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FRAME REPRESENTATIONS FOR GROUP-LIKE UNITARY OPERATOR SYSTEMS

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Abstract. A group-like unitary system $\mathcal{U}$ is a set of unitary operators such that the group generated by the system is contained in $\mathbb{T}\mathcal{U}$, where $\mathbb{T}$ denotes the unit circle. Every frame representation for a group-like unitary system is (unitarily equivalent to) a subrepresentation of its left regular representation and the norm of a normalized tight frame vector determines the redundancy of the representation. In the case that a group-like unitary system admits enough Bessel vectors, the commutant of the system can be characterized in terms of the analysis operators associated with all the Bessel vectors. This allows us to define a natural quantity (the frame redundancy) for the system which will determine when the system admits a cyclic vector. A simple application of this leads to an elementary proof to the well-known time-frequency density theorem in Gabor analysis.

Keywords: *Group-like unitary systems, Gabor systems, frame vectors, von Neumann algebras, frame representations, analysis operators.*

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1. PRELIMINARIES

Motivated by the incompleteness property in Gabor analysis, we investigate the frame representations for group-like unitary systems.

Let A and B be two $d \times d$ invertible real matrices. A Gabor family is a collection of functions $g_{Am,Bn}$ obtained by translating and modulating a fixed function g in $L^2(\mathbb{R}^d)$,

$$g_{Am,Bn}(t) = \mathrm{e}^{2\pi\mathrm{i}\langle Am,t\rangle} g(t - Bn), \quad n, m \in \mathbb{Z}^d.$$

Most of the investigations on Gabor representations concern the problem of expanding $L^2(\mathbb{R}^d)$-functions f in terms of families $\{g_{Am,Bn} : n, m \in \mathbb{Z}^d\}$ for a fixed function g in $L^2(\mathbb{R}^d)$. This problem was mainly studied in the context of *frames* (cf. [2], [11], [6], [7], [18], [19] etc.). We refer to the book [8] by Feichtinger and Strohmer, and a survey paper [3] by Pete Casazza for some recent developments in Gabor analysis.

One of the important questions in Gabor analysis is the so-called density (or incompleteness) problem: Under what conditions can we find a function $g \in L^2(\mathbb{R}^d)$ such that the Gabor family $\{g_{Am,Bn}\}$ is a complete set of $L^2(\mathbb{R}^d)$? In the one dimensional case, the answer is well known (cf. [5]): There is a function f in $L^2(\mathbb{R})$ such that $\{f_{n\alpha,m\beta} : n, m \in \mathbb{Z}\}$ (where $A = \alpha, B = \beta$ with $\alpha, \beta > 0$) is a complete sequence in $L^2(\mathbb{R})$ if and only if $\alpha\beta \leqslant 1$. In higher dimensions, analogous necessary condition have been established, see [19] and [17]. Recently the second author and Y. Wang ([11]) proved the sufficiency by studying a problem concerning lattice tiling in $\mathbb{R}^d$.

A Gabor family can be viewed as a sequence obtained by applying a system of unitary operators to a particular window function. Recall that ([4]) a *unitary system* is a countable set of unitary operators $\mathcal{U}$ containing the identity operator and acting on a separable Hilbert space H. Letting group$(\mathcal{U})$ be the group generated by $\mathcal{U}$ and $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, we call $\mathcal{U}$ a *group-like unitary system* if

$$\mathrm{group}(\mathcal{U}) \subset \mathbb{T}\mathcal{U} := \{\lambda u : \lambda \in \mathbb{T},\, u \in \mathcal{U}\}$$

and $\mathcal{U}$ is linearly independent in the sense that $\mathbb{T}u \neq \mathbb{T}v$ whenever u and v are different elements of $\mathcal{U}$. We note that, in quantum mechanics theory a group-like, unitary system is simply the image of a projective unitary representation for some countable group.

If we consider the unitary system

$$\mathcal{U}_{A,B} := \{U_{Am}V_{Bn} : m, n \in \mathbb{Z}^d\},$$

where U_x and V_y for $x, y \in \mathbb{R}^d$ are defined by

$$(U_x f)(t) = \mathrm{e}^{2\pi \mathrm{i}\langle x,t\rangle} f(t) \quad \text{and} \quad (V_y f)(t) = f(t-y)$$

for all $f \in L^2(\mathbb{R}^d)$, then it is easy to check that $\mathcal{U}_{A,B}$ is a group-like unitary system.

Let H be a separable Hilbert space. A family of vectors $\{x_i\}_{i\in\mathcal{I}}$ is called a *frame* if there are two positive constants C_1 and C_2 such that

$$C_1\|x\|^2 \leqslant \sum_{i\in\mathcal{I}} |\langle x, x_i\rangle|^2 \leqslant C_2\|x\|^2$$

holds for all $x \in H$. The two optimal constants are called the *frame bounds*. When $C_1 = C_2 = 1$, it is called *normalized tight*. The family is called *Bessel* if we can allow $C_1 = 0$. For a unitary system $\mathcal{U}$, a vector $\xi \in H$ is called *a frame vector* (respectively *normalized tight frame vector* or *Bessel vector*) for $\mathcal{U}$ if $\{u\xi\}_{u\in\mathcal{U}}$ is a frame (respectively normalized tight frame or Bessel family) for $[\mathcal{U}\xi]$, where $[\,\cdot\,]$ denotes the closed linear span. When $[\mathcal{U}\xi] = H$, then the frame vector (respectively normalized frame vector) is said to be *complete*. In case that $\{u\xi : u \in \mathcal{U}\}$ is an orthonormal basis (respectively Riesz basis) for H, ξ is said to be a *complete wandering vector* (respectively *complete Riesz vector*) for $\mathcal{U}$.

Concerning the incompleteness question for Gabor analysis, we can ask:

Question. Under what conditions do we have a vector $x \in H$ such that $\mathcal{U}x$ is complete (respectively a frame) for H?

By investigating the frame representations and analyzing the commutant of the system, we will provide an easily computable quantity which will lead to a necessary condition for the above question. In particular, we are able to recapture the incompleteness result in Gabor analysis, to answer a problem asked in Chapter 4 of [10] and to obtain some other new results. We remark that besides applications, frame representations for group or group-like unitary systems are interesting and natural objects to study from the operator algebra point of view (cf. [10]).

Let $\mathcal{U}$ be a group-like unitary system. There exists a function $f : \mathrm{group}(\mathcal{U}) \to \mathbb{T}$ and a mapping $\sigma : \mathrm{group}(\mathcal{U}) \to \mathcal{U}$ such that $w = f(w)\sigma(w)$ for all $w \in \mathrm{group}(\mathcal{U})$. To see that f and σ are well defined, let $w = \lambda_1 u_1 = \lambda_2 u_2$ with $u_1, u_2 \in \mathcal{U}$ ($\lambda_1, \lambda_2 \in \mathbb{T}$). Then $u_1 = u_2$ and $\lambda_1 = \lambda_2$ since $\mathcal{U}$ is an independent set. Hence both f and σ are well defined. We will need the following basic properties for the mappings f and σ:

Proposition 1.1. *Let $\mathcal{U}$, f and σ be as above.*

(i) $f(u\sigma(vw))f(vw) = f(\sigma(uv)w)f(uv)$, $u, v, w \in \mathrm{group}(\mathcal{U})$.

(ii) $\sigma(u\sigma(vw)) = \sigma(\sigma(uv)w)$, $u, v, w \in \mathrm{group}(\mathcal{U})$.

(iii) $\sigma(u) = u$ *and* $f(u) = 1$ *if* $u \in \mathcal{U}$.

(iv) *If* $v, w \in \mathrm{group}(\mathcal{U})$, *then*

$$\begin{aligned}\mathcal{U} &= \{\sigma(uv) : u \in \mathcal{U}\} = \{\sigma(vu^{-1}) : u \in \mathcal{U}\} \\ &= \{\sigma(vu^{-1}w) : u \in \mathcal{U}\} = \{\sigma(v^{-1}u) : u \in \mathcal{U}\}.\end{aligned}$$

(v) *Let* $v, w \in \mathcal{U}$. *Then the following mappings from* $\mathcal{U}$ *to* $\mathcal{U}$ *are injective:*

$$u \mapsto \sigma(vu) \quad (\textit{respectively } \sigma(uv), \sigma(uv^{-1}), \sigma(v^{-1}u), \sigma(vu^{-1}), \sigma(u^{-1}v), \sigma(vu^{-1}w)).$$

A *unitary representation* π on H for a group-like unitary system $\mathcal{U}$ is a one-to-one mapping from $\mathcal{U}$ into the set of unitary operators on some Hilbert space K such that

$$\pi(u)\pi(v) = f(uv)\pi(\sigma(uv)), \quad \pi(u)^{-1} = f(u^{-1})\pi(\sigma(u^{-1})),$$

where f and σ are the corresponding mappings associated with $\mathcal{U}$.

Let e_u be the element in $l^2(\mathcal{U})$ which takes value 1 at u and zero everywhere else. Then $\{e_u : u \in \mathcal{U}\}$ is the standard orthonormal basis for $l^2(\mathcal{U})$. For each fixed $u \in \mathcal{U}$, we define $L_u \in B(l^2(\mathcal{U}))$ such that

$$L_u e_v = f(uv)e_{\sigma(uv)}, \quad v \in \mathcal{U}.$$

Then L_u is a well-defined unitary operator and the mapping $L : u \to L_u$ is a unitary representation for $\mathcal{U}$, which will be called the *left regular representation* for $\mathcal{U}$. Similarly, the right regular representation of $\mathcal{U}$ can be defined by

$$R_u e_v = f(vu^{-1})e_{\sigma(vu^{-1})}, \quad v \in \mathcal{U}.$$

It is easy to check the following:

$$R_u R_v = \overline{f(uv)}R_{\sigma(uv)}, \quad (R_u)^{-1} = \overline{f(u^{-1})}R_{\sigma(u^{-1})}.$$

Then, by our definition of unitary representations for group-like unitary systems, the right regular representation is not necessarily a unitary representation for $\mathcal{U}$. We can call it a *conjugate unitary representation*. The reason for considering the conjugate unitary representation is that we need the identity: $L_u R_v = R_v L_u$.

