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WHEN IS THE PRODUCT OF HANKEL OPERATORS A HANKEL OPERATOR?

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Abstract. In this paper we characterize when the product of two block Hankel operators on the vector-valued Hardy space is a Hankel operator. We also describe when a block Toeplitz and a block Hankel operator commute. These characterizations extend results in two recent papers by T. Yoshino and R.A. Martínez-Avendaño respectively.

Keywords: *Block Hankel operator, Toeplitz operator, Hardy space.*

MSC (2000): 47B35.

1. INTRODUCTION

This work is motivated by two recent papers [9] and [7]. In [9], T. Yoshino answers the question of when the products of two Hankel operators (on the Hardy space) is also a Hankel operator. In [7], R.A. Martínez-Avendaño characterizes when a Toeplitz operator and a Hankel operator commute.

In this paper we obtain generalizations of these two results for block Hankel and Toeplitz operators on vector-valued Hardy space. We also characterize when the product of a Toeplitz and Hankel operator is a Hankel. These results are motivated by the classical results in [2] where Brown and Halmos characterized when the product of two Toeplitz operators (on the Hardy space) is also a Toeplitz operator and when two Toeplitz operators commute.

The aim is to describe the algebraic properties of the Toeplitz and Hankel operators via the properties of their symbols. Our basic idea is to reduce operator equations involving Toeplitz and Hankel operators on the Hardy space to some operator equations on the subspaces of constant functions. These equations on the subspaces can be readily reduced to the relations on the symbols of these operators. This idea has been used in the author's paper [5] to show that if the product of six Toeplitz operators is zero, then one of them has to be zero. This idea provides an unified framework for these algebraic problems of Toeplitz and Hankel operators.

We give an outline of the paper. In Section 2 we answer some algebraic questions for Toeplitz and Hankel operators in the setting of an abstract Hilbert space. This abstract approach does not use the symbols of Toeplitz and Hankel operators, which demonstrates its generality and simplicity. In Section 3 we apply the results of Section 2 to answer the question of when a block Hankel operator and a block Toeplitz operator on vector-valued Hardy space commute. We also characterize when the product of a block Hankel and a block Toeplitz operator is a Hankel operator. In Section 4, by using the results of Section 2 we show that if the product of two block Hankel operators on the Hardy space is also a Hankel operator, then their symbols have to be simple rational functions.

In Section 5, we note that if the product of three or more Hankel operators is a Hankel operator, their symbols can be quite general, however if all the products of three Hankel operators (by varying their positions in the products) are Hankel operators, then their symbols are quite special and related to inner functions. This result is motivated by recent work of Xia and Zheng ([8]) and the author ([5]) where the questions when all the products of three Hankel operators are zero or of finite rank were discussed. We end this section by stating a conjecture concerning when all the products of a finite number of Hankel operators are Hankel operators.

2. Hankel and Toeplitz operators on a Hilbert space

Let K be a complex separable Hilbert space. Let S be a pure isometry on K (i.e., $S^*S = I$ and $S^{*n} \to 0$ strongly as $n \to \infty$). Set

$$\Delta = I - SS^* \quad \text{and} \quad K_0 = \Delta K.$$

Note that $\Delta = P_{K_0}$ where P_{K_0} is the projection onto K_0. The subspace K_0 is called the wandering subspace of S and the whole space K can be decompose as

$$K = K_0 \oplus SK_0 \oplus S^2K_0 \oplus \cdots .$$

See page 125 in [3] for more details. A linear bounded operator T on K is a Toeplitz operator if T satisfies the operator equation $S^*TS = T$ and an operator H is said to be a Hankel operator if $HS = S^*H$.

In this section we discuss several algebraic problems for Hankel and Toeplitz operators on K. The basic idea is to reduce operator equations involving Toeplitz and Hankel operators on the space K to operator equations on the subspace K_0. Those operators equations on K_0 can be readily reduced to relations on the symbols of these operators as will be done in the subsequent sections. The abstract approach of this section does not use the symbols of Toeplitz and Hankel operators, which demonstrates its generality and simplicity.

LEMMA 2.1. *If H_1 and H_2 are two Hankel operators and T_1 and T_2 are two Toeplitz operators on K, then*

$$S^*(T_1H_1 - H_2T_2) - (T_1H_1 - H_2T_2)S = S^*T_1\Delta H_1 + H_2\Delta T_2 S.$$

Proof. Note that

$$\begin{aligned} S^*T_1H_1 &= S^*T_1(SS^* + \Delta)H_1 \\ &= S^*T_1SS^*H_1 + S^*T_1\Delta H_1 \\ &= T_1H_1S + S^*T_1\Delta H_1 \\ &= (T_1H_1 - H_2T_2)S + H_2T_2S + S^*T_1\Delta H_1 \\ &= (T_1H_1 - H_2T_2)S + H_2(SS^* + \Delta)T_2S + S^*T_1\Delta H_1 \\ &= (T_1H_1 - H_2T_2)S + H_2SS^*T_2S + H_2\Delta T_2S + S^*T_1\Delta H_1 \\ &= (T_1H_1 - H_2T_2)S + S^*H_2T_2 + H_2\Delta T_2S + S^*T_1\Delta H_1. \end{aligned}$$

The lemma follows from the above equations. ∎

The following theorem essentially reduces the operator equation $TH = HT$ on the whole space K to some operator equations on the subspace K_0.

THEOREM 2.2. *Let H be a Hankel operator and T be a Toeplitz operator on K. Then $TH = HT$ if and only if*

$$H\Delta TS + S^*T\Delta H = 0 \tag{2.1}$$

$$(TH - HT)\Delta = 0. \tag{2.2}$$

Proof. The necessity is clear from the above lemma. (With $H = H_1 = H_2$ and $T = T_1 = T_2$).

Now we prove the sufficiency. Let $D = TH - HT$. By assumption (2.1) and the above lemma

$$S^*D = DS.$$

Therefore

$$\ker(D) \supset S\ker(S^*D) \supset S\ker(D).$$

Thus $\ker(D)$ is invariant for S. By assumption (2.2), $\ker(D) \supset K_0$. Therefore, by the invariance of $\ker(D)$ for S,

$$\ker(D) \supset SK_0.$$

By iteration, $\ker(D) \supset S^nK_0$ for $n \geqslant 0$. We conclude that $D = 0$. That is, $TH = HT$. ∎

THEOREM 2.3. *Let $H_{1,}H_2$ be two Hankel operators. Then H_1H_2 is also a Hankel operator if and only if*

$$H_1\Delta H_2S - S^*H_1\Delta H = 0 \tag{2.3}$$

$$(H_1H_2S - S^*H_1H_2)\Delta = 0 \tag{2.4}$$

$$(H_1H_2S - S^*H_1H_2)^*\Delta = 0. \tag{2.5}$$

Proof. Note that

$$H_1H_2S = H_1SS^*H_2S + H_1\Delta H_2S = S^*(H_1H_2S)S + H_1\Delta H_2S$$
$$S^*H_1H_2 = S^*H_1SS^*H_2 + S^*H_1\Delta H_2 = S^*(S^*H_1H_2)S + S^*H_1\Delta H_2.$$

Let $D = H_1H_2S - S^*H_1H_2$. Subtracting the above two equations gives

$$D = S^*DS + H_1\Delta H_2S - S^*H_1\Delta H_2.$$

Assume H_1H_2 is a Hankel operator. Thus $D = 0$. The necessity of condition (2.3) follows from the above equation.

