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CHARACTERIZATIONS OF ESSENTIAL IDEALS AS OPERATOR MODULES OVER C^*-ALGEBRAS

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ABSTRACT. In this paper we give characterizations of essential left ideals of a C^*-algebra A in terms of their properties as operator A-modules. Conversely, we seek C^*-algebraic characterizations of those ideals J in A such that A is an essential extension of J in various categories of operator modules. In the case of two-sided ideals, we prove that all the above concepts coincide. We obtain results, analogous to M. Hamana's results, which characterize the injective envelope of a C^*-algebra as a maximal essential extension of the C^*-algebra, but with completely positive maps replaced by completely bounded module maps. By restricting to one-sided ideals, module actions reveal clear differences which do not show up in the two-sided case. Throughout this paper, module actions are crucial.

KEYWORDS: *Essential ideals, left ideals, C^*-module extensions.*

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1. INTRODUCTION

Let A denote a C^*-algebra. In the C^*-algebra literature a two-sided ideal, J, of A is called *essential* if $aJ = \{0\}$ implies $a = 0$. Given a category and an object X contained in an object Y, then Y is called an *essential extension* of X provided that every morphism of Y that restricts to be an isomorphism of X must necessarily be an isomorphism of Y. In this paper we study the relationships between these two different approaches to essentiality as we vary the category.

Every left ideal can be regarded as an object in the category of operator left A-modules, with morphisms either the completely contractive or completely bounded left A-module maps. Thus, it is interesting to know the relationships between an ideal J being essential in A, in the C^*-algebraic sense, and A being an essential extension of J in these categories. For two-sided ideals we show that all three concepts coincide. Moreover, these conditions are equivalent to the injective envelopes, $I(J)$ and $I(A)$, being completely isometrically isomorphic via a map that restricts to be the identity on J.

However, for left ideals these concepts differ. It is still true that if $I(J)$ and $I(A)$ are completely isometrically isomorphic via a map that restricts to be the identity on J, then J is an essential left ideal in A. But the converse does not necessarily hold.

In this paper, we characterize the essential left ideals of A via their properties as modules in the above categories. We also give some characterizations of the left ideals for which A is an essential extension in some of these categories in C^*-algebraic terms. But there are some categories for which we are unable to give C^*-algebraic characterizations of these families of left ideals.

As in the earlier work of M. Frank and the second author ([5]), we find that when the morphisms are the completely contractive maps, then our questions reduce to the case of completely positive maps and rely mostly on the earlier results of M. Hamana. However, when the morphisms are taken to be the completely bounded maps or in the case of one-sided ideals, then we are dealing with a completely isomorphic situation which requires new techniques or the recent results of [5] and [3].

Two of M. Hamana's original characterizations of the injective envelope $I(A)$ of A were as the minimal injective object containing A and as a maximal essential extension of A in the category of operator systems and completely positive maps. In the work of M. Frank and the second author ([5]), it was shown that $I(A)$ could also be characterized as the minimal injective containing A in the category of operator left A-modules and completely bounded maps. Completing this set of ideas we show that $I(A)$ can also be characterized as the maximal essential extension of A in this latter category.

Throughout this paper, C^*-algebras are not necessarily unital. In Section 3 we cite a result of [5] where "unital" is assumed. But as is remarked there, many results, including this one, follow in the case of a non-unital C^*-algebra A, by taking its minimal unitization A^1 and observing that A-modules are naturally A^1-modules.

2. DEFINITIONS

We begin by recalling a remark from [5]. When dealing with a category whose morphisms are vector spaces of bounded linear maps, then requiring that an object is injective only guarantees that every bounded linear map has a bounded extension, but not necessarily of the same norm. When we want to insist that extensions also have the same norm, then we shall refer to the objects as *tight* injectives. It is easy to see that tight injectivity is equivalent, by scaling, to requiring that every contractive linear map has a contractive linear extension. Thus tight injectivity is the same as saying that the object is injective in a category where the morphisms are all contractive linear maps. Throughout this paper we shall refer to situations as tight, or simply omit the word tight, instead of constantly referring to a change of categories. But of course, both viewpoints are useful.

We start with definitions of several notions many of which are already known, but perhaps not in our particular setting.

Definition 2.1. Let A and B be operator algebras.

(1) An *operator A-B-bimodule* X is an operator space which is also an A-B-bimodule with

$$\|(a_{lm})(x_{lm})\| \leqslant \|(a_{lm})\| \, \|(x_{lm})\| \quad \text{and} \quad \|(x_{lm})(b_{lm})\| \leqslant \|(x_{lm})\| \, \|(b_{lm})\|$$

$\forall (x_{lm}) \in \mathbb{M}_n(X)$, $\forall (a_{lm}) \in \mathbb{M}_n(A)$, $\forall (b_{lm}) \in \mathbb{M}_n(B)$, $\forall n \in \mathbb{N}$. If A (respectively, B) contains an identity 1_A (respectively, 1_B), we always assume that $\forall x, 1_A x = x$ (respectively, $\forall x$, $x 1_B = x$). We also say an *operator left A-module* (respectively, *operator right A-module*, *operator A-bimodule*) for an operator A-$\mathbb{C}$-bimodule (respectively, operator $\mathbb{C}$-A-bimodule, operator A-A-bimodule).

(2) A (respectively, *tight*) *A-B-rigid extension* of an operator A-B-bimodule X is a pair (Y, ι) consisting of an operator A-B-bimodule Y and a completely bounded (respectively, completely isometric) A-B-bimodule map $\iota : X \to Y$ which is an isomorphism onto its range (a one-to-one completely bounded linear map whose inverse $\iota(X) \to X$ is also completely bounded) such that the identity map on Y is the only completely bounded (respectively, completely contractive) A-B-bimodule map on Y which fixes each element of $\iota(X)$.

(3) A (respectively, *tight*) *A-B-essential extension* of an operator A-B-bimodule X is a pair (Y, ι) consisting of an operator A-B-bimodule Y and a completely boundedly (respectively, completely isometric) A-B-bimodule map $\iota : X \to Y$ which is an isomorphism onto its range, such that for any operator A-B-bimodule W and for any completely bounded (respectively, completely contractive) A-B-bimodule map $\phi : Y \to W$, ϕ is a completely bounded isomorphism (respectively, a complete isometry) whenever $\phi|\iota(X)$ is. If there is no proper embedding of Y by a completely boundedly isomorphic (respectively, completely isometric) A-B-bimodule map into any (respectively, tight) A-B-essential extension of X which is an isomorphism onto its range (or, equivalently, there is no proper (respectively, tight) A-B-essential extension of Y), then we say (Y, ι) is a maximal (respectively, tight) A-B-essential extension of X.

(4) An operator A-B-bimodule W is (respectively, *tight*) *A-B-injective*, if for any operator A-B-bimodules Y and Z with $Y \subset Z$ (Y has a matrix norm and a module action inherited from Z) any completely bounded A-B-bimodule map $Y \to W$ extends to a completely bounded A-B-bimodule map $Z \to W$ (respectively, preserving the completely bounded norm).

(5) A (respectively, *tight*) *A-B-injective extension* of an operator A-B-bimodule X is a pair (Y, ι) consisting of an operator A-B-bimodule Y and a completely bounded (respectively, completely isometric) A-B-bimodule map $\iota : X \to Y$ which is an isomorphism onto its range, with Y (respectively, tight) A-B-injective.

(6) An *A-B-injective envelope* of an operator A-B-bimodule X is a pair (Y, ι) consisting of an operator A-B-bimodule Y and a completely isometric A-B-bimodule map $\iota : X \to Y$ with Y a minimal tight A-B-injective operator A-B-bimodule containing $\iota(X)$ (i.e. for any tight A-B-injective operator A-B-bimodule Z with $\iota(X) \subset Z \subset Y$, necessarily $Y = Z$ holds, where Z has a matrix norm and a module action inherited from Y). We denote this Y by $I(X)$. Frequently we identify X with $\iota(X)$ and regard it as $X \subset I(X)$.