Let $B(H)$ be the operator algebra of all the bounded linear operators on H. A *von Neumann algebra* $\mathcal{M}$ is a $*$-subalgebra of $B(H)$ such that $I \in \mathcal{M}$ and $\mathcal{M}$ is closed in the weak operator (or strong operator) topology. By the double commutant theorem, a $*$-subalgebra $\mathcal{M}$ is a von Neumann algebra if and only if $\mathcal{M} = \mathcal{M}''$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. If $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$, then $\mathcal{M}$ is called a *factor*. A von Neumann algebra is said to be *finite* if every isometry in the algebra is unitary. Two projections P and Q in a von Neumann algebra $\mathcal{M}$ are said to be *equivalent* if there is an operator $T \in \mathcal{M}$ such that $TT^* = P$ and $T^*T = Q$. So $\mathcal{M}$ is finite if there is no proper subprojection of I which is equivalent to I. If a von Neumann algebra $\mathcal{M}$ admits a faithful trace, i.e., there is a linear functional Φ on $\mathcal{M}$ with the properties that $\Phi(AB) = \Phi(BA)$ and $\Phi(AA^*) \geqslant 0$ for all $A, B \in \mathcal{M}$, and $\Phi(A^*A) = 0$ only when $A = 0$, then $\mathcal{M}$ must be finite. We refer to [14] for more information about von Neumann algebra theory. A vector $x \in H$ is called *cyclic for* $\mathcal{M}$ if $[\mathcal{M}x] = H$, and *separating* if the mapping $\mathcal{M} \to H : A \mapsto Ax$ is injective. For any subset $\mathcal{S}$ of $B(H)$, we use $\mathrm{w}^*(\mathcal{S})$ to denote the von Neumann algebra generated by $\mathcal{S}$.

Let $\mathcal{L}$ (respectively $\mathcal{R}$) be the von Neumann algebra generated by $\{L_u : u \in \mathcal{U}\}$ (respectively $\{R_u : u \in \mathcal{U}\}$). Then, similarly to the group case, the following is true and will be used in the this paper:

PROPOSITION 1.2. (i) *The von Neumann algebra $\mathcal{R}$ is the commutant of $\mathcal{L}$.*

(ii) *Both $\mathcal{L}$ and $\mathcal{R}$ are finite von Neumann algebras.*

(iii) *Suppose that $\mathcal{U}$ is a group-like unitary system. If either $\{\sigma(vuv^{-1}) : v \in \mathcal{U}\}$ is infinite or $\{f(vuv^{-1}) : v \in \mathcal{U}\}$ is infinite for each $u \neq I$, then both $\mathcal{L}$ and $\mathcal{R}$ are factors. This extends the* ICC *condition for* ICC *groups to group-like unitary systems.*

Although Proposition 1.2 has an analogue in the group case, its proof is not just a trivial generalization and sometimes tricky because of the involvement of the mappings f and σ associated with the group-like unitary system. For the reader's convenience, we will include the sketched proofs for these Propositions 1.1 and 1.2 as an Appendix at the end of this paper.

2. FRAME REPRESENTATIONS

In this section we discuss those unitary representations that admit frame vectors. A *frame representation* is a unitary representation which admits a complete frame vector. Suppose that $\mathcal{S} \subset B(H)$ and $x \in H$. The *local commutant* $\mathrm{C}_x(\mathcal{S})$ (see [4]) of $\mathcal{U}$ at x is defined to be the set

$$\{T \in B(H) : TSx = STx,\ S \in \mathcal{S}\}.$$

PROPOSITION 2.1. *Suppose that $\mathcal{U}$ is a group-like unitary system and x is cyclic for $\mathcal{U}$. Then $\mathrm{C}_x(\mathcal{U}) = \mathcal{U}'$. Moreover, if ψ is a complete wandering vector for $\mathcal{U}$, then a vector η is a complete wandering vector for $\mathcal{U}$ if and only if there is a (unique) unitary operator $S \in \mathcal{U}'$ such that $\eta = S\psi$.*

Proof. Let $\mathcal{A}$ be the linear span of $\mathcal{U}$. Then $\mathcal{A}$ is an algebra and x is also cyclic for $\mathcal{A}$. Thus $\mathrm{C}_x(\mathcal{U}) = \mathrm{C}_x(\mathcal{A})$. If $T \in \mathrm{C}_x(\mathcal{A})$, then $TABx = ABTx = ATBx$ for all $A, B \in \mathcal{A}$. Hence $T \in \mathcal{A}'$ since x is cyclic. So $\mathrm{C}_x(\mathcal{U}) = \mathrm{C}_x(\mathcal{A}) = \mathcal{A}' = \mathcal{U}'$. The last statement follows from Proposition 2.3 in [4]. ∎

PROPOSITION 2.2. *Suppose that $\mathcal{U}$ is a group-like unitary system. Then, up to unitary equivalence, there is only one unitary representation π of $\mathcal{U}$ such that $\pi(\mathcal{U})$ has a complete wandering vector.*

Proof. We know that the left regular representation has complete wandering vectors. Assume that π_1 and π_2 are two unitary representations of $\mathcal{U}$ such that both $\pi_1(\mathcal{U})$ and $\pi_2(\mathcal{U})$ have complete wandering vectors. Choose ψ_i as a complete wandering vector for $\pi_i(\mathcal{U})$, $i = 1, 2$. We can define a unitary operator W by

$$W\pi_1(u)\psi_1 = \pi_2(u)\psi_2, \quad u \in \mathcal{U}.$$

Then for any $u, v \in \mathcal{U}$,

$$\begin{aligned} W\pi_1(u)\pi_1(v)\psi_1 &= f(uv)W\pi_1(\sigma(uv))\psi_1 = f(uv)\pi_2(\sigma(uv))\psi_2 \\ &= \pi_2(u)\pi_2(v)\psi_2 = \pi_2(u)W\pi_1(v)\psi_1. \end{aligned}$$

Thus $W\pi_1(u) = \pi_2(u)W$, as expected. ∎

COROLLARY 2.3. *Assume that $\mathcal{U}$ is a group-like unitary system which has a complete wandering vector. Then, every complete frame vector for $\mathcal{U}$ is a Riesz vector. In particular, every complete normalized tight frame vector is a complete wandering vector.*

Proof. By Proposition 2.2 and Proposition 1.2, we have that $\mathcal{U}'$ is a finite von Neumann algebra. Fix a complete wandering vector ψ for $\mathcal{U}$. Let η be a complete frame vector for $\mathcal{U}$. Define A by

$$Ax = \sum_{u \in \mathcal{U}} \langle x, u\eta \rangle u\psi, \quad x \in H.$$

Then A is injective and $A^* \in \mathrm{C}_\psi(\mathcal{U}) = \mathcal{U}'$. So $A \in \mathcal{U}'$. Hence A is invertible since $\mathcal{U}'$ is finite. Note that $u\eta = A^* u\psi$ for all $u \in \mathcal{U}$. Thus η is a Riesz vector for $\mathcal{U}$. ∎

We recall from [10] that two unitary systems $\mathcal{U}$ and $\mathcal{V}$ are called *isomorphic* if there is a bijection $h : \mathcal{U} \to \mathcal{V}$ such that $h(uv) = h(u)h(v)$ whenever $u, v, uv \in \mathcal{U}$. A unitary system $\mathcal{U}$ on H is said to have the *dilation property* if for every complete normalized tight frame vector (we can assume that it is not a wandering vector since, in that case, we do not need to dilate it) η for $\mathcal{U}$, there exists a Hilbert space K and a unitary system $\mathcal{V}$ on K such that $\mathcal{V}$ is isomorphic to $\mathcal{U}$ (say, by h) and the unitary system $\{u \oplus h(u) : u \in \mathcal{U}\}$ has a complete wandering vector $\eta \oplus x$ on $H \oplus K$. It is known that unitary groups and Gabor type unitary systems have dilation property (cf. [10]). The following is an natural extension of this result.

PROPOSITION 2.4. *Let $\mathcal{U}$ be a group-like unitary system on H. If $\mathcal{U}$ has a cyclic Bessel vector, then $\mathcal{U}$ has a normalized tight frame vector for H. Moreover, $\mathcal{U}$ is then unitarily equivalent to a subrepresentation of the left regular representation of $\mathcal{U}$, and thus $\mathrm{w}^*(\mathcal{U})$ and $\mathcal{U}'$ are finite von Neumann algebras.*

Proof. Let η be a cyclic Bessel vector for $\mathcal{U}$. Define $W : H \to l^2(\mathcal{U})$ by

$$Wx = \sum_{u\in\mathcal{U}} \langle x, u\eta\rangle e_u, \quad x \in H.$$

Then W is bounded, and we also have $Wux = L_u Wx$ for all $x \in H$. The partial isometry in the polar decomposition of W will be a unitary intertwining operator from H onto the closed subspace generated by WH. Therefore $\mathcal{U}$ is unitarily equivalent to the subrepresentation $L|P$ of the left regular representation of $\mathcal{U}$. Note that every subrepresentation $L|P$ admits a complete normalized tight frame vector (for example, Pe_I). Hence $\mathcal{U}$ has a normalized tight frame vector, as claimed. ∎

PROPOSITION 2.5. *Let $\mathcal{U}$ be a group-like unitary system on H. Suppose that $\eta, \xi \in H$ are cyclic vectors for $\mathcal{U}$ and that*

$$\sum_{u\in\mathcal{U}} |\langle \xi, u\eta\rangle|^2 < \infty.$$

Then $\mathcal{U}$ has a complete normalized tight frame vector.

Proof. For any $x, y \in H$, we write $c_{xy} = \{\langle y, ux\rangle\}_{u\in\mathcal{U}}$. Let

$$\mathcal{D} = \{x \in H : c_{\eta x} \in l^2(\mathcal{U})\} \quad \text{and} \quad \Gamma = \{x \oplus c_{\eta x} : x \in \mathcal{D}\}.$$

If $x_n \in \mathcal{D}$ are such that $x_n \to x$ and $c_{\eta x_n} \to c = \{c_u\} \in l^2(\mathcal{U})$. Then for each u, $\langle x_n, u\eta\rangle \to c_u$. Hence $\langle x, u\eta\rangle = c_u$, which implies that $c_{\eta x} \in l^2(\mathcal{U})$. Thus Γ is a closed subspace of $H \oplus l^2(\mathcal{U})$. Let $x \in \mathcal{D}$ and $u \in \mathcal{U}$. Then $ux \in \mathcal{D}$ and $(u \oplus L_u)(x \oplus c_{\eta x}) = ux \oplus c_{\eta(ux)} \in \Gamma$. Thus Γ is invariant under $u \oplus L_u$ for all $u \in \mathcal{U}$.

