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TAKESAKI DUALITY FOR WEIGHTS ON LOCALLY COMPACT QUANTUM GROUP COVARIANT SYSTEMS

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Abstract. Takesaki duality for weights on a covariant system based on an action of a locally compact quantum group (in the sense of Kustermans and Vaes) on a von Neumann algebra is shown. In the course of proving the duality, we also introduce a notion of a Radon-Nikodym derivative with respect to an action and a weight.

Keywords: *Locally compact quantum group, action, von Neumann algebra, Takesaki duality.*

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INTRODUCTION

In [14], Takesaki showed the remarkable duality for crossed products of von Neumann algebras by actions of locally compact abelian groups. As is well-known, it played an essential part in the structure theorem of von Neumann algebras of type III. Since then, the Takesaki duality has been largely extended even to the case of actions of the so-called "quantum groups" (cf. [3], [15]). This duality theory generally involves three objects: (von Neumann) algebras, actions and weights. In the literature cited above, while the duality for both algebras and actions was completely described, the duality for weights was not treated in a satisfactory manner. As one can see from the papers [2], [4], [5], [3] and [15], the main concern on weights so far was only the construction of dual weights and identification of their modular objects. To the best of author's knowledge, the most recent result which treats a "true" duality for weights (i.e., a relationship between a weight and its bidual weight) can be found in the paper by Strătilă, Voiculescu and Zsidó (see Theorem III.3.1 in [12] and also Theorem 19.18 in [11]). According to their result, the duality for weights was phrased as follows: let α be an action of a locally compact group G on a von Neumann algebra A, and ϕ be a faithful normal semifinite weight on A. Then the bidual weight $\tilde{\tilde{\phi}}$ satisfies

$$(D\tilde{\tilde{\phi}} : D(\phi \otimes \mathrm{Tr}))_t = (1 \otimes \Delta_G^{\mathrm{i}t})U_t, \quad t \in \mathbb{R},$$

where Tr is the usual trace on $B(L^2(G))$, Δ_G is the modular function of G, and U_t is the unitary in $A \otimes L^\infty(G)$ defined by

$$g \in G \mapsto U_t(g) := (D(\phi \circ \alpha_g) : D\phi)_t \in A.$$

Even in their paper, they were engaged only in the case of a locally compact group action, and in the case of a group coaction. Neddless to say, the case of a general quantum group action was left untouched. It seems that the main difficulty in dealing with these untouched cases is that we do have elements of the form α_g around when we treat an action α of a group G, while we do not have such an object any more in the general case. So, for example, the unitary $(D(\phi\circ\alpha_g) : D\phi)_t$, which we believe plays a key role in duality for weights, no longer makes sense in the general situation. Also, analysis such as is done for the proof of Theorem III.3.1 from [12] appears to be inapplicable to the general case. Therefore, in order to obtain a general duality for weights, it seems that one has to find a new proof which can be equally applied to every situation.

The purpose of this paper is to establish a complete duality for weights on covariant systems based on actions of locally compact quantum groups. Our approach to achieve this goal is new and based heavily on Connes' theory of spatial derivatives.

The organization of the paper is as follows. Section 1 is concerned with terminology and notation used throughout this paper. We review fundamental facts on locally compact quantum groups (in the von Neumann algebraic setting) introduced by Kustermans and Vaes ([10]). Actions of locally compact quantum groups are discussed too. We also give a quick review on theory of spatial derivatives. In Section 2, we prepare some results that will be applied in the following sections. In Section 3, to every pair of an action of a locally compact quantum group $\mathbb{G}$ on a von Neumann algebra and a faithful normal semifinite weight on the algebra where $\mathbb{G}$ acts, we associate a family of unitaries, called the Radon-Nikodym derivative with respect to the action and the weight. In the final section, we prove the Takesaki duality for weights in full generality.

1. TERMINOLOGY AND NOTATION

This section is devoted to introduction of terminology and notation used in the sections that follow. We mainly discuss two things: locally compact quantum groups recently introduced by Kustermans and Vaes and Connes' spatial derivatives.

Given a von Neumann algebra A and a faithful normal semifinite weight ϕ on A, we introduce the subsets $\mathfrak{n}_\phi$, $\mathfrak{m}_\phi$ and $\mathfrak{m}_\phi^+$ of A by

$$\mathfrak{n}_\phi = \{x \in A : \phi(x^*x) < \infty\}, \quad \mathfrak{m}_\phi = \mathfrak{n}_\phi^* \mathfrak{n}_\phi, \quad \mathfrak{m}_\phi^+ = \mathfrak{m}_\phi \cap A_+.$$

We denote by π_ϕ the standard (GNS) representation associated with ϕ. Its representation space is denoted by H_ϕ. We use the symbol Λ_ϕ for the canonical embedding of $\mathfrak{n}_\phi$ into H_ϕ. Let $\mathfrak{a}_\phi := \mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*$ and set $\mathfrak{A}_\phi := \Lambda_\phi(\mathfrak{a}_\phi)$, which is the full left Hilbert algebra associated with ϕ. For a left bounded vector $\xi \in H_\phi$ with respect to the left Hilbert algebra $\mathfrak{A}_\phi$, we write $\pi_{\mathrm{l}}(\xi)$ for the left multiplication operator corresponding to ξ. For a right bounded vector η, we use $\pi_{\mathrm{r}}(\eta)$ for the corresponding right multiplication operator. The Tomita algebra associated to the left Hilbert algebra $\mathfrak{A}_\phi$ is denoted by $\mathfrak{T}_\phi$. The other relevant modular objects associated to ϕ are denoted by J_ϕ, ∇_ϕ, S_ϕ, F_ϕ, $\sigma^\phi, \ldots$. (Since we follow the notation employed in [15], the symbol ∇ will be used to denote the modular operator of a weight.) Since $\pi_\phi(A)' = J_\phi \pi_\phi(A) J_\phi$, the equation

$$\phi'(y) := \phi(\pi_\phi^{-1}(J_\phi y J_\phi)), \quad y \in (\pi_\phi(A)')_+$$

defines a faithful normal semifinite weight ϕ' on $\pi_\phi(A)'$.

For a linear operator T on a vector space, $\mathfrak{D}(T)$ designates the domain of T. We let $B(H)$ stand for the algebra of all bounded operators on a Hilbert space H.