In the above (2), (3), (5), and (6), often we simply write Y for (Y, ι).

REMARK 2.2. (1) There are still some difficulties involved in trying to define injective envelopes in the non-tight setting. In (6) above we really are defining injective envelopes only in the tight setting.

(2) The following is a well-known argument ([8]). Let A and B be C^*-algebras. For any operator A-B-bimodule, take any tight A-B-injective extension (Y, ι) of X. (Such an extension exists even if X is an operator A-B-bimodule, since X can be considered as $X \subset \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ by the representation theorem for operator modules ([4], [2], [3]) and $\mathbb{B}(\mathcal{H})$ is a tight A-B-injective by the bimodule version of G. Wittstock's extension theorem (Theorem 4.1 in [19]. See also [15].) Given a minimal $\iota(X)$-projection on Y (the existence of a minimal projection can be shown in the same way as [8]), then it follows that "(Y, ι) is an A-B-injective envelope of $X \Leftrightarrow Y$ is the range of a minimal $\iota(X)$-projection on some A-B-injective operator A-B-bimodule containing $\iota(X)$". In particular, for any operator A-B-bimodule X, its tight A-B-injective envelope exists.

(3) When A and B are C^*-algebras, as is proved in Corollary 2.6 in [3], an operator A-B-bimodule is tight A-B-injective if and only if it is $\mathbb{C}$-$\mathbb{C}$-injective, and also when X is an operator A-B-bimodule, Y is an A-B-injective envelope of X if and only if Y is a $\mathbb{C}$-$\mathbb{C}$-injective envelope of X. For this reason, when X is an operator A-B-bimodule we frequently call $I(X)$ just an *injective envelope of* X. Also this fact justifies that we use the simple notation $I(X)$ without mentioning A and B.

3. ESSENTIAL EXTENSIONS WITH MODULE ACTIONS

In this section we obtain the necessary characterizations of injective envelopes as maximal essential extensions. In the tight situation, these are basically restatements of results of M. Hamana. However, in the non-tight situation, these characterizations rely on recent work of M. Frank and the second author ([5]) which extends Hamana's rigidity results to completely bounded module maps. Analogous results to the known results in the category of modules over a ring (Theorem 2.10.20 in [14]) were obtained by M. Hamana in the cases of Banach modules, C^*-algebras, operator systems and operator spaces ([6], [7], [8], [9]). Without any intrinsic changes in his proofs, similar results hold for the case of operator modules over C^*-algebras. We summarize the key facts below.

THEOREM 3.1. *Let A, B be C^*-algebras, and X an operator A-B-bimodule. Then the following are equivalent:*

(i) (Y, ι) *is a maximal tight A-B-essential extension of X;*

(ii) (Y, ι) *is a tight A-B-injective and tight A-B-essential extension of X;*

(iii) (Y, ι) *is an A-B-injective envelope of X;*

(iv) (Y, ι) *is a tight A-B-injective and tight A-B-rigid extension of X.*

Moreover, such a Y is complete.

Proof. First, we suppose that X is an operator A-B-bimodule.

That (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) immediately follows from the operator module over C^*-algebra version of Lemma 2.11 in [9], but this result relies on several earlier results. For the convenience of the reader, we also include an outline of the proof of this part.

(ii) $\Rightarrow$ (iii) We only need to show minimality of Y. Let Z be a tight A-B-injective operator A-B-bimodule such that $\iota(X) \subset Z \subset Y$. Then id_Z extends to a completely contractive A-B-bimodule map $\phi : Y \to Z$. By the assumption (ii), ϕ has to be a complete isometry, so that $Y = Z$.

(iii) $\Rightarrow$ (iv) The same as the proof of Lemma 3.6 in [8]. When applying this proof, note that, if ϕ is an $\iota(X)$-projection on Y (i.e. a completely contractive A-B-bimodule map on Y with $\phi^2 = \phi$ and $\phi|\iota(X) = \mathrm{id}_{\iota(X)}$), then $\mathrm{Im}\,\phi \subset Y$ is a tight A-B-injective operator A-B-bimodule, so that $\mathrm{Im}\,\phi = Y$ by minimality of Y. Hence ϕ $(= \mathrm{id}_Y)$ is a minimal $\iota(X)$-projection on Y and Y is its range.

(iv) $\Rightarrow$ (ii) The same as the proof (necessity) of Theorem 3.7. in [8].

(i) $\Rightarrow$ (ii) Let $(I(Y), \kappa)$ be an A-B-injective envelope of Y. By (iii) $\Rightarrow$ (ii), $(I(Y), \kappa)$ is a tight A-B-essential extension of Y, so that $(I(Y), \kappa \circ \iota)$ is a tight A-B-essential extension of X, hence $I(Y) = \kappa(Y)$ by maximality. Thus Y is tight A-B-injective.

(ii) $\Rightarrow$ (i) Let (Z, κ) be a tight A-B-essential extension of Y. By injectivity of Y, $\kappa^{-1} : \kappa(Y) \to Y$ extends to a completely contractive A-B-bimodule map $\phi : Z \to Y$. But by essentiality of Z, ϕ has to be a complete isometry. Thus $\kappa(Y) = Z$.

Finally, we show completeness. Let Y satisfy the equivalent conditions (i)–(iv) and let $(\widetilde{Y}, \kappa)$ be its completion. Then $\widetilde{Y}$ is an A-B-bimodule and obviously a tight A-B-essential extension of Y. By injectivity of Y, $\kappa(Y) \to Y$ extends to a completely contractive A-B-bimodule map $\phi : \widetilde{Y} \to Y$ with $\phi|\kappa(Y)$ completely isometric. But by an essentiality of $\widetilde{Y}$, ϕ has to be a complete isometry, so that $\widetilde{Y} = \kappa(Y)$. ∎

REMARK 3.2. (1) From this theorem, in particular, for any operator A-B-bimodule X, a maximal tight A-B-essential extension exists, since an A-B-injective envelope exists (Remark 2.2 (2)).

(2) It immediately follows from rigidity in the above theorem that the A-B-injective envelope is unique up to completely isometric A-B-bimodule isomorphisms.

(3) In particular, if $X = A$ with A a unital C^*-algebra, $I(A)$ is completely isometrically isomorphic to M. Hamana's injective envelope C^*-algebra $I_{C^*}(A)$ ([7]) of A. This is easily seen by observing unital completely positive maps are just the same as unital completely contractive maps, and using the rigidity of $I(A)$ and $I_{C^*}(A)$. If A is a non-unital C^*-algebra, then $I(A)$ is completely isometrically isomorphic to $I_{C^*}(A^1)$, via a map that restricts to be the identity on A, by Proposition 2.8 in [3], where A^1 is the minimal unitization of A. Namely, $I(A^1)$ is completely isometrically isomorphic to $I(A)$. Actually the above identification of $I(A)$ with $I_{C^*}(A)$ or $I_{C^*}(A^1)$ is also as A-A-, A-$\mathbb{C}$- and $\mathbb{C}$-A-bimodules by Remark 2.2 (3).

In the non-tight case a similar result holds thanks to Corollary 2.2 in [5] which is the generalization of “rigidity” to the non-tight case in the presence of a module action.

Theorem 3.3. *Let A be a C^*-algebra, $I(A)$ its injective envelope and let E be an operator left A-module with $A \subset E$. Then the following are equivalent:*

(i) *E is a maximal A-$\mathbb{C}$-essential extension of A;*

(ii) *E is an A-$\mathbb{C}$-injective and A-$\mathbb{C}$-essential extension of A;*

(iii) *E is a minimal A-$\mathbb{C}$-injective extension of A;*

(iv) *E is completely boundedly isomorphic to $I(A)$ as left A-modules, via a map that restricts to the identity on A;*

(v) *E is an A-$\mathbb{C}$-injective and A-$\mathbb{C}$-rigid extension of A.*

The analogous result holds for the operator right A-module case and the operator A-bimodule case.

