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FUGLEDE'S THEOREM, THE BICOMMUTANT THEOREM AND p-MULTIPLIER OPERATORS FOR THE CIRCLE

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Abstract. Two classical results concerning normal operators in a Hilbert space are the Fuglede commutativity theorem and von Neumann's bicommutant theorem. Analogues of these results are established for Fourier multiplier operators acting in L^p-spaces of the circle group, for $1 < p < \infty$. The arguments used are a combination of techniques coming from harmonic analysis, functional analysis and operator theory.

Keywords: *Fourier multipliers, Fuglede's theorem, spectral operators, bicommutant.*

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1. INTRODUCTION

The purpose of this note is to extend two well known results from operator theory in Hilbert spaces, namely Fuglede's theorem and von Neumann's bicommutant theorem (both for normal operators) to the setting of Fourier p-multiplier operators in the Banach spaces $L^p(\mathbb{T})$, $1 < p < \infty$, where $\mathbb{T}$ is the circle group.

Recall a classical result of B. Fuglede ([8]) which states if T is a bounded normal operator in a Hilbert space H and A is any bounded linear operator on H satisfying $TA = AT$, then also $AT^* = T^*A$. For an elegant proof we refer to [17]. An abstract formulation proceeds along the following lines. Let X be a Banach space and $L(X)$ be the algebra of all continuous linear operators of X into itself. Suppose that T belongs to a subalgebra of $L(X)$ which has an involution $*$. The question arises of whether $AT^* = T^*A$ whenever $A \in L(X)$ commutes with T? For instance, let T be any scalar-type spectral operator in a Banach space X (see Chapter XV of [6]). Then $T = \int_{\sigma(T)} z \, \mathrm{d}P(z)$ and we can take $T^* := \int_{\sigma(T)} \bar{z}\,\mathrm{d}P(z)$. Here $P(\cdot)$ is a projection-valued measure on the Borel sets of the spectrum, $\sigma(T)$, which is σ-additive for the strong operator topology. Fuglede's theorem is known to hold in this case; this follows easily from Corollary 7 of [6], p. 1935 . A similar result holds for the slightly larger class of operators called *quasispectral* (Theorem 1.2 in [1]) and also for the class of scalar-type prespectral operators (Theorem 5.12 in [5]). I. Colojoară and C. Foias ([2]) introduced the extensive class of generalized scalar operators $T \in L(X)$, namely those possessing a spectral distribution $\varphi : C^\infty(\mathbb{C}) \to L(X)$ for which $T = \varphi(\mathrm{id}_\mathbb{C})$, where $\mathrm{id}_\mathbb{C}$ is the identity map on $\mathbb{C}$. Then, with $T^* := \varphi(\overline{\mathrm{id}_\mathbb{C}})$, Fuglede's theorem holds whenever T is regular generalized scalar (p. 98 in [2]). Criteria for T to be regular can be found in pp. 100 and 103 of [2].

One of our main aims is to establish Fuglede's theorem for the class of all Fourier p-multiplier operators in $L^p(\mathbb{T})$, where $1 \leqslant p < \infty$, see Theorem 3.1. Of course, when $p = 2$ such operators are normal in $L^2(\mathbb{T})$ and the original Fuglede theorem applies.

Concerning our other main result (i.e. the bicommutant theorem) let H be a Hilbert space and $T \in L(H)$. Let $\overline{\langle I, T, T^* \rangle}^{\mathrm{wo}}$ be the closed subalgebra of $L(H)$ generated by the identity operator I, T and T^* with respect to the weak operator topology. Recall that the commutant $\mathfrak{U}^{\mathrm{c}}$ of a subset $\mathfrak{U} \subset L(H)$ is given by $\mathcal{U}^{\mathrm{c}} = \{A \in L(H) : AS = SA, \forall S \in \mathfrak{U}\}$ and that its bicommutant $\mathfrak{U}^{\mathrm{cc}}$ is given by $(\mathfrak{U}^{\mathrm{c}})^{\mathrm{c}}$. A classical result of J. von Neumann states that a $*$-algebra of operators in $L(H)$ coincides with its bicommutant if and only if it is closed for the weak operator topology (Chapter IX in [6]). It follows from this result and Fuglede's theorem that $\overline{\langle I, T, T^* \rangle}^{\mathrm{wo}} = \{T\}^{\mathrm{cc}}$ for every normal operator $T \in L(X)$.

Scalar-type spectral operators in Banach spaces have an integral representation akin to that for normal operators in Hilbert spaces. Moreover, by the Mackey-Wermer theorem, every scalar-type spectral operator T in a Hilbert space H is essentially normal as it is of the form $S\widetilde{T}S^{-1}$, where $\widetilde{T}$ is a normal operator and S is a selfadjoint operator which is invertible in $L(H)$ (Theorem 8.3 of [5]). So, does von Neumann's bicommutant theorem for normal operators also extend to the class of scalar-type spectral p-multiplier operators in $L^p(\mathbb{T})$? It is shown in Section 5 that this is indeed the case; see Theorem 5.4. The underlying reason for this is that the resolution of the identity associated with any such scalar-type spectral p-multiplier operator is necessarily purely atomic. This is due to the fact that such operators are in a one-to-one correspondence with Littlewood-Paley decompositions of $\mathbb{Z}$ (relative to $L^p(\mathbb{T})$); see Theorem 4.1 and Remark 4.2. These facts and other important properties for such operators are established in Section 4. For instance, it is shown that translation operators corresponding to elements of infinite order (necessarily unitary operators in the L^2-setting) are not spectral for $p \neq 2$; the proof is quite different to that of T.A. Gillespie ([10]), where the result is established for arbitrary locally compact abelian groups. In contrast to the case of L^2, it turns out that scalar-type spectral multiplier operators in the setting $p \neq 2$ can never have a cyclic vector. This is due to the fact that multiplicity theory for Boolean algebras of p-multiplier projections exhibits some curious features when $p \neq 2$; see Lemma 5.1.

In conclusion, we point out that the obvious analogues of the results presented here (and their proofs) remain valid when $\mathbb{T}$ is replaced with any compact abelian metrizable group G, in which case the spaces $L^p(G)$, $1 < p < \infty$, are still separable and the dual group is countable and discrete.

2. PRELIMINARIES

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denote the circle group, identified with $(-\pi, \pi]$ in the usual way. The Fourier transform

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}nt} f(t) \,\mathrm{d}t, \quad n \in \mathbb{Z},$$

is defined for all $f \in L^1(\mathbb{T})$. According to Hausdorff-Young's inequality, the Fourier transform $f \mapsto \widehat{f}$ is bounded from $L^p(\mathbb{T})$ into $l^q(\mathbb{Z})$, where q is the dual exponent, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, and $1 \leqslant p \leqslant 2$. An element $T \in L(L^p(\mathbb{T}))$, $1 \leqslant p \leqslant \infty$, is called a *p-multiplier operator* if it commutes with each translation operator τ_w, $w \in \mathbb{T}$, where $\tau_w f(z) = f(zw^{-1})$. This is the case if and only if there exists $\psi \in l^\infty(\mathbb{Z})$, necessarily unique, such that $\widehat{Tf} = \psi \widehat{f}$, for $f \in L^p \cap L^2(\mathbb{T})$. The function ψ is called a *p-multiplier for* $\mathbb{T}$. Since p will always be clearly identified, the operator T is denoted simply by T_ψ. The space of all p-multipliers is denoted by $M_p(\mathbb{T})$. The inequality

$$\|\psi\|_\infty \leqslant \|T_\psi\|_{L(L^p(\mathbb{T}))}, \quad \psi \in M_p(\mathbb{T}), \tag{2.1}$$

is well known. If we equip $M_p(\mathbb{T})$ with the norm $|\!|\!|\psi|\!|\!|_p := \|T_\psi\|_{L(L^p(\mathbb{T}))}$, then $M_p(\mathbb{T})$ becomes a commutative unital Banach algebra with respect to pointwise multiplication. Moreover, for each $\psi \in M_p(\mathbb{T})$, the function $\overline{\psi}$ also belongs to $M_p(\mathbb{T})$. Since $M_p(\mathbb{T})$ is isometrically isomorphic to $M_q(\mathbb{T})$, where q is dual to p and $1 < p < \infty$, we will restrict our attention to $1 < p \leqslant 2$. The space $\{T_\psi : \psi \in M_p(\mathbb{T})\}$ of all p-multiplier operators (which is isomorphic to $M_p(\mathbb{T})$), will be denoted by $O_p(\mathbb{T})$.