1.1. Locally compact quantum groups.

Definition 1.1.1. Following [10] (see [9] also), we say that a quadruple $\mathbb{G} = (M, \Delta, \varphi, \psi)$ is a *locally compact quantum group* (in the von Neumann algebra setting) or a *von Neumann algebraic quantum group* if

(i) M is a von Neumann algebra;

(ii) Δ is a unital normal injective $*$-homomorphism from M into $M \otimes M$ satisfying $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$;

(iii) φ is a faithful normal semifinite weight on M, called the *left invariant weight* of $\mathbb{G}$, such that

$$\varphi((\omega \otimes \mathrm{id})(\Delta(x))) = \varphi(x)\omega(1) \quad \forall \omega \in M_*^+, \forall x \in \mathfrak{m}_\varphi^+;$$

(iv) ψ is a faithful normal semifinite weight on M, called the *right invariant weight* of $\mathbb{G}$, such that

$$\psi((\mathrm{id} \otimes \omega)(\Delta(x))) = \psi(x)\omega(1), \quad \forall \omega \in M_*^+, \forall x \in \mathfrak{m}_\psi^+.$$

Let us fix a locally compact quantum group $\mathbb{G} = (M, \Delta, \varphi, \psi)$ throughout the rest of this section. We will always think of M as represented on the GNS-Hilbert space H_φ obtained from φ. By the left invariance of φ, one gets a unitary $W(\mathbb{G})$ on $H_\varphi \otimes H_\varphi$ characterized by

$$W(\mathbb{G})^*(\Lambda_\varphi(x) \otimes \Lambda_\varphi(y)) = \Lambda_{\varphi\otimes\varphi}(\Delta(y)(x \otimes 1)), \quad x, y \in \mathfrak{n}_\varphi.$$

This unitary is called the *Kac-Takesaki operator* of $\mathbb{G}$, and is denoted simply by W if there is no danger of confusion. The modular operator and the modular conjugation of φ will be denoted simply by J and ∇.

According to [10], there canonically exists another locally compact quantum group $\widehat{\mathbb{G}} = (\widehat{M}, \widehat{\Delta}, \widehat{\varphi}, \widehat{\psi})$, called the *locally compact quantum group dual to* $\mathbb{G}$ such that $\{\widehat{M}, H_\varphi\}$ is a standard representation. So we always regard $\widehat{M}$ as acting on H_φ. There is a canonical identification (= the Fourier transform) of $H_{\widehat{\varphi}}$ with H_φ. So we consider the modular operator and the modular conjugation of $\widehat{\varphi}$, denoted by $\widehat{\nabla}$ and $\widehat{J}$, as acting on H_φ. For the definitions of locally compact quantum groups such as the commutant $\mathbb{G}'$, the opposite $\mathbb{G}^{\mathrm{op}}$ etc., we refer the readers to Section 4 of [10].

A (left) action of $\mathbb{G}$ on a von Neumann algebra A is a normal injective unital $*$-homomorphim α from A into $M \otimes A$ satisfying $(\mathrm{id}_M \otimes \alpha) \circ \alpha = (\Delta \otimes \mathrm{id}_M) \circ \alpha$ ([15]). If one wishes to work with *right* actions, with which we believe most of the readers are familiar, then all the results established in [15] need to be translated in terms of right actions. Roughly speaking, this can be done by substituting the opposite $\mathbb{G}^{\mathrm{op}}$ for $\mathbb{G}$ in the results of [15].

Fix an action α of $\mathbb{G}$ on a von Neumann algebra A. By Proposition 1.3 from [15], the equation

$$T_\alpha(a) := (\psi \otimes \mathrm{id})(\alpha(a)), \quad a \in A_+$$

defines a faithful normal operator valued weight T_α from A onto $A^\alpha := \{a \in A : \alpha(a) = 1 \otimes a\}$, the fixed-point algebra A^α of α. We call T_α the operator valued weight associated to the action α.

The crossed product of A by the action α is the von Neumann algebra generated by $\alpha(A)$ and $\widehat{M} \otimes \mathbb{C}$. We denote the crossed product by $\mathbb{G} {}_\alpha\!\ltimes A$. By Proposition 2.2 from [15], there exists a unique action $\widehat{\alpha}$ of $\widehat{\mathbb{G}}^{\mathrm{op}}$ on $\mathbb{G} {}_\alpha\!\ltimes A$, called the *dual action* of α, such that

$$\widehat{\alpha}(\alpha(a)) = 1 \otimes \alpha(a), \quad \widehat{\alpha}(z \otimes 1) = \widehat{\Delta}^{\mathrm{op}}(z) \otimes 1 \quad a \in A,\, z \in \widehat{M}.$$

We have $(\mathbb{G} {}_\alpha\!\ltimes A)^{\widehat{\alpha}} = \alpha(A)$. For every faithful normal semifinite weight ϕ on A, the equation

$$\widetilde{\phi} := \phi \circ \alpha^{-1} \circ T_{\widehat{\alpha}}$$

defines a faithful normal semifinite weight $\widetilde{\phi}$ on $\mathbb{G} {}_\alpha\!\ltimes A$. The weight $\widetilde{\phi}$ is called the *dual weight* of ϕ. In [15], Vaes studied dual weights in great detail. The Hilbert space $H_{\widetilde{\phi}}$ is identified with $H_\varphi \otimes H_\phi$. The unitary U_ϕ on $H_\varphi \otimes H_\phi$ defined by

$$U_\phi := J_{\tilde{\phi}}(\widehat{J} \otimes J_\phi)$$

is called the *canonical implementation* of α (see [15]). It satisfies

$$\alpha(a) = U_\phi(1 \otimes a)U_\phi^*, \quad a \in A.$$

By Theorem 2.6 of [15], there is a unital $*$-isomorphism Θ from the double crossed product $\widehat{\mathbb{G}}^{\mathrm{op}} {}_{\widehat{\alpha}}\!\ltimes (\mathbb{G} {}_\alpha\!\ltimes A)$ onto $B(H_\varphi) \otimes A$, and an action γ of $\mathbb{G}$ on $B(H_\varphi) \otimes A$ such that

$$\gamma := \mathrm{Ad}\,(\Sigma V^* \Sigma \otimes 1) \circ (\sigma \otimes \mathrm{id}_A) \circ (\mathrm{id}_{B(H_\varphi)} \otimes \alpha), \quad (\mathrm{Ad}\,(J\widehat{J}) \otimes \Theta\,) \circ \widehat{\widehat{\alpha}} = \gamma \circ \Theta,$$

where $V := (\widehat{J}\otimes\widehat{J})\Sigma W^*\Sigma(\widehat{J}\otimes\widehat{J})$, and $\Sigma : H_\varphi\otimes H_\varphi \to H_\varphi\otimes H_\varphi$ is the flip. The above result is usually referred to as (the Takesaki) duality for crossed products (see Theorem 2.6 in [15] for example). The bidual weight $\widetilde{\widetilde{\phi}}$ of ϕ is by definition defined on $\widehat{\mathbb{G}}^{\rm op}{}_{\widehat{\alpha}}\ltimes(\mathbb{G}\,{}_\alpha\!\ltimes A)$, but, through the above isomorphism Θ in the Takesaki duality, we may view it as a weight on $B(H_\varphi)\otimes A$. In what follows, we often take this point of view. Hence we have

