of B(H) is a positive linear functional of norm 1 whose kernel contains the compact operators on H. By the Hahn-Banach Theorem, an operator in $B(H)$ that is in the kernel of all singular states must be a compact operator. This fact will be used in Lemma 2.9. Note that $W_{\mathrm{e}}(x_1, x_2, \dots, x_n)$ is a convex subset of $\mathbb{C}^n$ and, if each x_i is self-adjoint, then $W_{\mathrm{e}}(x_1, x_2, \dots, x_n) \subset \mathbb{R}^n$.

NOTATION 2.3. If $x_1, \dots, x_n$ are operators on a Hilbert space H and $\eta \in H$, we write

$$\vec{v}(x_1, \dots, x_n, \eta) = (\langle x_1\eta, \eta\rangle, \dots, \langle x_n\eta, \eta\rangle)$$

for the vector in $\mathbb{C}^n$ (or $\mathbb{R}^n$ if the x_i's are self-adjoint) which these elements determine.

LEMMA 2.4. *With notation as above, if F is a finite orthonormal set of vectors in H, $\vec{v} = (\lambda_1, \lambda_2, \dots, \lambda_n) \in W_{\mathrm{e}}(x_1, x_2, \dots, x_n)$ and $\varepsilon > 0$, then there is a unit vector $\eta \in H$ such that:*

(i) *η is orthogonal to each vector in F;*

(ii) $\|\vec{v} - \vec{v}(x_1, \dots, x_n, \eta)\| < \varepsilon$.

Proof. By definition of the joint essential numerical range, there is a singular state f such that $f(x_i) = \lambda_i$ for $i = 1, \dots, n$. Let p denote the projection onto

$F^{\perp}$. Since F is finite, and f is a singular state, $f(pap) = f(a)$ for each a in $B(H)$, so we may regard f as a singular state on $B(pH)$. By a theorem of Glimm ([2], Theorem 2, p. 216) there is a sequence $\{\xi_k\}$ of unit vectors in pH such that

$$\lim_{k\to\infty}\langle x_i\xi_k, \xi_k\rangle = f(x_i) = \lambda_i \quad \text{for } i = 1, \ldots, n.$$

Now select an integer k such that

$$|\langle x_i\xi_k, \xi_k\rangle - f(x_i)| < \frac{\varepsilon}{\sqrt{n}}$$

for each $i = 1, \ldots, n$, and let $\eta = \xi_k$. Since η lies in pH , part (i) of the lemma is satisfied. Further, we have

$$\begin{aligned}\|\vec{v} - \vec{v}(x_1, \ldots, x_n, \eta)\| &= \|(\langle x_1\xi_k, \xi_k\rangle, \ldots, \langle x_n\xi_k, \xi_k\rangle) - (\lambda_1, \ldots, \lambda_n)\| \\ &= \|(\langle x_1\xi_k, \xi_k\rangle - f(x_1), \ldots, \langle x_n\xi_k, \xi_k\rangle - f(x_n))\| < \varepsilon\end{aligned}$$

and so part (ii) of the lemma is true. ∎

LEMMA 2.5. *If $a_1, \ldots, a_n$ are self-adjoint operators in $B(H)$, $\vec{v}_1, \ldots, \vec{v}_{n+1}$ are vectors in $W_{\mathrm{e}}(a_1, \ldots, a_n)$, F_0 is a finite orthonormal set and $\varepsilon > 0$, then there are orthonormal vectors $\eta_1, \ldots, \eta_{n+1}$ and finite orthonormal sets $F_1, \ldots F_{n+1}$ in H such that:*

(i) *$F_0 \subset F_1 \subset \cdots \subset F_{n+1}$;*
(ii) *$\eta_i \in F_i \cap F_{i-1}^{\perp}$, $1 \leqslant i \leqslant n+1$;*
(iii) *$\{a_1\eta_i, a_2\eta_i, \ldots, a_n\eta_i\} \subset \mathrm{span}(F_i)$, $1 \leqslant i \leqslant n+1$;*
(iv) *$\|\vec{v}_i - \vec{v}(a_1, \ldots, a_n, \eta_i)\| < \varepsilon$ for $i = 1, \ldots, n+1$; and*
(v) *if $j \neq k$, then $\langle a_i\eta_j, \eta_k\rangle = 0$, $1 \leqslant i \leqslant n$, $1 \leqslant j,\ k \leqslant n+1$.*