Let Q be the operator with domain H defined by

$$x \to x \oplus 0 \to x' \oplus c_{\eta x'} \to c_{\eta x'},$$

where the second arrow is the orthogonal projection from $H \oplus l^2(\mathcal{U})$ onto Γ. If $Qx = 0$, then $c_{\eta x'} = 0$. Hence $x' = 0$ since η is cyclic for $\mathcal{U}$. Therefore x is orthogonal to $\mathcal{D}$. Note that the closure of $\mathcal{D}$ is H since $v\xi \in \mathcal{D}$ for all $v \in \mathcal{U}$ and ξ is cyclic for $\mathcal{U}$. Thus $x = 0$, which implies that Q is injective on H.

Now we check that Q intertwines u and L_u for every $u \in \mathcal{U}$. In fact, for $x \in H$, we have $(ux)' = ux'$ since Γ is invariant under $u \oplus L_u$. Thus

$$\begin{aligned} Qux = c_{\eta(ux)'} &= \sum_{v\in\mathcal{U}} \langle ux', v\eta\rangle e_v = \sum_{v\in\mathcal{U}} \overline{f(u^{-1}v)}\langle x', \sigma(u^{-1}v)\eta\rangle e_v \\ &= L_u \sum_{v\in\mathcal{U}} \langle x', \sigma(u^{-1}v)\eta\rangle e_{\sigma(u^{-1}v)} = L_u Qx. \end{aligned}$$

Therefore Q intertwines u and L_u. From the polar decomposition of Q we obtain an isometric intertwining operator from H to the closure of QH. Thus $\mathcal{U}$ is unitarily equivalent to the subrepresentation $L|P$, where $P \in \mathcal{R}$ is the orthogonal projection onto $[QH]$, the closure of QH. Hence $\mathcal{U}$ has a complete normalized tight frame vector. ∎

By Proposition 2.4, frame representations can be viewed as subrepresentations of the left regular representation. Suppose that a frame representation π of $\mathcal{U}$ is unitarily equivalent to two subrepresentations $L|P$ and $L|Q$ for some projections $P, Q \in \mathcal{R}$. Then P and Q are equivalent projections in $\mathcal{R}$. So each frame representation π corresponds to an equivalent class, say $[\pi]$, of projections in $\mathcal{R}$. Two frame representations π_1 and π_2 are unitarily equivalent if and only if $[\pi_1] = [\pi_2]$.

The following is an extension of Lemma 6.2 in [10] which tells us when a direct sum of frame representations is also a frame representation.

PROPOSITION 2.6. *Suppose that π_i, $i = 1, 2, \ldots, n < \infty$, are frame representations of $\mathcal{U}$. Then $\pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_n$ is a frame representation if and only if there exist $P_i \in [\pi_i]$ such that $P_i \perp P_j$ when $i \neq j$.*

Proof. First assume that there exist $P_i \in [\pi_i]$ such that $P_i \perp P_j$ when $i \neq j$. Then $P_1 e_I \oplus \cdots \oplus P_n e_I$ is a complete normalized tight frame vector for $L|P_1 \oplus \cdots \oplus L|P_n$. Hence $\pi_1 \oplus \cdots \oplus \pi_n$ is a frame representation due to the unitary equivalence.

For the converse, let $\eta_1 \oplus \cdots \oplus \eta_n$ be a complete normalized tight frame vector for the direct sum representation. Let

$$T_i x_i = \sum_{u\in\mathcal{U}} \langle x_i, u\eta_i\rangle u e_I, \quad x_i \in H_i.$$

Then $P_i \in [\pi_i]$, where P_i is the orthogonal projection from $l^2(\mathcal{U})$ onto $T_i H_i$. By Theorem 2.8 in [10], we have that $P_i \perp P_j$ if $i \neq j$. ∎

COROLLARY 2.7. *Let π be a frame representation for $\mathcal{U}$. Then*

(i) $\bigoplus_{i=1}^{\infty} \pi$ *is not a frame representation;*

(ii) $\bigoplus_{i=1}^{n} \pi$, $n < \infty$, *is a frame representation if and only if there are $P_i \in [\pi]$ such that $P_i \perp P_j$, $i \neq j$;*

(iii) $\bigoplus_{i=1}^{n} \pi$, $n < \infty$, *has a complete wandering vector if and only if there exists an orthogonal family of projection $\{P_i\}_{i=1}^{n} \subset [\pi]$ with the property that $\sum_{i=1}^{n} P_i = I$.*

Proof. Statements (ii) and (iii) follow from Proposition 2.6. For (i), if $\bigoplus_{i=1}^{\infty} \pi$ is a frame representation, then, by Proposition 2.6, there is an orthogonal sequence $\{P_i\}_{i=1}^{\infty}$ of projections in $[\pi]$. This implies that $\sum_{i=1}^{\infty} P_i$ is equivalent to a proper subprojection $\sum_{i=1}^{\infty} P_{2i}$, which is impossible since $\mathcal{R}$ is a finite von Neumann algebra. ∎

Now we give the characterization (in terms of analysis operators) of $\pi(\mathcal{U})'$ for unitary representations that admit enough Bessel vectors. Let B_π be the set of all Bessel vectors for a unitary representation π on a Hilbert space K. Then B_π is a linear subspace invariant under $\pi(\mathcal{U})$ and $\pi(\mathcal{U})'$. For $\xi \in B_\pi$, the *analysis operator* with respect to ξ is defined by:

$$T_\xi(x) = \sum_{u \in \mathcal{U}} \langle x, \pi(u)\xi \rangle e_u, \quad x \in K.$$

Then $T_\xi^* e_u = \pi(u)\xi$ for all $u \in \mathcal{U}$, and ξ is a normalized tight frame vector if and only if $T_\xi^* T_\xi$ is a projection. Moreover, ξ is a complete frame vector (respectively complete normalized tight frame vector) if and only if T_ξ is injective with closed range (respectively isometry). We give a characterization of the frame representations in term of the special structure of the commutants.

THEOREM 2.8. *Let π be a unitary representation of $\mathcal{U}$ with the representing Hilbert space K. Then π is a frame representation if and only if $\pi(\mathcal{U})' = \{T_\xi^* T_\eta : \xi, \eta \in B_\pi\}$.*

Proof. For each $v \in \mathcal{U}$, we have

$$\begin{aligned} T_\xi^* T_\eta \pi(v) x &= \sum_{u \in \mathcal{U}} \langle \pi(v)x, \pi(u)\eta \rangle \pi(u)\xi = \sum_{u \in \mathcal{U}} \langle x, \pi(v)^{-1}\pi(u)\eta \rangle \pi(u)\xi \\ &= \sum_{u \in \mathcal{U}} \langle x, f(v^{-1})\, f(\sigma(v^{-1})u)\pi(\sigma(\sigma(v^{-1})u))\eta \rangle \pi(u)\xi \\ &= \pi(v) \sum_{u \in \mathcal{U}} \langle x, f(u^{-1}) f(\sigma(v^{-1})u)\pi(\sigma(\sigma(v^{-1})u))\eta \rangle \pi(v)^{-1}\pi(u)\xi \\ &= \pi(v) \sum_{u \in \mathcal{U}} \langle x, \pi(\sigma(\sigma(v^{-1})u))\eta \rangle \pi(\sigma(\sigma(v^{-1})u))\xi = \pi(v) T_\xi^* T_\eta x, \end{aligned}$$

where $x \in K$, $\xi, \eta \in B_\pi$. So we always have that $T_\xi^* T_\eta \in \pi(\mathcal{U})'$ for all $\xi, \eta \in B_\pi$.

Now assume that π is a frame representation and let η be a complete normalized tight frame vector. Let $A \in \pi(\mathcal{U})'$. Then $A\eta \in B_\pi$ and

$$T_{A\eta}^* T_\eta x = \sum_{u \in \mathcal{U}} \langle x, \pi(u)\eta \rangle \pi(u) A\eta = A \sum_{u \in \mathcal{U}} \langle x, \pi(u)\eta \rangle \pi(u)\eta = Ax, \quad x \in K.$$

Hence $T_{A\eta}^* T_\eta = A$ and thus $\pi(\mathcal{U})' = \{T_\xi^* T_\eta : \xi, \eta \in B_\pi\}$.

For the sufficiency, since $I = T_\xi^* T_\eta$ for some $\xi, \eta \in B_\pi$, we have that

$$x = \sum_{u \in \mathcal{U}} \langle x, \pi(u)\eta \rangle \pi(u)\xi, \quad x \in K.$$

Thus both η and ξ are cyclic vectors for π. Hence π is a frame representation by Proposition 2.5. ∎

For a unitary representation π, if $A \in \pi(\mathcal{U})'$ and $\xi, \eta \in B_\pi$, then $AT_\xi^* T_\eta = T_{A\xi}^* T_\eta$ and $T_\xi^* T_\eta A = T_\xi^* T_{A^*\eta}$. Thus $\mathrm{span}\{T_\xi^* T_\eta : \xi, \eta \in B_\pi\}$ is a subalgebra (in fact a two-sided ideal) of $\pi(\mathcal{U})'$. The above result tells us that if π is a frame representation, then $\{T_\xi^* T_\eta : \xi, \eta \in B_\pi\}$ is a linear subspace. But this is not true in general. For instance, let π be as in the following Corollary 2.9 such that it is not a frame representation. Then Theorem 2.8 and Corollary 2.9 below imply that $\{T_\xi^* T_\eta : \xi, \eta \in B_\pi\}$ is not a linear subspace.

COROLLARY 2.9. *Assume that* $\pi = \bigoplus_{i=1}^{n} \pi_i$, $n < \infty$, *where each* π_i *is a frame representation. Then*

$$\pi(\mathcal{U})' = \{T_{\xi_1}^* T_{\eta_1} + \cdots + T_{\xi_n}^* T_{\eta_n} : \xi_i, \eta_i \in B_\pi\}.$$

Proof. Take a complete normalized tight frame vector η_i for π_i. Then $T_{\eta_1}^* T_{\eta_1} + \cdots + T_{\eta_n}^* T_{\eta_n} = I$, where I is the identity operator on the direct sum Hilbert space. Hence, for each $A \in \pi(\mathcal{U})'$, we have:

$$A = A(T_{\eta_1}^* T_{\eta_1} + \cdots + T_{\eta_n}^* T_{\eta_n}) = T_{A\eta_1}^* T_{\eta_1} + \cdots + T_{A\eta_n}^* T_{\eta_n}. \qquad \blacksquare$$

For general unitary representations we have:

THEOREM 2.10. *Let* π *be a unitary representation of* $\mathcal{U}$. *Then* $\pi(\mathcal{U})'$ *is the closure (in the weak operator topology) of* $\mathrm{span}\{T_\xi^* T_\eta : \xi, \eta \in B_\pi\}$ *if and only if* B_π *is dense in* H.