$$\widetilde{\widetilde{\phi}} = \widetilde{\phi}\circ T_\gamma. \tag{1.1.1}$$

2.1 Connes' spatial derivatives. Let A be a von Neumann algebra acting on a Hilbert space K. Let ϕ be a faithful normal semifinite weight on A. A vector $\xi\in K$ is said to be ϕ-bounded if there exists a positive constant C such that

$$\|a\xi\| \leqslant C\|\Lambda_\phi(a)\| \quad \forall a\in\mathfrak{n}_\phi.$$

We let $D(K,\phi)$ be the subspace of ϕ-bounded vectors in K. Then, for any $\xi\in D(K,\phi)$, there exists a unique bounded operator $R^\phi(\xi)$ from H_ϕ into K such that

$$R^\phi(\xi)\Lambda_\phi(a) = a\xi, \quad a\in\mathfrak{n}_\phi.$$

The operator $R^\phi(\xi)R^\phi(\eta)^*$, $\xi,\eta\in D(K,\phi)$, then belongs to the commutant A' of A in $B(K)$. For a faithful normal semifinite weight Φ on A', the equation

$$q(\xi) := \Phi(R^\phi(\xi)R^\phi(\xi)^*), \quad \xi\in D(K,\phi)$$

defines a lower semicontinuous quadratic form on K with dense domain $\mathfrak{D}(q) := \{\xi\in D(K,\phi) : q(\xi)<\infty\}$. By the general theory of quadratic forms on Hilbert spaces (see Appendix A.10 in [11] for example), there exists a greatest nonsingular positive self-adjoint operator S on K such that

(i) $\mathfrak{D}(q)\subseteq\mathfrak{D}(S^{1/2})$;
(ii) $q(\xi) = \|S^{1/2}\xi\|^2 \ \forall\,\xi\in\mathfrak{D}(q)$.

The operator S is called the *spatial derivative of* Φ with respect to ϕ, and denoted by $\frac{\mathrm{d}\Phi}{\mathrm{d}\phi}$ (see [1] for fundamental properties of spatial derivatives).

By using the theory of spatial derivatives, it is well-konwn that Haagerup's order-reversing bijection $T\mapsto T^{-1}$ (Theorem 6.13 in [6]) of operator valued weights can be constructed in a canonical manner (cf. Remarque in p. 164 of [7], Section 1.2 of [8] or Corollary 12.11 of [11]). Let $A\subseteq B$ be von Neumann algebras acting on a Hilbert space K, and $B'\subseteq A'$ be their commutants in $B(K)$. The set of all faithful normal semifinite operator valued weights from B to A is denoted by $P(B,A)$. The set $P(A',B')$ is defined similarly. Then there exists a bijection

$$T\in P(B,A)\mapsto T^{-1}\in P(A',B')$$

uniquely determined by the property that

$$\frac{\mathrm{d}\psi}{\mathrm{d}(\phi\circ T)} = \frac{\mathrm{d}(\psi\circ T^{-1})}{\mathrm{d}\phi},$$

where ψ and ϕ are any faithful normal semifinite weights on A and B', respectively. This fact will be crucial in the discussion that follows.

2. PRELIMINARY RESULTS

This section is devoted to establishing a result that will be needed to prove our main theorem. The result is concerned with the tensor product formula of two spatial derivatives. It might be known to specialists, but the author was unable to find a proof in the literature of the relevant subject. So we include a proof for the formula here for readers' convenience. We also include a proposition which states a relationship between operator valued weights $(S \otimes T)^{-1}$ and $S^{-1} \otimes T^{-1}$. This result too might be known to specialists.

Proposition 2.1. *Let A and B be von Neumann algebras acting on Hilbert spaces H and K, respectively. Suppose that ω (respectively ψ) is a faithful normal weight on A (respectively B), and that Ω (respectively Ψ) is a faithful normal semifinite weight on A' (respectively B'). Then we have*

$$\frac{\mathrm{d}(\Omega \otimes \Psi)}{\mathrm{d}(\omega \otimes \psi)} = \frac{\mathrm{d}\Omega}{\mathrm{d}\omega} \otimes \frac{\mathrm{d}\Psi}{\mathrm{d}\psi}.$$

Proof. Let $a := \frac{\mathrm{d}\Omega}{\mathrm{d}\omega}$, $b := \frac{\mathrm{d}\Psi}{\mathrm{d}\psi}$ and $c := \frac{\mathrm{d}(\Omega\otimes\Psi)}{\mathrm{d}(\omega\otimes\psi)}$. Denote by q the lower semicontinuous quadratic form on $H \otimes K$ defined by

$$q(\zeta) := (\Omega \otimes \Psi)(R^{\omega\otimes\psi}(\zeta)R^{\omega\otimes\psi}(\zeta)^*), \quad \zeta \in D(H \otimes K, \omega \otimes \psi).$$

It is easy to see that $D(H \otimes K, \omega \otimes \psi)$ contains the algebraic tensor product $D(H, \omega) \odot D(K, \psi)$. Set

$$D_\Omega := \{\xi \in D(H, \omega) : R^\omega(\xi)R^\omega(\xi)^* \in \mathfrak{m}_\Omega^+\},$$
$$D_\Psi := \{\eta \in D(K, \psi) : R^\psi(\eta)R^\psi(\eta)^* \in \mathfrak{m}_\Psi^+\}.$$

If $\zeta = \sum_{j=1}^{n} \xi_j \otimes \eta_j$ is in $D_\Omega \odot D_\Psi$, then, by the density theorem for left Hilbert algebras, we have $R^{\omega\otimes\psi}(\zeta) = \sum_{j=1}^{n} R^\omega(\xi_j) \otimes R^\psi(\eta_j)$. This implies that ζ belongs to the domain $\mathfrak{D}(q)$ of the quadratic form q and

$$q(\zeta) = \|(a^{1/2} \otimes b^{1/2})\zeta\|^2.$$

Hence we get

$$\|c^{1/2}\zeta\|^2 = q(\zeta) = \|(a^{1/2} \otimes b^{1/2})\zeta\|^2, \quad \forall\zeta \in D_\Omega \odot D_\Psi.$$

Since $D_\Omega \odot D_\Psi$ is a core for $a^{1/2} \otimes b^{1/2}$ (see equation (6) on p. 95 of [11]), $\mathfrak{D}(a^{1/2} \otimes b^{1/2})$ is contained in $\mathfrak{D}(c^{1/2})$, and we obtain

$$\|c^{1/2}\zeta\| = \|(a^{1/2} \otimes b^{1/2})\zeta\|, \quad \forall\zeta \in \mathfrak{D}(a^{1/2} \otimes b^{1/2}). \tag{2.1.1}$$