Proof. Applying Lemma 2.4 we may select η_1 in H such that

$$\|\vec{v}_1 - \vec{v}(a_1, \ldots, a_n, \eta_1)\|_2 < \varepsilon$$

and $\eta_1 \perp F_0$. Now select a finite orthonormal set F_1 such that $F_0 \cup \{\eta_1\} \subset F_1$ and

$$\{a_1\eta_1, a_2\eta_1, \ldots, a_n\eta_1\} \subset \mathrm{span}(F_1).$$

Continuing by induction, suppose that vectors $\eta_1, \ldots, \eta_i$ and finite orthonormal sets $F_1, \ldots, F_i$ satisfying conditions (i), (ii), (iii) and (iv) have been selected for some $1 \leqslant i < n+1$. Applying Lemma 2.4 once more to F_i we get a unit vector η_{i+1} such that $\eta_{i+1} \perp F_i$ and $\|\vec{v}_{i+1} - \vec{v}(a_1, \ldots, a_n, \eta_{i+1})\| < \varepsilon$. Now select a finite orthonormal set F_{i+1} containing η_i and F_i and such that

$$\{a_1\eta_{i+1}, a_2\eta_{i+1}, \ldots, a_n\eta_{i+1}\} \subset \mathrm{span}(F_{i+1}).$$

This completes the induction argument.

Finally, observe that by construction if $j < k \leqslant n+1$, then $\langle a_i\eta_j, \eta_k\rangle = 0$, $1 \leqslant i \leqslant n$ and since each a_i is self-adjoint these equalities also hold for $j > k$. Hence, condition (v) is true. ∎

It is useful to record some facts about n-simplices before proceeding. Recall that an *n-simplex* in $\mathbb{R}^n$ is a convex set of the form

$$S = \mathrm{conv}(\vec{v}_1, \ldots, \vec{v}_{n+1})$$

such that its interior is not empty. The vectors $\vec{v}_1, \ldots, \vec{v}_{n+1}$ are called the *vertices* of S. The following fact is well known and easy to prove.

LEMMA 2.6. *Every vector in an n-simplex has a unique representation as a convex combination of its vertices.*

The coefficients in this representation are called the *barycentric coordinates* of the vector.

Lemma 2.8 below is the key to the proof of the main theorem in this section (Theorem 2.10). We are grateful to Nik Weaver for showing us an argument that considerably shortens our original proof. We shall employ the following notation. If S is a subset of $\mathbb{R}^n$ and $\varepsilon > 0$, then the ε-*interior* of S is by definition the set of all vectors $\vec{v} \in S$ such that S contains the closed ε-ball centered at $\vec{v}$. The proof of the next result is routine.

LEMMA 2.7. *If C is a convex subset of $\mathbb{R}^n$ and $\vec{v}$ is a vector in the interior of C, then there are $\varepsilon > 0$ and vectors $\vec{v}_1, \ldots, \vec{v}_{n+1}$ in C such that:*
(i) *$S = \operatorname{conv}(\vec{v}_1, \ldots, \vec{v}_{n+1}) \subset C$ is an n-simplex and*
(ii) *$\vec{v}$ is in the ε-interior of S.*

LEMMA 2.8. *If $S = \operatorname{conv}(\vec{v}_1, \ldots, \vec{v}_{n+1})$ and $S' = \operatorname{conv}(\vec{v}_1', \ldots \vec{v}_{n+1}')$ are n-simplices in $\mathbb{R}^n$, $\varepsilon > 0$, and $\|\vec{v}_i - \vec{v}_i'\| < \varepsilon$ for $i = 1, \ldots, n+1$, then S' contains the ε-interior of S. In particular, if $\vec{v}$ is in the ε-interior of S, then $\vec{v}$ is in each S' that satisfies the hypotheses.*