Fix $p \in (1, 2]$. A decomposition $\{\Delta_j\}_{j \in J}$ of $\mathbb{Z}$ is said to be a *Littlewood-Paley p-decomposition* if there exist positive constants c_p and C_p such that, with $S_j f(x) = \sum\limits_{k \in \Delta_j} \widehat{f}(k) \mathrm{e}^{\mathrm{i}kx}$, we have

$$c_p \|f\|_r \leqslant \left\| \left(\sum_{j \in J} |S_j f|^2 \right)^{1/2} \right\|_r \leqslant C_p \|f\|_r,$$

for all r in the interval $[p, q]$. By Khinchin's inequality this is equivalent to the requirement that for all sequences ε_j of ± 1's the operator $T_\varepsilon = \sum\limits_{j \in J} \varepsilon_j S_j$ is bounded on $L^r(\mathbb{T})$, for $p \leqslant r \leqslant q$, with norm depending only on p and the decomposition $\{\Delta_j\}_{j \in J}$ (see [7]). We point out that whenever $1 \leqslant p_1 < p_2 \leqslant 2$, there exist Littlewood-Paley p_2-decompositions which are not p_1-decompositions ([13]).

If X is a Banach space and $T \in L(X)$, we denote by $\sigma_{\mathrm{pt}}(T), \sigma_{\mathrm{r}}(T)$ and $\sigma_{\mathrm{c}}(T)$ the point, residual and continuous spectra of T, respectively (p. 580 in [6]). The spectrum $\sigma(T)$ of T is the complement (in $\mathbb{C}$) of all points λ for which $(\lambda I - T)^{-1}$

exists as an element of $L(X)$. Of course, $\sigma(T)=\sigma_{\mathrm{pt}}(T)\cup\sigma_{\mathrm{r}}(T)\cup\sigma_{\mathrm{c}}(T)$. If f is a function analytic in a neighbourhood of $\sigma(T)$, then a bounded operator $f(T)$ can be defined by

$$f(T):=\frac{1}{2\pi\mathrm{i}}\int_{\Gamma} f(\lambda)(\lambda I-T)^{-1}\,\mathrm{d}\lambda, \tag{2.2}$$

for a suitable contour Γ in the domain of f which surrounds $\sigma(T)$. For the basic properties of the analytic calculus $f\mapsto f(T)$ we refer to Chapter VII of [6]. A subset F of $\sigma(T)$ which is both open and closed in $\sigma(T)$ is called a *spectral set.* There is then an analytic function f which is identically 1 on F and 0 on the rest of $\sigma(T)$. Then $P(F):=f(T)$ is a continuous projection operator, called the *spectral projection* of T corresponding to F; it depends only on T and not on f with the above properties.

The following result collects together some basic spectral properties of p-multiplier operators (for the circle group).

Lemma 2.1. *Let $1<p\leqslant 2$ and $\psi\in M_p(\mathbb{T})$.*

(i) *T_ψ is injective if and only if $0\notin\psi(\mathbb{Z})$.*

(ii) *$\lambda\in\sigma_{\mathrm{pt}}(T_\psi)$ if and only if $\lambda\in\psi(\mathbb{Z})$.*

(iii) *$\sigma_{\mathrm{r}}(T_\psi)=\emptyset$.*

(iv) *$\lambda\in\sigma_{\mathrm{c}}(T_\psi)$ if and only if $\psi^{-1}(\{\lambda\})=\emptyset$ and $(\psi-\lambda)^{-1}\notin M_p(\mathbb{T})$.*

(v) *If $\lambda\in\sigma(T_\psi)$ is an isolated point, then $\lambda\in\sigma_{\mathrm{pt}}(T_\psi)$. Moreover, the spectral projection $P(\{\lambda\})$ of T_ψ corresponding to the spectral set $\{\lambda\}$ belongs to $O_p(\mathbb{T})$. In fact, the characteristic function $\chi_\lambda:=\chi_{\psi^{-1}(\{\lambda\})}\in M_p(\mathbb{T})$ and $P(\{\lambda\})=T_{\chi_\lambda}$. The range of $P(\{\lambda\})$ is precisely $\ker(T_{\psi-\lambda I})$.*

For the proof note that (i) and (ii) are easy to check. For (iii) see Lemma 2.4 in [18]. To establish (iv) let $\lambda\in\sigma_{\mathrm{c}}(T_\psi)$. Then $T_{\psi-\lambda}$ is injective and so $\psi^{-1}(\{\lambda\})=\emptyset$ by (i). If $(\psi-\lambda)^{-1}\in M_p(\mathbb{T})$, then $T_{\psi-\lambda}$ is invertible in $L(L^p(\mathbb{T}))$ contradicting the fact that $\sigma_{\mathrm{c}}(T_\psi)\subset\sigma(T_\psi)$. Conversely, suppose that $\psi^{-1}(\{\lambda\})=\emptyset$ and $(\psi-\lambda)^{-1}\notin M_p(\mathbb{T})$. Then $\psi^{-1}(\{\lambda\})=\emptyset$ implies $\lambda\notin\sigma_{\mathrm{pt}}(T_\psi)$ and $(\psi-\lambda)^{-1}\notin M_p(\mathbb{T})$ implies $\lambda\in\sigma(T_\psi)$. Since $\sigma_{\mathrm{r}}(T_\psi)=\emptyset$ it follows that $\lambda\in\sigma_{\mathrm{c}}(T_\psi)$.

Finally the proof of (v). By Theorem 2.3 of [18] we have $\lambda\in\sigma_{\mathrm{pt}}(T_\psi)$. Let f be analytic in a neighbourhood of $\sigma(T_\psi)$ and suppose f is 1 in a disc D with centre λ and 0 in a neighbourhood of $\sigma(T_\psi)\setminus\{\lambda\}$ which does not intersect D. Then (2.2) implies that

$$P(\{\lambda\})=\frac{1}{2\pi\mathrm{i}}\int_{\Gamma} f(\gamma)T_{(\gamma-\psi)^{-1}}\,\mathrm{d}\gamma, \tag{2.3}$$

for any contour Γ surrounding $\sigma(T_\psi)$ and chosen such that the part of Γ surrounding $\{\lambda\}$ is contained in D and has index 1. Since each translation operator $\tau_w, w\in\mathbb{T}$, commutes with $T_{(\gamma-\psi)^{-1}}$, for $\gamma\in\Gamma$, and the integral in (2.3) can be approximated by Riemann sums (in the operator norm), it is clear that $P(\{\lambda\})$ commutes with translations, and hence, that $P(\{\lambda\})\in O_p(\mathbb{T})$. It follows from (2.3) and the properties of f that, for $g\in L^p(\mathbb{T})$, we have

$$P(\{\lambda\})g=\frac{1}{2\pi\mathrm{i}}\int_{\Gamma\cap D} T_{(\gamma-\psi)^{-1}}g\,\mathrm{d}\gamma,$$

where the integral exist as a Bochner integral. An approximation argument (using vector-valued simple functions) and Hausdorff-Young's inequality yields

$$(P(\{\lambda\})g)\widehat{\ }(n) = \frac{1}{2\pi\mathrm{i}} \int\limits_{\Gamma\cap D} \frac{\widehat{g}(n)}{\gamma - \psi(n)}\,\mathrm{d}\gamma,$$

as an equality in $l^q(\mathbb{Z})$. For $n \in \mathbb{Z}$ with $\psi(n) \neq \lambda$ we have $\psi(n) \in \sigma_{\mathrm{pt}}(T_\psi) \setminus \{\lambda\}$ and so $\psi(n)$ is not inside $\Gamma \cap D$. However, if $\psi(n) = \lambda$, then $\psi(n)$ is inside $\Gamma \cap D$. So, the evaluation of the contour integral above shows $P(\{\lambda\}) = T_{\chi_\lambda}$.