For any $y \in A \otimes B$, by Theorem 9 of [1], we have

$$\begin{aligned}(a^{-1} \otimes b^{-1})^{\mathrm{i}t}\sigma_t^{\omega\otimes\psi}(y) &= (a^{-\mathrm{i}t} \otimes b^{-\mathrm{i}t})(\sigma_t^\omega \otimes \sigma_t^\psi)(y) \\ &= (a^{-\mathrm{i}t} \otimes b^{-\mathrm{i}t})(a^{\mathrm{i}t} \otimes b^{\mathrm{i}t})y(a^{-\mathrm{i}t} \otimes b^{-\mathrm{i}t}) \\ &= y(a^{-1} \otimes b^{-1})^{\mathrm{i}t}.\end{aligned}$$

From this and Theorem 13 of [1], it results that there is a unique faithful normal semifinite weight Θ on $A' \otimes B'$ such that

$$a \otimes b = \frac{\mathrm{d}\Theta}{\mathrm{d}(\omega \otimes \psi)}.$$

Since, for all $z \in A' \otimes B'$

$$\sigma_t^{\Omega\otimes\Psi}(z) = (\sigma_t^{\Omega} \otimes \sigma_t^{\Psi})(z) = \mathrm{Ad}(a \otimes b)^{\mathrm{i}t}(z) = \mathrm{Ad}\left[\frac{\mathrm{d}\Theta}{\mathrm{d}(\omega \otimes \psi)}\right]^{\mathrm{i}t}(z) = \sigma_t^{\Theta}(z)$$

there exists a nonsingular positive self-adjoint operator k affiliated with the center $Z(A \otimes B)$ of $A \otimes B$ such that

$$(D(\Omega \otimes \Psi) : D\Theta)_t = k^{\mathrm{i}t}, \quad \Omega \otimes \Psi = \Theta(k\,\cdot\,).$$

By Theorem 9 from [1], we have

$$c^{\mathrm{i}t} = k^{\mathrm{i}t}(a \otimes b)^{\mathrm{i}t}. \tag{2.1.2}$$

Meanwhile, from Proposition 8 of [1] and the fact that $k^{\mathrm{i}t}$ belongs to $Z(A \otimes B)$, one has

$$k^{\mathrm{i}t}(a \otimes b)k^{-\mathrm{i}t} = k^{\mathrm{i}t}\frac{\mathrm{d}\Theta}{\mathrm{d}(\omega \otimes \psi)}k^{-\mathrm{i}t} = \frac{\mathrm{d}\Theta(k^{-\mathrm{i}t}\,\cdot\,k^{\mathrm{i}t})}{\mathrm{d}(\omega \otimes \psi)} = \frac{\mathrm{d}\Theta}{\mathrm{d}(\omega \otimes \psi)} = a \otimes b.$$

Hence $(a\otimes b)^{\mathrm{i}s}$ commutes with $k^{\mathrm{i}t}$ for any $s, t \in \mathbb{R}$. In other words, k commutes with $a\otimes b$ in the sense of Exercise 9.24 in [13]. Let $k = \int_0^\infty \lambda\,\mathrm{d}p(\lambda)$ and $a\otimes b = \int_0^\infty \lambda\,\mathrm{d}e(\lambda)$ be the spectral decompositions of k and $a\otimes b$. Set $p_n := \int_0^n \lambda\,\mathrm{d}p(\lambda)$ and $e_n := \int_0^n \lambda\,\mathrm{d}e(\lambda)$, $n = 1, 2, \ldots$. Then put

$$\mathfrak{D}_0 := \bigcup_{n,m=1}^{\infty} p_n e_m (H \otimes K),$$

which is a dense subspace of $H \otimes K$. If we define a linear operator $k(a \otimes b)$ with $\mathfrak{D}(k(a \otimes b)) = \mathfrak{D}_0$ by

$$k(a \otimes b)\zeta := k((a \otimes b)\zeta), \quad \zeta \in \mathfrak{D}_0,$$

then, by Appendix A.6 of [11], $k(a \otimes b)$ is preclosed and the closure $\overline{k(a \otimes b)}$ is a positive self-adjoint operator on $H \otimes K$ such that $\overline{k(a \otimes b)} = \overline{(a \otimes b)k}$ and

$$\left(\overline{k(a \otimes b)}\right)^{\mathrm{i}t} = k^{\mathrm{i}t}(a \otimes b)^{\mathrm{i}t}, \quad t \in \mathbb{R}.$$

It then follows from (2.1.2) that $c = \overline{k(a \otimes b)}$. Similarly, one can show that the closure $\overline{k^{1/2}(a \otimes b)^{1/2}}$ is $c^{1/2}$. In particular, $\mathfrak{D}((a \otimes b)^{1/2})$ is a core for $c^{1/2}$. But then, Equation (2.1.1) entails that $\mathfrak{D}((a \otimes b)^{1/2})$ equals $\mathfrak{D}(c^{1/2})$. Therefore, we obatin

$$\|c^{1/2}\zeta\| = \|(a^{1/2} \otimes b^{1/2})\zeta\|, \quad \forall \zeta \in \mathfrak{D}(a^{1/2} \otimes b^{1/2}) = \mathfrak{D}(c^{1/2}).$$

By the uniqueness of the polar decomposition, we conclude that $c^{1/2} = (a \otimes b)^{1/2}$, i.e., $c = a \otimes b$. ∎

PROPOSITION 2.2. *Let A and B be von Neumann algebras acting on Hilbert spaces H and K, respectively. Let C (respectively D) be a (unital) von Neumann subalgebra of A (respectively B), and S (respectively T) a faithful normal semifinite operator valued weight from A onto C (respectively from B onto D). Then we have*

$$(S \otimes T)^{-1} = S^{-1} \otimes T^{-1}.$$

Proof. Let ω (respectively ψ) be a faithful normal semifinite weight on C (respectively D) and Ω (respectively Ψ) a faithful normal semifinte weight on C' (respectively D'). Then, by Proposition 2.1 and Theorem 5.5 from [5], we have

$$\begin{aligned}\frac{\mathrm{d}(\Omega \otimes \Psi) \circ (S \otimes T)^{-1}}{\mathrm{d}(\omega \otimes \psi)} &= \frac{\mathrm{d}(\Omega \otimes \Psi)}{\mathrm{d}(\omega \otimes \psi) \circ (S \otimes T)} = \frac{\mathrm{d}(\Omega \otimes \Psi)}{\mathrm{d}(\omega \circ S \otimes \psi \circ T)} \\ &= \frac{\mathrm{d}\Omega}{\mathrm{d}\omega \circ S} \otimes \frac{\mathrm{d}\Psi}{\mathrm{d}\psi \circ T} = \frac{\mathrm{d}\Omega \circ S^{-1}}{\mathrm{d}\omega} \otimes \frac{\mathrm{d}\Psi \circ T^{-1}}{\mathrm{d}\psi} \\ &= \frac{\mathrm{d}(\Omega \circ S^{-1} \otimes \Psi \circ T^{-1})}{\mathrm{d}(\omega \otimes \psi)} = \frac{\mathrm{d}(\Omega \otimes \Psi) \circ (S^{-1} \otimes T^{-1})}{\mathrm{d}(\omega \otimes \psi)}.\end{aligned}$$