Assume now conditions (2.3), (2.4) and (2.5) hold. The above equation implies that $D = S^*DS$. By assumption (2.4), $\ker(D) \supset K_0$, thus

$$\ker(S^*DS) = \ker(D) \supset K_0.$$

Therefore

$$DSK_0 \subset \ker(S^*) = K_0.$$

On the other hand, the assumption (2.5) $\ker(D^*) \supset K_0$ implies that

$$DSK_0 \subset \text{range}(D) \subset [\ker(D^*)]^\perp \subset K_0^\perp.$$

Thus $\ker(D) \supset SK_0$. Repeating the above argument with the fact that $\ker(D) \supset SK_0$, we see that $\ker(D) \supset S^2K_0$. By iteration, $\ker(D) \supset S^nK_0$ for $n \geqslant 0$. That is, $D = 0$. Therefore H_1H_2 is a Hankel operator. ∎

PROPOSITION 2.4. *Let $H_{1,}H_2$ and H_3 be three Hankel operators. Then $H_1H_2H_3$ is also a Hankel operator if and only if*

$$S^*H_1H_2\Delta H_3 = H_1\Delta H_2H_3S.$$

Proof. Note that

$$\begin{aligned} H_1H_2H_3S &= H_1SS^*H_2H_3S + H_1\Delta H_2H_3S \\ &= S^*H_1H_2SS^*H_3 + H_1\Delta H_2H_3S \\ &= S^*H_1H_2(I-\Delta)H_3 + H_1\Delta H_2H_3S \\ &= S^*H_1H_2H_3 - S^*H_1H_2\Delta H_3 + H_1\Delta H_2H_3S. \end{aligned}$$

The lemma follows immediately from the above identity. ∎

THEOREM 2.5. *Let H_1, H_2, H_3 and H_4 be four Hankel operators on K. Then $H_1H_2 = H_3H_4$ if and only if*

$$H_1\Delta H_2 - H_3\Delta H_4 = 0$$
$$(H_1H_2 - H_3H_4)\Delta = 0$$
$$(H_1H_2 - H_3H_4)^*\Delta = 0.$$

Proof. Note that

$$H_1H_2 = H_1SS^*H_2 + H_1\Delta H_2 = S^*(H_1H_2)S + H_1\Delta H_2$$
$$H_3H_4 = H_3SS^*H_4 + H_3\Delta H_4 = S^*(H_3H_4)S + H_3\Delta H_4.$$

Let $D = H_1H_2 - H_3H_4$. Subtracting the above two equations gives

$$D = S^*DS + H_1\Delta H_2 - H_3\Delta H_4.$$

The rest of the proof is similar to that of Theorem 2.3. ∎

PROPOSITION 2.6. *Let H be a Hankel operator and T be a Toeplitz operator on K. Then HT is a Hankel operator if and only if*

$$H\Delta TS = 0.$$

Proof. Note that

$$HTS = H(SS^* + \Delta)TS = HSS^*TS + H\Delta TS = S^*HT + H\Delta TS.$$

The result follows immediately from the above identity. ∎

3. COMMUTING TOEPLITZ AND HANKEL OPERATORS

Let L^2 be the space of Lebesgue square integrable functions on the unit circle and L^∞ be the space of essentially bounded functions on the unit circle. The Hardy space $H^2(D)$ is the closed linear span of analytic polynomials in L^2. In this section we apply Theorem 2.2 to Toeplitz and Hankel operators on the vector valued Hardy space $H^2_n(D)$ which is the direct sum of n copies of (scalar) Hardy space $H^2(D)$. Let $L^\infty_{n\times n}$ be the space of all $n \times n$ matrices with entries in L^∞. Let P be the projection from $L^2_n(D)$ onto $H^2_n(D)$. The Toeplitz operator T_Φ with symbol $\Phi \in L^\infty_{n\times n}$ is defined by

$$T_\Phi h = P(\Phi h), \quad h \in H^2_n(D)$$

and the Hankel operator H_Ψ with symbol $\Psi \in L^\infty_{n\times n}$ is defined by

$$H_\Psi h = P[\overline{z}\Psi(\overline{z})h(\overline{z})], \quad h \in H^2_n(D).$$

It is clear that H_Ψ depends only on $(I-P)\Psi$. So unless otherwise stated, if Ψ is the symbol of the Hankel operator H_Ψ, we will assume Ψ is such that $(I-P)\Psi = \Psi$. Let S be the unilateral shift on $H^2_n(D)$, or $S = T_{zI}$ where I stands for the $n \times n$ identity matrix. For $F(z) \in L^\infty_{n\times n}$, let F^* be the adjoint of F and $\widetilde{F} = F^*(\overline{z})$. It is well known (and easy to see that) $T_\Phi T_G = T_{\Phi G}$ and $H_\Psi T_G = H_{\Psi G}$ if $G \in H^\infty_{n\times n}$; $T^*_\Phi = T_{\Phi^*}$ and $H^*_\Psi = H_{\widetilde{\Psi}}$.

We will make use of the following definition.

DEFINITION 3.1. Let Φ be an $n \times n$ matrix with entries in L^2. The matrix-valued function Φ is said to be *regular* if the n column vectors of Φ are linearly independent in $L^2_n(D)$.

THEOREM 3.2. *Assume both $P[\overline{z}\Phi(z)]$ and $P[\overline{z}\Phi^*(z)]$ are regular. Then $T_\Phi H_\Psi = H_\Psi T_\Phi$ if and only if $\Psi = (I-P)[\Phi(\overline{z})A]$ for some constant $n \times n$ matrix A with Φ such that both $\Phi(\overline{z})A + A\Phi(z)$ and $\Phi(\overline{z})A\Phi(z)$ are analytic.*

Proof. Let $e_0, \dots, e_{n-1}$ be the standard basis of the n dimensional Euclidean space C^n. Note

$$\Delta = I - SS^* = I - T_{zI}T^*_{zI} = e_0 \otimes e_0 + e_1 \otimes e_1 + \cdots + e_{n-1} \otimes e_{n-1}.$$

We first prove the necessity. We will assume $n = 2$ since the proof for the general case is analogous. By Theorem 2.2, $T_\Phi H_\Psi = H_\Psi T_\Phi$ implies that

$$H_\Psi(e_0 \otimes e_0 + e_1 \otimes e_1)T_\Phi S + S^*T_\Phi(e_0 \otimes e_0 + e_1 \otimes e_1)H_\Psi = 0.$$

Equivalently

$$H_\Psi e_0 \otimes S^*T^*_\Phi e_0 + H_\Psi e_1 \otimes S^*T^*_\Phi e_1 = -S^*T_\Phi e_0 \otimes H^*_\Psi e_0 - S^*T_\Phi e_1 \otimes H^*_\Psi e_1. \tag{3.1}$$