Proof. Throughout the proof, without loss of generality, we may assume that $I(A) = I_{C^*}(A)$ (Remark 3.2 (3)).

That (iii) $\Rightarrow$ (iv) was proven in Theorem 3.3 in [5].

That (iv) $\Rightarrow$ (iii) follows by noting that any A-$\mathbb{C}$-injective extension of A which is completely boundedly isomorphic to a minimal A-$\mathbb{C}$-injective extension as left A-modules is also minimal and that $I(A)$ is a minimal A-$\mathbb{C}$-injective extension by Theorem 3.1 in [5].

(iv) $\Rightarrow$ (i) Without loss of generality we may assume that $E = I(A)$. Let W be any operator left A-module and $\phi : I(A) \to W$ a completely bounded left A-module map which is a completely bounded isomorphism on A. Let $\psi : \phi(A) \to A$ be the inverse of this map and extend ψ to a completely bounded left A-module map $\widetilde{\psi} : W \to I(A)$. Since $\widetilde{\psi} \circ \phi : I(A) \to I(A)$ is the identity on A, it is the identity on $I(A)$ by rigidity (Corollary 2.2 in [5]). This shows that $\phi(I(A))$ is completely boundedly isomorphic to $I(A)$. Maximality: Suppose that (F, ι) is an A-$\mathbb{C}$-essential extension of $I(A)$. A completely boundedly isomorphic left A-module map $\iota^{-1} : \iota(I(A)) \to I(A)$ extends to a completely bounded left A-module map $\rho : F \to I(A)$. By essentiality of F, ρ has to be a completely bounded isomorphism, so that $F = \iota(I(A))$.

(i) $\Rightarrow$ (iv) We may assume that $A \subset E \subset \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ by the representation theorem for operator modules ([4], [2], [3]). Extend the identity map on A to a completely bounded left A-module map $\psi : \mathbb{B}(\mathcal{H}) \to I(A)$, then $\psi(E) \subset I(A)$ is completely boundedly isomorphic to E. The map $\psi^{-1} : \psi(E) \to E$ extends to a completely bounded left A-module map $\phi : I(A) \to \mathbb{B}(\mathcal{H})$. By rigidity (Corollary 2.2 in [5]), $\psi \circ \phi$ is the identity on $I(A)$. Hence we have $A \subset E \subset \phi(I(A))$ with $\phi(I(A))$ completely boundedly isomorphic to $I(A)$. This makes $\phi(I(A))$ an operator A-$\mathbb{C}$-essential extension of A by applying (iv) $\Rightarrow$ (i) that we have already proved. Thus, by maximality of E, $E = \phi(I(A))$ so that $I(A)$ is completely boundedly isomorphic to E as left A-modules by ϕ.

Now we have that (i) $\Rightarrow$ (ii), since clearly (i) and (iv) imply (ii).

Next, we show that (ii) $\Rightarrow$ (iv). By the assumption (ii), the identity map on A extends to a completely bounded left A-module map $\phi : E \to I(A)$ and the map $\phi(E) \to E$ extends to a completely bounded left A-module map $\psi : I(A) \to E$. Then $\psi \circ \phi$ is the identity on E. But by rigidity (Corollary 2.2 in [5]), $\phi \circ \psi$ is the identity on $I(A)$.

(iv) $\Rightarrow$ (v) Without loss of generality, we may assume that $E = I(A)$. Then (v) immediately follows from Corollary 2.2 in [5].

(v) $\Rightarrow$ (iv) By injectivity of E, the identity map on A extends to a completely bounded left A-module map $\phi : I(A) \to E$. Similarly, by injectivity of $I(A)$, the identity map on A extends to a completely contractive A-B-bimodule map $\psi : E \to I(A)$. But $\psi\circ\phi = \mathrm{id}_{I(A)}$ and $\phi\circ\psi = \mathrm{id}_E$ by rigidity of $I(A)$ (Corollary 2.2 in [5]) and E, respectively. ∎

As is seen in the above proof, Corollary 2.2 in [5] played a crucial role.

Before moving on to the main results, let us introduce some definitions.

4. ESSENTIAL LEFT IDEALS

Recall that a closed two-sided ideal K in a C^*-algebra A is *essential*, if $aK = \{0\}$ implies $a = 0$, or, equivalently, $Ka = \{0\}$ implies $a = 0$, or, equivalently, $K \cap K' \neq \{0\}$ for all non-zero closed two-sided ideals K' in A, where $a \in A$. We now generalize these ideas to the one-sided case. Note that a subset $J \subset A$ is a closed left ideal in A if and only if J^* is a closed right ideal in A. Let J be a closed left ideal in A. As is well known (say, [10], [12], [16]), J has a contractive right approximate identity $\{e_\alpha\}$. Namely, $\{e_\alpha\} \subset (J \cap J^*)_+$ is an increasing net and $\lim\limits_{\alpha\to\infty} je_\alpha = j \ \forall j \in J$ and $\|e_\alpha\| \leqslant 1 \ \forall\alpha$. Clearly, it is also a contractive left approximate identity of J^*.

LEMMA 4.1. *Let A be a C^*-algebra, I a closed right ideal in A, and J a closed left ideal in A. Then*

$$I \cap J = IJ,$$

where $IJ := \overline{\mathrm{span}}\{ij : i \in I,\, j \in J\}$, here the closure is taken in A.

Proof. $I \cap J \supset IJ$ is clear. We show $I \cap J \subset IJ$. Let $i \in I \cap J$ and $\{e_\alpha\}$ a right approximate identity of J, then $i = \lim\limits_{\alpha\to\infty} ie_\alpha \in IJ$. ∎

PROPOSITION 4.2. *Let A be a C^*-algebra and J a closed left ideal in A. Then the following are equivalent:*

(i) *JJ^* is an essential two-sided ideal in A;*

(ii) *if $a \in A$ and $aJ = \{0\}$, then $a = 0$;*

(iii) *$I \cap J \neq \{0\}$ for any non-zero closed right ideal I in A;*

(iv) *$K \cap J \neq \{0\}$ for any non-zero closed two-sided ideal K in A.*

Proof. (i) $\Rightarrow$ (ii) Suppose $aJ = \{0\}$ for $a \in A$, then $aJJ^* = \{0\}$, so that $a = 0$.

(ii) $\Rightarrow$ (iii) Let I be a closed right ideal in A with $I \cap J = \{0\}$, then $IJ = \{0\}$ by Lemma 4.1, so that $I = \{0\}$ by the assumption (ii).

(iii) $\Rightarrow$ (iv) Clear.

(iv) $\Rightarrow$ (i) Obviously JJ^* is a closed two-sided ideal in A, so we only show essentiality. Let $K := \{a \in A : aJJ^* = \{0\}\}$, then K is a closed two-sided ideal in A. Suppose that $a \in K$ then $aJJ^* = \{0\}$ and $0 = ajj^*a^* = aj(aj)^*$ for all $j \in J$, so that $aJ = \{0\}$. Hence $K \cap J = KJ = \{0\}$. Thus assumption (iv) implies that $K = \{0\}$ and so JJ^* is an essential ideal. ∎

One obtains a parallel result for a right ideal I by setting $J = I^*$.

REMARK 4.3. Let A be a C^*-algebra, let J be a closed left ideal and let $\{e_\alpha\}$ be a right approximate identity of J.

(1) JJ^* is the two-sided ideal in A generated by J in A. Also $JJ^* = JA$ since, for any $a \in A$, $ja = \lim_{\alpha\to\infty}(je_\alpha)a = j \cdot \lim_{\alpha\to\infty} e_\alpha a$.