Proof. Let $\mathcal{M}$ be the weak closure of $\mathrm{span}\{T_\xi^* T_\eta : \xi, \eta \in B_\pi\}$. Then $\mathcal{M}$ is a weakly closed $*$-subalgebra of $B(H)$. Thus it is reflexive in the sense that $\mathcal{M} = \{T \in B(H) : Tx \in [\mathcal{M}x], x \in H\}$, where $[\cdot]$ denotes the norm closure.

For necessity, let $x \in H$ be arbitrary. By our assumption, I is in the weak closure of $\mathcal{A} := \mathrm{span}\{T_\xi^* T_\eta : \xi, \eta \in B_\pi\}$. Thus there is a net $\{A_\lambda\} \in \mathcal{A}$ such that A_λ converges to I in the weak operator topology. In particular, $A_\lambda x$ converges to x weakly. Note that if $\xi, \eta \in B_\pi$, then $T_\xi^* T_\eta x = \sum_{u \in \mathcal{U}} \langle x, \pi(u)\eta\rangle \pi(u)\xi$. Thus $A_\lambda x$ is in the closure of B_π. So x is in the weak closure (which is the norm closure) of B_π. Therefore B_π is dense in H.

Conversely, suppose that B_π is dense in H. Let $A \in \pi(\mathcal{U})'$. We need to show that $A \in \mathcal{M}$. By the reflexivity of $\mathcal{M}$, it suffices to prove that $Ax \in [\mathcal{M}x]$ for every $x \in H$. Choose $x_n \in B_\pi$ such that $x_n \to x$. Let H_n be the closed subspace generated by $\pi(\mathcal{U})x_n$. Then, by Proposition 2.5, π has a normalized tight frame vector η_n such that $H_n = [\pi(\mathcal{U})x_n]$. So we have that

$$T_{A\eta_n}^* T_{\eta_n} x_n = \sum_{u \in \mathcal{U}} \langle x_n, \pi(u)\eta_n\rangle \pi(u) A\eta_n = A \sum_{u \in \mathcal{U}} \langle x_n, \pi(u)\eta_n\rangle \pi(u)\eta_n = Ax_n.$$

Note that $\|T_{A\eta_n}^* T_{\eta_n}\| \leqslant \|A\|$. Since the unit ball of $\mathcal{M}$ is compact in the weak operator topology, we can assume that $T_{A\eta_n}^* T_{\eta_n}$ converges to an operator $T \in \mathcal{M}$. Thus $Tx = Ax$, as expected. $\blacksquare$

We remark that it is possible for B_π to be $\{0\}$. For example, if $\mathcal{U}$ is the cyclic group generated by M_z, where M_z is the multiplication (by z) unitary operator on $L^2(\mathbb{T}, \mu)$ such that μ is singular with respect to the Lebesgue measure, then zero is the only Bessel vector for $\mathcal{U}$.

3. REDUNDANCY AND INCOMPLETENESS PROPERTY

In this section we investigate the frame redundancy for group-like unitary representations and use it to answer the question asked in Section 1.

Let $\mathcal{A}$ be a von Neumann algebra. A trace τ on $\mathcal{A}$ is said to be *normal* if it is continuous in the ultra-weak operator topology. For a faithful trace τ, $\tau(I)^{-1}\tau$ is a *normalized* faithful trace, i.e. it takes the value 1 at I.

If π is a frame representation for a group-like unitary system $\mathcal{U}$, then, by Proposition 2.4, both $\mathrm{w}^*(\pi(\mathcal{U}))$ and $\pi(\mathcal{U})'$ are finite von Neumann algebras, and thus admit faithful traces. We can construct a faithful normalized trace for $\mathcal{U}'$ in the following way: Let ξ be a complete normalized tight frame vector for $\pi(\mathcal{U})$ and let $T_\xi : H \to l^2(\mathcal{U})$ be the analysis operator defined by

$$T_\xi x = \sum_{u\in\mathcal{U}} \langle x, \pi(u)\xi\rangle e_u, \quad x \in H.$$

Let P be the orthogonal projection from $l^2(\mathcal{U})$ onto $T_\xi H$. Then P is in $\mathcal{R}$ and $\pi(\mathcal{U})$ is unitarily equivalent to $L|P$, and $T_\xi^* Pe_u = \pi(u)\xi$. Let $\eta = Pe_I/\|Pe_I\|$. Then $\tau(A) = \langle A\eta, \eta\rangle$ is a faithful trace for $P\mathcal{R}P$. Consider T_ξ as a unitary from H onto $T_\xi H$. Then $\pi(\mathcal{U})' = T_\xi^* P\mathcal{R}PT_\xi$. Hence we can get a faithful trace for $\mathcal{U}'$ by simply letting $\Phi(A) = \langle T_\xi A T_\xi^* \eta, \eta\rangle = \frac{1}{\|\xi\|^2}\langle A\xi, \xi\rangle$. Thus we have:

PROPOSITION 3.1. *Suppose that π is a frame representation of $\mathcal{U}$. Then for each complete normalized tight frame vector ξ of $\pi(\mathcal{U})$, $\tau(A) = \frac{1}{\|\xi\|^2}\langle A\xi, \xi\rangle$ defines a normalized faithful trace for $\pi(\mathcal{U})'$.*

COROLLARY 3.2. *Let π be a frame representation of $\mathcal{U}$. Then $\mathrm{tr}(T_x^* T_y) = \langle x, y\rangle$, $x, y \in B_\pi$, defines a faithful normal trace for $\pi(\mathcal{U})'$.*

Proof. Let ξ be a complete normalized tight frame vector for $\pi(\mathcal{U})$. By Proposition 3.1 and Theorem 2.8, it suffices to check that $\langle T_x^* T_y \xi, \xi\rangle = \langle x, y\rangle$ if $x, y \in B_\pi$. This is true since

$$\begin{aligned}
\langle T_x^* T_y \xi, \xi\rangle &= \langle T_y\xi, T_x\xi\rangle = \sum_{u\in\mathcal{U}} \langle \xi, uy\rangle\langle ux, \xi\rangle = \sum_{u\in\mathcal{U}} \langle x, u^{-1}\xi\rangle\langle u^{-1}\xi, y\rangle \\
&= \left\langle \sum_{u\in\mathcal{U}} \langle x, f(u^{-1})\sigma(u^{-1})\xi\rangle f(u^{-1})\sigma(u^{-1})\xi, y \right\rangle \\
&= \left\langle \sum_{u\in\mathcal{U}} \langle x, u\xi\rangle u\xi, y \right\rangle = \langle x, y\rangle
\end{aligned}$$

where f and σ are the corresponding mappings associated with $\mathcal{U}$. ∎

Note that Corollary 3.2 implies that all complete normalized tight frame vectors for $\pi(\mathcal{U})$ have the same norm $\sqrt{\mathrm{tr}(I)}$. Corollary 3.2 can be extended considerably.

PROPOSITION 3.3. *Suppose that π is a unitary representation of $\mathcal{U}$ with the representing Hilbert space K such that B_π is dense in K. Then* $\mathrm{tr}(T_\xi^* T_\eta) = \langle \xi, \eta \rangle$ *defines a faithful trace on* $\mathrm{span}\{T_\xi^* T_\eta : \xi, \eta \in B_\pi\}$.

Proof. First, we show that $\mathrm{tr}(\cdot)$ is well-defined. Assume that $T_{\xi_1}^* T_{\eta_1} + \cdots + T_{\xi_n}^* T_{\eta_n} = 0$. It suffices to show that $\langle \xi_1, \eta_1 \rangle + \cdots + \langle \xi_n, \eta_n \rangle = 0$. Let x be a (not necessarily complete) normalized tight frame vector for $\pi(\mathcal{U})$, and let P_x be the orthogonal projection from K onto $H_x := [\pi(\mathcal{U})x]$. Then

$$\begin{aligned}
\Big\langle \sum_{i=1}^n T_{\xi_i}^* T_{\eta_i} x, x \Big\rangle &= \sum_{i=1}^n \Big(\sum_{u \in \mathcal{U}} \langle x, \pi(u)\eta_i \rangle \langle \pi(u)\xi_i, x \rangle \Big) \\
&= \sum_{i=1}^n \Big(\sum_{u \in \mathcal{U}} \langle \pi(u)^{-1}x, \eta_i \rangle \langle \xi_i, \pi(u)^{-1}x \rangle \Big) \\
&= \sum_{i=1}^n \Big(\sum_{u \in \mathcal{U}} \langle \xi_i, \pi(\sigma(u^{-1}))x \rangle \pi(\sigma(u^{-1}))x, \eta_i \rangle \Big) \\
&= \sum_{i=1}^n \Big(\sum_{u \in \mathcal{U}} \langle \xi_i, \pi(u)x \rangle \pi(u)x, \eta_i \rangle \Big) = \sum_{i=1}^n \langle P_x \xi_i, \eta_i \rangle.
\end{aligned}$$

Since B_π is dense in K and K is separable, we can find a (possibly finite) sequence $\{x_i\}$ of normalized tight frame vector such that $H_{x_i} \perp H_{x_j}$, $i \neq j$, and $\sum_j P_{x_j} = I$. Thus $\sum_{i=1}^n \langle \xi_i, \eta_i \rangle = 0$ if $\sum_{i=1}^n T_{\xi_i}^* T_{\eta_i} = 0$. Hence $\mathrm{tr}(\cdot)$ is well defined.

If $A := \sum_{i=1}^n T_{\xi_i} T_{\eta_i}$ is a non-zero positive operator, then there is a normalized tight frame vector x for $\pi(\mathcal{U})$ such that $\langle Ax, x \rangle > 0$. Choose normalized tight frame vectors $\{x_j\}$ such that $x_1 = x$ and $\sum_{j=1}^\infty P_{x_j} = I$. Then $\sum_{i=1}^n \langle P_{x_1}\xi_i, \eta_i \rangle = \langle Ax, x \rangle > 0$ and $\sum_{i=1}^n \langle P_{x_j}\xi_i, \eta_i \rangle = \langle Ax_j, x_j \rangle \geqslant 0$ for all $j \geqslant 2$. Therefore

$$\mathrm{tr}(A) = \sum_{i=1}^n \langle \xi_i, \eta_i \rangle = \sum_{i=1}^n \Big\langle \sum_{j=1}^\infty P_{x_j} \xi_i, \eta_i \Big\rangle = \sum_{j=1}^\infty \langle Ax_j, x_j \rangle > 0.$$

So, $\mathrm{tr}(\cdot)$ is faithful.