Suppose that $P(\{\lambda\})h = h$, i.e. h is in the range of $P(\{\lambda\}) \neq 0$. Applying the Fourier transform one shows that $\chi_{\psi^{-1}(\{\lambda\})}\widehat{h} = \widehat{h}$, that is, $\psi(k) = \lambda$ for all k such that $\widehat{h}(k) \neq 0$. Hence $(\psi - \lambda)\widehat{h} = 0$, that is, $h \in \ker(T_{\psi-\lambda})$. It follows that λ is a pole of the resolvent function of T_ψ of order $\nu = 1$ (Theorem 18 in [6], p. 573) and hence, the range of $P(\{\lambda\})$ is precisely $\ker(T_{\psi-\lambda})$ (Theorem 24 in [6], p. 576). ∎

It is well known that

$$\overline{\psi(\mathbb{Z})} \subset \sigma(T_\psi), \quad \psi \in M_p(\mathbb{T}); \tag{2.4}$$

see, e.g. Lemma 2.1 of [18]. The containment (2.4) is proper in general ([11]), even if ψ vanishes at infinity ([19]). Multipliers ψ for which (2.4) is an equality are said to have the spectral mapping property. For the notion of a decomposable operator (in an arbitrary Banach space) we refer to [2].

LEMMA 2.2. *Let $1 < p \leqslant 2$ and $\psi \in M_p(\mathbb{T})$.*

(i) *If T_ψ is decomposable, then $\sigma(T_\psi) = \overline{\psi(\mathbb{Z})}$.*

(ii) *If $\sigma(T_\psi)$ is a totally disconnected subset of $\mathbb{C}$, then T_ψ is decomposable.*

Proof. Part (i) follows from (2.4) and Corollary 3.3 of [1]. Concerning part (ii) we observe that T_ψ is a U-scalar operator (in the sense of Definition 1.18 from [2]; see Example 1.20 in [2]. Then Theorem 1.19 of [2] implies T_ψ is decomposable. ∎

3. THE FUGLEDE THEOREM

The aim of this section is to establish the following result concerning Fourier multiplier operators for the circle.

THEOREM 3.1. *Let $1 \leqslant p \leqslant 2$ and $\psi \in M_p(\mathbb{T})$. If $A \in L(L^p(\mathbb{T}))$ satisfies $AT_\psi = T_\psi A$, then also $AT_{\overline{\psi}} = T_{\overline{\psi}}A$.*

Proof. For $p = 2$ the result follows from the classical Fuglede theorem. So, we may assume $1 \leqslant p < 2$. Let S be the convolution operator corresponding to the multiplier $n \mapsto \exp(-|n|)$. Note that the convolution kernel of S is a smooth function, therefore S maps $L^p(\mathbb{T})$ into $L^p(\mathbb{T})$ and $L^p(\mathbb{T})$ continuously into $L^2(\mathbb{T})$. Let $A_2 : L^2(\mathbb{T}) \to L^p(\mathbb{T})$ denote the restriction of A to the subspace $L^2(\mathbb{T})$ of $L^p(\mathbb{T})$. Then the composition SA_2 maps $L^2(\mathbb{T}) \to L^2(\mathbb{T})$ continuously. Fix $f \in L^2(\mathbb{T})$. Since T_ψ acts identically in L^2, where we denote it by $\widetilde{T}_\psi$, as in L^p,

$$(SA_2)\widetilde{T}_\psi f = S(A_2\widetilde{T}_\psi f) = S(AT_\psi f) = S(T_\psi Af) = S(T_\psi A_2 f).$$

But, $ST_\psi = \widetilde{T}_\psi S$ as bounded operators from L^p to L^2 and we obtain that $(SA_2)\widetilde{T}_\psi f = \widetilde{T}_\psi(SA_2)f$ for $f \in L^2$. By Fuglede's theorem applied to the normal operator $\widetilde{T}_\psi \in L(L^2(\mathbb{T}))$ it follows that

$$(SA_2)\widetilde{T}_{\overline{\psi}} = \widetilde{T}_{\overline{\psi}}(SA_2) \tag{3.1}$$

as elements of $L(L^2(\mathbb{T}))$. But, for each $f \in L^2(\mathbb{T})$, the right-hand-side of (3.1) equals $ST_{\overline{\psi}}Af$ since $ST_{\overline{\psi}} = \widetilde{T}_{\overline{\psi}}S$ as bounded operators from L^p to L^2. It follows from (3.1) that $SA_2\widetilde{T}_{\overline{\psi}}f = ST_{\overline{\psi}}A_2 f$ for $f \in L^2(\mathbb{T})$, and again since $\widetilde{T}_{\overline{\psi}}$ and $T_{\overline{\psi}}$ agree on L^2, that $SA_2T_{\overline{\psi}}f = ST_{\overline{\psi}}A_2f$, for $f \in L^2(\mathbb{T})$. The injectivity of S and the definition of A_2 then imply that $AT_{\overline{\psi}}f = T_{\overline{\psi}}Af$, for $f \in L^2(\mathbb{T})$. Finally, since both $AT_{\overline{\psi}}$ and $T_{\overline{\psi}}A$ are bounded on L^p and L^2 is dense in L^p, we conclude that A and $T_{\overline{\psi}}$ commute. ∎

4. SCALAR-TYPE SPECTRAL OPERATORS

Let $1 < p \leqslant 2$. An operator $T \in L(L^p(\mathbb{T}))$ which is spectral in the sense of N. Dunford (Chapter XV in [6]) has a decomposition $T = S + N$ where S is a scalar-type spectral (briefly, scalar) operator and N is a quasinilpotent operator commuting with S. If $T \in O_p(\mathbb{T})$, then necessarily $N = 0$, (Proposition 2.1 in [1]). That is, the only spectral operators T in $O_p(\mathbb{T})$ are scalar ones. This means that T has an integral representation of the form $T = \int\limits_{\sigma(T)} \lambda \, \mathrm{d}P(\lambda)$, where P is a spectral measure (called the *resolution of the identity of* T) defined on the Borel subsets of $\mathbb{C}$ (actually, $\operatorname{supp}(P) = \sigma(T)$). More specifically, $P(\emptyset) = 0$ and $P(\sigma(T)) = I$, and P is multiplicative (i.e. $P(E \cap F) = P(E)P(F)$) and σ-additive with respect to the strong operator topology in $L(L^p(\mathbb{T}))$. For more precise details we refer to [6]. In this section we make a detailed analysis of scalar p-multiplier operators in $L(L^p(\mathbb{T}))$. The main result is the characterization of such operators via Littlewood-Paley p-decompositions of $\mathbb{Z}$; see Theorem 4.1. Several consequences are deduced from this result. In particular, it is shown (cf. Remark 4.2) that the resolution of the identity of any scalar p-multiplier operator is atomic; this will play an important role in the final section.