(2) Any $j \in J$ satisfies $j = \lim_{\alpha\to\infty} je_\alpha$, hence $J \subset JJ^*$ even if A is non-unital. Similarly, $J^* \subset JJ^*$.

DEFINITION 4.4. Let A be a C^*-algebra and let J be a closed left ideal in A. When any of the equivalent statements in Proposition 4.2 hold, we say that J is an *essential left ideal* in A. We say that a right ideal I is an *essential right ideal* when I^* is an essential left ideal.

5. MAIN RESULTS

In this section we present our main results, showing the relationships between the various notions of "essential" for left ideals. We apologize to the reader in advance for the rather long list of conditions that are equivalent to a left ideal being essential in the C^*-algebraic sense. However, it is convenient to know that all of these are equivalent. We have presented the results in this particular fashion so that one can more easily contrast the tight and non-tight cases and the cases of one-sided and two-sided ideals.

THEOREM 5.1. *Let A be a C^*-algebra, J a closed left ideal in A, and $\{e_\alpha\}$ a contractive right approximate identity of J. Then the statements in* (I) *and* (II) *are equivalent, respectively, each statement in* (I) *implies* (5) *and* (5) *implies each statement in* (II).

(I) $\left\{\right.$

(1) *$I(J)$ is completely isometrically isomorphic to $I(A)$ via a map that restricts to the identity on J;*

(2) *A is a tight $\mathbb{C}$-$\mathbb{C}$-essential extension of J;*

(3) *A is a tight A-$\mathbb{C}$-essential extension of J;*

(4) $\|(a_{lm})\| = \sup_\alpha \|(a_{lm}e_\alpha)\|$ *for all $(a_{lm}) \in \mathbb{M}_n(A)$ and for all $n \in \mathbb{N}$;*

(5) *there exists $n \in \mathbb{N}$ such that*

$\|(a_{lm})\| = \sup_\alpha \|(a_{lm}e_\alpha)\|$ *for all $(a_{lm}) \in \mathbb{M}_n(A)$;*

(II)

(6) $\|(a_{lm})\| = \sup\{\|(a_{lm})(j_{lm})\| : (j_{lm}) \in \mathbb{M}_n(J),\ \|(j_{lm})\| \leqslant 1\}$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$ *and for all* $n \in \mathbb{N}$*;*

(7) *there exists* $n \in \mathbb{N}$ *such that* $\|(a_{lm})\| = \sup\{\|(a_{lm})(j_{lm})\| : (j_{lm}) \in \mathbb{M}_n(J),\ \|(j_{lm})\| \leqslant 1\}$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$*;*

(8) *there exists* $c \geqslant 1$ *such that* $\|(a_{lm})\| \leqslant c \cdot \sup\{\|(a_{lm})(j_{lm})\| : (j_{lm}) \in \mathbb{M}_n(J),\ \|(j_{lm})\| \leqslant 1\}$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$ *and for all* $n \in \mathbb{N}$*;*

(9) *there exist* $c \geqslant 1$ *and* $n \in \mathbb{N}$ *such that* $\|(a_{lm})\| \leqslant c \cdot \sup\{\|(a_{lm})(j_{lm})\| : (j_{lm}) \in \mathbb{M}_n(J),\ \|(j_{lm})\| \leqslant 1\}$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$*;*

(10) *A is a tight $\mathbb{C}$-J-essential extension of J;*

(11) *A is a $\mathbb{C}$-J-essential extension of J;*

(12) *A is a tight A-J-essential extension of J;*

(13) *A is an A-J-essential extension of J;*

(14) *$\mathbb{M}_n(J)$ is an essential left ideal in $\mathbb{M}_n(A)$ for all $n \in \mathbb{N}$;*

(15) *$\mathbb{M}_n(J)$ is an essential left ideal in $\mathbb{M}_n(A)$ for some $n \in \mathbb{N}$;*

(16) *$\mathbb{M}_n(A)$ is canonically $*$-isomorphically embedded in $\mathcal{M}(\mathbb{M}_n(JJ^*))$ for all $n \in \mathbb{N}$;*

(17) *$\mathbb{M}_n(A)$ is canonically $*$-isomorphically embedded in $\mathcal{M}(\mathbb{M}_n(JJ^*))$ for some $n \in \mathbb{N}$;*

where $\mathcal{M}(\mathbb{M}_n(JJ^*))$ is the multiplier algebra of $\mathbb{M}_n(JJ^*)$.

Proof. (1) $\Rightarrow$ (2) Without loss of generality, we may assume that $A \subset I(J) = I(A)$. Let W be any operator space, $\phi : A \to W$ any complete contraction which is completely isometric on J. Since $I(J)$ is tight $\mathbb{C}$-$\mathbb{C}$-injective, $\phi^{-1} : \phi(J) \to J$ extends to a complete contraction $\psi : W \to I(J)$. Then, again, by tight $\mathbb{C}$-$\mathbb{C}$-injectivity of $I(J)$, $\psi \circ \phi$ extends to a complete contraction $\rho : I(J) \to I(J)$ with $\rho | J = \mathrm{id}_J$. By rigidity (Theorem 3.1), $\rho = \mathrm{id}_{I(J)}$, hence ϕ has to be a complete isometry.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (4) Let us define $\wp : A \to \mathbb{R}_+$ by $\wp(a) := \sup\limits_{\alpha} \|a e_\alpha\|$, $\forall a \in A$. And let $\widetilde{A} := A / \mathrm{Ker}\, \wp$, then $\widetilde{A}$ is an operator space with a well-defined matrix norm $\|\!| \cdot |\!\|$, where $\|\!|(a_{lm} + \mathrm{Ker}\, \wp)|\!\| := \sup\limits_{\alpha} \|(a_{lm} e_\alpha)\|$, $\forall (a_{lm}) \in \mathbb{M}_n(A)$, $\forall n \in N$. Define $\phi : A \to \widetilde{A}$ by $\phi(a) := a + \mathrm{Ker}\, \wp$. Moreover, since $\mathrm{Ker}\wp$ is a left A-module, $\widetilde{A}$ is an operator A-$\mathbb{C}$-bimodule. The map ϕ is a completely contractive A-$\mathbb{C}$-bimodule map which is completely isometric on J, so that, ϕ is completely isometric on A by the assumption (3). Hence $\|(a_{lm})\| = \|\!|(a_{lm})|\!\| = \sup\limits_{\alpha} \|(a_{lm} e_\alpha)\|$, $\forall (a_{lm}) \in \mathbb{M}_n(A)$, $\forall n \in N$.

(4) $\Rightarrow$ (1) Since the assumption remains valid if we adjoin an identity to A, we may assume that A is unital. Consider the canonical operator system $\mathcal{S} = \begin{pmatrix} \mathbb{C} & J \\ J^* & \mathbb{C} \end{pmatrix} \subset M_2(A)$ associated with J. Keeping the notation of [3], we have that $I(\mathcal{S}) = \begin{pmatrix} I_{11} & I(J) \\ I(J)^* & I_{22} \end{pmatrix}$.

By [8], the identity map on $\mathcal{S}$ extends to a $*$-homomorphism, $\pi : C^*(\mathcal{S}) \to I(\mathcal{S})$. Note that there will exist maps such that $\pi((b_{i,j})) = \begin{pmatrix} \pi_{1,1}(b_{1,1}) & \pi_{1,2}(b_{1,2}) \\ \pi_{2,1}(b_{2,1}) & \pi_{2,2}(b_{2,2}) \end{pmatrix}$. We may extend π to a completely positive map $\Phi : M_2(A) \to I(\mathcal{S})$. Since Φ is a $*$-homomorphism on $C^*(\mathcal{S})$ it will be a bimodule map over this C^*-algebra, that is $\Phi(bxc) = \pi(b)\Phi(x)\pi(c)$ for $b, c \in C^*(\mathcal{S})$.