Proof. Suppose that there is a vector $\vec{v}$ in the ε-interior of S such that $\vec{v}$ is not in S'. Since S' is convex, there is a hyperplane P that strictly separates $\vec{v}$ and S'. Let $\vec{x}$ denote the unique vector in P that is closest to $\vec{v}$ and write $\vec{y} = \vec{v} - \vec{x}$. We have that $\vec{y}$ is orthogonal to P and "points away" from S'. Now set

$$\vec{w} = \vec{v} + \frac{\varepsilon}{\|\vec{y}\|}\vec{y}.$$

Since $\|\vec{w}-\vec{v}\| = \varepsilon$ and S contains the closed ε-ball about $\vec{v}$, $\vec{w} \in S$. By construction, $\vec{x}$ is the unique vector in P that is closest to $\vec{w}$. Thus, if $\vec{z} \in P$, then we have

$$\|\vec{w} - \vec{z}\| \geqslant \|\vec{w} - \vec{x}\| = \|\vec{w} - \vec{v} + \vec{y}\| = \left(1 + \frac{\varepsilon}{\|\vec{y}\|}\right)\|\vec{y}\| = \|\vec{y}\| + \varepsilon > \varepsilon$$

and so $\operatorname{dist}(\vec{w}, S') > \varepsilon$.

On the other hand, as $\vec{w} \in S$, by Lemma 2.6 we have

$$\vec{w} = \sum_{i=1}^{n+1} t_i \vec{v}_i,$$

where the t_i's are the barycentric coordinates of $\vec{w}$. If we now write

$$\vec{w}\,' = \sum_{i=1}^{m} t_i \vec{v}_i',$$

then we have $\vec{w}\,' \in S'$ and

$$\|\vec{w} - \vec{w}\,'\| \leqslant \sum_{i=1}^{m} t_i \|\vec{v}_i - \vec{v}_i'\| < \varepsilon,$$

which is a contradiction. ∎

Lemma 2.9. *If $a_1, \dots, a_n$ are bounded self-adjoint linear operators acting on a separable Hilbert space H, then $a_1, \dots, a_n, 1$ are linearly independent in the Calkin algebra if and only if $W_{\mathrm{e}}(a_1, \dots, a_n)$ has nonempty interior in $\mathbb{R}^n$.*

Proof. Suppose that $W_{\mathrm{e}}(a_1, \dots, a_n)$ has empty interior. This means that the dimension of $W_{\mathrm{e}}(a_1, \dots, a_n)$ is less than n and so $W_{\mathrm{e}}(a_1, \dots, a_n) \subset \vec{v} + W$, where $\vec{v} \in \mathbb{R}^n$ and W is a subspace of dimension less than n. Let $\vec{x} = (t_1, \dots, t_n)$ denote a unit vector that is orthogonal to W. If f is a singular state on $B(H)$ so that

$$(f(a_1), \dots, f(a_n)) = \vec{v} + \vec{w}$$

for some $\vec{w} \in W$, and we write $t_{n+1} = -\langle \vec{v}, \vec{x} \rangle$, then we have

$$\begin{aligned} f(t_1 a_1 + \dots + t_n a_n + t_{n+1} I) &= \langle (f(a_1), \dots, f(a_n)), \vec{x} \rangle - \langle \vec{v}, \vec{x} \rangle \\ &= \langle \vec{v} + \vec{w}, \vec{x} \rangle - \langle \vec{v}, \vec{x} \rangle = 0 \end{aligned}$$

and therefore, as noted in the Remark 2.2, $t_1 a_1 + \dots + t_n a_n + t_{n+1} I$ is compact .