Hence we obtain $(\Omega \otimes \Psi) \circ (S \otimes T)^{-1} = (\Omega \otimes \Psi) \circ (S^{-1} \otimes T^{-1})$. From this and Lemma 4.8 of [4], it follows that $(S \otimes T)^{-1} = S^{-1} \otimes T^{-1}$. ∎

3. THE RADON-NIKODYM DERIVATIVE ASSOCIATED TO AN ACTION

In the remainder of the paper, we fix a locally compact quantum group $\mathbb{G} = (M, \Delta, \varphi, \psi)$ and an action α of $\mathbb{G}$ on a von Neumann algebra A. We will freely employ the notation introduced in Section 1. We also fix a faithful normal semifinite weight ϕ on A once and for all. Let $\widetilde{\phi}$ be the dual weight of ϕ on the crossed product $\mathbb{G}\,{}_\alpha\!\ltimes A$. We write $\widetilde{S}, \widetilde{J}, \widetilde{\nabla}, \ldots$ for $S_{\tilde{\phi}}, J_{\tilde{\phi}}, \nabla_{\tilde{\phi}}, \ldots$. (I.e., the modular objects of $\widetilde{\phi}$.) We also denote by U the canonical implementation of α on $H_\varphi \otimes H_\phi$. Hence we have $U = \widetilde{J}(\widehat{J} \otimes J_\phi)$.

LEMMA 3.1. *For any $t \in \mathbb{R}$, the unitary $\widetilde{\nabla}^{\mathrm{it}}(\widehat{\nabla}^{-\mathrm{it}} \otimes 1)$ belongs to $M \otimes B(H_\phi)$.*

Proof. As in the proof of Proposition 3.12 in [15], let $\{\mu_t\}_{t \in \mathbb{R}}$ be the one-parameter automorphism group of M defined by $\mu_t(x) := J\widehat{\nabla}^{\mathrm{it}} JxJ\widehat{\nabla}^{-\mathrm{it}} J$. From the proof of Proposition 3.12 in [15], we have

$$\widetilde{\nabla}^{\mathrm{it}}(JxJ \otimes 1)\widetilde{\nabla}^{-\mathrm{it}} = J\mu_t(x)J \otimes 1, \quad \forall x \in M, \forall t \in \mathbb{R},$$

which is equivalent to

$$\widetilde{\nabla}^{\mathrm{it}}(y \otimes 1)\widetilde{\nabla}^{-\mathrm{it}} = \widehat{\nabla}^{\mathrm{it}} y \widehat{\nabla}^{-\mathrm{it}} \otimes 1, \quad \forall y \in M', \forall t \in \mathbb{R}. \tag{3.1.1}$$

From this, it follows that, for any $y \in M'$,

$$\begin{aligned}\widetilde{\nabla}^{\mathrm{it}}(\widehat{\nabla}^{-\mathrm{it}} \otimes 1)(y \otimes 1) &= \widetilde{\nabla}^{\mathrm{it}}(\widehat{\nabla}^{-\mathrm{it}} y \widehat{\nabla}^{\mathrm{it}} \otimes 1)(\widehat{\nabla}^{-\mathrm{it}} \otimes 1) \\ &= \widetilde{\nabla}^{\mathrm{it}} \cdot \widetilde{\nabla}^{-\mathrm{it}}(y \otimes 1)\widetilde{\nabla}^{\mathrm{it}} \cdot (\widehat{\nabla}^{-\mathrm{it}} \otimes 1) = (y \otimes 1)\widetilde{\nabla}^{\mathrm{it}}(\widehat{\nabla}^{-\mathrm{it}} \otimes 1).\end{aligned}$$

This completes the proof. ∎

Lemma 3.2. *We have*

$$\widetilde{J}\alpha(a)\widetilde{J} = 1 \otimes J_\phi a J_\phi$$

for all $a \in A$.

Proof. The assertion immediately follows from the identities $\alpha(a) = U(1 \otimes a)U^*$ and $U = \widetilde{J}(\widehat{J} \otimes J_\phi)$. ∎

Lemma 3.3. *Let* $b \in A'$. *Then we have*

$$\widetilde{\nabla}^{\mathrm{i}t}(1 \otimes b)\widetilde{\nabla}^{-\mathrm{i}t} = 1 \otimes \nabla_\phi^{\mathrm{i}t}\, b\, \nabla_\phi^{-\mathrm{i}t}$$

for any $t \in \mathbb{R}$.

Proof. Let $\eta \in \mathfrak{T}_\phi$. Then we have

$$\begin{aligned}\widetilde{\nabla}^{\mathrm{i}t}(1 \otimes \pi_{\mathrm{r}}(\eta))\widetilde{\nabla}^{-\mathrm{i}t} &= \widetilde{\nabla}^{\mathrm{i}t}(1 \otimes J_\phi \pi_{\mathrm{l}}(J_\phi \eta) J_\phi)\widetilde{\nabla}^{-\mathrm{i}t} = \widetilde{\nabla}^{\mathrm{i}t}\widetilde{J}\alpha(\pi_{\mathrm{l}}(J_\phi\eta))\widetilde{J}\widetilde{\nabla}^{-\mathrm{i}t} \\ &= \widetilde{J}\sigma_t^{\tilde{\phi}}(\alpha(\pi_{\mathrm{l}}(J_\phi\eta)))\widetilde{J} = \widetilde{J}\alpha(\sigma_t^{\phi}(\pi_{\mathrm{l}}(J_\phi\eta)))\widetilde{J} \\ &= \widetilde{J}\alpha(\pi_{\mathrm{l}}(\nabla_\phi^{\mathrm{i}t}J_\phi\eta))\widetilde{J} = 1 \otimes J_\phi \pi_{\mathrm{l}}(\nabla_\phi^{\mathrm{i}t}J_\phi\eta)J_\phi \\ &= 1 \otimes \pi_{\mathrm{r}}(\nabla_\phi^{\mathrm{i}t}\eta) = 1 \otimes \nabla_\phi^{\mathrm{i}t}\pi_{\mathrm{r}}(\eta)\nabla_\phi^{-\mathrm{i}t}.\end{aligned}$$

The second and the sixth steps are due to the previous lemma. The fourth step is guaranteed by Proposition 3.7 of [15]. Since $\pi_{\mathrm{r}}(\mathfrak{T}_\phi)$ is dense in A' in the strong-operator topology, we obtain the desired identity. ∎

Proposition 3.4. *For any* $t \in \mathbb{R}$, *the unitary* $\widetilde{\nabla}^{\mathrm{i}t}(\widehat{\nabla}^{-\mathrm{i}t} \otimes \nabla_\phi^{-\mathrm{i}t})$ *belongs to* $M \otimes A$.