We also have that there exist maps ϕ_1, ϕ_2 and ϕ, such that

$$\Phi((a_{i,j})) = \begin{pmatrix} \phi_1(a_{1,1}) & \phi(a_{1,2}) \\ \phi(a_{2,1}^*)^* & \phi_2(a_{2,2}) \end{pmatrix}.$$

Note that $\begin{pmatrix} 0 & 0 \\ 0 & J^*J \end{pmatrix} \subset C^*(\mathcal{S})$. Hence, for any $a \in A$ and any α, we have that

$$\begin{aligned}
\begin{pmatrix} 0 & ae_\alpha \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & \phi(ae_\alpha) \\ 0 & 0 \end{pmatrix} = \Phi(\begin{pmatrix} 0 & ae_\alpha \\ 0 & 0 \end{pmatrix}) = \Phi(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix}) \\
&= \Phi(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix})\pi(\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix}) = \begin{pmatrix} 0 & \phi(a) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \pi_{2,2}(e_\alpha) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \phi(a)\pi_{2,2}(e_\alpha) \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Thus, we find that $ae_\alpha = \phi(a)\pi_{2,2}(e_\alpha)$ and so we have that

$$\|\phi(a)\| \geqslant \sup_\alpha \|\phi(a)\pi_{2,2}(e_\alpha)\| = \sup_\alpha \|ae_\alpha\| = \|a\|,$$

from which it follows that ϕ is an isometry of A into $I(J)$. The proof that ϕ is a complete isometry follows similarly.

Since $I(J)$ is injective, ϕ extends to a completely contractive map from $I(A)$ to $I(J)$, which we still denote by ϕ.

Since $I(A)$ is injective, $(\phi|A)^{-1} : \phi(A) \to A$ extends to a completely contractive map $\psi : I(J) \to I(A)$. Then $\psi \circ \phi$ and $\phi \circ \psi$ are completely contractive maps, that restrict to be the identity on A and J, respectively. Thus, $\psi \circ \phi = \mathrm{id}_{I(A)}$ and $\phi \circ \psi = \mathrm{id}_{I(J)}$ by rigidity (Theorem 3.1). Hence ϕ and ψ are onto complete isometries, so that $I(J)$ is completely isometrically isomorphic to $I(A)$, via a map that is the identity on J.

It is clear that (4) $\Rightarrow$ (5) $\Rightarrow$ (7) $\Rightarrow$ (9) $\Rightarrow$ (15), that (6) $\Rightarrow$ (7), and that (6) $\Rightarrow$ (8) $\Rightarrow$ (9).

(6) $\Rightarrow$ (10) Let W be any operator right J-module and $\phi : A \to W$ any completely contractive right J-module map which is completely isometric on J. Then $\forall n \in \mathbb{N}$ and $\forall (a_{lm}) \in \mathbb{M}_n(A)$,

$$\begin{aligned}
\|(a_{lm})\| &= \sup\{\|(a_{lm})(j_{lm})\| : (j_{lm}) \in \mathbb{M}_n(J),\ \|(j_{lm})\| \leqslant 1\} \\
&= \sup\{\|\phi_n((a_{lm})(j_{lm}))\| : (j_{lm}) \in \mathbb{M}_n(J),\ \|(j_{lm})\| \leqslant 1\} \\
&= \sup\{\|(\phi(a_{lm}))(j_{lm})\| : (j_{lm}) \in \mathbb{M}_n(J),\ \|(j_{lm})\| \leqslant 1\} \\
&\leqslant \|(\phi(a_{lm}))\|,
\end{aligned}$$

which shows that ϕ is completely isometric on A. Where $\phi_n : \mathbb{M}_n(A) \to \mathbb{M}_n(W)$ is defined by $\phi_n((a_{lm})) := (\phi(a_{lm}))$.

(10) $\Rightarrow$ (12) Clear.

(12) $\Rightarrow$ (15) Let $K := \{a \in A : aJ = \{0\}\}$, then K is a closed two-sided ideal in A and $K \cap JJ^* = KJJ^* = \{0\}$. Let A/K be equipped with the quotient norm that makes A/K a C^*-algebra, hence an operator A-J-bimodule with induced matrix norms and natural module actions. Let $\pi : A \to A/K$ be the quotient map that is a $*$-homomorphism and also a completely contractive A-J-bimodule map. Since $K \cap JJ^* = \{0\}$, π is one-to-one on JJ^*, hence $*$-isometric on JJ^*. Thus π is completely isometric on JJ^*, especially on J. By the assumption (12), π is one-to-one on A, so that $K = \{0\}$, which means J is an essential left ideal.

(6) $\Rightarrow$ (11) Let W be any operator right J-module and $\phi : A \to W$ any completely bounded right J-module map which is completely boundedly isomorphic on J. Then there exists $c > 0$ such that $\|(j_{lm})\| \leqslant c \cdot \|(\phi(j_{lm}))\|$, $\forall (j_{lm}) \in \mathbb{M}_n(J)$ with $\|(j_{lm})\| \leqslant 1$, $\forall n \in \mathbb{N}$. Hence $\forall n \in \mathbb{N}$ and $\forall (a_{lm}) \in \mathbb{M}(A)$,

$$\begin{aligned} \|(a_{lm})\| &= \sup \|(a_{lm})(j_{lm})\| : (j_{lm}) \in \mathbb{M}_n(J),\ \|(j_{lm})\| \leqslant 1\} \\ &\leqslant c \cdot \sup\{\|\phi_n((a_{lm})(j_{lm}))\| : (j_{lm}) \in \mathbb{M}_n(J),\ \|(j_{lm})\| \leqslant 1\} \\ &= c \cdot \sup\{\|(\phi(a_{lm}))(j_{lm})\| : (j_{lm}) \in \mathbb{M}_n(J),\ \|(j_{lm})\| \leqslant 1\} \\ &\leqslant c \cdot \|(\phi(a_{lm}))\|. \end{aligned}$$

Thus, ϕ is a completely bounded isomorphism.

(11) $\Rightarrow$ (13) Clear.

(13) $\Rightarrow$ (15) Just the same as that (12) $\Rightarrow$ (15).

(15) $\Rightarrow$ (14) Suppose $aJ = \{0\}$, $\forall a \in A$. Let $(a_{lm}) \in \mathbb{M}_n(A)$ be the matrix such that $a_{11} = a$ and all other entries are 0. Then, by the assumption, $(a_{lm}) \cdot \mathbb{M}_n(J) = \{0\}$ which implies $(a_{lm}) = (0)$ from (15), so that $a = 0$. Hence J is an essential left ideal in A, so that $\mathbb{M}_{n'}(J)$ is an essential left ideal in $\mathbb{M}_{n'}(A)$, $\forall n' \in \mathbb{N}$.

(14) $\Rightarrow$ (16) Fix $n \in \mathbb{N}$. Since $\mathbb{M}_n(J)$ is an essential left ideal in $\mathbb{M}_n(A)$, the canonical embedding $\varphi : \mathbb{M}_n(A) \to \mathcal{M}(\mathbb{M}_n(JJ^*))$ is one-to-one (Proposition 4.2), hence it is a $*$-isomorphism.

(16) $\Rightarrow$ (17) Clear.