Conversely, if there are real scalars $t_1, \dots, t_n$ and t such that $t_1 a_1 + \dots + t_n a_n + t1$ is compact, then for every singular state f we have

$$\sum_{i=1}^{n} t_i f(a_i) = -t$$

and so $W_{\mathrm{e}}(a_1, \dots, a_n)$ lies in an hyperplane. Thus this set has empty interior. ∎

Theorem 2.10. (The Infinite Rank Theorem) *If $a_1, \dots, a_n$ are bounded self-adjoint linear operators on a separable infinite dimensional Hilbert space H such that $a_1, \dots, a_n$ and 1 are linearly independent in the Calkin algebra and*

$$\vec{v} = (\lambda_1, \dots, \lambda_n)$$

is a vector in the interior of $W(a_1, \dots, a_n)$, then there is a projection p of infinite rank on H such that

$$p a_i p = \lambda_i p$$

for each index i.

Proof. As $\vec{v}$ is in the interior of $W(a_1, \dots, a_n)$, we may apply Lemma 2.7 to find $\varepsilon > 0$ and vectors $\vec{v}_1, \dots, \vec{v}_{n+1}$ in $W(a_1, \dots, a_n)$ such that if $S = \operatorname{conv}(\vec{v}_1, \dots, \vec{v}_{n+1})$, then S is an n-simplex, $S \subset W(a_1, \dots, a_n)$ and $\vec{v}$ is in the ε-interior of S.

We shall now apply Lemma 2.5 in an inductive construction. In each application we shall use the operators $a_1, \dots, a_n$, the vectors $\vec{v}_1, \dots, \vec{v}_{n+1}$ and the number ε found in the previous paragraph. The finite orthonormal set used in each application will be constructed by the inductive process.

Let us begin by setting $F_0 = \emptyset$ and applying Lemma 2.5 to get orthonormal vectors $\eta_{11}, \dots, \eta_{1,n+1}$ and finite orthonormal sets $F_{11}, \dots, F_{1,n+1}$ satisfying conditions (i), (ii), (iii), (iv) and (v) of this lemma.

Now let us proceed by induction. Suppose that for some integer $k \geqslant 1$, sequences $\eta_{11}, \dots, \eta_{1,n+1}, \dots, \eta_{k1}, \dots, \eta_{k,n+1}$ and finite orthonormal sets

$$F_{11} \subset \dots \subset F_{1,n+1} \subset \dots \subset F_{k1} \subset \dots \subset F_{k,n+1}$$

have been selected such that if $1 \leqslant j < k$, then conditions (i), (ii), (iii), (iv) and (v) of Lemma 2.5 hold for $F_{j,n+1}$, $F_{j+1,1}, \dots, F_{j+1,n+1}$ and $\eta_{j+1,1}, \dots, \eta_{j+1,n+1}$.

To complete the inductive argument, apply Lemma 2.5 once more to $F_{k,n+1}$ to get $F_{k+1,1}, \ldots, F_{k+1,n+1}$ and $\eta_{k+1,1}, \ldots, \eta_{k+1,n+1}$ satisfying conditions (i), (ii), (iii), (iv) and (v) of Lemma 2.5.

This argument produces an infinite orthonormal sequence

$$\eta_{11}, \ldots, \eta_{1,n+1}, \eta_{21}, \ldots, \eta_{2,n+1}, \ldots, \eta_{k1}, \ldots, \eta_{k,n+1}, \ldots$$

such that

$$\|\vec{v}_i - \vec{v}(a_1, \ldots, a_n, \eta_{ki})\| < \varepsilon, \quad i = 1, 2, \ldots, n+1,\ k = 1, 2, \ldots$$

and

$$\langle a_i \eta_{jk}, \eta_{lm} \rangle = 0, \quad 1 \leqslant i \leqslant n,\ \text{if } j \neq l \text{ or } k \neq m. \tag{2.1}$$

Next, for each index k write

$$S_k = \operatorname{conv}(\vec{v}(a_1, \ldots, a_n, \eta_{k1}), \vec{v}(a_1, \ldots, a_n, \eta_{k2}), \ldots, \vec{v}(a_1, \ldots, a_n, \eta_{k,n+1})).$$