By assumption, $P[\overline{z}\Phi^*(z)]$ are regular, thus $S^*T^*_\Phi e_0$ and $S^*T^*_\Phi e_1$ are linearly independent. By equating the ranges of the operators on two sides of the above equation, we see that there exists a constant 2×2 matrix $A = (a_{ij})$ such that

$$[\, H_\Psi e_0 \quad H_\Psi e_1 \,] = [\, S^*T_\Phi e_0 \quad S^*T_\Phi e_1 \,] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{3.2}$$

Plugging the above equation into (3.1), we have

$$\begin{aligned} &(a_{11}S^*T_\Phi e_0 + a_{21}S^*T_\Phi e_1) \otimes S^{*2}T^*_\Phi e_0 + (a_{12}S^*T_\Phi e_0 + a_{22}S^*T_\Phi e_1) \otimes S^{*2}T^*_\Phi e_1 \\ &= S^*T_\Phi e_0 \otimes (\overline{a_{11}}S^{*2}T^*_\Phi e_0 + \overline{a_{12}}S^{*2}T^*_\Phi e_1) + S^*T_\Phi e_1 \otimes (\overline{a_{21}}S^{*2}T^*_\Phi e_0 + \overline{a_{22}}S^{*2}T^*_\Phi e_1) \\ &= -S^*T_\Phi e_0 \otimes S^*_z H^*_\Psi e_0 - S^*T_\Phi e_1 \otimes S^*_z H^*_\Psi e_1. \end{aligned}$$

By assumption, $S^*T_\Phi e_0$ and $S^*T_\Phi e_1$ are also linearly independent, thus

$$\begin{aligned} [\, H^*_\Psi e_0 \quad H^*_\Psi e_1 \,] &= -[\, S^*T^*_\Phi e_0 \quad S^*T^*_\Phi e_1 \,] \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{bmatrix} \\ &= -[\, S^*T^*_\Phi e_0 \quad S^*T^*_\Phi e_1 \,]\, A^*. \end{aligned} \tag{3.3}$$

Now equation (3.2) reads as

$$\begin{aligned} [\, H_\Psi e_0 \quad H_\Psi e_1 \,] &= \left[\, P\left[\overline{z}\Psi(\overline{z})\begin{bmatrix}1\\0\end{bmatrix}\right] \quad P\left[\overline{z}\Psi(\overline{z})\begin{bmatrix}0\\1\end{bmatrix}\right] \,\right] = P[\overline{z}\Psi(\overline{z})] \\ &= [\, S^*T_\Phi e_0 \quad S^*T_\Phi e_1 \,] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = P[\overline{z}\Phi(z)A], \end{aligned}$$

equivalently,

$$\Psi = (I-P)[\Phi(\overline{z})A] \quad \text{or} \quad H_\Psi = H_{\Phi(\overline{z})A}.$$

Next by equation (3.3) and $H^*_\Psi = H^*_{\Phi(\overline{z})A} = H_{A^*\Phi(z)^*}$, we have

$$\begin{aligned} [\, H^*_\Psi e_0 \quad H^*_\Psi e_1 \,] &= [\, H_{A^*\Phi(z)^*}e_0 \quad H_{A^*\Phi(z)^*}e_1 \,] = P[\overline{z}A^*\Phi^*(\overline{z})] \\ &= -[\, S^*T^*_\Phi e_0 \quad S^*T^*_\Phi e_1 \,]\, A^* = -P[\overline{z}\Phi^*(z)A^*]. \end{aligned}$$

Thus

$$P[\overline{z}A^*\Phi^*(\overline{z})] = -P[\overline{z}\Phi^*(z)A^*].$$

That is, $P[\overline{z}(A^*\Phi^*(\overline{z}) + \overline{z}\Phi^*(z)A^*)] = 0$. This implies that $A^*\Phi^*(\overline{z}) + \Phi^*(z)A^*$ is conjugate analytic. Equivalently, $\Phi(\overline{z})A + A\Phi(z)$ is analytic.

Now by equation (2.2) and $H_\Psi = H_{\Phi(\overline{z})A}$,

$$\begin{aligned}
0 &= [\,(T_\Phi H_\Psi - H_\Psi T_\Phi)e_0 \quad (T_\Phi H_\Psi - H_\Psi T_\Phi)e_1\,] \\
&= [\,(T_\Phi H_{\Phi(\overline{z})A} - H_{\Phi(\overline{z})A} T_\Phi)e_0 \quad (T_\Phi H_{\Phi(\overline{z})A} - H_{\Phi(\overline{z})A} T_\Phi)e_1\,] \\
&= P[\Phi P[\overline{z}\Phi(z)A]] - P[\overline{z}\Phi(z)A[P\Phi](\overline{z})] \\
&= P[\overline{z}\Phi z P[\overline{z}\Phi(z)A]] - P[\overline{z\alpha}\Phi(z)A[P\Phi](\overline{z})] \\
&= P[\overline{z}\Phi(z)[zP[\overline{z}\Phi(z)A] - A[P\Phi](\overline{z})]] \\
&= -P[\overline{z}\Phi(z)A\Phi(\overline{z})].
\end{aligned}$$

In the last equality we note that

$$zP[\overline{z}\Phi(z)A] - A[P\Phi](\overline{z}) = -P[A\Phi(\overline{z})] - (I-P)[A\Phi(\overline{z})] = -A\Phi(\overline{z}),$$

since

$$A[P\Phi](\overline{z}) = (I-P)[A\Phi(\overline{z})] + A\Phi(0)$$

and $\Phi(z)A + A\Phi(\overline{z})$ being conjugate analytic implies that

$$zP[\overline{z}\Phi(z)A] = -zP[\overline{z}A\Phi(\overline{z})] = -P[A\Phi(\overline{z})] + A\Phi(0).$$

Thus $\Phi(z)A\Phi(\overline{z})$ is conjugate analytic. Replacing z by $\overline{z}$, we see that $\Phi(\overline{z})A\Phi(z)$ is analytic.

The sufficiency is clear from the above arguments and Theorem 2.2. This completes the proof. ∎

REMARK 3.3. If Ψ is regular and $T_\Phi H_\Psi = H_\Psi T_\Phi$, then, by equation (3.1), $P[\overline{z}\Phi(z)] = 0$ implies that $P[\overline{z}\Phi^*(z)] = 0$. Without the assumption that both $P[\overline{z}\Phi(z)]$ and $P[\overline{z}\Phi^*(z)]$ are regular, the characterization for $T_\Phi H_\Psi = H_\Psi T_\Phi$ is more complicated.

In the scalar case the above theorem reduces to the following result of R.A. Martínez-Avendaño ([7]).