(16) $\Rightarrow$ (6) Let an $n \in \mathbb{N}$. Let us consider the following maps.

$$\mathbb{M}_n(A) \overset{\varphi}{\hookrightarrow} \mathcal{M}(\mathbb{M}_n(JJ^*)) \overset{\tilde{\rho}}{\overset{\sim}{\to}} \mathcal{M}(\mathbb{K}(\mathbb{M}_n(J))) \overset{\tau}{\overset{\sim}{\to}} \mathbb{B}(\mathbb{M}_n(J)).$$

Where we regard $\mathbb{M}_n(J)$ as a right Hilbert C^*-module over $\mathbb{M}_n(J^*J)$ with the inner product $\langle \cdot | \cdot \rangle : \mathbb{M}_n(J) \times \mathbb{M}_n(J) \to \mathbb{M}_n(J^*J)$ defined by $\langle j | j' \rangle := j^* j'$, $\forall j$, $\forall j' \in \mathbb{M}_n(J)$.

$\mathbb{B}(\mathbb{M}_n(J))$ is the set of adjointable maps on $\mathbb{M}_n(J)$.

$\mathbb{K}(\mathbb{M}_n(J))$ is the set of compact adjointable operators on $\mathbb{M}_n(J)$. Namely, $\mathbb{K}(\mathbb{M}_n(J)) := \overline{\operatorname{span}}\{\theta_{j,j'} : j, j' \in \mathbb{M}_n(J)\}$, where $\theta_{j,j'} : \mathbb{M}_n(J) \to \mathbb{M}_n(J)$ is defined by $\theta_{j,j'}(j'') := j{j'}^* j''$ for $\forall j$, $\forall j'$, $\forall j'' \in \mathbb{M}_n(J)$, and the closure is taken in $\mathbb{B}(\mathbb{M}_n(J))$.

The map φ is the canonical embedding.

The map $\rho : \mathbb{M}_n(JJ^*) \to \mathbb{K}(\mathbb{M}_n(J))$ is defined in the following way:

$$\rho\Big(\sum_m (j_m {j'_m}^*)\Big) = \sum_m \theta_{j_m, j'_m}, \quad \forall j_m,\ \forall j'_m \in \mathbb{M}_n(J)$$

is a well-defined $*$-homomorphism and extends to an onto $*$-homomorphism $\mathbb{M}_n(JJ^*) \to \mathbb{K}(\mathbb{M}_n(J))$. By Proposition 4.2, $\mathbb{M}_n(J)$ is an essential left ideal in $\mathbb{M}_n(JJ^*)$, thus ρ is one-to-one and hence a $*$-isomorphism. Therefore, we obtain a $*$-isomorphism $\widetilde{\rho} : \mathcal{M}(\mathbb{M}_n(JJ^*)) \to \mathcal{M}(\mathbb{K}(\mathbb{M}_n(J)))$.

The map $\tau^{-1} : \mathbb{B}(\mathbb{M}_n(J)) \to \mathcal{M}(\mathbb{K}(\mathbb{M}_n(J)))$ is defined in the following way: $\tau^{-1}(T) := (T_1, T_2)$, where $T_1(\theta_{j,j'}) := T \cdot \theta_{j,j'}$, $T_2(\theta_{j,j'}) := \theta_{j,j'} \cdot T$, $\forall T \in \mathbb{B}(\mathbb{M}_n(J))$. For a proof that τ^{-1} is a $*$-isomorphism, see [11] and [18].

We can easily see that $\tau \circ \widetilde{\rho} \circ \varphi(a) = T_a$, where $T_a(j) := aj$, $\forall a \in \mathbb{M}_n(A)$, $\forall j \in \mathbb{M}_n(J)$. Hence (6) follows.

(17) $\Rightarrow$ (7) The same as that (16) $\Rightarrow$ (6). ∎

REMARK 5.2. (i) We will show later in this paper that the equivalent conditions (II) do not imply (5). We do not know whether or not (5) implies the equivalent conditions (I). It can be seen that (5) implies that there is an n-isometric A-$\mathbb{C}$-bimodule map of A into $I(J)$.

(ii) The quantity $\sup_\alpha \|(a_{lm}e_\alpha)\|$ does not depend on the choice of contractive right approximate identities. In fact, take another contractive right approximate identity f_β. Then, by noting that $\{e_\alpha\} \subset J \cap J^*$ and f_β is also a contractive left approximate identity of J^* (Remark 4.3), $\lim_{\beta\to\infty} e_\alpha f_\beta = e_\alpha = \lim_{\beta\to\infty} f_\beta e_\alpha$, $\forall\alpha$. Hence,

$$\begin{aligned}\sup_\alpha \|(a_{lm}e_\alpha)\| &= \sup_\alpha \lim_{\beta\to\infty} \|(a_{lm}e_\alpha f_\beta)\| = \sup_\alpha \lim_{\beta\to\infty} \|(a_{lm}f_\beta e_\alpha)\| \\ &\leqslant \sup_\alpha \sup_\beta \|(a_{lm}f_\beta e_\alpha)\| \leqslant \sup_\beta \|(a_{lm}f_\beta)\|.\end{aligned}$$

Similarly, the other inequality holds. Similar reasoning applies to the approximate identities that appear in Theorem 5.3, Corollary 5.4 and Corollary 5.5.

(iii) It is interesting to note that injective envelopes are related to tight "$\mathbb{C}$-$\mathbb{C}$"-, or, "A-$\mathbb{C}$"-essential extensions, while essential left ideals are related to tight "$\mathbb{C}$-J"-, or, "A-J"-essential extensions.

(iv) In (4) and (5) of Theorem 5.1, we can replace $\|(a_{lm}e_\alpha)\|$ by $\|(e_\alpha a_{lm})\|$. In fact,

$$\|(a_{lm})\| = \|(a_{lm})^*\| = \sup_\alpha \|(a^*_{ml}e_\alpha)\| = \sup_\alpha \|(e_\alpha a_{lm})\|.$$

Similar considerations hold in Theorem 5.3, Corollary 5.4, and Corollary 5.5.

(v) Theorem 5.1 remains true if (5) is replaced by the weakest condition, namely, that $\|a\| = \sup_\alpha \|ae_\alpha\|$ for all $a \in A$. Similar considerations hold in Theorem 5.3, Corollary 5.4, and Corollary 5.5.

(vi) The statement (9) with $n = 1$ is equivalent to saying that the left A-module J is c-faithful in the terminology of [2]. Hence we see that if we regard left ideals of A as operator left A-modules, then J being faithful (this is equivalent to saying that J is an essential left ideal in A in our definition (Definition 4.4)) and J being c-faithful are equivalent for any $c \geqslant 1$.

(vii) In the proof that (16) $\Rightarrow$ (6) and that (17) $\Rightarrow$ (7), we used the Hilbert C^*-module theory. We can give an alternative proof of these parts that looks easier but is less informative (the involution is not given explicitly). First, note that, that (16) $\Rightarrow$ (14) and that (17) $\Rightarrow$ (15) easily follow. So it suffices to show that (14) $\Rightarrow$ (6). We use an already known principle ([17], [1]): Any contractive homomorphism from C^*-algebra to Banach algebra is a $*$-homomorphism with a certain involution in the range. In fact, $\mathbb{B}(\mathbb{M}_n(J))$ is a Banach algebra and the canonical embedding $\mathbb{M}_n(A) \to \mathbb{B}(\mathbb{M}_n(J))$ is a contractive homomorphism. (14) implies that this embedding is one-to-one, hence a $*$-isomorphism, so that (6) follows.

In the non-tight case, we could not connect essential extensions with injective envelopes. This is mainly because we can not say $I(J)$ is a (non-tight) A-$\mathbb{C}$-rigid extension of J. But still a similar result holds.