Now assume that $A, B \in \pi(\mathcal{U})'$ such that $B = \sum_{i=1}^n T_{\xi_i}^* T_{\eta_i}$ for some $\xi_i, \eta_i \in B_\pi$. Then $AB = \sum_{i=1}^n T_{A\xi_i}^* T_{\eta_i}$ and $BA = \sum_{i=1}^n T_{\xi_i}^* T_{A^*\eta_i}$. Thus

$$\mathrm{tr}(AB) = \sum_{i}^n \langle A\xi_i, \eta_i \rangle = \sum_{i=1}^n \langle \xi_i, A^*\eta_i \rangle = \mathrm{tr}(BA),$$

as expected. ∎

From the proof of Proposition 3.3 we have:

COROLLARY 3.4. *Suppose that π is a unitary representation for $\mathcal{U}$ such that B_π is dense in K. Then there exist frame representations π_i such that $\pi = \bigoplus_{i=1}^{n} \pi_i$ (where n can be ∞).*

In general, $\mathrm{tr}(\cdot)$ may not be extended to a faithful trace to $\pi(\mathcal{U})'$. For example, if π_1 is a frame representation of $\mathcal{U}$ on K, then $\pi(u) = \pi_1(u) \otimes I$ defines a unitary representation of $\mathcal{U}$ on $K \otimes l^2(\mathbb{N})$ such that B_π is dense in $K \otimes l^2(\mathbb{N})$. But $\pi(\mathcal{U})' = \pi_1(\mathcal{U})' \otimes B(l^2(\mathbb{N}))$, which is not a finite von Neumann algebra. Hence, in this case, it is impossible to extend $\mathrm{tr}(\cdot)$ as a faithful trace on $\pi(\mathcal{U})'$.

Let x_i be normalized tight frame vector for π such that $H_{x_i} \perp H_{x_j}$ when $i \neq j$ and K be generated by $\{H_{x_i} : i \in \mathbb{N}\}$. Then $I = \sum_{i=1}^{\infty} T_{x_i}^* T_{x_i}$, where the convergence is in the strong operator topology. We claim that $\sum_{i=1}^{\infty} \|x_i\|^2$ is independent of the choices of x_i. In fact, let $I = \sum_{i=1}^{\infty} T_{y_i}^* T_{y_i}$ for $y_i \in B_\pi$. Write $A_n = \sum_{i=1}^{n} T_{y_i}^* T_{y_i}$ and $P_n = \sum_{i=1}^{n} T_{x_i}^* T_{x_i}$. Then

$$\mathrm{tr}(A_n P_m) = \sum_{i=1}^{m} \langle A_n x_i, x_i \rangle = \sum_{j=1}^{n} \langle P_m y_j, y_j \rangle.$$

Thus

$$\sum_{i=1}^{\infty} \langle A_n x_i, x_i \rangle = \sum_{j=1}^{n} \langle y_j, y_j \rangle.$$

Note that $\{A_n\}$ is bounded (by the uniformly bounded principle) sequence of positive operators. A very elementary argument shows that

$$\sum_{i=1}^{\infty} \langle x_i, x_i \rangle \ = \ \sum_{i=1}^{\infty} \langle y_i, y_i \rangle.$$

Therefore we can always use $\mathrm{tr}(I)$ to denote $\sum_{i=1}^{\infty} \|x_i\|^2$. Certainly $\mathrm{tr}(I)$ can be infinity in our notation. However if $\mathrm{tr}(I) < \infty$, then, for any $A \in \pi(\mathcal{U})'$, we can define $\phi(A) = \sum_{i=1}^{\infty} \langle Ax_i, x_i \rangle$, where $\{x_i\}$ are fixed normalized tight frame vectors as above. So ϕ is continuous in the ultra-weak operator topology. The above computation tells us that ϕ agrees with $\mathrm{tr}(\cdot)$ on $\mathrm{span}\{T_x^* T_y : x, y \in B_\pi\}$. Thus ϕ is a normal faithful trace on $\pi(\mathcal{U})$, which is independent of the choices of $\{x_i\}$. So we can use $\mathrm{tr}(A)$ as a substitute for $\phi(A)$. Moreover, since $\mathrm{tr}(\cdot)$ is ultra-weakly continuous, we have that $\mathrm{tr}(A) = \sum_{i=1}^{\infty} \langle \xi_i, y_i \rangle$ if $A = \sum_{i=1}^{\infty} T_{\xi_i}^* T_{y_i}$ for $x_i, y_i \in B_\pi$.

Consider a unitary representation π of $\mathcal{U}$. If B_π is dense in K, then we will define the *redundancy* of π, denoted by $r(\pi)$, to be $\frac{1}{\mathrm{tr}(I)}$, where $\mathrm{tr}(I) = \sum_i \|x_i\|^2$ when $\sum_i T_{x_i}^* T_{x_i} = I$. For our convenience, we will say that the redundancy of π is zero if B_π is not a dense subset. The following is immediate from the definition and the fact that a normalized tight frame vector is a wandering vector if and only if it has norm one.

COROLLARY 3.5. (i) *Let ξ be a complete normalized tight frame vector for a frame representation π of $\mathcal{U}$ on K. Then $r(\pi) = \frac{1}{\|\xi\|^2} \geqslant 1$.*

(ii) *A frame representation π of $\mathcal{U}$ has a complete wandering vector if and only if $r(\pi) = 1$.*

We remark that the converse of Corollary 3.5 (i) is not true. For instance, let $\mathcal{U}$ be an abelian group and let L be its left regular representation. Choose a projection $P \in \mathcal{R}$ such that $\langle Pe_I, e_I\rangle \leqslant \frac{1}{2}$. Then $\pi = L|P \oplus L|P$ is a unitary representation of $\mathcal{U}$ such that $r(\pi) \geqslant 1$. We claim that π is not a frame representation. Assume, to the contrary, that π has a complete normalized tight frame vector $\xi \oplus \eta$. Let H be the range of P. Then ξ and η are complete normalized tight frame vectors for $L|P$. Note that Pe_I is also a complete normalized tight frame vector for $L|P$. Then, by Proposition 3.13 in [10], there exist unitary operators A and B in the commutant of $L|P(\mathcal{U})$ such that $\xi = APe_I$ and $\eta = BPe_I$. Then $\pi(\mathcal{U})(\xi \oplus \eta) = \{(A \oplus B)(Pe_u \oplus Pe_u) : u \in \mathcal{U}\}$ which can not generate $H \oplus H$. So $\xi \oplus \eta$ can not be cyclic for π, which leads to a contradiction.

The following result tells us that if $\mathcal{U}$ is a group-like unitary system such that $r(\pi) < 1$, then $\mathcal{U}x$ can never be complete in H.

THEOREM 3.6. *Let π be a unitary representation of a group-like unitary system $\mathcal{U}$ such that B_π is dense in K. If $\pi(\mathcal{U})$ has a cyclic vector, then $r(\pi) \geqslant 1$.*

Proof. Let $\psi_i \in B_\pi$ such that $\psi_i \to \psi$, where ψ is a cyclic vector for $\pi(\mathcal{U})$. By Proposition 2.5, for each i there is a normalized tight frame vector ξ_i such that $[\mathcal{U}\xi_i] = [\mathcal{U}\psi_i]$. Note that $P_i := T^*_{\xi_i} T_{\xi_i}$ is the orthogonal projection from H onto $[\mathcal{U}\xi_i]$. Thus $P_i\psi_i = \psi_i$.

Let $I = \sum\limits_{i=1}^{\infty} T^*_{x_i} T_{x_i}$ such that x_i are normalized tight frame vectors with the property that $H_{x_i} \perp H_{x_j}$, $i \neq j$. Write $A_j = T^*_{x_j} T_{x_j}$. Then, as discussed before,

$$\begin{aligned}\langle P_i x_j, x_j\rangle &= \mathrm{tr}(T^*_{P_i x_j} T_{x_j}) = \mathrm{tr}(P_i T^*_{x_j} T_{x_j}) = \mathrm{tr}(T^*_{x_j} T_{x_j} P_i)\\ &= \mathrm{tr}(T^*_{A_j \xi_i} T_{\xi_i}) = \langle T^*_{x_j} T_{x_j} \xi_i, \xi_i\rangle.\end{aligned}$$

Write $Q_m = \sum\limits_{i=1}^{m} T^*_{x_i} T_{x_i}$. Then Q_m is an orthogonal projection and

$$\sum_{j=1}^{m} \langle P_i x_j, x_j\rangle = \langle Q_m \xi_i, \xi_i\rangle \leqslant 1.$$

Thus

$$\sum_{j=1}^{m} \|x_j\|^2 = \lim_{i\to\infty} \sum_{j=1}^{m} \langle P_i x_j, x_j\rangle \leqslant 1.$$

So $\mathrm{tr}(I) = \sum\limits_{j=1}^{\infty} \|x_j\|^2 \leqslant 1$, and thus $r(\pi) \geqslant 1$, as expected. ∎

4. SOME APPLICATIONS IN GABOR ANALYSIS

The incompleteness property (Theorem 3.6) can have many simple applications in different concrete situations. In this section we discuss some applications in Gabor analysis. We first restrict ourselves to one dimensional Gabor systems. Let $\alpha, \beta > 0$. Recall that U_α and V_β are unitary operators defined by

$$(U_\alpha g)(x) = \mathrm{e}^{2\pi \mathrm{i} \alpha x} g(x) \quad \text{and} \quad (V_\beta g)(x) = g(x-\beta)$$

for $g \in L^2(\mathbb{R})$. Let π be the identity representation of $\mathcal{U}_{\alpha,\beta}$, i.e. $\pi(u) = u$ for every $u \in \mathcal{U}_{\alpha,\beta}$.