Proof. Thanks to Lemma 3.1, it suffices to show that $\widetilde{\nabla}^{\mathrm{i}t}(\widehat{\nabla}^{-\mathrm{i}t} \otimes \nabla_\phi^{-\mathrm{i}t})$ belongs to $B(H_\varphi) \otimes A$. Let $b \in A'$. Then, by the previous lemma, we have

$$\begin{aligned}\widetilde{\nabla}^{\mathrm{i}t}(\widehat{\nabla}^{-\mathrm{i}t} \otimes \nabla_\phi^{-\mathrm{i}t})(1 \otimes b) &= \widetilde{\nabla}^{\mathrm{i}t}(1 \otimes \nabla_\phi^{-\mathrm{i}t}\, b\, \nabla_\phi^{\mathrm{i}t})(\widehat{\nabla}^{-\mathrm{i}t} \otimes \nabla_\phi^{-\mathrm{i}t}) \\ &= (1 \otimes b)\widetilde{\nabla}^{\mathrm{i}t}(\widehat{\nabla}^{-\mathrm{i}t} \otimes \nabla_\phi^{-\mathrm{i}t}).\end{aligned}$$

This completes the proof. ∎

Definition 3.5. With the notation as above, we set

$$(\mathrm{d}\phi \circ \alpha : \mathrm{d}\phi)_t := \widetilde{\nabla}^{\mathrm{i}t}(\widehat{\nabla}^{-\mathrm{i}t} \otimes \nabla_\phi^{-\mathrm{i}t}), \quad t \in \mathbb{R}.$$

We call this strong-operator continous function $(\mathrm{d}\phi \circ \alpha : \mathrm{d}\phi)_t$ on $\mathbb{R}$ into the unitary group of $M \otimes A$ *the Radon-Nikodym derivative* derived from the action α and the weight ϕ. Thus we may regard $(\mathrm{d}\phi \circ \alpha : \mathrm{d}\phi)$ as an element of $M \otimes A \otimes L^\infty(\mathbb{R})$.

Remark. If the locally compact quantum group M comes from a locally compact group G, namely, $M = L^\infty(G)$, then the action α is an action of G on A in the ordinary sense. In this case, it can be easily checked that the Radon-Nikodym derivative $(\mathrm{d}\phi\circ\alpha : \mathrm{d}\phi)$, considered as belonging to $L^\infty(G)\otimes A \otimes L^\infty(\mathbb{R}) = L^\infty(G \times \mathbb{R}, A)$, is the function

$$(g, t) \in G \times \mathbb{R} \mapsto (D\phi \circ \alpha_g : D\phi)_t \in A,$$

where $(D\phi \circ \alpha_g : D\phi)_t$ of course stands for Connes' Radon-Nikodym derivative of $\phi \circ \alpha_g$ with respect to ϕ. This justifies our terminology in some sense.

4. TAKESAKI DUALITY FOR WEIGHTS

We retain the notation established in the preceding section.

Let Φ be the faithful normal semifinite weight on $B(H_\varphi) \otimes A$ defined by $\Phi := \mathrm{Tr}(\widehat{\nabla}\,\cdot\,) \otimes \phi$, where Tr is the usual trace on $B(H_\varphi)$. With this notation, we have

LEMMA 4.1. *The function* $t \in \mathbb{R} \mapsto (\mathrm{d}\phi \circ \alpha : \mathrm{d}\phi)_t \in M \otimes A \subseteq B(H_\varphi) \otimes A$ *is a* σ^Φ*-cocycle, i.e.,*

$$(\mathrm{d}\phi \circ \alpha : \mathrm{d}\phi)_{s+t} = (\mathrm{d}\phi \circ \alpha : \mathrm{d}\phi)_s \sigma_s^\Phi((\mathrm{d}\phi \circ \alpha : \mathrm{d}\phi)_t), \quad s, t \in \mathbb{R}.$$

Proof. Since $\sigma_t^\Phi = \mathrm{Ad}\,(\widehat{\nabla}^{\mathrm{i}t} \otimes \nabla_\phi^{\mathrm{i}t})$, the assertion follows from a direct computation. ∎

From the above lemma, there uniquely exists a faithful normal semifinite weight Ψ on $B(H_\varphi) \otimes A$ such that

$$(D\Psi : D\Phi)_t = (\mathrm{d}\phi \circ \alpha : \mathrm{d}\phi)_t, \quad t \in \mathbb{R}.$$

LEMMA 4.2. *The modular automorphism group* σ^Ψ *of the weight* Ψ *is given by* $\sigma_t^\Psi = \mathrm{Ad}\,\widetilde{\nabla}^{\mathrm{i}t}|B(H_\varphi) \otimes A$. *In particular, we have* $\sigma_t^\Psi\big|\, \mathbb{G}\,_\alpha\!\ltimes A = \sigma_t^{\tilde{\phi}}$ *for any* $t \in \mathbb{R}$.

Proof. By the definition of Ψ, we have

$$\begin{aligned}(D\Psi : D(\mathrm{Tr} \otimes \phi))_t &= (D\Psi : D\Phi)_t (D\Phi : D(\mathrm{Tr} \otimes \phi))_t \\ &= (\mathrm{d}\phi \circ \alpha : \mathrm{d}\phi)_t (\widehat{\nabla}^{\mathrm{i}t} \otimes 1) = \widetilde{\nabla}^{\mathrm{i}t}(1 \otimes \nabla_\phi^{-\mathrm{i}t}).\end{aligned}$$

From this, we get

$$\sigma_t^\Psi = \mathrm{Ad}\,\widetilde{\nabla}^{\mathrm{i}t}(1 \otimes \nabla_\phi^{-\mathrm{i}t}) \circ (\mathrm{id} \otimes \sigma_t^\phi) = \mathrm{Ad}\,\widetilde{\nabla}^{\mathrm{i}t}|B(H_\varphi) \otimes A.$$

The last assertion is now obvious. ∎

COROLLARY 4.3. *The modular automorphism* σ_t^Ψ *of* Ψ *is characterized by the following identities:*

(4.3.1) $$\sigma_t^\Psi(X) = \sigma_t^{\tilde{\phi}}(X), \qquad X \in \mathbb{G}\,_\alpha\!\ltimes A;$$

(4.3.2) $$\sigma_t^\Psi(y \otimes 1) = \widehat{\nabla}^{\mathrm{i}t} y \widehat{\nabla}^{-\mathrm{i}t} \otimes 1, \quad y \in M'.$$

Proof. The first identity is due to the preceding lemma. The second identity follows from (3.1.1). By Theorem 2.6 of [15], we have $\mathbb{G}\,_\alpha\!\ltimes A \vee M' \otimes \mathbb{C} = B(H_\varphi) \otimes A$. Hence σ_t^Ψ is determined by the above identities. ∎

LEMMA 4.4. *There uniquely exists a faithful normal semifinite operator valued weight* P *from* $B(H_\varphi) \otimes A$ *onto* $\mathbb{G}\,_\alpha\!\ltimes A$ *such that* $\Psi = \widetilde{\phi} \circ P$.