Fix $1 < p \leqslant 2$ and let $\{\Delta_j : j \in \mathbb{Z}\}$ be a Littlewood-Paley p-decomposition for $\mathbb{Z}$. Given $\xi \in l^\infty(\mathbb{Z})$ define $\widetilde{\xi} : \mathbb{Z} \to \mathbb{C}$ by $\widetilde{\xi} = \sum\limits_{j\in\mathbb{Z}} \xi_j \, \chi_{\Delta_j}$, in which case $\widetilde{\xi} \in M_p(\mathbb{T})$ and $|\!|\!|\widetilde{\xi}|\!|\!|_p \leqslant C_p \|\widetilde{\xi}\|_{l^\infty(\mathbb{Z})}$ for some $C_p > 0$ depending only on p and the decomposition $\{\Delta_j : j \in \mathbb{Z}\}$. Let Σ denote the σ-algebra of all subsets of $\mathbb{Z}$. For each $E \in \Sigma$, the idempotent $\psi_E = \sum\limits_{j\in E} \chi_{\Delta_j}$ belongs to $M_p(\mathbb{T})$ and satisfies $|\!|\!|\psi_E|\!|\!|_p \leqslant C_p$. An argument as in the proof of Proposition 8 from [16] shows that $R : \Sigma \to L(L^p(\mathbb{T}))$ defined by $R(E) = T_{\psi_E}$ is a spectral measure. Then the p-multiplier operator $T_{\tilde{\xi}} = \int\limits_{\mathbb{Z}} \xi(n) \, \mathrm{d}R(n)$ is a scalar operator with spectrum $\sigma(T_{\tilde{\xi}})$ equal to the R-essential range of ξ (pp. 2188–2191 in [6]). Since R has no non-trivial null sets (an R-null set $F \in \Sigma$ means $R(E) = 0$ for all $E \in \Sigma$

with $E \subset F$), the R-essential range of ξ is precisely the closure $\overline{\widetilde{\xi}(\mathbb{Z})}$. Accordingly, $\sigma(T_{\tilde{\xi}}) = \overline{\widetilde{\xi}(\mathbb{Z})}$. Let $\{\omega_n\}_{n\in\mathbb{N}}$ be an enumeration of the distinct points in $\{\xi_j : j \in \mathbb{Z}\}$ and for each $n \in \mathbb{N}$, define $P(\{\omega_n\}) = \sum_{\omega_n=\xi_j} R(\{j\})$. Then the resolution of the identity $P : \mathfrak{B}(\sigma(T_{\tilde{\xi}})) \to L(L^p(\mathbb{T}))$ of $T_{\tilde{\xi}}$, where $\mathfrak{B}(\sigma(T_{\tilde{\xi}}))$ denotes the σ-algebra of all Borel subsets of $\sigma(T_{\tilde{\xi}})$, is given by $P(F) = \sum_{n=1}^{\infty} \delta_{\omega_n}(F)P(\{\omega_n\})$. Here δ_{ω_n} is the Dirac measure at ω_n. We point out that every compact set $K \subset \mathbb{C}$ is the spectrum of some scalar operator $T \in O_p(\mathbb{T})$. Indeed, take any Littlewood-Paley p-decomposition $\{\Delta_j : j \in \mathbb{Z}\}$ with infinitely many terms and choose $\{\xi_j\}_{j\in\mathbb{Z}}$ to be any enumeration of a countable dense subset of K. Then $T = T_{\tilde{\xi}}$ has the required properties.

It turns out that the above construction of scalar operators in $O_p(\mathbb{T})$ is a paradigm. That is, there are no others, as made precise in the following result.

THEOREM 4.1. *Let $1 < p \leqslant 2$ and $T_\psi \in O_p(\mathbb{T})$. Then T_ψ is scalar if and only if $\{\psi^{-1}(\{\omega\}) : \omega \in \psi(\mathbb{Z})\}$ is a Littlewood-Paley p-decomposition of $\mathbb{Z}$.*

Proof. Suppose that $\Delta = \{\psi^{-1}(\{\omega\}) : \omega \in \psi(\mathbb{Z})\}$ has the Littlewood-Paley p-property. If $\psi(\mathbb{Z})$ is finite, then T_ψ is a finite linear combination of projections from $O_p(\mathbb{T})$ and hence, is a scalar operator. Otherwise, let $\{\xi_j\}_{j\in\mathbb{Z}}$ be an enumeration (indexed by $\mathbb{Z}$) of the distinct points of $\psi(\mathbb{Z})$. Then $\xi = (\xi_j)_{j\in\mathbb{Z}}$ is an element of $l^\infty(\mathbb{Z})$. Let $T_{\tilde{\xi}}$ be defined as above. Since $\widetilde{\xi} = \psi$ it is clear that $T_\psi = T_{\tilde{\xi}}$ and hence, T_ψ is scalar.

Conversely, suppose that T_ψ is a scalar operator. Since T_ψ is then decomposable (p. 33 in [2]) it follows from Lemma 2.2 that $\sigma(T_\psi) = \overline{\psi(\mathbb{Z})}$. Fix $\lambda \in \psi(\mathbb{Z})$. Let $P : \mathfrak{B}(\sigma(T_\psi)) \to L(L^p(\mathbb{T}))$ be the resolution of the identity of T_ψ. Since $\lambda \in \sigma_{\mathrm{pt}}(T)$ (see Lemma 2.1 (ii)) the non-zero projection $P(\{\lambda\})$ satisfies

$$P(\{\lambda\})(L^p(\mathbb{T})) = \{f \in L^p(\mathbb{T}) : T_\psi f = \lambda f\} = \ker(T_{\psi-\lambda}), \tag{4.1}$$

Theorem 2 of [6], p. 1955. Moreover, $P(\{\lambda\})$ commutes with every bounded operator commuting with T_ψ (Corollary 7 in [6], p. 1935) and hence, in particular, it commutes with all translations. Accordingly, $P(\{\lambda\}) \in O_p(\mathbb{T})$ and so equals $T_{\chi_{F(\lambda)}}$ for some set $F(\lambda) \subset \mathbb{Z}$. For each $n \in \psi^{-1}(\{\lambda\})$ the function $f(z) = z^n$ satisfies $T_\psi f = \lambda f$. Then (4.1) implies that $P(\{\lambda\}f = f$, that is, $T_{\chi_{F(\lambda)}} f = f$. Take Fourier transforms and evaluate at n shows that $n \in F(\lambda)$. Accordingly, $\psi^{-1}(\{\lambda\}) \subset F(\lambda)$. On the other hand, choose $n \in F(\lambda)$ and let f be as above. Then $T_{\chi_{F(\lambda)}} f = f$, that is, f belongs to the range of $T_{\chi_{F(\lambda)}} = P(\{\lambda\})$. It follows from (4.1), upon taking Fourier transforms, that $\psi(n) = \lambda$. Hence $F(\lambda) \subset \psi^{-1}(\{\lambda\})$. This establishes that $\chi_{\psi^{-1}(\{\lambda\})} \in M_p(\mathbb{T})$, for $\lambda \in \psi(\mathbb{Z})$.

Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be an enumeration of $\psi(\mathbb{Z})$. Then the sets $E(n) := \psi^{-1}(\{\lambda_n\})$, for $n \in \mathbb{N}$, form a disjoint covering of $\mathbb{Z}$. To show that the $E(n)$'s have the Littlewood-Paley p-property we need to show, for all sequences ε_n of ± 1's and

$N \in \mathbb{N}$, that the functions $g = \sum_{n=1}^{N} \varepsilon_n \chi_{E(n)} \in M_p(\mathbb{T})$ uniformly. Now, we have

$$T_g = \sum_{n=1}^{N} \varepsilon_n T_{\chi_{E(n)}} = \sum_{n=1}^{N} \varepsilon_n P(\{\lambda_n\}) = \int_{\sigma(T)} \widetilde{g} \, \mathrm{d}P,$$

where $\widetilde{g} : \mathbb{C} \to \mathbb{C}$ is the function $\sum_{n=1}^{N} \varepsilon_n \chi_{\{\lambda_n\}}$. Since $P(\{\lambda_n\}) \neq 0$, for each $n \in \mathbb{N}$, the P-essential supremum norm $|\widetilde{g}|_P$ of $\widetilde{g}$ (see p. 2187 in [6]) equals $\|g\|_{l^\infty(\mathbb{Z})} = 1$. It follows that

$$|\!|\!|g|\!|\!|_p \leqslant 4 \sup\{\|P(E)\| : E \in \mathfrak{B}(\sigma(T_\psi))\} < \infty$$