THEOREM 5.3. *Let A be a C^*-algebra, J a closed left ideal and let $\{e_\alpha\}$ be a contractive right approximate identity for J. Then* (2a)$\Rightarrow$ (3a) $\Rightarrow$ (4a) $\Rightarrow$ (5a) $\Rightarrow$ *"each statement of* (II) *in Theorem* 5.1*".*

(2a) *A is a $\mathbb{C}$-$\mathbb{C}$-essential extension of J;*

(3a) *A is an A-$\mathbb{C}$-essential extension of J;*

(4a) *There exists $c \geqslant 1$ such that $\|(a_{lm})\| \leqslant c \cdot \sup_\alpha \|(a_{lm}e_\alpha)\|$ for all $(a_{lm}) \in \mathbb{M}_n(A)$ and all $n \in \mathbb{N}$;*

(5a) *There exist $c \geqslant 1$ and $n \in \mathbb{N}$ such that $\|(a_{lm})\| \leqslant c \cdot \sup_\alpha \|(a_{lm}e_\alpha)\|$ for all $(a_{lm}) \in \mathbb{M}_n(A)$.*

Proof. (2a) $\Rightarrow$ (3a) Clear.

(3a) $\Rightarrow$ (4a) Similar to that (3) $\Rightarrow$ (4) in Theorem 5.1.

That (4a) $\Rightarrow$ (5a) $\Rightarrow$ "(9) in Theorem 5.1" is clear. ∎

Note that the statement (2a) says just that $A = J$. In fact, if $J \subsetneq A$, then we can take a J-projection ϕ on A (i.e. a completely bounded linear map on A with $\phi^2 = \phi$ and $\phi|J = \mathrm{id}_J$) such that $J \subset \operatorname{Im}\phi \subsetneq A$ with the codimension of $\operatorname{Im}\phi$ in A is 1.

Together with the right ideal versions of Theorem 5.1 and Theorem 5.3, and by noting that, when J is a two-sided ideal in A, $\mathbb{C}$-J- or A-J- in (10)—(13) of Theorem 5.1 can be replaced by $\mathbb{C}$-A- or A-A- with trivial modifications in the proof, the next corollary immediately follows.

COROLLARY 5.4. *Let A be a C^*-algebra, K a closed two-sided ideal in A, and $\{e_\alpha\}$ a contractive approximate identity of K. Then the following are equivalent:*

(1) *$I(K)$ is completely isometrically isomorphic to $I(A)$, via a map that restricts to the identity on K;*

(2) *A is a tight $\mathbb{C}$-$\mathbb{C}$-essential extension of K;*

(3) *A is a tight A-$\mathbb{C}$-essential extension of K;*

(3a) *A is an A-$\mathbb{C}$-essential extension of K;*

(4) $\|(a_{lm})\| = \sup_\alpha \|(a_{lm}e_\alpha)\|$ *for all $(a_{lm}) \in \mathbb{M}_n(A)$ and for all $n \in \mathbb{N}$;*

(4a) *There exists $c \geqslant 1$ such that $\|(a_{lm})\| \leqslant c \cdot \sup_\alpha \|(a_{lm}e_\alpha)\|$ for all $(a_{lm}) \in \mathbb{M}_n(A)$ and for all $n \in \mathbb{N}$;*

(5) *There exists* $n \in \mathbb{N}$ *such that* $\|(a_{lm})\| = \sup_\alpha \|(a_{lm}e_\alpha)\|$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$;

(5a) *There exist* $c \geqslant 1$ *and* $n \in \mathbb{N}$ *such that* $\|(a_{lm})\| \leqslant c \cdot \sup_\alpha \|(a_{lm}e_\alpha)\|$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$;

(6) $\|(a_{lm})\| = \sup\{\|(a_{lm})(k_{lm})\| : (k_{lm}) \in \mathbb{M}_n(K), \|(k_{lm})\| \leqslant 1\}$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$ *and for all* $n \in \mathbb{N}$;

(7) *There exists* $n \in \mathbb{N}$ *such that* $\|(a_{lm})\| = \sup\{\|(a_{lm})(j_{lm})\| : (k_{lm}) \in \mathbb{M}_n(K), \|(k_{lm})\| \leqslant 1\}$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$;

(8) *There exists* $c \geqslant 1$ *such that* $\|(a_{lm})\| \leqslant c \cdot \sup\{\|(a_{lm})(k_{lm})\| : (k_{lm}) \in \mathbb{M}_n(K), \|(k_{lm})\| \leqslant 1\}$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$ *and for all* $n \in \mathbb{N}$;

(9) *There exist* $c \geqslant 1$ *and* $n \in \mathbb{N}$ *such that* $\|(a_{lm})\| \leqslant c \cdot \sup\{\|(a_{lm})(k_{lm})\| : (k_{lm}) \in \mathbb{M}_n(K), \|(k_{lm})\| \leqslant 1\}$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$;

(10′) *A is a tight $\mathbb{C}$-A-essential extension of K;*

(11′) *A is a $\mathbb{C}$-A-essential extension of K;*

(12′) *A is a tight A-A-essential extension of K;*

(13′) *A is an A-A-essential extension of K;*

(14) $\mathbb{M}_n(K)$ *is an essential ideal in* $\mathbb{M}_n(A)$ *for all* $n \in \mathbb{N}$;

(15) $\mathbb{M}_n(K)$ *is an essential ideal in* $\mathbb{M}_n(A)$ *for some* $n \in \mathbb{N}$;

(16) $\mathbb{M}_n(A)$ *is $*$-isomorphically embedded in* $\mathcal{M}(\mathbb{M}_n(K))$ *for all* $n \in \mathbb{N}$;

(17) $\mathbb{M}_n(A)$ *is $*$-isomorphically embedded in* $\mathcal{M}(\mathbb{M}_n(K))$ *for some* $n \in \mathbb{N}$.

Thus, in the two-sided case, all the concepts of essentiality that we have introduced are equivalent.

From the above corollary and by observing that "$\mathbb{M}_n(J)$ is an essential left ideal in $\mathbb{M}_n(A) \Leftrightarrow \mathbb{M}_n(JJ^*)$ is an essential two-sided ideal in $\mathbb{M}_n(A)$" (Definition 4.4), the following also holds.

Corollary 5.5. *Let A be a C^*-algebra, let J be a closed left ideal of A and let $\{u_\beta\}$ be a contractive approximate identity of JJ^*. Then each statement of* (II) *in Theorem* 5.1 *is equivalent to each of the following:*

(1b) *$I(JJ^*)$ is completely isometrically isomorphic to $I(A)$, via a map that restricts to the identity on JJ^*;*

(2b) *A is a tight $\mathbb{C}$-$\mathbb{C}$-essential extension of JJ^*;*

(3b) *A is a tight A-$\mathbb{C}$-essential extension of JJ^*;*

(3ab) *A is an A-$\mathbb{C}$-essential extension of JJ^*;*

(4b) $\|(a_{lm})\| = \sup_\beta \|(a_{lm}u_\beta)\|$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$ *and for all* $n \in \mathbb{N}$;

(4ab) *There exists* $c \geqslant 1$ *such that* $\|(a_{lm})\| \leqslant c \cdot \sup_\beta \|(a_{lm}u_\beta)\|$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$ *and for all* $n \in \mathbb{N}$;

(5b) *There exists* $n \in \mathbb{N}$ *such that* $\|(a_{lm})\| = \sup_\beta \|(a_{lm}u_\beta)\|$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$;

(5ab) *There exist* $c \geqslant 1$ *and* $n \in \mathbb{N}$ *such that* $\|a\| \leqslant c \cdot \sup_{\beta} \|ae_\beta\|$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$*;*

(6b) $\|(a_{lm})\| = \sup\{\|(a_{lm})(k_{lm})\| : (k_{lm}) \in \mathbb{M}_n(JJ^*),\ \|(k_{lm})\| \leqslant 1\}$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$ *and for all* $n \in \mathbb{N}$*;*

(7b) *There exists* $n \in \mathbb{N}$ *such that* $\|(a_{lm})\| = \sup\{\|(a_{lm})(k_{lm})\| : (k_{lm}) \in \mathbb{M}_n(JJ^*),\ \|(k_{lm})\| \leqslant 1\}$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$*;*