Proof. The assertion follows from the preceding corollary and Theorem 5.1 in [6] ∎

In what follows, t_K stands for the faithful state on $\mathbb{C}1_K := \{c \cdot 1_K : c \in \mathbb{C}\}$ defined by $t_K(c \cdot 1_K) := c$, where K is a (separable) Hilbert space and 1_K is the identity operator on K.

LEMMA 4.5. *Let P be the operator valued weight obtained in Lemma 4.4. Then we have*

$$\widetilde{\phi}\,' = (t_{H_\varphi} \otimes \phi') \circ P^{-1}.$$

We also have

$$\widetilde{\widetilde{\nabla}} = \frac{\mathrm{d}\Psi}{\mathrm{d}(t_{H_\varphi} \otimes \phi')}.$$

Proof. By Lemma 4.2 and Theorem 13 of [1], there exists a faithful normal semifinite weight ρ on A' such that, with the notation introduced before this lemma,

$$\widetilde{\widetilde{\nabla}} = \frac{\mathrm{d}\Psi}{\mathrm{d}(t_{H_\varphi} \otimes \rho)}.$$

Then we have

$$\begin{aligned}\frac{\mathrm{d}\widetilde{\phi}\,'}{\mathrm{d}\widetilde{\phi}} &= \widetilde{\widetilde{\nabla}}^{-1} = \frac{\mathrm{d}(t_{H_\varphi} \otimes \rho)}{\mathrm{d}\Psi} = \frac{\mathrm{d}(t_{H_\varphi} \otimes \rho)}{\mathrm{d}(\widetilde{\phi} \circ P)} \\ &= \frac{\mathrm{d}(t_{H_\varphi} \otimes \rho) \circ P^{-1}}{\mathrm{d}\widetilde{\phi}}.\end{aligned}$$

Thus we have obtained $\widetilde{\phi}\,' = \rho \circ P^{-1}$. In the meantime, by Theorem 9 of [1] and Proposition 2.1, we have

$$\begin{aligned}\left[\frac{\mathrm{d}\Psi}{\mathrm{d}(t_{H_\varphi} \otimes \phi')}\right]^{\mathrm{i}t} &= (D\Psi : D\Phi)_t \left[\frac{\mathrm{d}\Phi}{\mathrm{d}(t_{H_\varphi} \otimes \phi')}\right]^{\mathrm{i}t} = (\mathrm{d}\phi \circ \alpha : \mathrm{d}\phi)_t \left[\frac{\mathrm{d}(\mathrm{Tr}(\widehat{\nabla}\cdot\,) \otimes \phi)}{\mathrm{d}(t_{H_\varphi} \otimes \phi')}\right]^{\mathrm{i}t} \\ &= (\mathrm{d}\phi \circ \alpha : \mathrm{d}\phi)_t (\widehat{\nabla}^{\mathrm{i}t} \otimes \nabla_\phi^{\mathrm{i}t}) = \widetilde{\widetilde{\nabla}}^{\mathrm{i}t} = \left[\frac{\mathrm{d}\Psi}{\mathrm{d}(t_{H_\varphi} \otimes \rho)}\right]^{\mathrm{i}t}.\end{aligned}$$

From this, it follows that $\rho = \phi'$. ∎

As usual, let $T_{\widehat{\alpha}}$ be the faithful normal semifinite operator valued weight from $\mathbb{G}\,{}_\alpha\!\ltimes A$ onto $\alpha(A)$ associated to the dual action $\widehat{\alpha}$. Since $B(H_\varphi) \otimes A$ is the basic extension of the inclusion $\alpha(A) \subseteq \mathbb{G}\,{}_\alpha\!\ltimes A$, i.e., $B(H_\varphi) \otimes A = \widetilde{J}\alpha(A)'\widetilde{J}$ on $H_\varphi \otimes H_\phi$, it follows from Section 5 of [15] that there exists a unique faithful normal semifinite operator valued weight S from $B(H_\varphi) \otimes A$ onto $\mathbb{G}\,{}_\alpha\!\ltimes A$ such that

$$\frac{\mathrm{d}(\psi \circ S)}{\mathrm{d}\omega'} = \frac{\mathrm{d}\psi}{\mathrm{d}(\omega \circ T_{\widehat{\alpha}})'}$$

for all faithful normal semifinite weights ψ on $\mathbb{G}\,{}_\alpha\!\ltimes A$ and ω on $\alpha(A)$, where η' in general denotes the faithful normal semifinite weight on $\widetilde{J}B\widetilde{J}$ given by the formula $\eta'(x) := \eta(\widetilde{J}x\widetilde{J})$ for all $x \in (\widetilde{J}B\widetilde{J})_+$, whenever η is a faithful normal semifinite weight on a von Neumann algebra $B \subseteq B(H_\varphi \otimes H_\phi)$. Indeed, one can easily show that S is given by

$$S(X) = \widetilde{J}(T_{\widehat{\alpha}})^{-1}(\widetilde{J}X\widetilde{J})\widetilde{J}, \quad \forall X \in (B(H_\varphi) \otimes A)_+.$$

For any faithful normal semifinite operator valued weight T from $\mathbb{G}\,_\alpha\ltimes A$ onto $\alpha(A)$ or from $\alpha(A)'$ onto $(\mathbb{G}\,_\alpha\ltimes A)'$, put

$$\widetilde{T}(x) := \widetilde{J}T(\widetilde{J}x\widetilde{J})\widetilde{J}.$$