COROLLARY 3.4. ([7]) *Assume φ and ψ are scalar functions and φ is not a constant. $T_\varphi H_\psi = H_\varphi T_\psi$, if and only if $\psi(z) = \alpha(I-P)[\varphi(\overline{z})]$ for some constant α and both $\varphi(\overline{z}) + \varphi(z)$ and $\varphi(\overline{z})\varphi(z)$ are analytic.*

A description of functions φ and ψ with properties as in the above corollary was given by Lemma 9 in [7]. We are not successful in giving an explicit description of matrix-valued functions with those properties as in Theorem 3.2. We now state the following characterization for commuting Toeplitz and Hankel operators which is Theorem 10 in [7]. Let m be the Lebesgue measure on the unit circle ∂D. For subsets E, E_1 and E_2 of ∂D, let

$$E^* = \{\overline{z} : z \in E\}, \quad E^{\mathrm{c}} = \partial D \setminus E \quad \text{and} \quad E_1 \triangle E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1).$$

Let κ_E denote the characteristic function of the set E.

THEOREM 3.5. ([7]) *Let $\varphi \in L^\infty$. The nonzero Hankel operator H commutes with the Toeplitz operators T_φ (not a multiple of the identity operator) if and only if $H = H_{\alpha\varphi(\overline{z})}$ where α is a constant and $\varphi(z) = a\kappa_E + b$, with constants a and b, and $E \subset \partial D$ such that $m(E^* \triangle E^c) = 0$.*

It is well-known that for scalar functions φ and ψ, $H_\psi T_{\varphi(z)} = T_{\varphi(\overline{z})} H_\psi$ if and only if φ is analytic, see a proof of this in [7]. Indeed this fact plays an important role in the proof of Corollary 3.4 above by Martínez-Avendaño's approach. There seems no such an analogous result for block Toeplitz and Hankel operators because of the non-commutativity of matrices. If φ is analytic, then $H_\psi T_{\varphi(z)} = H_{\psi\varphi(z)}$. A related question is to ask when the product of a Hankel and Toeplitz operator is also a Hankel operator. The following proposition answers this question.

PROPOSITION 3.6. *If Φ is regular, then $H_\Phi T_\Psi$ is a Hankel operator if and only if Ψ is analytic.*

Proof. If Ψ is analytic, it is clear that $H_\Phi T_\Psi = H_{\Phi\Psi}$. So we need to prove the necessity. We again prove the result by assuming $n = 2$. By Proposition 2.6, $H_\Phi T_\Psi$ is a Hankel operator implies that

$$H_\Phi(e_0 \otimes e_0 + e_1 \otimes e_1)T_\Psi S = 0.$$

Equivalently

$$H_\Phi e_0 \otimes S^* T_\Psi^* e_0 e_0 + H_\Phi e_1 \otimes S^* T_\Psi^* e_1 = 0.$$

By the assumption, Φ is regular, thus $H_\Phi e_0$ and $H_\Phi e_1$ are linearly independent. Therefore

$$0 = [\, S^* T_\Psi^* e_0 \quad S^* T_\Psi^* e_1 \,] = P[\overline{z}\Psi^*(z)].$$

Therefore Ψ is analytic. This completes the proof. ∎

4. PRODUCT OF TWO HANKEL OPERATORS

If A is a constant $n \times n$ matrix whose eigenvalues are inside the unit disk, then $(I - zA)^{-1}$ is analytic inside the disk and has the following power series expansion

$$(I - zA)^{-1} = \sum_{i=0}^{\infty} A^i z^i. \tag{4.1}$$

To see this, let A_0 be the Jordan form of A and C is an invertible matrix such that

$$CAC^{-1} = A_0 = D + E,$$

where D is a diagonal matrix and E is a matrix such that $E^n = 0$. Then for $i > n$

$$A^i = C^{-1} A_0^i C = C^{-1}(D+E)^i C = C^{-1}\left[\sum_{j=0}^{n-1} \left[\binom{i}{j}\right] D^{i-j} E^j\right] C.$$

Since D is a strict contraction, we see that the series in (4.1) is indeed convergent in $|z| \leqslant 1$. In the scalar case, for $|\alpha| < 1$, $(1 - \alpha z)^{-1}$ is an eigenvector of S^*. In the vector-valued case, $(I - zA)^{-1}$ acts like an eigenvector of S^* since

$$S^*(I - zA)^{-1} = P[\overline{z}(I - zA)^{-1}] = P\left[\overline{z}\left(\sum_{i=0}^{\infty} A^i z^i\right)\right] = \sum_{i=0}^{\infty} A^i z^{i-1} = A(I - zA)^{-1}.$$

By the above discussion, it is easy to verify the following lemma.

Lemma 4.1. *Let A be a constant $n\times n$ matrices whose eigenvalues are inside the unit disk. Let $F(z)=\sum_{i=0}^{\infty}F_iz^i\in H^{\infty}_{n\times n}$. Then*

$$P[F(\overline{z})(I-zA)^{-1}]=F(A)(I-zA)^{-1}$$

where $F(A)=\sum_{i=0}^{\infty}F_iA^i$.

Now we prove the main result of this section.

Theorem 4.2. *Assume Φ and Ψ are regular. $H_\Phi H_\Psi$ is a Hankel operator if and only if there exist constant $n\times n$ matrices A, L and R such that the eigenvalues of A are inside the unit disk and*

$$\Phi(z)=\overline{z}L(I-\overline{z}A)^{-1},\quad \Psi(z)=\overline{z}(I-\overline{z}A)^{-1}R. \tag{4.2}$$

If this is the case, then $H_\Phi H_\Psi=H_\Theta$ where

$$\Theta(z)=\overline{z}L(I-\overline{z}A)^{-1}(I-A^2)^{-1}R. \tag{4.3}$$

Proof. We first prove the necessity. Without loss of generality, we assume $n=2$. Note

$$\Delta=I-SS^*=e_0\otimes e_0+e_1\otimes e_1.$$

By Theorem 2.3, $H_\Phi H_\Psi$ is a Hankel operator implies that

$$H_\Phi(e_0\otimes e_0+e_1\otimes e_1)H_\Psi S=S^*H_\Phi(e_0\otimes e_0+e_1\otimes e_1)H_\Psi.$$

Equivalently

$$H_\Phi e_0\otimes S^*H_\Psi^*e_0+H_\Phi e_1\otimes S^*H_\Psi^*e_1=S^*H_\Phi e_0\otimes H_\Psi^*e_0+S^*H_\Phi e_1\otimes H_\Psi^*e_1. \tag{4.4}$$

By equating the ranges of the operators on two sides and using the independence of $H_\Psi^*e_0$ and $H_\Psi^*e_1$, we see that there exists a constant 2×2 matrix $A=(a_{ij})$ such that

$$[\,S^*H_\Phi e_0\quad S^*H_\Phi e_1\,]=[\,H_\Phi e_0\quad H_\Phi e_1\,]\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}. \tag{4.5}$$

We now show that all the eigenvalues of A have modulus less than 1. We will do this by proving that any eigenvalue of A is also an eigenvalue of S^* (as an operator on H^2). Without loss of generality, assume

$$C^{-1}AC=\begin{bmatrix}\alpha&0\\1&\alpha\end{bmatrix}$$

for some invertible matrix C. Write

$$[\,H_\Phi e_0\quad H_\Phi e_1\,]\,C=\begin{bmatrix}b_{11}(z)&b_{12}(z)\\b_{21}(z)&b_{22}(z)\end{bmatrix}$$

and note that by assumption the column vectors of the above matrix is linearly independent. Multiplying equation (4.5) by C, we have

$$\begin{bmatrix}S^*b_{11}(z)&S^*b_{12}(z)\\S^*b_{21}(z)&S^*b_{22}(z)\end{bmatrix}=\begin{bmatrix}\alpha b_{11}(z)+b_{12}(z)&\alpha b_{12}(z)\\\alpha b_{21}(z)+b_{22}(z)&\alpha b_{22}(z)\end{bmatrix}.$$

Therefore α is an eigenvalue of S^* (as an operator on H^2).