(8b) *There exists* $c \geqslant 1$ *such that* $\|(a_{lm})\| \leqslant c \cdot \sup\{\|(a_{lm})(k_{lm})\| : (k_{lm}) \in \mathbb{M}_n(JJ^*),\ \|(k_{lm})\| \leqslant 1\}$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$ *and for all* $n \in \mathbb{N}$*;*

(9b) *There exist* $c \geqslant 1$ *and* $n \in \mathbb{N}$ *such that* $\|(a_{lm})\| \leqslant c \cdot \sup\{\|(a_{lm})(k_{lm})\| : (k_{lm}) \in \mathbb{M}_n(JJ^*), \|(k_{lm})\| \leqslant 1\}$ *for all* $(a_{lm}) \in \mathbb{M}_n(A)$*;*

(10′b) A *is a tight* $\mathbb{C}$-A*-essential extension of* JJ^**;*

(11′b) A *is a* $\mathbb{C}$-A*-essential extension of* JJ^**;*

(12′b) A *is a tight* A-A*-essential extension of* JJ^**;*

(13′b) A *is an* A-A*-essential extension of* JJ^**;*

(14b) $\mathbb{M}_n(JJ^*)$ *is an essential ideal in* $\mathbb{M}_n(A)$ *for all* $n \in \mathbb{N}$*;*

(15b) $\mathbb{M}_n(JJ^*)$ *is an essential ideal in* $\mathbb{M}_n(A)$ *for some* $n \in \mathbb{N}$.

6. EXAMPLES AND APPLICATIONS

It is easy to construct an example of a left but not two-sided ideal which satisfies all the statements in Theorem 5.1.

EXAMPLE 6.1. Let A be any C^*-algebra which properly contains an essential two-sided ideal K. Set

$$J := \begin{pmatrix} A & K \\ A & K \end{pmatrix}.$$

Then J is a left but not two-sided ideal in $\mathbb{M}_2(A)$ and $\mathbb{M}_2(K) \subset J \subset \mathbb{M}_2(A)$, hence we can make $I(\mathbb{M}_2(K)) \subset I(J) \subset I(\mathbb{M}_2(A))$ with an injective envelope $(I(\mathbb{M}_n(K)), \iota)$ of $\mathbb{M}_n(K)$. The identity map on $I(\mathbb{M}_2(K))$ extends to a completely contractive linear map $\phi : I(\mathbb{M}_2(A)) \to I(\mathbb{M}_2(K))$. $\mathbb{M}_2(A)$ is a tight A-$\mathbb{C}$-essential extension of $\mathbb{M}_2(K)$ by Corollary 5.4, and $I(\mathbb{M}_2(A))$ is a tight A-$\mathbb{C}$-essential extension of $\mathbb{M}_2(A)$ by Theorem 3.1, so that $I(\mathbb{M}_2(A))$ is a tight A-$\mathbb{C}$-essential extension of $\mathbb{M}_2(K)$. Hence by rigidity (Theorem 3.1), ϕ is a complete isometry since it fixes each element of $\iota(\mathbb{M}_2(K))$. Thus $I(\mathbb{M}_2(A)) = I(J) = I(\mathbb{M}_2(K))$.

EXAMPLE 6.2. This example shows that each statement in (II) in Theorem 5.1 does not necessarily imply (5) in Theorem 5.1 or (5a) in Theorem 5.3. Especially, J is an essential left ideal in A is equivalent to saying that A is a (tight) $\mathbb{C}$-J-essential extension of J, but those do not necessarily imply that A is a (tight) A-$\mathbb{C}$-essential extension of J.

Let $A := \mathbb{M}_2$ and

$$J := \begin{pmatrix} \mathbb{C} & O \\ \mathbb{C} & O \end{pmatrix}.$$

Then A is a C^*-algebra and J is a closed left ideal in A with a contractive right identity. $e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. It is easy to see $JJ^* = A$, so especially, JJ^* is an essential two-sided ideal in A, hence J is an essential left ideal in A by Definition 4.4. Let $a := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and let $(a_{lm}) \in \mathbb{M}_n(A)$ be such that $a_{11} = a$, $a_{lm} = 0$ if $l \neq 1$ or $m \neq 1$. Then $\|(a_{lm})\| = 1$, while $\|(a_{lm}e)\| = 0$.

REMARK 6.3. If A is a C^*-algebra and J is a closed left ideal in A, and if A is a (respectively, tight) A-$\mathbb{C}$-essential extension of J, with J properly contained in A, then J can not have a (respectively, contractive) right identity. In fact, suppose that J has a (respectively, contractive) right identity e. Then $(a_{lm}) \longmapsto (a_{lm}e)\ \forall (a_{lm}) \in \mathbb{M}_n(A), \forall n \in \mathbb{N}$ defines a completely bounded (respectively, completely contractive) left A-module map $\phi : A \to J$ which is completely isometric on J, so that ϕ is a completely bounded isomorphism (respectively, a complete isometry). But $0 \neq (a - ae) \in \operatorname{Ker} \phi$ for $a \in A \setminus J$, hence a contradiction.

Thus, in particular, for any finite dimensional left ideal $J \subsetneq A$, A can not be a (tight) A-$\mathbb{C}$-essential extension of J.

We close this section with another application of a part of Theorem 5.1. As is defined in [13], for two C^*-algebras A and B with $A, B \subset \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, we say that A *norms* B when the following equation holds for each $n \geqslant 1$ and for each $X \in \mathbb{M}_n(B)$.

$$\|X\| = \sup\{\|RXC\| : R \in \mathrm{Row}_n(A),\ C \in \mathrm{Col}_n(A),\ \|R\|,\ \|C\| \leqslant 1\},$$

where $\mathrm{Row}_n(A)$ and $\mathrm{Col}_n(A)$ are, respectively, row and column matrices over A. We show that, if A is an essential two-sided ideal in B, then A norms B. By Lemma 2.4 in [13] (they were requiring all algebras to be unital, but this is not essential), it suffices to show that, for each $n \geqslant 1$ and for each $X \in \mathbb{M}_n(B)$,

$$\|X\| = \sup\{\|XC\| : C \in \mathrm{Col}_n(A),\ \|C\| \leqslant 1\}.$$

But $\mathrm{Col}_n(A)$ is (readily seen to be) an essential left ideal in $\mathbb{M}_n(B)$, so the equation follows from Theorem 5.1 (14) $\Rightarrow$ (6).

7. CONCLUSIONS AND QUESTIONS

(1) Compared with the tight case, the non-tight case is generally unknown. The difficulty in the non-tight case mainly comes from the lack of a rigidity result that does not need some module actions. Even in the Banach space case, there are a few deep results, but the entire picture is unclear. We do not know the existence of a "non-tight injective envelope", namely a minimal injective extension, for an arbitrary operator space. Also the lack of rigidity makes it difficult to connect essential extensions with even tight injective envelopes. As a result, we do not know if either of the following implications is true: A is a tight A-$\mathbb{C}$-essential extension of $J \overset{?}{\Leftrightarrow} A$ is an A-$\mathbb{C}$-essential extension of J.

(2) The implications (5) $\overset{?}{\Rightarrow}$ (4) in Theorem 5.1 and (5a) $\overset{?}{\Rightarrow}$ (4a) $\overset{?}{\Rightarrow}$ (3a) in Theorem 5.3 are still unknown.

(3) As one possible generalization of the results in this paper, one can consider replacing C^*-algebras by operator algebras. In such a case, the difficulty comes from the fact that we still do not know whether or not the representation $\mathbb{B}(\mathcal{H})$ of an operator A-B-bimodule ([4], [2], [3]) is A-B-injective. Consequently, we do not know the existence of any A-B-injective operator modules. Such modules play an important role in characterizing and constructing essential extensions.

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