(see p. 2181 in [6]). ∎

REMARK 4.2. If $T_\psi \in O_p(\mathbb{T})$ is scalar, then its resolution of the identity $P : \mathfrak{B}(\sigma(T_\psi)) \to L(L^p(\mathbb{T}))$ is given by

$$P(F) = \sum_{\lambda_n \in F} T_{\chi_{E(n)}}, \quad F \in \mathfrak{B}(\sigma(T_\psi)),$$

and

$$T_\psi = \int_{\sigma(T_\psi)} \lambda \, \mathrm{d}P(\lambda) = \sum_{n=1}^{\infty} \lambda_n T_{\chi_{E(n)}}.$$

To see this define, for each $E \in \mathfrak{B}(\sigma(T_\psi))$, the projection

$$Q(E) = \sum_{n=1}^{\infty} \delta_{\lambda_n}(E) P(\{\lambda_n\}) = \sum_{n=1}^{\infty} \delta_{\lambda_n}(E) T_{\chi_{E(n)}}. \tag{4.2}$$

This formula and the σ-additivity of P imply that

$$Q(\sigma(T_\psi)) = \sum_{n=1}^{\infty} P(\lambda_n) = \sum_{n=1}^{\infty} T_{\chi_{E(n)}}, \tag{4.3}$$

where the series converges unconditionally in the strong operator topology of $L(L^p(\mathbb{T}))$. Using the fact that $\{E(n)\}_{n\in\mathbb{N}}$ is a decomposition of $\mathbb{Z}$, it is obvious that the right-hand-side of (4.3) applied to any trigonometric polynomial f just gives f again. It follows that $Q(\sigma(T_\psi)) = I$. Using the identities $\delta_{\lambda_n}(E \cap F) = \delta_{\lambda_n}(E)\delta_{\lambda_n}(F)$, for $n \in \mathbb{N}$ and $E, F \in \mathfrak{B}(\sigma(T_\psi))$, and $T_{\chi_{E(n)}} T_{\chi_{E(m)}} = 0$ whenever $n \neq m$, it follows from (4.2) that $Q(E \cap F) = Q(E)Q(F)$. The σ-additivity of P implies that

$$\langle Q(E)f, g\rangle = \sum_{n=1}^{\infty} \langle T_{\chi_{E(n)}} f, g\rangle \delta_{\lambda_n}(E), \quad E \in \mathfrak{B}(\sigma(T_\psi)), \tag{4.4}$$

for $f \in L^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$, where the series is unconditionally convergent in $\mathbb{C}$. The Vitali-Hahn-Saks theorem ensures that the right-hand-side of (4.4) is a σ-additive complex measure. So, Q is σ-additive for the weak (hence, also strong) operator topology. Accordingly, Q is a spectral measure. Moreover, the identities

$\operatorname{supp}(Q) = \operatorname{supp}(P)$ — clear from (4.2) — and $\operatorname{supp}(P) = \sigma(T_\psi)$ (Corollary 11 (ii) of [6], p. 2191) show that $\operatorname{supp}(Q) = \sigma(T_\psi)$. Finally, it follows from (4.4) that the operator

$$\int_{\sigma(T_\psi)} \lambda \, \mathrm{d}Q(\lambda) = \sum_{n=1}^{\infty} \lambda_n P(\{\lambda_n\}) = \sum_{n=1}^{\infty} \lambda_n T_{\chi_{E(n)}}$$

coincides with $\int_{\sigma(T_\psi)} \lambda \, \mathrm{d}P(\lambda)$ on trigonometric polynomials and hence, on all of $L^p(\mathbb{T})$. Hence, Q is a resolution of the identity for T_ψ. By uniqueness (Corollary 8 in [6], p. 1935) $Q = P$. ∎

We proceed to deduce some consequences of Theorem 4.1. First a result, via quite different methods, due to T.A. Gillespie ([9]).

Corollary 4.3. *Let $u \in \mathbb{T}$ be an element of infinite order. Then, for $p \in (1, 2)$, the translation operator τ_u is not scalar.*

Proof. Note that $\tau_u = T_\psi$, where $\psi(n) = u^n$ for $n \in \mathbb{Z}$. Since ψ is injective, it follows that $\{\psi^{-1}(\{\omega\}) : \omega \in \psi(\mathbb{Z})\}$ is the family of all singleton subsets of $\mathbb{Z}$, which does not have the Littlewood-Paley p-property. ∎

A slightly more general result is the following one.

Corollary 4.4. *Let $1 < p < 2$ and $\psi \in M_p(\mathbb{T})$. If ψ is injective on $[N, \infty) \cap \mathbb{Z}$ or $(-\infty, N] \cap \mathbb{Z}$, for some $N \in \mathbb{Z}$, then T_ψ is not a scalar operator.*

Proof. We consider only $E := [N, \infty) \cap \mathbb{Z}$ as the other case is similar. Suppose T_ψ is scalar. Since $\chi_E \in M_p(\mathbb{T})$, the projection T_{χ_E} is surely scalar. By a result of C.A. McCarthy ([12]) the family of scalar operators in $O_p(\mathbb{T})$ is an algebra and hence, the p-multiplier operator $T_{\psi\chi_E} = T_\psi T_{\chi_E}$ is also scalar. If $\psi_N := \psi\chi_E$, then $\{\psi_N^{-1}(\{\omega\}) : \omega \in \psi_N(\mathbb{Z})\}$ contains all singleton sets $\{n\}$, for $n \geqslant N$ and so cannot have the Littlewood-Paley p-property. Hence, T_{ψ_N} is not scalar which is a contradiction. So, the original assumption that T_ψ is scalar is false. ∎

Corollary 4.4 is a version, for the circle group, of a similar result known for p-multiplier operators T_ψ on groups other than $\mathbb{T}$. For instance, if $\psi : \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{C}$ has certain local monotonicity properties, then this is the case (see Propositions 2.4 and 2.5 in [1]).

Lemma 4.5. *Let $1 < p \leqslant 2$ and $\psi \in M_p(\mathbb{T})$. Suppose that T_ψ is a compact operator. Then 0 is the only possible limit point of $\psi(\mathbb{Z})$ and $\psi^{-1}(\{\lambda\})$ is a finite subset of $\mathbb{Z}$, for each non-zero $\lambda \in \psi(\mathbb{Z})$. In particular, $\sigma(T_\psi) = \overline{\psi(\mathbb{Z})}$.*

Proof. It is well known that $\sigma(T_\psi)$ is countable with 0 as only possible limit point. Lemma 2.2 applies to show that $\sigma(T_\psi) = \overline{\psi(\mathbb{Z})}$. Hence, 0 is the only possible limit point of $\psi(\mathbb{Z})$. Accordingly, each $\lambda \in \psi(\mathbb{Z}) \setminus \{0\}$ is an isolated point of $\sigma(\mathbb{T}_\psi)$. By Lemma 2.1 (v) λ is an eigenvalue of T_ψ, the characteristic function χ_λ of the set $\psi^{-1}(\{\lambda\})$ belongs to $M_p(\mathbb{T})$ and the range of the spectral projection $P(\{\lambda\}) = T_{\chi_\lambda}$ of T_ψ corresponding to the spectral set $\{\lambda\}$ is $\ker(T_\psi - \lambda I)$. Since λ is a pole of T_ψ of order 1 (see the proof of Lemma 2.1 (v)), the compactness of T_ψ implies that $\ker(T_\psi - \lambda I)$ is finite dimensional ([6], p. 579). But if $n \in \psi^{-1}(\{\lambda\})$, then z^n belongs to $\ker(T_\psi - \lambda I)$. Hence, $\psi^{-1}(\{\lambda\})$ is finite. ∎

REMARK 4.6. (i) The condition that 0 is the only possible limit point of $\psi(\mathbb{Z})$ and $\psi^{-1}(\{\lambda\})$ is a finite subset of $\mathbb{Z}$, for each non-zero $\lambda \in \psi(\mathbb{Z})$, is equivalent to $\lim_{|n|\to\infty} \psi(n) = 0$, i.e. $\psi \in c_0(\mathbb{Z})$.