Plugging equation (4.5) into (4.4), we have

$$\begin{aligned} &H_\Phi e_0 \otimes S^* H_\Psi^* e_0 + H_\Phi e_1 \otimes S^* H_\Psi^* e_1 \\ &\qquad = S^* H_\Phi e_0 \otimes H_\Psi^* e_0 + S^* H_\Phi e_1 \otimes H_\Psi^* e_1 \\ &\qquad = (a_{11} H_\Phi e_0 + a_{21} H_\Phi e_1) \otimes H_\Psi^* e_0 + (a_{12} H_\Phi e_0 + a_{22} H_\Phi e_1) \otimes H_\Psi^* e_1 \\ &\qquad = H_\Phi e_0 \otimes (\overline{a_{11}} H_\Psi^* e_0 + \overline{a_{12}} H_\Psi^* e_1) + H_\Phi e_1 \otimes (\overline{a_{21}} H_\Psi^* e_0 + \overline{a_{22}} H_\Psi^* e_1). \end{aligned}$$

Therefore, by the assumption that $H_\Phi e_0$ and $H_\Phi e_1$ are linearly independent,

$$[\, S^* H_\Psi^* e_0 \quad S^* H_\Psi^* e_1 \,] = [\, H_\Psi^* e_0 \quad H_\Psi^* e_1 \,]\, A^*. \tag{4.6}$$

It follows from equation (4.5) that

$$\begin{aligned} [\, S^* H_\Phi e_0 \quad S^* H_\Phi e_1 \,] &= [\, H_{z\Phi} e_0 \quad H_{z\Phi} S e_1 \,] = P[\overline{z}^2 \Phi(\overline{z})] \\ &= [\, H_\Phi e_0 \quad H_\Phi e_1 \,]\, A = P[\overline{z} \Phi(\overline{z}) A]. \end{aligned}$$

That is

$$P[\overline{z} \Phi(\overline{z})(\overline{z} I - A)] = 0. \tag{4.7}$$

We claim there exists a constant $n \times n$ matrix L such that

$$\Phi(z) = \overline{z} L (I - \overline{z} A)^{-1}. \tag{4.8}$$

Let $F(z) = \sum_{i=0}^{\infty} F_i z^i := \overline{z} \Phi(\overline{z})$. $F(z)$ is analytic since we are assuming $\Phi(z) = (I-P)[\Phi(z)]$. Now equation (4.7) becomes

$$P[\overline{z} F(z)] = \sum_{i=1}^{\infty} F_i z^{i-1} = \sum_{i=0}^{\infty} F_{i+1} z^i = P[F(z) A] = \sum_{i=0}^{\infty} F_i A z^i.$$

Therefore $F_{i+1} = F_i A$ for $i \geqslant 0$. So we have

$$F(z) := \overline{z} \Phi(\overline{z}) = \sum_{i=0}^{\infty} F_0 A^i z^i = F_0 (I - zA)^{-1}.$$

This proves our claim (4.8) with $L = F_0$.

Similarly, equation (4.6) implies that

$$P[\overline{z}^2 \Psi^*(z)] = P[\overline{z} \Psi^*(z) A^*].$$

Therefore, there exists a constant matrix R such that

$$\Psi(z) = \overline{z} (I - \overline{z} A)^{-1} R.$$

Next we prove the sufficiency. Assume Φ and Ψ can be represented as in (4.2). The above arguments show that equation (2.3) holds. To verify equation (2.4), we note that

$$[\, H_\Psi S e_0 \quad H_\Psi S e_1 \,] = P[\overline{z} \Psi(\overline{z}) \overline{z} I] = P[\overline{z} (I - zA)^{-1} R] = (I - zA)^{-1} A R$$

and

$$\begin{aligned} [\, H_\Phi H_\Psi S e_0 \quad H_\Phi H_\Psi S e_1 \,] &= P[\overline{z} \Phi(\overline{z}) A (I - \overline{z} A)^{-1} R] \\ &= P[L (I - zA)^{-1} (I - \overline{z} A)^{-1} A R] \\ &= L (I - zA)^{-1} (I - A^2)^{-1} A R, \end{aligned}$$

where the last equality follows from Lemma 4.1. Similarly

$$\begin{aligned}[\, S^* H_\Phi H_\Psi e_0 \quad S^* H_\Phi H_\Psi e_1 \,] &= P[\bar{z} L (I - zA)^{-1} (I - \bar{z}A)^{-1} R] \\ &= L(I - zA)^{-1}(I - A^2)^{-1} A R.\end{aligned}$$

Equation (2.5) can be established analogously. By Theorem 2.3, $H_\Phi H_\Psi$ is a Hankel operator. Set $H_\Phi H_\Psi = H_\Theta$.

$$\begin{aligned}P[\bar{z}\Theta(\bar{z})] &= [\, H_\Theta e_0 \quad H_\Theta e_1 \,] = [\, H_\Phi H_\Psi e_0 \quad H_\Phi H_\Psi e_1 \,] \\ &= L(I - zA)^{-1}(I - A^2)^{-1} R.\end{aligned}$$

Thus Θ is represented as in (4.3). This completes the proof. ∎

THEOREM 4.3. *Assume Φ_1 and Φ_2 are regular. $H_{\Phi_1} H_{\Psi_1} = H_{\Psi_2} H_{\Phi_2}$ if and only if $\Psi_2 = \Phi_1 A$ and $\Psi_1 = A \Phi_2$ for some constant matrix A.*

Proof. We will prove the theorem for $n = 2$. By Theorem 2.5, $H_{\Phi_1} H_{\Psi_1} = H_{\Psi_2} H_{\Phi_2}$ implies that

$$H_{\Phi_1}(e_0 \otimes e_0 + e_1 \otimes e_1) H_{\Psi_1} = H_{\Psi_2}(e_0 \otimes e_0 + e_1 \otimes e_1) H_{\Phi_2}.$$

Equivalently

$$H_{\Phi_1} e_0 \otimes H^*_{\Psi_1} e_0 + H_{\Phi_1} e_1 \otimes H^*_{\Psi_1} e_1 = H_{\Psi_2} e_0 \otimes H^*_{\Phi_2} e_0 + H_{\Psi_2} e_1 \otimes H^*_{\Phi_2} e_1. \tag{4.9}$$

Since by assumption $H^*_{\Phi_2} e_0$ and $H^*_{\Phi_2} e_1$ are linearly independent, there exists a constant matrix A such that

$$[\, H_{\Psi_2} e_0 \quad H_{\Psi_2} e_1 \,] = P[\bar{z}\Psi_2(\bar{z})] = \bar{z}\Psi_2(\bar{z}) = [\, H_{\Phi_1} e_0 \quad H_{\Phi_1} e_1 \,] A = \bar{z}\Phi_1(\bar{z}) A.$$

Therefore $\Psi_2 = \Phi_1 A$. Plugging this relation into equation (4.9) and using the linear independence of $H_{\Phi_1} e_0$ and $H_{\Phi_1} e_1$, we conclude that

$$\left[\, H^*_{\Psi_1} e_0 \quad H^*_{\Psi_1} e_1 \,\right] = P[\bar{z}\Psi_1^*(z)] = \bar{z}\Psi_1^*(z) = \left[\, H^*_{\Phi_2} e_0 \quad H^*_{\Phi_2} e_1 \,\right] A^* = \bar{z}\Phi_2^*(z) A,$$

thus $\Psi_1(z) = A\Phi_2(z)$.