LEMMA 4.1. *The redundancy of π is $\frac{1}{\alpha\beta}$.*

Proof. It suffice to find normalized tight frame vectors ξ_i such that $I = \sum_{i=1}^{k} T_{\xi_i}^* T_{\xi_i}$ and $\sum_{i=1}^{k} \|\xi_i\|^2 = \alpha\beta$. If $\alpha\beta \leqslant 1$, then $\xi = \sqrt{\alpha}\chi_{[0,\beta]}$ is a complete normalized tight frame vector. Thus $T_\xi^* T_\xi = I$. Clearly $\|\xi\|^2 = \alpha\beta$. If $\alpha\beta > 1$, choose continuous disjoint intervals E_j, $j = 1, \ldots, k$, such that the length of each E_j is less than or equal to $\frac{1}{\alpha}$ and $\bigcup_{j=1}^{k} E_j = [0, \beta)$. Let $\xi_i = \sqrt{\alpha}\chi_{E_j}$. Then each ξ_i is a normalized tight frame vector (but not complete) and $\sum_{j=1}^{k} T_{\xi_j}^* T_{\xi_j} = I$. Also $\sum_{j=1}^{k} \|\xi_j\|^2 = \alpha\beta$. ∎

From Theorem 3.6 and Lemma 4.1, we immediately have:

THEOREM 4.2. *The following statements are equivalent:*
(i) $\alpha\beta \leqslant 1$*;*
(ii) $\mathcal{U}_{\alpha,\beta}$ *has a complete normalized tight frame vector;*
(iii) $\mathcal{U}_{\alpha,\beta}$ *has a complete frame vector;*
(iv) $\mathcal{U}_{\alpha,\beta}$ *has a cyclic Bessel vector;*
(v) $\mathcal{U}_{\alpha,\beta}$ *has a cyclic vector.*

For a unitary representation π of a group-like unitary system $\mathcal{U}$, the *cyclic multiplicity* of π is defined to be the minimal cardinality of the sets $\mathcal{S} \subset K$ such that $\operatorname{span}\{\pi(u)x : u \in \mathcal{U}x, x \in \mathcal{S}\}$ is dense in K. Note that for each $x \in K$, $[\pi(\mathcal{U})x]$ is invariant under $\pi(\mathcal{U})$. So if π has cyclic multiplicity n, then there exist vectors $x_1, \ldots, x_n \in K$ such that $[\pi(\mathcal{U})x_i]$ are orthogonal each other and generate the whole Hilbert space K.

COROLLARY 4.3. *The following are equivalent:*
(i) $n - 1 < \alpha\beta \leqslant n$*;*
(ii) $\mathcal{U}_{\alpha,\beta}$ *has multiplicity* n.
In this case, $\mathcal{U}_{\alpha,\beta}$ *has normalized tight frame vectors* $\xi_1, \ldots, \xi_n$ *such that* $T_{x_1}^* T_{\xi_1} + \cdots + T_{x_n}^* T_{x_n} = I$.

Proof. From the proof of Lemma 4.1, we have that if $\alpha\beta \leqslant n$, then π has cyclic multiplicity at most n. On the other hand, let us assume that π has cyclic multiplicity n. Let $\{x_1, \ldots, x_n\}$ be a cyclic set for π such that $H_i := [\pi(\mathcal{U})x_i]$

are orthogonal each other. Then, by Theorem 3.6, $\mathrm{tr}(P_i) \leqslant 1$, where P_i is the orthogonal projection onto H_i which is in the commutant of $\pi(\mathcal{U})$. Thus $\alpha\beta = \mathrm{tr}(I) = \mathrm{tr}\Big(\sum_{i=1}^{n} P_i\Big) = \sum_{i=1}^{n} \mathrm{tr}(P_i) \leqslant n$. Hence (i) and (ii) are equivalent. The last statement follows from the proof of Lemma 4.1. ∎

Now we consider the higher dimensional case. Let A and B be $d\times d$ invertible real matrices. We associate with them the unitary system

$$\mathcal{U}_{A,B} := \{U_{Ak}V_{Bl} : k, l \in \mathbb{Z}^d\}.$$

As in the one dimensional case we need to compute the redundancy of the system.

LEMMA 4.4. *The redundancy of* $\mathcal{U}_{A,B}$ *is* $\frac{1}{|\det AB|}$.

Proof. Let W be the unitary operator $f(\xi) \to \sqrt{|\det B^{-1}|}f(B^{-1}\xi)$. Then $W^*\mathcal{U}_{A,B}W = \mathcal{U}_{B^{\mathrm{t}}A,I}$, where B^{t} is the transpose of B. So it suffices to consider the case when $B = I$. Let $\Omega = \{(x_1, \ldots, x_d) : 0 \leqslant x_i < 2\pi,\ i = 1, \ldots, d\}$ and let $G_n = A^{\mathrm{t}}\Omega \cap (\Omega + 2n\pi)$ for every $n \in \mathbb{Z}^d$. Note that $\{\Omega + 2n\pi : n \in \mathbb{Z}^d\}$ is a partition of $\mathbb{R}^d$. Thus $\bigcup_{n\in\mathbb{Z}^d} G_n = A^{\mathrm{t}}\Omega$, i.e., $\bigcup_{n\in\mathbb{Z}^d} (A^{\mathrm{t}})^{-1}G_n = \Omega$. Write $E_n = (A^{\mathrm{t}})^{-1}G_n$. Then $\{E_n : n \in \mathbb{Z}^d\}$ is a partition of Ω. Let $f_n = \sqrt{|\det A|}\chi_{E_n}$. Since $A^{\mathrm{t}}E_n - 2n\pi = G_n - 2n\pi \subseteq \Omega$, we can show that $\{\mathrm{e}^{\mathrm{i}\langle Ak,\xi\rangle}f_n : k \in \mathbb{Z}^d\}$ is a normalized tight frame for $L^2(E_n)$. In fact, for any $f \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}
\langle f, \mathrm{e}^{-\mathrm{i}\langle Ak,\xi\rangle}f_0\rangle &= \sqrt{|\det A|}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle k,A^{\mathrm{t}}\xi\rangle}\chi_{E_0}(\xi)f(\xi)\,\mathrm{d}\xi \\
&= \frac{1}{\sqrt{|\det A|}}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle k,\xi\rangle}\chi_{E_0}((A^{\mathrm{t}})^{-1}\xi)f((A^{\mathrm{t}})^{-1}\xi)\,\mathrm{d}\xi \\
&= \frac{1}{\sqrt{|\det A|}}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle k,\xi\rangle}\chi_{A^{\mathrm{t}}E_0}(\xi)f((A^{\mathrm{t}})^{-1}\xi)\,\mathrm{d}\xi \\
&= \frac{1}{\sqrt{|\det A|}}\int_{\Omega} \mathrm{e}^{\mathrm{i}\langle k,\xi\rangle}\chi_{A^{\mathrm{t}}E_0}(\xi)f((A^{\mathrm{t}})^{-1}\xi)\,\mathrm{d}\xi
\end{aligned}$$

since $A^{\mathrm{t}}E_0 = G_0 \subset \Omega$. Thus

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^d} |\langle\chi_{E_0}f, \mathrm{e}^{-\mathrm{i}\langle Ak,\xi\rangle}\rangle|^2 &= \frac{1}{|\det A|}\int_{\Omega} |\chi_{A^{\mathrm{t}}E_0}(\xi)f((A^{\mathrm{t}})^{-1}\xi)|^2\,\mathrm{d}\xi \\
&= \frac{1}{|\det A|}\int_{A^{\mathrm{t}}\Omega} |f((A^{\mathrm{t}})^{-1}\xi)|^2\,\mathrm{d}\xi \\
&= \int_{E_0} |f(\xi)|^2\,\mathrm{d}\xi = \|f\|^2_{L^2(E_0)}.
\end{aligned}$$

So $\{\mathrm{e}^{\mathrm{i}\langle Ak,\xi\rangle}f_0 : k \in \mathbb{Z}^d\}$ is a normalized tight frame for $L^2(E_0)$ and similarly for the other E_n. Therefore $\mathcal{U}_{A,I}f_n$ is a normalized tight frame for $L^2\Big(\bigcup_{k\in\mathbb{Z}^d}(E_n + 2k\pi)\Big)$.

Let $F_n = \bigcup_{k\in\mathbb{Z}^d} (E_n + 2\pi k)$. Then $\{F_n\}$ is a partition of $\mathbb{R}^d$. Thus $I = \sum_{n\in\mathbb{Z}^d} T^*_{f_n} T_{f_n}$, which implies that

$$\mathrm{tr}(I) = \sum_{n\in\mathbb{Z}^d} \|f_n\|^2 = |\det A| \sum_{n\in\mathbb{Z}^d} \mu(E_n) = |\det A|\mu(\Omega) = |\det A|,$$

where μ is the Lebesgue measure on $\mathbb{R}^d$. Therefore the redundancy of $\mathcal{U}_{A,I}$ is $\frac{1}{|\det A|}$. ∎

From Theorem 3.6 and Lemma 4.4 we have:

COROLLARY 4.5. *If* $|\det AB| > 1$, *then* $\mathcal{U}_{A,B}$ *does not have any cyclic vector.*

We can also consider the direct sum representations of a group-like unitary system which are related to the concept of strongly disjoint frames introduced in [10] (or, equivalently, to the concept of orthogonal multiframes introduced by R. Bălan in [1]). In particular, we can answer one question posed in [10].

COROLLARY 4.6. *Suppose that* A_i *and* B_i *are invertible* $d \times d$ *real matrices such that* $A_i^{\mathrm{t}} B_i = A_j^{\mathrm{t}} B_j$ *for all* i, j. *Let* $\mathcal{U} = \{U_{A_1k}V_{B_1l} \oplus \cdots \oplus U_{A_nk}V_{B_nl} : k, l \in \mathbb{Z}^d\}$. *If* $|\det A_1B_1| > \frac{1}{n}$, *then* $\mathcal{U}$ *does not admit any cyclic vector in* $H = L^2(\mathbb{R}^d) \oplus \cdots \oplus L^2(\mathbb{R}^d)$.

Proof. By our assumption $\mathcal{U}$ is a group-like unitary system. Let I_H denote the identity operator on H. Then, by Lemma 4.4, $\mathrm{tr}(I_H) = \sum_{i=1}^{n} |\det A_iB_i| = n|\det A_1B_1|$. Hence, by Theorem 3.6, $\mathcal{U}$ does not admit any cyclic vector. ∎

REMARK. In the case that $A_i^{\mathrm{t}} B_i \neq A_j^{\mathrm{t}} B_j$ for some $i \neq j$, the direct sum unitary system $\mathcal{U}$ in Corollary 4.5 is not a group-like unitary system. Thus Lemma 4.5 can not be applied in this situation.

However, we still have the following weaker result:

PROPOSITION 4.7. *Assume that* $\sum_{i=1}^{n} |\det A_iB_i| > 1$. *Then the unitary system* $\mathcal{U} = \{U_{A_1k}V_{B_1l} \oplus \cdots \oplus U_{A_nk}V_{B_nl} : k, l \in \mathbb{Z}^d\}$ *does not admit any complete normalized tight frame vector for* $H = L^2(\mathbb{R}^d) \oplus \cdots \oplus L^2(\mathbb{R}^d)$.