(ii) Lemma 4.5 does not characterize the compact operators in $O_p(\mathbb{T})$.

Proof. As noted earlier, M. Zafran exhibited p-multipliers $\psi \in c_0(\mathbb{Z})$ for which the inclusion (2.4) is proper. By Lemma 4.5 we see that T_ψ cannot be compact in this case. ∎

COROLLARY 4.7. *Let $1 < p \leqslant 2$. A scalar operator $T_\psi \in O_p(\mathbb{T})$ is a compact if and only if $\psi \in c_0(\mathbb{Z})$.*

Proof. As noted above, compactness of T_ψ alone ensures $\psi \in c_0(\mathbb{Z})$. So, suppose $\psi \in c_0(\mathbb{Z})$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be an enumeration of $\psi(\mathbb{Z}) \setminus \{0\}$. By Remark 4.2 we have, with $E(n) := \psi^{-1}(\{\lambda_n\})$, that $T_\psi = \sum_{n\in\mathbb{N}} \lambda_n T_{\chi_{E(n)}}$, where the series is unconditionally convergent in the strong operator topology. Since the resolution of the identity for T_ψ has no non-trivial null sets, it follows from Theorem 10 of [6], p. 2189, that there exists a constant $K > 0$ such that

$$\left\| T_\psi - \sum_{n=1}^{N} \lambda_n T_{\chi_{E(n)}} \right\| \leqslant K \sup\{|\lambda_n| : n > N\}, \quad N \in \mathbb{N}. \tag{4.5}$$

Since $\psi \in c_0(\mathbb{Z})$ each set $E(n)$ is finite so $T_{\chi_{E(n)}}$ is a finite rank projection. Moreover, since the right-hand-side of (4.5) converges to 0 we see that T_ψ is compact. ∎

REMARK 4.8. It is straightforward to exhibit compact multiplier operators which are not scalar. Take for instance $\psi(n) = (1 + n^2)^{-1}, n \in \mathbb{Z}$. Then $T_\psi \in M_p(\mathbb{T})$, $1 \leqslant p \leqslant 2$, is compact, as it is easily seen that T_ψ can be approximated in the operator norm topology by finite rank operators. But, T_ψ is not scalar; see Corollary 4.4. ∎

COROLLARY 4.9. *Let $1 < p < 2$ and $T_\psi \in O_p(T)$ have totally disconnected spectrum. For each spectral set E of T_ψ, let $P(E)$ denote the corresponding spectral projection. Then T_ψ is scalar if and only if*

$$\sup\{\|P(E)\|_{L(L^p(\mathbb{T}))} : E \text{ a spectral set of } T_\psi\} < \infty. \tag{4.6}$$

Proof. If T_ψ is scalar, then each spectral projection for T_ψ belongs to the range of the resolution of the identity of T_ψ. It was noted in the proof of Theorem 4.1 that this range is a uniformly bounded subset of $L(L^p(\mathbb{T}))$ from which (4.6) follows.

Conversely, suppose (4.6) holds. Lemma 2.2 shows that $\sigma(T_\psi) = \overline{\psi(\mathbb{Z})}$. Given any spectral set $E \subset \sigma(T_\psi)$, an argument as in the proof of Lemma 2.1 (v), with the singleton set $\{\lambda\}$ replaced by E and the contour Γ suitably chosen to surround E (within the domain of f), shows that $\chi_{\psi^{-1}(E)} \in M_p(\mathbb{T})$ and the associated spectral projection $P(E)$ equals $T_{\chi_{\psi^{-1}(E)}}$.

Let $T'_\psi \in O_q(\mathbb{T})$ denote the dual operator to T_ψ. Then $\sigma(T_\psi) = \sigma(T'_\psi)$. Moreover, since $(T'_\psi)' = T_\psi$ and $\sigma_{\mathrm{r}}(T'_\psi) = \emptyset$ (see Lemma 2.1) it follows from

Proposition 1.14 of [5] that $\sigma_{\mathrm{pt}}(T_\psi) = \sigma_{\mathrm{pt}}(T'_\psi)$. It follows from Proposition 1.2 and Theorem 1.19 (iv) of [5] that T_ψ and T'_ψ have the same spectral sets. In addition, if $P(E)$ is a spectral projection of T_ψ, then $P(E)'$ is a spectral projection of T'_ψ associated with the spectral set E. Combining these comments with the fact that $L^p(\mathbb{T})$ is a reflexive Banach space, Theorem 5.36 of [5] applied to $T = T'_\psi$ implies that T_ψ is prespectral of class $L^q(\mathbb{T})$. That is, there exists a spectral measure $\widetilde{P} : \mathfrak{B}(\sigma(T_\psi)) \to L(L^p(\mathbb{T}))$ of class $L^q(\mathbb{T})$ satisfying Definition 5.5 of [5] and agreeing with the finitely additive (see p. 1930 in [6]) spectral measure P on the algebra of all spectral subsets of $\sigma(T_\psi)$. Accordingly, T_ψ is actually a spectral operator (Theorem 6.5 in [5]) and $\widetilde{P}$ is its resolution of the identity. As noted at the start of the section, T_ψ is then necessarily scalar. ∎

Remark 4.10. Some comments concerning (4.6) are in order. Let $T_\psi \in O_p(\mathbb{T})$ have totally disconnected spectrum. According to Theorem 4.1 and Corollary 4.9 we see that (4.6) is satisfied if and only if $\Delta = \{\psi^{-1}(\{\lambda\}) : \lambda \in \psi(\mathbb{Z})\}$ is a Littlewood-Paley p-decomposition of $\mathbb{Z}$. Let χ_λ denote the characteristic function of $\psi^{-1}(\{\lambda\}), \lambda \in \psi(\mathbb{Z})$. Then Δ is a Littlewood-Paley p-decomposition if and only if

$$\sup\left\{\Big\|\sum_{\lambda\in F} T_{\chi_\lambda}\Big\|_{L(L^p(\mathbb{T}))} : F \subset \psi(\mathbb{Z}), F \text{ finite}\right\} < \infty. \tag{4.7}$$

So, (4.6) and (4.7) are equivalent. However, the two families of operators in (4.6) and (4.7) can be quite different. Recall that $\sigma(T_\psi) = \overline{\psi(\mathbb{Z})}$. If each $\lambda \in \psi(\mathbb{Z})$ is an isolated point of the spectrum, that is, $\{\lambda\}$ is a spectral set of T_ψ, then

$$\left\{\sum_{\lambda\in F} T_{\chi_\lambda} : F \subset \psi(\mathbb{Z}), F \text{ finite}\right\} \subset \{P(E) : E \text{ a spectral set of } T_\psi\} \tag{4.8}$$

and so clearly (4.6) implies (4.7). However, if some of the limit points of $\psi(\mathbb{Z})$ are eigenvalues of T_ψ (in which case these limit points actually belong to $\psi(\mathbb{Z})$; see Lemma 2.1), then (4.8) fails to be satisfied. Of course, if there are only finitely many limit points of $\psi(\mathbb{Z})$ belonging to $\sigma_{\mathrm{pt}}(T_\psi)$, then it can still be argued that (4.7) follows from (4.6). However, if there are infinitely many such limit points, then it is not clear why (4.7) follows from (4.6); this is precisely what the proof of Corollary 4.9 argues. ∎

5. VON NEUMANN'S BICOMMUTANT THEOREM

As noted in the introduction, if T is a normal operator in a Hilbert space, then a result of J. von Neumann states that $\{T\}^{\mathrm{cc}} = \overline{\langle I, T, T^*\rangle}^{\mathrm{wo}}$. If $T_\psi \in O_p(\mathbb{T})$, for some $1 < p < 2$, is a scalar operator (the natural counterpart in $L^p(\mathbb{T})$ of a (normal) multiplier operator in $L^2(\mathbb{T})$) and we interpret $T_{\overline{\psi}}$ as T^*_ψ, then an analogue of von Neumann's bicommutant theorem would be that $\{T_\psi\}^{\mathrm{cc}} = \overline{\langle I, T_\psi, T_{\overline{\psi}}\rangle}^{\mathrm{wo}}$. The aim of this section is to show that this is indeed the case. As a by-product of the arguments involved we show, for $1 < p < 2$, that no scalar operator in $O_p(\mathbb{T})$ can have a cyclic vector. This is in contrast to the case $p = 2$ where a cyclic vector does exist whenever ψ is injective.