On the other hand, if $\Psi_2 = \Phi_1 A$ and $\Psi_1 = A\Phi_2$, then

$$H_{\Phi_1} H_{\Psi_1} = H_{\Phi_1} H_{A\Phi_2} = H_{\Phi_1} T_A H_{\Phi_2} = H_{\Phi_1 A} H_{\Phi_2} = H_{\Psi_2} H_{\Phi_2}.$$

This completes the proof. ∎

COROLLARY 4.4. *Assume Φ is regular.*

(i) *$H_\Phi H_\Psi = H_\Psi H_\Phi$ if and only if $\Psi = \Phi A = A \Phi$ for some constant matrix A.*

(ii) *$H_\Phi H_\Psi = 0$ if and only if $\Psi = 0$.*

COROLLARY 4.5. ([2]) *Let φ and ψ be two scalar functions. $H_\varphi H_\psi = 0$ if and only if either $H_\varphi = 0$ or $H_\psi = 0$.*

5. PRODUCT OF THREE HANKEL OPERATORS

In this section we discuss when the product of three Hankel operators is also a Hankel operator. This question can be answer readily by using Proposition 2.4.

THEOREM 5.1. *Assume Φ_1 and Φ_3 is regular. $H^*_{\Phi_1}H_{\Phi_2}H^*_{\Phi_3}$ is a Hankel operator if and only if there exist a constant matrix A such that*

$$[\Phi_2^*(z) - A^*]\Phi_1(z), \quad \Phi_3(z)[\Phi_2^*(z) - A^*] \in H^2_{n\times n}. \tag{5.1}$$

In this case, if we write

$$F_1(z) := [\Phi_2^*(z) - A^*]\Phi_1(z) \tag{5.2}$$

$$F_2(z) := \Phi_3(z)[\Phi_2^*(z) - A^*], \tag{5.3}$$

then

$$H^*_{\Phi_1}H_{\Phi_2}H^*_{\Phi_3} = H_{\Phi_1^*(\overline{z})F_2^*(\overline{z})} = H_{F_1^*(\overline{z})\Phi_3^*(\overline{z})}. \tag{5.4}$$

Proof. Again we assume $n = 2$. By Theorem 2.2, $H^*_{\Phi_1}H_{\Phi_2}H^*_{\Phi_3}$ is a Hankel operator implies that

$$H^*_{\Phi_1}(e_0\otimes e_0 + e_1\otimes e_1)H_{\Phi_2}H^*_{\Phi_3}S = S^*H^*_{\Phi_1}H_{\Phi_2}(e_0\otimes e_0 + e_1\otimes e_1)H^*_{\Phi_3}.$$

By the assumption of the linear independence of $H^*_{\Phi_1}e_0$ and $H^*_{\Phi_1}e_1$, and $H_{\Phi_3}e_0$ and $H_{\Phi_3}e_1$, there exists a constant matrix A such that

$$\begin{aligned}\left[\, S^*H^*_{\Phi_1}H_{\Phi_2}e_0 \quad S^*H^*_{\Phi_1}H_{\Phi_2}e_1 \,\right] &= \left[\, H^*_{\Phi_1}e_0 \quad H^*_{\Phi_1}e_1 \,\right] A\\ \left[\, S^*H_{\Phi_3}H^*_{\Phi_2}e_0 \quad S^*H_{\Phi_3}H^*_{\Phi_2}e_1 \,\right] &= \left[\, H_{\Phi_3}e_0 \quad H_{\Phi_3}e_1 \,\right] A^*.\end{aligned}$$

Equivalently

$$\begin{aligned}P[\overline{z}\Phi_1^*(z)\Phi_2(z) - \overline{z}\Phi_1^*(z)A] &= 0\\ P[\overline{z}\Phi_3(\overline{z})\Phi_2^*(\overline{z}) - \overline{z}\Phi_3(\overline{z})A^*] &= 0.\end{aligned}$$

That is, (5.1) holds.

Assume now equations (5.2) and (5.3) hold. Multiplying (5.2) by $\Phi_3(z)$ on the left and applying (5.3), we see that

$$\Phi_3(z)F_1(z) = F_2(z)\Phi_1(z).$$

This shows that $H_{\Phi_1^*(\overline{z})F_2^*(\overline{z})} = H_{F_1^*(\overline{z})\Phi_3^*(\overline{z})}$.

To show that $H^*_{\Phi_1}H_{\Phi_2}H^*_{\Phi_3} = H_{\Phi_1^*(\overline{z})F_2^*(\overline{z})}$, we note that by (5.3)

$$\begin{aligned}\left[\, H_{\Phi_2}H^*_{\Phi_3}e_0 \quad H^*_{\Phi_2}H_{\Phi_3}e_1 \,\right] &= \left[\, H_{\Phi_2-A}H^*_{\Phi_3}e_0 \quad H_{\Phi_2-A}H^*_{\Phi_3}e_1 \,\right]\\ &= P[(\Phi_2(\overline{z}) - A)\Phi_3^*(\overline{z})] = F_2^*(\overline{z}).\end{aligned}$$

Therefore

$$\begin{aligned}&\left[\, H^*_{\Phi_1}H_{\Phi_2}H^*_{\Phi_3}e_0 \quad H^*_{\Phi_1}H_{\Phi_2}H^*_{\Phi_3}e_1 \,\right]\\ &\qquad = P[\overline{z}\Phi_1^*(z)F_2^*(z)] = \left[\, H_{\Phi_1^*(\overline{z})F_2^*(\overline{z})}e_0 \quad H_{\Phi_1^*(\overline{z})F_2^*(\overline{z})}e_1 \,\right].\end{aligned}$$

This completes the proof. ∎

If Φ_1 and Φ_3 are matrix-valued inner functions, the conditions in the above theorem can be expressed in terms of the divisibilities of matrix-valued analytic functions. We state this result for the scalar case.

Corollary 5.2. *Assume θ_1 and θ_3 are scalar inner functions. $H^*_{\overline{\theta}_1} H_{\overline{\theta}_2} H^*_{\overline{\theta}_3}$ is a Hankel operator if and only if for some constant α*

$$\theta_1|(\theta_2(z)-\alpha) \quad \textit{and} \quad \theta_3(z)|(\theta_2(z)-\alpha). \tag{5.5}$$

Recall an $n \times n$ matrix-valued analytic function Θ is inner if $\Theta^*(z)\Theta(z) = I_{n\times n}$. If Θ is inner, we also have $\Theta(z)\Theta^*(z) = I_{n\times n}$.