Thus we get a faithful normal semifinite operator valued weight $\widetilde{T}$ from $(\mathbb{G}\,_\alpha\ltimes A)'$ onto $\widetilde{J}\alpha(A)\widetilde{J} = \mathbb{C}\otimes A'$ or from $\widetilde{J}\alpha(A)'\widetilde{J} = B(H_\varphi)\otimes A$ onto $\mathbb{G}\,_\alpha\ltimes A$. By Lemma 1.3 of [8], one has

$$(\widetilde{T})^{-1} = \widetilde{T^{-1}}.$$

LEMMA 4.6. *With the notation as above and the one in Section* 1.1*, we have* $S = T_\gamma$*. Namely,*

$$T_\gamma = (\widetilde{T_{\widehat{\alpha}}})^{-1} = (\widetilde{T_{\widehat{\alpha}})^{-1}}.$$

Proof. First we apply Proposition 5.12 from [15] to the dual action $\widehat{\alpha}$ of $\widehat{\mathbb{G}}^{\mathrm{op}}$ on $N := \mathbb{G}\,_\alpha\ltimes A$. Then we see that the $*$-homomorphism ρ in Theorem 5.3 of [15] is faithful. By Proposition 5.7 of [15], S can be identified with the operator valued weight $T_{\widetilde{\widetilde{\alpha}}}$ associated with the bidual action $\widetilde{\widetilde{\alpha}}$ through the isomorphism ρ. Hence it will suffice to show that ρ is exactly the isomorphism Θ introduced in Section 1.1. By Theorem 2.6 of [15] and Theorem 5.3 of [15], ρ and Θ are equal on $\widehat{\alpha}(N)$. Suppose for the moment that the canonical implementation of $\widehat{\alpha}$ obtained from the dual weight $\widetilde{\phi}$ is $W(\widehat{\mathbb{G}}^{\mathrm{op}})^*\otimes 1$. Then, by Theorem 2.6 of [15] and Theorem 5.3 in [15] again, ρ and Θ equal on $M'\otimes\mathbb{C}\otimes\mathbb{C}$. Thus we can conclude that ρ coincides with Θ. Therefore we have only to prove that the canonical implementation of $\widehat{\alpha}$ is $W(\widehat{\mathbb{G}}^{\mathrm{op}})^*\otimes 1$. From Proposition 2.5 of [15], we see that $\widetilde{\phi}$ is a $\widehat{\delta}$-invariant weight. Hence, by virtue of Proposition 4.3 from [15], the unitary $V_{\tilde{\phi}}$ constructed by applying Proposition 2.4 of [15] to the pair $(\widehat{\alpha}, \widetilde{\phi})$ is the desired canonical implementation. But one can easily check with the aid of Proposition 7.2 from [15] that $V_{\tilde{\phi}}$ is exactly $W(\widehat{\mathbb{G}}^{\mathrm{op}})^*\otimes 1$. The details are left to the readers. ∎

LEMMA 4.7. *As in Section* 1.1*, regard the bidual weight* $\widetilde{\widetilde{\phi}}$ *as a weight on* $B(H_\varphi)\otimes A$*. Then, with the notation so far, we have* $\widetilde{\widetilde{\phi}} = \Psi$*. In particular, one has*

$$\widetilde{\nabla} = \frac{\mathrm{d}\widetilde{\widetilde{\phi}}}{\mathrm{d}(t_{H_\varphi}\otimes\phi')}$$

Proof. To prove the lemma, we freely use the formulae obtained in the proof of Lemma 1.3 from [8]. First we consider the weight $\phi\circ\alpha^{-1}$ on $\alpha(A)$. Note that $\widetilde{J}\alpha(A)\widetilde{J} = \mathbb{C}\otimes A'$. For any $b\in(A')_+$, we have

$$\begin{aligned}(\phi\circ\alpha^{-1})'(1\otimes b) &= \phi\circ\alpha^{-1}(\widetilde{J}(1\otimes b)\widetilde{J}) = \phi\circ\alpha^{-1}(\alpha(J_\phi bJ_\phi))\\ &= \phi'(b) = (t_{H_\varphi}\otimes\phi')(1\otimes b).\end{aligned}$$

Thus we have $(\phi \circ \alpha^{-1})' = t_{H_\varphi} \otimes \phi'$. In the meantime, by (1.1.1) and Lemma 4.6, one has

$$(\widetilde{\phi}' \circ (T_{\widehat{\alpha}})^{-1})' = \widetilde{\phi} \circ (\widetilde{T_{\widehat{\alpha}}})^{-1} = \widetilde{\phi} \circ T_\gamma = \widetilde{\widetilde{\phi}}.$$

Hence we obtain

$$\begin{aligned}\widetilde{\nabla}^{-1} &= \widetilde{J}\widetilde{\nabla}\widetilde{J} = \widetilde{J}\,\frac{\mathrm{d}\widetilde{\phi}}{\mathrm{d}\widetilde{\phi}'}\,\widetilde{J} = \widetilde{J}\,\frac{\mathrm{d}(\phi \circ \alpha^{-1}) \circ T_{\widehat{\alpha}}}{\mathrm{d}\widetilde{\phi}'}\,\widetilde{J} \\ &= \widetilde{J}\,\frac{\mathrm{d}(\phi \circ \alpha^{-1})}{\mathrm{d}(\widetilde{\phi}' \circ (T_{\widehat{\alpha}})^{-1})}\,\widetilde{J} = \frac{\mathrm{d}(\phi \circ \alpha^{-1})'}{\mathrm{d}(\widetilde{\phi}' \circ (T_{\widehat{\alpha}})^{-1})'} = \frac{\mathrm{d}(t_{H_\varphi} \otimes \phi')}{\mathrm{d}\widetilde{\widetilde{\phi}}}.\end{aligned}$$

By Lemma 4.5, we have $\Psi = \widetilde{\widetilde{\phi}}$. ∎

Theorem 4.8. (Takesaki duality for weights) *Let α be an action of a locally compact quantum group $\mathbb{G} = (M, \Delta, \varphi, \psi)$ on a von Neumann algebra A. Suppose that ϕ is a faithful normal semifinite weight on A. Let $\widetilde{\widetilde{\phi}}$ be the bidual weight of ϕ, regarded as a weight on $B(H_\varphi) \otimes A$ by the Takesaki duality for crossed products. Namely, with the notation in Section* 1.1, *$\widetilde{\widetilde{\phi}} = \widetilde{\phi} \circ T_\gamma$. Then it satisfies*

$$(D\widetilde{\widetilde{\phi}} : D(\mathrm{Tr}(\widehat{\nabla}\,\cdot\,) \otimes \phi))_t = (\mathrm{d}\phi \circ \alpha : \mathrm{d}\phi)_t, \quad t \in \mathbb{R}.$$

Proof. Recall that the weight Ψ was chosen so that $(D\Psi : D\Phi)_t = (\mathrm{d}\phi \circ \alpha : \mathrm{d}\phi)_t$. But, by the previous lemma, we now know that $\Psi = \widetilde{\widetilde{\phi}}$. Therefore we obtain the required result. ∎

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