Let X be a Banach space. For the notion of a Boolean algebra (briefly, B.a.) of projections $\mathfrak{M}$ in $L(X)$ we refer to Chapter 5 of [5] or Chapter XVII of [6]. Of particular relevance is the concept of $\mathfrak{M}$ being σ-complete (Chapter XVII in [6]). The range of any spectral measure defined on a σ-algebra of sets is a σ-complete B.a. of projections; see p. 2204 in [6]. A non-zero projection G in a B.a. of projections $\mathfrak{M} \subset L(X)$ is called an *atom* if, whenever $H \in \mathfrak{M}$ satisfies $H \leqslant G$ (meaning $HG = H$ or, equivalently, that $HX \subset GX$), then either $H = 0$ or $H = G$. The B.a. $\mathfrak{M}$ is said to be *countably atomic* if there exists a sequence of atoms $\{G_n\}_{n\in\mathbb{N}}$ in $\mathfrak{M}$ such that, whenever $G \in \mathfrak{M}$, there is a subset $A \subset \mathbb{N}$ such that $G = \sum_{n\in A} G_n$, where the series converges unconditionally in the strong operator topology. We say that $\mathfrak{M}$ is generated by $\{G_n\}_{n\in\mathbb{N}}$.

Multiplicity theory for a complete B.a. of projections $\mathfrak{M}$ in a Banach space X is due to W.G. Bade; see Section 3 of Chapter XVIII in [6]. For X separable (which we now assume) it is known that every $P \in \mathfrak{M}$ satisfies the countable chain condition (see p. 2266 in [6] for the definition). Moreover, $\mathfrak{M}$ is complete if and only if it is σ-complete (Lemma 21 of [6], p. 2215). The multiplicity $m(P)$ of P is defined to be the smallest cardinal power of a set of vectors $A \subset PX$ such that $PX = \overline{\operatorname{span}}\{\mathfrak{M}[x] : x \in A\}$, where $\mathfrak{M}[x] := \overline{\operatorname{span}}\{Qx : Q \in \mathfrak{M}\}$ is the cyclic space spanned by x with respect to $\mathfrak{M}$. A projection $P \in \mathfrak{M}$ is said to *have uniform multiplicity* $n \in \mathbb{N} \cup \{\aleph_0\}$ if $m(Q) = n$ whenever $0 \neq Q \leqslant P$. If $P \in \mathfrak{M}$ is an atom and $0 \neq x \in PX$, then $\mathfrak{M}[x]$ is the 1-dimensional space spanned by x. Hence, in this case $m(P) = \dim(PX)$ and, in particular, P has uniform multiplicity equal to $\dim(PX)$. There is a unique family $\{P_n : n \in \mathbb{N} \cup \{\aleph_0\}\}$ of pairwise disjoint elements in $\mathfrak{M}$ such that $I = \bigvee P_n$ (the supremum is formed in the B.a. $\mathfrak{M}$) and, if $P_n \neq 0$, then P_n has uniform multiplicity n (Theorem 3 in [6], p. 2265). We say that $\mathfrak{M}$ has finite uniform multiplicity if there exists an integer $N \in \mathbb{N}$ such that $P_n = 0$ for all $n \in \{\aleph_0\} \cup \{N+1, N+2, \ldots\}$.

The following result shows that σ-complete B.a.'s of p-multiplier projections for the circle group have a curious structure.

Lemma 5.1. *Let $1 < p < 2$ and $\mathfrak{M} \subset O_p(\mathbb{T})$ be any σ-complete B.a. of projections.*

(i) $\mathfrak{M}$ *is countably atomic.*

(ii) $\mathfrak{M}$ *does not have finite uniform multiplicity.*

Proof. (i) Since $L^p(\mathbb{T})$ is separable, there is a scalar operator $T \in L(L^p(\mathbb{T}))$ whose resolution of the identity P has range precisely $\mathfrak{M}$ (Proposition 2 in [14]). Since $P(E)$ commutes with every translation, for each $E \in \mathfrak{B}(\sigma(T))$, a standard approximation argument (p. 2190 in [6]) shows that $T = T_\psi$ for some $\psi \in M_p(\mathbb{T})$. It is clear from the formula for P given in Remark 4.2 that $\mathfrak{M}$ is generated by the countable family $P(\{\lambda_n\}) = T_{\chi_{E(n)}}$, for $n \in \mathbb{N}$, where $\{\lambda_n\}_{n\in\mathbb{N}}$ is an enumeration of $\psi(\mathbb{Z})$ and $E(n) = \psi^{-1}(\{\lambda_n\})$. Since each set $\{\lambda_n\}$ is a singleton it is immediate that $P(\{\lambda_n\})$ is an atom of $\mathfrak{M}$. Accordingly, $\mathfrak{M}$ is countably atomic.

(ii) If $\mathfrak{M}$ contains an atom with infinite dimensional range then, as noted prior to the lemma, this atom has uniform multiplicity $\aleph_0$. Then $\mathfrak{M}$ cannot have finite uniform multiplicity. So, suppose the other possibility occurs, i.e. all atoms

$P(\{\lambda_n\})$ have finite dimensional range. If it were the case that $\mathfrak{M}$ has finite uniform multiplicity N, then

$$N = \sup\{\dim(P(\{\lambda_n\})(L^p(\mathbb{T})) : n \in \mathbb{N}\} < \infty.$$

By Theorem 4.1 the family $\Delta = \{E(n)\}_{n\in\mathbb{N}}$ is a Littlewood-Paley p-decomposition of $\mathbb{Z}$. It is routine to verify that

$$T_{\chi_E}(L^p(\mathbb{T})) = \{g \in L^p(\mathbb{T}) : \operatorname{supp}(\widehat{g}) \subset E\} \tag{5.1}$$

whenever $E \subset \mathbb{Z}$ is a set for which $\chi_E \in M_p(\mathbb{T})$. Accordingly, each set $E(n)$ is finite, for $n \in \mathbb{N}$, and no set has more than N elements. Choose one element e_n^1 from each set $E(n)$ and call the collection of all these elements F_1. Now choose one element e_n^2 from each set $E(n) \setminus \{e_n^1\}$ which happens to be non-empty and let F_2 denote the set of all of these elements. Continue the process to produce sets $F_1, \ldots, F_N$ which are pairwise disjoint and have union $\mathbb{Z}$. Let $q > 2$ be the dual exponent to p. We will see that each set F_n is a $\Lambda(q)$-set and, since a finite union of $\Lambda(q)$-sets is again a $\Lambda(q)$-set, this produces a contradiction since $\mathbb{Z}$ is not a $\Lambda(p)$-set. To prove this claim we show F_1 is a $\Lambda(q)$-set: define $\widehat{Q_n f} = \chi_{\{e_n^1\}}\widehat{f}$ and $(P_{E(n)}f)\widehat{\ } = \chi_{E(n)}\widehat{f}$. One may employ, for instance, the vector-valued version of the L^p-boundedness of the Hilbert transform, to see that

$$\Big\|\Big(\sum_n |Q_n g_n|^2\Big)^{1/2}\Big\|_p \leqslant C_p \,\Big\|\Big(\sum_n |g_n|^2\Big)^{1/2}\Big\|_p.$$

If we plug in $g_n = P_{E(n)}f$ and use $Q_n P_{E(n)} = Q_n$, then the fact that $\{E(n)\}_{n\in\mathbb{N}}$ is a Littlewood-Paley p-decomposition implies

$$\sum_{m\in F_1} |\widehat{f}(m)|^2 \leqslant C_p \Big\|\Big(\sum_n |P_{E(n)}f|^2\Big)^{1/2}\Big\|_p^2 \leqslant C_p' \|f\|_p^2.$$