Proposition 5.3. *Assume Θ_1 and Θ_3 are matrix-valued inner functions and $\Theta_2 \in H^\infty_{n\times n}$. $H^*_{\Theta_1^*} H_{\Theta_2^*} H^*_{\Theta_3^*}$ is zero if and only if there exist a constant matrix A and $F(z) \in H^\infty_{n\times n}$ such that*

$$\Theta_2(z) - A = \Theta_3(z)F(z)\Theta_1(z).$$

Proof. Without loss of generality, we assume $\Theta_1(0) = \Theta_3(0) = 0$. Otherwise applying the previous theorem to $H^*_{\Theta_1^*-\Theta_1^*(0)} H_{\Theta_2^*} H^*_{\Theta_3^*-\Theta_3^*(0)}$ $(= H^*_{\Theta_1^*} H_{\Theta_2^*} H^*_{\Theta_3^*})$. By the previous theorem, $H^*_{\Theta_1^*} H_{\Theta_2^*} H^*_{\Theta_3^*}$ is zero implies that

$$F_2(z) := \Theta_3^*(z)[\Theta_2(z) - A] \in H^\infty_{n\times n}$$

for some constant matrix A. Multiplying the above equation on the left by $\Theta_3(z)$, we have

$$[\Theta_2(z) - A] = \Theta_3(z)F_2(z).$$

Now $H^*_{\Theta_1^*} H_{\Theta_2^*} H^*_{\Theta_3^*} = H_{\Theta_1(\overline{z})F_2^*(\overline{z})} = 0$ implies that $\Theta_1(\overline{z})F_2^*(\overline{z}) = F^*(\overline{z})$ for some $F(z) \in H^\infty_{n\times n}$. Thus $F_2(z) = F(z)\Theta_1(z)$. Combining this and the above equation, we prove the necessity. The sufficiency follows by a direct application of Theorem 5.1. ∎

An immediate corollary is the following result from Xia and Zheng ([8]).

Corollary 5.4. ([8]) *Assume θ_1, θ_2 and θ_3 are scalar inner functions. $H^*_{\overline{\theta}_1} H_{\overline{\theta}_2} H^*_{\overline{\theta}_3}$ is zero if and only if for some constant α*

$$\theta_1\theta_3|(\theta_2(z)-\alpha).$$

It is interesting to compare the condition for $H^*_{\overline{\theta}_1} H_{\overline{\theta}_2} H^*_{\overline{\theta}_3}$ being a Hankel operator from Corollary 5.2 to the conditions for $H^*_{\overline{\theta}_1} H_{\overline{\theta}_2} H^*_{\overline{\theta}_3}$ being zero from the above corollary.

For the product of two Hankel operators to be a Hankel operator, their symbols have to take a very special form as seen from the last section. However for the product of three Hankel operators to be a Hankel, their symbols can be quite general as shown in Theorem 5.1. A similar phenomenon occurs in the question of when the product of several Hankel operators is zero. If the product of two Hankel operators is zero, then Brown and Halmos (Corollary 4.5 above) showed that one of them has to be zero. But there are three nonzero Hankel operators whose product is zero as seen from Proposition 5.3 above. However the following result was proved by Xia and Zheng ([8]).

THEOREM 5.5. ([8]) *Let φ_1, φ_2 and φ_3 be three functions. If $H^*_{\varphi_1} H_{\varphi_2} H^*_{\varphi_3}$, $H^*_{\varphi_1} H_{\varphi_2} H^*_{\varphi_3}$ and $H^*_{\varphi_1} H_{\varphi_2} H^*_{\varphi_3}$ are all zero, then one of $H_{\varphi_1}, H_{\varphi_2}$ and H_{φ_3} is zero.*

Yet another similar situation is when the product of several Hankel operators is of finite rank. Axler, Chang and Sarason ([1]) showed that if the product of two Hankel operators is of finite rank, then one of them has to be of finite rank. Furthermore, in this case, the rank of the product of the two Hankel operator is equal to the minimum of the ranks of the two Hankel operators. For an operator A, let $\operatorname{rank}(A)$ denote the rank of A. The following analogue was obtained by the author ([4]).

THEOREM 5.6. ([4]) *Let φ_1, φ_2 and φ_3 be three scalar functions. Then*

$$\begin{aligned}&\max\{\operatorname{rank}(H^*_{\varphi_1} H_{\varphi_2} H^*_{\varphi_3}), \operatorname{rank}(H^*_{\varphi_2} H_{\varphi_1} H^*_{\varphi_3}), \operatorname{rank}(H^*_{\varphi_1} H_{\varphi_3} H^*_{\varphi_2})\}\\ &\qquad = \min\{\operatorname{rank}(H_{\varphi_1}), \operatorname{rank}(H_{\varphi_2}), \operatorname{rank}(H_{\varphi_3})\}.\end{aligned}$$

In other words if $H^*_{\varphi_1} H_{\varphi_2} H^*_{\varphi_3}, H^*_{\varphi_2} H_{\varphi_1} H^*_{\varphi_3}$ and $H^*_{\varphi_1} H_{\varphi_3} H^*_{\varphi_2}$ are all of finite rank, then one of $H_{\varphi_1}, H_{\varphi_2}$ and H_{φ_3} is of finite rank. In fact a version of the above result for products of an arbitrary finite number of Hankel operators was proved by the author ([6]) by using a result on the kernel of the product of several Hankel operators. Inspired by these recent results we prove the following theorem.

THEOREM 5.7. *Let $\varphi_1, \varphi_2, \varphi_3 \in \overline{zH^\infty}$. Then $H^*_{\varphi_1} H_{\varphi_2} H^*_{\varphi_3}, H^*_{\varphi_2} H_{\varphi_1} H^*_{\varphi_3}$ and $H^*_{\varphi_1} H_{\varphi_3} H^*_{\varphi_2}$ are all Hankel operators if and only if*

$$\varphi_1 = \alpha[\overline{\theta} - \theta(0)], \quad \varphi_2 = \beta[\overline{\theta} - \theta(0)], \quad \varphi_3 = \gamma[\overline{\theta} - \theta(0)] \tag{5.6}$$

for some constants α, β, γ and inner function θ. In this case

$$H^*_{\varphi_1} H_{\varphi_2} H^*_{\varphi_3} = H^*_{\varphi_1} H_{\varphi_2} H^*_{\varphi_3} = H^*_{\varphi_1} H_{\varphi_2} H^*_{\varphi_3} = \overline{\alpha}\beta\overline{\gamma} H^*_{\overline{\theta}}.$$

Proof. If φ_1, φ_2 and φ_3 are given by (5.6), then

$$H^*_{\varphi_1} H_{\varphi_2} H^*_{\varphi_3} = \overline{\alpha}\beta\overline{\gamma} H^*_{\overline{\theta}} H_{\overline{\theta}} H^*_{\overline{\theta}} = \overline{\alpha}\beta\overline{\gamma} H^*_{\overline{\theta}},$$

since $H^*_{\overline{\theta}} H_{\overline{\theta}}$ is the projection onto the range of $H^*_{\overline{\theta}}$. This proves the sufficiency.