Proof. Suppose, on the contrary that $\mathcal{U}$ has a complete normalized tight frame vector $F = (f_1, \ldots, f_n)$. Then f_i is a complete normalized tight frame vector for $\mathcal{U}_{A_i,B_i}$. Thus, by Lemma 4.5, $|\det A_iB_i| = \|f_i\|^2$. Hence $\sum_{i=1}^{n} |\det A_iB_i| = \sum_{i=1}^{n} \|f_i\|^2 = \|F\|^2 \leqslant 1$, which contradicts our assumption. ∎

We still do not know, even in the one dimensional case, whether the condition $\sum_{i=1}^{n} |\det A_iB_i| > 1$ imply that $\mathcal{U}$ does not admit any cyclic vector.

Now we answer a question asked in Remark 4.11 of [10].

COROLLARY 4.8. *Suppose that $\alpha_i, \beta_i > 0$ such that $\alpha_i\beta_i = \alpha_j\beta_j$ for all i, j. Let $\mathcal{U} = \{U_{\alpha_1 k}V_{\beta_1 l} \oplus \cdots \oplus U_{\alpha_n k}V_{\beta_n l} : k, l \in \mathbb{Z}\}$. Then the following are equivalent:*

(i) $\sum_{i=1}^{n} \alpha_i\beta_i \leqslant 1$;
(ii) *$\mathcal{U}$ has a complete normalized tight frame vector;*
(iii) *$\mathcal{U}$ has a cyclic Bessel vector;*
(iv) *$\mathcal{U}$ has a cyclic vector.*

Moreover, $\mathcal{U}$ has a complete wandering vector if and only if $\sum_{i=1}^{n} \alpha_i\beta_i = 1$.

Proof. By Corollary 4.7, it suffices to check that (i) $\Rightarrow$ (ii). Due to the unitary equivalence, we can assume that $\alpha_1 = \cdots = \alpha_n = \alpha$ and $\beta_1 = \cdots = \beta_n = \beta$. Let $f_i = \sqrt{\alpha}\chi_{[(i-1)\beta, i\beta)}$ for $i = 1, \ldots, n$. Then $f_1 \oplus \cdots \oplus f_n$ is a complete normalized tight frame vector for $\mathcal{U}$, and it is a complete wandering vector when $\alpha\beta = \frac{1}{n}$. ∎

REMARK. After this paper was complete, we learned from R. Bălan that he and Z. Landau recently obtained the following result:

$\mathcal{U}_{\alpha,\beta}$ admits a maximal number of functions (windows) $g_1, \ldots, g_k$ (maximal with respect to k) such that $\mathcal{U}_{\alpha,\beta}g_1 \cup \cdots \cup \mathcal{U}_{\alpha,\beta}g_k$ forms an orthonormal set if and only if $[\alpha\beta] = k$, where $[\alpha\beta]$ denotes the integer part of $\alpha\beta$. Moreover, for the given $g_1, \ldots, g_k$, there is one complementary window function h such that $\mathcal{U}_{\alpha,\beta}h$ is a frame for the orthogonal complement of $\mathcal{U}_{\alpha,\beta}g_1 \cup \cdots \cup \mathcal{U}_{\alpha,\beta}g_k$.

We remark that this can also be obtained as a consequence of Lemma 4.1. We include a sketch of the proof here. First, assume that $[\alpha\beta] = k$. Then $\mathrm{tr}(I) = \alpha\beta < k+1$. Clearly there exist $f_1, \ldots, f_k$ such that $\mathcal{U}_{\alpha,\beta}f_1 \cup \cdots \cup \mathcal{U}_{\alpha,\beta}f_k$ forms an orthonormal set. (One can choose suitable (normalized) characteristic functions to play this role.) If there exist $g_1, \ldots, g_m$ such that $\mathcal{U}_{\alpha,\beta}g_1 \cup \cdots \cup \mathcal{U}_{\alpha,\beta}g_m$ forms an orthonormal set and $m > k$, then $I \geqslant T_{g_1}^*T_{g_1} + \cdots + T_{g_m}^*T_{g_m}$. Thus $\alpha\beta = \mathrm{tr}(I) \geqslant \mathrm{tr}(T_{g_1}^*T_{g_1} + \cdots + T_{g_m}^*T_{g_m}) = m$, which is a contradiction. Thus k is maximal.

Conversely, suppose that k is the maximal integer such that there exist $g_1, \ldots, g_m$ with the property that $\mathcal{U}_{\alpha,\beta}g_1 \cup \cdots \cup \mathcal{U}_{\alpha,\beta}g_k$ forms an orthonormal set. Let $n \leqslant \alpha\beta < n+1$. Then, by the above argument, k must be n. Hence $[\alpha\beta] = k$.

To get the last statement, let $g_1, \ldots, g_k$ be functions such that $\mathcal{U}_{\alpha,\beta}g_1 \cup \cdots \cup \mathcal{U}_{\alpha,\beta}g_k$ forms an orthonormal set with k maximal. Then $k = [\alpha\beta]$. Let $f_i = \sqrt{\alpha}\chi_{[(i-1)/\alpha, i/\alpha)}$ for $i = 1, \ldots, k$, and let $h = \sqrt{\alpha}\chi_{[k/\alpha, \beta]}$. Then $\mathcal{U}_{\alpha,\beta}f_1 \cup \cdots \cup \mathcal{U}_{\alpha,\beta}f_k$ forms an orthonormal set and $\mathcal{U}_{\alpha,\beta}h$ is a normalized tight frame for the space $L^2(\bigcup_{n\in\mathbb{Z}}([k/\alpha, \beta] + n\beta))$, which is the orthogonal complement of $\mathcal{U}_{\alpha,\beta}f_1 \cup \cdots \cup \mathcal{U}_{\alpha,\beta}f_k$. Let P and Q be the orthogonal projections onto the subspaces generated by $\mathcal{U}_{\alpha,\beta}f_1 \cup \cdots \cup \mathcal{U}_{\alpha,\beta}f_k$ and $\mathcal{U}_{\alpha,\beta}g_1 \cup \cdots \cup \mathcal{U}_{\alpha,\beta}g_k$, respectively. Define an operator $W : WU_\alpha^n V_\beta^m f_i = U_\alpha^n V_\beta^m g_i$ for all $n, m \in \mathbb{Z}, 1 \leqslant i \leqslant k$ and $Wf = 0$ for all $f \in P^\perp L^2(\mathbb{R})$. Then W is in the commutant of $\mathcal{U}_{\alpha,\beta}$ and has the property that $W^*W = P$ and $WW^* = Q$. Hence P and Q are equivalent projections in the finite von Neumann algebra $\mathcal{U}'_{\alpha,\beta}$. Therefore $P^\perp$ and $Q^\perp$ are also equivalent. Let $S \in \mathcal{U}'_{\alpha,\beta}$ be such that $S^*S = P^\perp$ and $SS^* = Q^\perp$. Then $\mathcal{U}_{\alpha,\beta}Sh$ is a normalized tight frame for $Q^\perp L^2(\mathbb{R})$, as expected.

5. APPENDIX

In this section we provide the sketch of proofs for Propositions 1.1 and 1.2 .

Proof of Proposition 1.1. The statement (iii) is trivial, while (i) and (ii) follow from the equality

$$f(u\sigma(vw))f(vw)\sigma(u\sigma(vw)) = u(vw) = (uv)w = f(uv)f(\sigma(uv)w)\sigma(\sigma(uv)w)$$

and the assumption that $\mathcal{U}$ is independent.

For each $g \in \mathcal{U}$, we have $\sigma(\sigma(gv^{-1})v) = \sigma(g\sigma(v^{-1}v)) = \sigma(g) = g$. Hence the first equality in (iv) holds. Note that

$$v\sigma(g^{-1}v)^{-1} = f(g^{-1}v)v(g^{-1}v)^{-1} = f(g^{-1}v)g.$$

Hence $\sigma(v\sigma(g^{-1}v)^{-1}) = g$ since $g \in \mathcal{U}$. Similarly, $\sigma(v\sigma(wg^{-1}v)^{-1}w) = g$. Thus the rest of (iv) holds.

For (v), assume that $\sigma(vu_1^{-1}w) = \sigma(vu_2^{-1}w)$ for some $u_1, u_2 \in \mathcal{U}$. Then

$$\overline{f(vu_1^{-1}w)}vu_1^{-1}w = \overline{f(vu_2^{-1}w)}vu_2^{-1}w.$$

Hence $f(vu_2^{-1}w)u_1 = f(vu_1^{-1}w)u_2$. So $u_1 = u_2$ by the independence of $\mathcal{U}$, which implies the injectivity of the mapping $u \to \sigma(vu^{-1}w)$. Similarly, one can easily check the injectivity for the other mappings in (v). ∎

Proof of Proposition 1.2. (i) By using Proposition 1.1, it is easy to check that $L_uR_v = R_vL_u$ for all $u, v \in \mathcal{U}$. Let $T \in \mathcal{R}'$ and $S \in \mathcal{L}'$. To show that $\mathcal{L}' = \mathcal{R}$, it suffices to prove that $ST = TS$. Write

$$Se_I = \sum_{u\in\mathcal{U}} a_u e_u, \quad Te_I = \sum_{u\in\mathcal{U}} b_u e_u.$$

Then for any $v \in \mathcal{U}$, we have

$$Se_v = S\overline{f(v)}L_v e_I = SL_v e_I = L_v S e_I = L_v \sum_{u\in\mathcal{U}} a_u e_u = \sum_{u\in\mathcal{U}} a_u f(vu) e_{\sigma(vu)}.$$

Replacing u by $\sigma(v^{-1}u)$ in the last expression, we get

$$Se_v = \sum_{u\in\mathcal{U}} a_{\sigma(v^{-1}u)} f(v\sigma(v^{-1}u)) e_{\sigma(v\sigma(v^{-1}u))} = \sum_{u\in\mathcal{U}} a_{\sigma(v^{-1}u)} f(v\sigma(v^{-1}u)) e_u.$$

Thus the equality $\langle S^*e_w, e_v\rangle = \langle e_w, Se_v\rangle$ implies that

$$S^*e_w = \sum_{u\in\mathcal{U}} \overline{a_{\sigma(u^{-1}w)}}\,\overline{f(u\sigma(u^{-1}w))} e_u$$

for all $w \in \mathcal{U}$.