Hence F_1 is a $\Lambda(q)$-set. Clearly, the same argument works for the other F_n's. ∎

REMARK 5.2. Let $p \in (1, 2]$ and $\mathfrak{N} \subset O_p(\mathbb{T})$ be any bounded B.a. of projections. Then $\mathfrak{N}$ has a purely atomic completion $\mathfrak{M} \subset O_p(\mathbb{T})$. Indeed, since a reflexive space cannot contain an isomorphic copy of the sequence space c_0, it is known that the closure $\mathfrak{M}$ of $\mathfrak{N}$ with respect to the strong operator topology in $L(L^p(\mathbb{T}))$ is a complete B.a. of projections, [10]. Clearly, $\mathfrak{M} \subset O_p(\mathbb{T})$. Then Lemma 5.1 ensures that $\mathfrak{M}$ is atomic. ∎

LEMMA 5.3. *Let X be a Banach space and $T \in L(X)$ be a scalar operator with resolution of the identity $P : \mathfrak{B}(\sigma(T)) \to L(X)$. Let $\mathfrak{M}$ denote the range of P and define $T^* := \int_{\sigma(T)} \overline{\lambda}\, \mathrm{d}P(\lambda)$. Then*

(i) $\{T\}^{\mathrm{cc}} = \mathfrak{M}^{\mathrm{cc}}$.
(ii) $\overline{\langle I, T, T^*\rangle}^{\mathrm{wo}} = \overline{\langle \mathfrak{M}\rangle}^{\mathrm{wo}}$.

Proof. It is known that $A \in L(X)$ satisfies $AT = TA$ if and only if $AP(E) = P(E)A$ for all $E \in \mathfrak{B}(\sigma(T))$ (Corollary 7 in [6], p. 1935), from which (i) is immediate. Part (ii) is also known; it follows from Proposition 4.2 of [4] applied to the strong operator topology closure $\overline{\mathfrak{M}}^{\mathrm{so}}$ of $\mathfrak{M}$, after noting that $\overline{\mathfrak{M}}^{\mathrm{so}}$ is complete (Lemma 23 in [6], p. 2216) and that $\overline{\langle\mathfrak{M}\rangle}^{\mathrm{wo}} = \overline{\langle\overline{\mathfrak{M}}^{\mathrm{so}}\rangle}^{\mathrm{wo}}$ and $\mathfrak{M}^{\mathrm{cc}} = (\overline{\mathfrak{M}}^{\mathrm{so}})^{\mathrm{cc}}$. ∎

We now have the main result of this section.

THEOREM 5.4. *Let* $1 < p \leqslant 2$ *and let* $T_\psi \in O_p(\mathbb{T})$ *be a scalar operator. Then*

$$\overline{\langle I, T_\psi, T_{\overline{\psi}} \rangle}^{\mathrm{wo}} = \{T_\psi\}^{\mathrm{cc}}.$$

Proof. It was noted at the beginning of this section that the range $\mathfrak{M}$ of the resolution of the identity of T_ψ is a σ-complete B.a. of projections. Moreover, Lemma 5.1 (i) shows that $\mathfrak{M}$ is countably atomic from which it follows that $\overline{\langle \mathfrak{M} \rangle}^{\mathrm{wo}} = \mathfrak{M}^{\mathrm{cc}}$ (Proposition 1 in [15]). Then Lemma 5.3 gives the desired conclusion. ∎

REMARK 5.5. The bicommutant result of Theorem 5.4 does not hold for all scalar operators in general Banach spaces. J. Dieudonné exhibited a σ-complete B.a. of projections $\mathfrak{M}$ in a separable, reflexive Banach space X such that the inclusion $\overline{\langle \mathfrak{M} \rangle}^{\mathrm{wo}} \subset \mathfrak{M}^{\mathrm{cc}}$ is strict ([3]). As noted in the proof of Lemma 5.1, separability of X implies that $\mathfrak{M}$ equals the range of the resolution of the identity of some scalar operator $T \in L(X)$. Lemma 5.3 shows that the inclusion $\overline{\langle I, T, T^* \rangle}^{\mathrm{wo}} \subset \{T\}^{\mathrm{cc}}$ is strict. ∎

Let $T \in L(X)$ be a scalar operator and $T^* := \int_{\sigma(T)} \overline{\lambda}\, \mathrm{d}P(\lambda)$, where P is the resolution of the identity of T. A vector $x \in X$ is said to be cyclic for T if the subspace $\{q(T, T^*)x : q \text{ a polynomial}\}$ is dense in X.

LEMMA 5.6. *Let* $T \in L(X)$ *be a scalar operator with resolution of the identity* P. *Then* $x \in X$ *is a cyclic vector for* T *if and only if the linear span of* $\{P(E)x : E \in \mathfrak{B}(\sigma(T))\}$ *is dense in* X.

Proof. Since both $\{q(T, T^*) : q \text{ a polynomial}\}$ and the linear span of $\{P(E) : E \in \mathfrak{B}(\sigma(T))\}$ are convex subsets of $L(X)$ it follows from Lemma 5.3 (ii) that

$$\overline{\langle I, T, T^* \rangle}^{\mathrm{so}} = \overline{\langle P(E) : E \in \mathfrak{B}(\sigma(T)) \rangle}^{\mathrm{so}}, \tag{5.2}$$

where "so" indicates the closed subalgebra generated with respect to the strong operator topology in $L(X)$. Moreover, the seminorm $\rho : S \mapsto \|Sx\|$, $S \in L(X)$, is continuous with respect to the strong operator topology. A routine approximation argument based on (5.2) and the property mentioned of ρ then gives the desired conclusion. ∎

We conclude with the following result.

THEOREM 5.7. (i) *Let* $1 < p < 2$ *and* $T_\psi \in O_p(\mathbb{T})$ *be a scalar operator. Then* T *has no cyclic vector.*

(ii) *Let* $\psi \in l^\infty(\mathbb{Z})$. *Then the normal operator* $T_\psi \in O_2(\mathbb{T})$ *has a cyclic vector if and only if* ψ *is injective.*

Proof. (i) Let $\mathfrak{M}$ denote the range of the resolution of the identity P of T_ψ. Suppose that a cyclic vector $f \in L^p(\mathbb{T})$ exists for T_ψ. Lemma 5.6 implies that $\mathfrak{M}[f] = L^p(\mathbb{T})$, from which it is clear that $\mathfrak{M}$ has uniform multiplicity one. This contradicts Lemma 5.1 (ii).

(ii) If T_ψ has a cyclic vector, then Lemma 5.6 implies that each atom in the countably atomic B.a. $\mathfrak{M}$ has 1-dimensional range. It follows from (5.1) that

$\psi^{-1}(\{\lambda\})$ must be a singleton subset of $\mathbb{Z}$, for each $\lambda \in \psi(\mathbb{Z})$. Hence, ψ is injective. Conversely, suppose that ψ is injective, in which case each atom T_{χ_λ} of $\mathfrak{M}$, for $\lambda \in \psi(\mathbb{Z})$, has rank 1. Recall that χ_λ is the characteristic function of $\psi^{-1}(\{\lambda\})$. Choose any $f \in L^2(\mathbb{T})$ with $\widehat{f}(k) \neq 0$ for all $k \in \mathbb{Z}$. Fix $n \in \mathbb{Z}$ and let $e_n(z) = z^n$. With $E = \{\psi(n)\}$ it turns out that $P(E)f = \widehat{f}(n)e_n$ and so e_n belongs to the span of $\{P(F)f : F \in \mathfrak{B}(\sigma(T_\psi))\}$. Accordingly, this span contains all trigonometric polynomials and so is dense in $L^2(\mathbb{T})$. By Lemma 5.6 we see f is a cyclic vector for T_ψ. ∎

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