Now we prove the necessity. By Proposition 2.4, $H^*_{\varphi_1} H_{\varphi_2} H^*_{\varphi_3}$ is a Hankel operator implies that

$$H^*_{\varphi_1}(e_0 \otimes e_0) H_{\varphi_2} H^*_{\varphi_3} S = S^* H^*_{\varphi_1} H_{\varphi_2}(e_0 \otimes e_0) H^*_{\varphi_3}.$$

Therefore there exists a constant a such that

$$\begin{aligned}P[\overline{z}\varphi_1^*(z)\varphi_2(z)] &= S^* H^*_{\varphi_1} H_{\varphi_2} e_0 = a H^*_{\varphi_1} e_0 = a P[\overline{z}\varphi_1^*(z)]\\ P[\overline{z}\varphi_3(\overline{z})\varphi_2^*(\overline{z})] &= S^* H_{\varphi_3} H^*_{\varphi_2} e_0 = \overline{a} H_{\varphi_3} e_0 = \overline{a} P[\overline{z}\varphi_3(\overline{z})].\end{aligned}$$

Equivalently there exists $y_1, y_2 \in H^2$ such that

$$\overline{\varphi_1(z)}\varphi_2(z) - a\overline{\varphi_1(z)} = \overline{y_1} \tag{5.7}$$

$$\varphi_3(\overline{z})\overline{\varphi_2(\overline{z})} - \overline{a}\varphi_3(\overline{z}) = \overline{y_2}. \tag{5.8}$$

Similarly, $H^*_{\varphi_2}H_{\varphi_1}H^*_{\varphi_3}$ and $H^*_{\varphi_1}H_{\varphi_3}H^*_{\varphi_2}$ are Hankel operators implies that there exist constants b and c, and $y_3, y_4, y_5, y_6 \in H^2$ such that

$$\overline{\varphi_2(z)}\varphi_1(z) - b\overline{\varphi_2(z)} = \overline{y_3} \tag{5.9}$$

$$\varphi_3(\overline{z})\overline{\varphi_1(\overline{z})} - \overline{b}\varphi_3(\overline{z}) = \overline{y_4} \tag{5.10}$$

$$\overline{\varphi_1(z)}\varphi_3(z) - c\overline{\varphi_1(z)} = \overline{y_5} \tag{5.11}$$

$$\varphi_2(\overline{z})\overline{\varphi_3(\overline{z})} - \overline{c}\varphi_2(\overline{z}) = \overline{y_6}. \tag{5.12}$$

Write equations (5.7) and (5.9) as

$$(\overline{\varphi_1(z)} - \overline{b})(\varphi_2(z) - a) = \overline{y_1} - \overline{b}(\varphi_2(z) - a) := \overline{x_1}$$
$$(\overline{\varphi_2(z)} - \overline{a})(\varphi_1(z) - b) = \overline{y_2} - \overline{a}(\varphi_1(z) - b) := \overline{x_2}$$

where $x_1, x_2 \in H^2$. Note that

$$x_1 = \overline{[(\overline{\varphi_1(z)} - \overline{b})(\varphi_2(z) - a)]} = (\overline{\varphi_2(z)} - \overline{a})(\varphi_1(z) - b) = \overline{x_2}.$$

Therefore x_1 and x_2 are constant functions. That is

$$(\overline{\varphi_1(z)} - \overline{b})(\varphi_2(z) - a) = d_1 \tag{5.13}$$

for some constant d_1. Similarly, equations (5.8) and (5.12), and equations (5.10) and (5.11) imply that

$$(\varphi_3(z) - c)(\overline{\varphi_2(z)} - \overline{a}) = d_2 \tag{5.14}$$

$$(\varphi_3(z) - c)(\overline{\varphi_1(z)} - \overline{b}) = d_3 \tag{5.15}$$

for some constants d_2 and d_3. It follows from equations (5.13), (5.14) and (5.15) that

$$(\varphi_3(z) - c) = \frac{d_2(\varphi_1(z) - b)}{\overline{d_1}}$$
$$(\varphi_2(z) - a) = \frac{d_1 d_2(\varphi_1(z) - b)}{\overline{d_1} d_3}.$$

Therefore by (5.13),

$$(\overline{\varphi_1(z)} - \overline{b})(\varphi_1(z) - b) = d$$

for some constant d. Since $\overline{\varphi_1(z)} \in zH^2$, $\overline{\varphi_1(z)} - \overline{b}$ is a constant multiple of an inner function θ. This completes the proof. ∎

We remark that in the above theorem the natural product is $H^*_{\varphi_1}H_{\varphi_2}H^*_{\varphi_3}$ instead of $H_{\varphi_1}H_{\varphi_2}H_{\varphi_3}$. Since $H^*_{\varphi_3}H_{\varphi_2}H^*_{\varphi_1}$, $H^*_{\varphi_1}H_{\varphi_2}H^*_{\varphi_3}$ and $H^*_{\varphi_1}H_{\varphi_2}H^*_{\varphi_3}$ are the adjoints of $H^*_{\varphi_1}H_{\varphi_2}H^*_{\varphi_3}$, $H^*_{\varphi_1}H_{\varphi_2}H^*_{\varphi_3}$ and $H^*_{\varphi_1}H_{\varphi_2}H^*_{\varphi_3}$. The condition that $H^*_{\varphi_1}H_{\varphi_2}H^*_{\varphi_3}$, $H^*_{\varphi_2}H_{\varphi_1}H^*_{\varphi_3}$ and $H^*_{\varphi_1}H_{\varphi_3}H^*_{\varphi_2}$ are all Hankel operators is equivalent to the condition that all possible six products of three Hankel operators are Hankel operators. It is curious to note that unlike the questions of when products of several Hankel operators are zero or of finite rank, the question of when the products of Hankel operators are Hankel operators admits different answers depending if the number of Hankel operators involved is even or odd. We make the following conjecture.

CONJECTURE 5.8. *If for any permutation σ of $\{1, 2, \ldots, 2n-1, 2n\}$, $H^*_{\varphi_{\sigma(1)}} H_{\varphi_{\sigma(2)}} \cdots H^*_{\varphi_{\sigma(2n-1)}} H_{\varphi_{\sigma(2n)}}$ is a Hankel operator, then*

$$\varphi_i = \alpha_i[\overline{\theta} - \theta(0)], \quad \theta = \frac{z - \alpha}{(1 - \overline{\alpha} z)}$$

for some constants α_i and α such that $|\alpha| < 1$. If for any permutation σ of $\{1, 2, \ldots, 2n, 2n+1\}$, $H^*_{\varphi_{\sigma(1)}} H_{\varphi_{\sigma(2)}} \cdots H_{\varphi_{\sigma(2n)}} H^*_{\varphi_{\sigma(2n+1)}}$ is a Hankel operator, then

$$\varphi_i = \alpha_i[\overline{\theta} - \theta(0)]$$

for some inner function θ and constants α_i.

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