Since $\|\vec{v}_i - \vec{v}(a_1, \ldots, a_n, \eta_{ki})\| < \varepsilon$ for each i and $\vec{v}$ is in the ε-interior of S, we have $\vec{v} \in S_k$ for each k by Lemma 2.8. Let $t_1, \ldots, t_{n+1}$ denote the barycentric coordinates of $\vec{v}$ as an element of S and write $s_{k1}, \ldots, s_{k,n+1}$ for the barycentric coordinates of $\vec{v}$ as an element of S_k so that we have

$$\vec{v} = \sum_{i=1}^{n+1} t_i \vec{v}_i = \sum_{i=1}^{n+1} s_{ki} \vec{v}(a_1, \ldots, a_n, \eta_{ki}).$$

Now write

$$\varphi_k = \sum_{i=1}^{n+1} \sqrt{s_{ki}}\, \eta_{ki}, \quad k = 1, 2, \ldots.$$

We have that $\{\varphi_k\}$ is an infinite orthonormal set by construction. Next let p denote the projection onto the span of $\varphi_1, \varphi_2, \ldots$. Observe that by (2.1) above, if $j \neq k$, then

$$\langle a_r \varphi_k, \varphi_j \rangle = \langle a_r \varphi_j, \varphi_k \rangle = 0, \quad r = 1, \ldots, n.$$

Also, we have

$$\langle a_r \varphi_k, \varphi_k \rangle = \sum_{i=1}^{n+1} s_{ki} \langle a_r \eta_{ki}, \eta_{ki} \rangle$$

because by (2.1) above the cross terms are 0 and this means that

$$\begin{aligned}
\vec{v}(a_1, \ldots, a_n, \varphi_k) &= (\langle a_1 \varphi_k, \varphi_k \rangle, \ldots, \langle a_n \varphi_k, \varphi_k \rangle) \\
&= \left(\sum_{i=1}^{n+1} s_{ki} \langle a_1 \eta_{ki}, \eta_{ki} \rangle, \ldots, \sum_{i=1}^{n+1} s_{ki} \langle a_n \eta_{ki}, \eta_{ki} \rangle \right) \\
&= \sum_{i=1}^{n+1} s_{ki} \vec{v}(a_1, \ldots, a_n, \eta_{ki}) = \sum_{i=1}^{n+1} t_i \vec{v}_i = \vec{v} = (\lambda_1, \ldots, \lambda_n).
\end{aligned}$$

In other words, for each k, $\langle a_r \varphi_k, \varphi_k \rangle = \lambda_r$, $r = 1, \ldots, n+1$. Hence, $p a_i p = \lambda_i p$ for $i = 1, \ldots, n$. ∎

EXAMPLES 2.11. (i) Let a denote any self-adjoint operator on H, suppose b is a compact positive operator with dense range and write

$$a_1 = a \quad \text{and} \quad a_2 = a + b.$$

Suppose that p is a projection of infinite rank such that $pa_1p = tp$. In this case we have

$$pa_2p = tp + pbp$$

and so if pa_2p is also a multiple of p, then pbp is a multiple of p. Since pbp is compact and p has infinite rank, we get that $pbp = 0$. But since b is positive and has dense range, this is impossible. Thus no such projection exists.

(ii) On the other hand, suppose $\{\eta_n\}$ denotes an orthonormal basis for H and we define the compact operator c by the formulas

$$c\eta_{2n-1} = \frac{1}{n} \quad \text{and} \quad c\eta_{2n} = -\frac{1}{n}, \quad n = 1, 2, \ldots.$$

If we then write

$$\xi_n = \frac{1}{\sqrt{2}}(\eta_{2n-1} + \eta_{2n})$$

and let p denote the projection onto the span of the ξ_n's, then we have $pcp = 0$ and so the conclusion of Theorem 2.10 holds for p and $p + c$.

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REFERENCES

1. C.A. AKEMANN, J. ANDERSON, *Lyapunov Theorems for Operator Algebras*, Mem. Amer. Math. Soc., vol. 458, 1991.
2. J. GLIMM, A Stone-Weierstrass theorem for operator algebras, *Ann. of Math.* **72** (1960), 216–244.
3. D. SAMET, Continuous selections for vector measures, *Math. Oper. Res.* **12**(1987), 536–543.

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