Now we compute Te_v. Assume that $\sigma(x^{-1}) = v$ for some $x \in \mathcal{U}$. Then, by the definition of σ and f, $x^{-1} = f(x^{-1})v$. Hence $v^{-1} = f(x^{-1})x$. This implies that $x = \sigma(v^{-1})$ and $f(v^{-1}) = f(x^{-1})$. Thus

$$Te_v = T\overline{f(x^{-1})}R_x e_I = \overline{f(x^{-1})}R_x T e_I = \sum_{u\in\mathcal{U}} b_u \overline{f(x^{-1})} f(ux^{-1}) e_{\sigma(ux^{-1})}.$$

Again, replacing u by $\sigma(ux)$, gives

$$\begin{aligned} Te_v &= \sum_{u\in\mathcal{U}} b_{\sigma(ux)}\overline{f(x^{-1})}f(\sigma(ux)x^{-1})e_u \\ &= \sum_{u\in\mathcal{U}} b_{\sigma(u\sigma(v^{-1}))}\overline{f(v^{-1})}f(\sigma(u\sigma(v^{-1}))\sigma(v^{-1})^{-1})e_u. \end{aligned}$$

Therefore,

$$T^*e_w = \sum_{u\in\mathcal{U}} \overline{b_{\sigma(w\sigma(u^{-1}))}}f(u^{-1})\overline{f(\sigma(w\sigma(u^{-1}))\sigma(u^{-1})^{-1})}e_u$$

for all $w \in \mathcal{U}$.

Note that $w\sigma(u^{-1}) = \overline{f(u^{-1})}wu^{-1} = \overline{f(u^{-1})}f(wu^{-1})\sigma(wu^{-1})$. Thus we have that $\sigma(w\sigma(u^{-1})) = \sigma(wu^{-1})$ and $f(w\sigma(u^{-1})) = \overline{f(u^{-1})}f(wu^{-1})$. Similarly

$$\sigma(u\sigma(v^{-1})) = \sigma(uv^{-1}), \quad f(u\sigma(v^{-1})) = \overline{f(v^{-1})}f(uv^{-1}).$$

Since (by Proposition 1.1)

$$f(v\sigma(v^{-1}u))f(v^{-1}u) = f(\sigma(vv^{-1})w)f(vv^{-1}) = f(w)f(I) = 1,$$

it follows that $f(v\sigma(v^{-1}u)) = \overline{f(v^{-1}u)}$. Similarly we have

$$\begin{aligned} &f(u\sigma(u^{-1}w)) = \overline{f(u^{-1}w)}, \\ &f(\sigma(u\sigma(v^{-1}))\sigma(v^{-1})^{-1}) = \overline{f(u\sigma(v^{-1}))} = f(v^{-1})\overline{f(uv^{-1})}, \\ &f(\sigma(w\sigma(u^{-1}))\sigma(u^{-1})^{-1}) = \overline{f(w\sigma(u^{-1}))} = f(u^{-1})\overline{f(wu^{-1})}. \end{aligned}$$

Therefore,

$$\langle TSe_v, e_w\rangle = \sum_{u\in\mathcal{U}} a_{\sigma(v^{-1}u)}b_{\sigma(wu^{-1})}\overline{f(v^{-1}u)f(wu^{-1})} \tag{5.1}$$

and

$$\langle STe_v, e_w\rangle = \sum_{u\in\mathcal{U}} a_{\sigma(u^{-1}w)}b_{\sigma(uv^{-1})}\overline{f(u^{-1}w)f(uv^{-1})}. \tag{5.2}$$

If we replace u by $\sigma(vu^{-1}w)$ in (5.1) and note that $\{\sigma(vu^{-1}w) :\ u \in \mathcal{U}\} = \mathcal{U}$, then we have

$$\langle TS_v, e_w\rangle = \sum_{u\in\mathcal{U}} a_{\sigma(u^{-1}w)}b_{\sigma(w\sigma(vu^{-1}w)^{-1})}\overline{f(v^{-1}\sigma(vu^{-1}w))f(w\sigma(vu^{-1}w)^{-1})}.$$

However, since

$$w\sigma(vu^{-1}w)^{-1} = wf(vu^{-1}w)w^{-1}uv^{-1} = f(vu^{-1}w)f(uv^{-1})\sigma(uv^{-1}),$$

it follows that

$$\sigma(w\sigma(vu^{-1}w)^{-1}) = \sigma(uv^{-1})$$

and

$$f(w\sigma(vu^{-1}w)^{-1}) = f(vu^{-1}w)f(uv^{-1}).$$

Similarly, we also have $f(v^{-1}\sigma(vu^{-1}w)) = \overline{f(vu^{-1}w)}f(u^{-1}w)$. So

$$\begin{aligned}\langle TSe_v, e_w\rangle &= \sum_{u\in\mathcal{U}} a_{\sigma(u^{-1}w)}b_{\sigma(uv^{-1})}\overline{f(u^{-1}w)f(uv^{-1})f(vu^{-1}w)}f(vu^{-1}w)\\ &= \sum_{u\in\mathcal{U}} a_{\sigma(u^{-1}w)}b_{\sigma(uv^{-1})}\overline{f(u^{-1}w)f(uv^{-1})}\end{aligned}$$

since $\overline{f(vu^{-1}w)}f(vu^{-1}w) = 1$. Hence $\langle TSe_v, e_w\rangle = \langle STe_v, e_w\rangle$, as required.

(ii) We first note that L (respectively R) can be easily extended to a mapping from group($\mathcal{U}$) to group($L_u; u \in \mathcal{U}$) (respectively group($R_u : u \in \mathcal{U}$)) with the property that $L_uL_v = f(uv)L_{\sigma(uv)}$ (respectively $R_uR_v = \overline{f(uv)}R_{\sigma(uv)}$) whenever $u, v \in$ group($\mathcal{U}$). Let $\phi(A) = \langle Ae_I, e_I\rangle$. We show that ϕ is a faithful trace for $\mathcal{L}$. Let $u, v \in$ group($\mathcal{U}$). Then

$$\phi(L_uL_v) = \langle L_uL_ve_I, e_I\rangle = f(uv)\langle L_{\sigma(uv)}e_I, e_I\rangle$$

and

$$\phi(L_vL_u) = \langle L_vL_ue_I, e_I\rangle = f(vu)\langle L_{\sigma(vu)}e_I, e_I\rangle.$$

Note that if $\sigma(uv) = I$, then $uv = f(uv)I$. So, by multiplying with v and v^{-1} on both sides, we have $vu = f(uv)I$. Thus $\sigma(uv) = I$ if and only if $\sigma(vu) = I$, and in this case we have $f(uv) = f(vu)$. Therefore $\phi(L_uL_v) = \phi(L_vL_u) = 0$ whenever $\sigma(uv) \neq I$ and $= 1$ whenever $\sigma(uv) = I$. By taking the weak-limit, we get $\phi(AB) = \phi(BA)$ for all $A, B \in \mathcal{L}$. If $A \in \mathcal{L}$ is positive and $\phi(A) = 0$, then $A^{\frac{1}{2}}e_I = 0$. For any $v \in \mathcal{U}$, let $\sigma(w^{-1}) = v$ for some $w \in \mathcal{U}$. Then

$$A^{\frac{1}{2}}e_v = A^{\frac{1}{2}}\overline{f(w^{-1})}R_we_I = \overline{f(w^{-1})}R_wA^{\frac{1}{2}}e_I = 0.$$

So $A^{\frac{1}{2}} = 0$, which implies that $A = 0$. Thus ϕ is a faithful trace. Similarly $\tau(T) = \langle Te_I, e_I\rangle$, $T \in \mathcal{R}$, is also a faithful trace for $\mathcal{R}$. Therefore both $\mathcal{L}$ and $\mathcal{R}$ are finite von Neumann algebras.

(iii) Let $T \in \mathcal{L} \cap \mathcal{R}$. Then $TL_v = L_vT$ and $TR_v = R_vT$ for all $v \in \mathcal{U}$. Write $Te_I = \sum_{u\in\mathcal{U}} a_ue_u$. Fix $v \in \mathcal{U}$ and let $\sigma(x^{-1}) = v$ for some $x \in \mathcal{U}$. Then $x = \sigma(v^{-1})$ and $f(x^{-1}) = f(v^{-1})$. Thus

$$\begin{aligned}TL_ve_I = Te_v &= \overline{f(x^{-1})}TR_xe_I = \overline{f(x^{-1})}R_xTe_I\\ &= \sum_{u\in\mathcal{U}} a_u\overline{f(x^{-1})}f(ux^{-1})e_{\sigma(ux^{-1})}\\ &= \sum_{u\in\mathcal{U}} a_{\sigma(ux)}\overline{f(x^{-1})}f(\sigma(ux)x^{-1})e_u\\ &= \sum_{u\in\mathcal{U}} a_{\sigma(uv^{-1})}\overline{f(v^{-1})}f(\sigma(u\sigma(v^{-1}))\sigma(v^{-1})^{-1})e_u\end{aligned}$$

and

$$L_vTe_I = \sum_{u\in\mathcal{U}} a_uf(vu)e_{\sigma(vu)} = \sum_{u\in\mathcal{U}} a_{\sigma(v^{-1}u)}f(v\sigma(v^{-1}u))e_u.$$

Thus

$$a_{\sigma(uv^{-1})}\overline{f(v^{-1})}f(\sigma(u\sigma(v^{-1}))\sigma(v^{-1})^{-1}) = a_{\sigma(v^{-1}u)}f(v\sigma(v^{-1}u)).$$

Replacing u by $\sigma(vu)$ in the above equality, we get that

$$a_{\sigma(vuv^{-1})}\overline{f(v^{-1})}f(\sigma(vuv^{-1})\sigma(v^{-1})^{-1}) = a_u f(vu).$$

Note that

$$\sigma(vuv^{-1})\sigma(v^{-1})^{-1} = \overline{f(vuv^{-1})}vuv^{-1}f(v^{-1})v = \overline{f(vuv^{-1})}f(v^{-1})f(vu)\sigma(vu).$$

Thus

$$f(\sigma(vuv^{-1})\sigma(v^{-1})^{-1}) = \overline{f(vuv^{-1})}f(v^{-1})f(vu).$$

Therefore

$$a_{\sigma(vuv^{-1})}\overline{f(vuv^{-1})} = a_u.$$

Assume that $u \neq I$. If $\{\sigma(vuv^{-1}) : v \in \mathcal{U}\}$ is infinite, then $a_u = 0$ since $\sum_{u\in\mathcal{U}} |a_u|^2 < \infty$. If $\{\sigma(vuv^{-1}) : v \in \mathcal{U}\}$ is finite, then $a_u = a_{\sigma(vuv^{-1})}$ for all but a finite number of v. Since $\{f(vuv^{-1}) : v \in \mathcal{U}\}$ is infinite, it follows that a_u must be zero. Hence $Te_I = a_I e_I$. This implies that $Te_v = L_v Te_I = a_I e_v$ for all $v \in \mathcal{U}$. So $T = a_I I$, as expected. ∎

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