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# NORM INEQUALITIES FOR OPERATORS WITH POSITIVE REAL PART

RAJENDRA BHATIA and XINGZHI ZHAN

*To Peter Rosenthal on his sixtieth birthday*

*Communicated by Norberto Salinas*

Abstract. Let $T = A + \mathrm{i}B$ with $A$ positive semidefinite and $B$ Hermitian. We derive a majorisation relation involving the singular values of $T, A$, and $B$. As a corollary, we show that $\|T\|_p^2 \leqslant \|A\|_p^2 + 2^{1-2/p}\|B\|_p^2$, for all $p \geqslant 2$; and that this inequality is sharp. When $1 \leqslant p \leqslant 2$ this inequality is reversed. For $p = 1$, we prove the sharper inequality $\|T\|_1^2 \geqslant \|A\|_1^2 + \|B\|_1^2$. Such inequalities are useful in studying the geometry of Schatten spaces, and our results include and improve upon earlier results proved in this context. Some related inequalities are also proved in the paper.

Keywords: *Positive operators, singular values, majorisation, Schatten p-norms, inequalities.*

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## 1. INTRODUCTION

Let $T$ be a bounded linear operator on a complex separable Hilbert space $\mathcal{H}$. We can write

$$T = A + \mathrm{i}B, \tag{1.1}$$

where $A$ and $B$ are Hermitian operators. Such a decomposition is unique; we have

$$A = \frac{1}{2}(T + T^*), \quad B = \frac{1}{2\mathrm{i}}(T - T^*). \tag{1.2}$$

The operators $A, B$ are called the *real* and *imaginary parts of* $T$, and the decomposition (1.1) is called the *Cartesian decomposition* of $T$.

If $T$ is in the Hilbert-Schmidt ideal and $\|T\|_2$ denotes its Hilbert-Schmidt norm, then

$$\|T\|_2^2 = \|A\|_2^2 + \|B\|_2^2. \tag{1.3}$$

For the usual operator norm $\|\cdot\|$, however, this relation is not valid. Using the triangle inequality and the arithmetic-geometric mean inequality, one can see that

$$\|T\|^2 \leqslant 2(\|A\|^2 + \|B\|^2). \tag{1.4}$$

The example

$$T = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix},$$

shows that this inequality is sharp.

Interesting improvements are, however, possible if $A$ or $B$ is chosen to be positive (semidefinite). If $A$ is positive, then

$$\|T\|^2 \leqslant \|A\|^2 + 2\|B\|^2, \tag{1.5}$$

and if both $A$ and $B$ are positive, then

$$\|T\|^2 \leqslant \|A\|^2 + \|B\|^2. \tag{1.6}$$

The inequality (1.5) was proved by Garling and Tomczak-Jaegermann ([6]), and the inequality (1.6) by Mirman [8]. The example

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$$

with small real values of $t$ shows that the inequality (1.5) is sharp ([6]).

Inequalities such as these are of wide interest in the analysis of operators ([2] and [7]) and in mathematical physics ([9]). The authors of [6] were motivated by their interest in the geometry of operator ideals. A similar motivation led Fack ([5]) to obtain a generalisation of (1.5). He proved that if $A$ is positive, then

$$\|T\|_p^2 \leqslant \|A\|_p^2 + 2\|B\|_p^2 \quad \text{for } 2 \leqslant p < \infty. \tag{1.7}$$

Here $\|T\|_p$ denotes the Schatten $p$-norm of $T$, and it is assumed that $T$ is in the ideal corresponding to this norm. (Actually, Fack proves this in the more general setting of the space $L^p(M)$ corresponding to a von Neumann algebra $M$.)

While the inequality (1.7) proved adequate for the purpose for which it was invented, it cries out for improvement in view of (1.3). The results proved in this paper include the following theorem that encompasses (1.3) and (1.5) in a natural way.

THEOREM 1.1. *Let $T = A + \mathrm{i}B$ be an operator in the Schatten $p$-class, with $A$ positive and $B$ Hermitian. Then*

$$\|T\|_p^2 \leqslant \|A\|_p^2 + 2^{1-2/p}\|B\|_p^2 \quad \textit{for } 2 \leqslant p \leqslant \infty, \tag{1.8}$$

*and*

$$\|T\|_p^2 \geqslant \|A\|_p^2 + 2^{1-2/p}\|B\|_p^2 \quad \textit{for } 1 \leqslant p \leqslant 2. \tag{1.9}$$

*For $p = 1$, the inequality* (1.9) *reduces to*

$$\|T\|_1^2 \geqslant \|A\|_1^2 + \frac{1}{2}\|B\|_1^2, \tag{1.10}$$

*proved in* [6].

The results presented here are related to our recent work [4] where we have proved that when both $A$ and $B$ are positive, then

$$\|T\|_p^2 \leqslant \|A\|_p^2 + \|B\|_p^2 \quad \text{for } 2 \leqslant p \leqslant \infty, \tag{1.11}$$

and

$$\|T\|_p^2 \geqslant \|A\|_p^2 + \|B\|_p^2 \quad \text{for } 1 \leqslant p \leqslant 2. \tag{1.12}$$

To complete the picture, we remark that when $A, B$ and $T$ are as in (1.1) with no further restriction, then

$$\|T\|_p^2 \leqslant 2^{1-2/p}(\|A\|_p^2 + \|B\|_p^2) \quad \text{for } 2 \leqslant p \leqslant \infty, \tag{1.13}$$

and

$$\|T\|_p^2 \geqslant 2^{1-2/p}(\|A\|_p^2 + \|B\|_p^2) \quad \text{for } 1 \leqslant p \leqslant 2. \tag{1.14}$$

In this way, each of the inequalities (1.4)-(1.6) is extended in a most natural way to all $p$-norms.

The inequalities (1.11) and (1.12) are obviously sharp. The $2 \times 2$ example given after (1.4) shows that (1.13) and (1.14) are also sharp. We will show that the inequality (1.8) is sharp too. It is, therefore, surprising that the inequality (1.9) is not sharp. We will prove the following result that improves upon (1.10).

Theorem 1.2. *Let $T = A + \mathrm{i}B$ be an operator in the trace class, with $A$ positive and $B$ Hermitian. Then*

$$\|T\|_1^2 \geqslant \|A\|_1^2 + \|B\|_1^2. \tag{1.15}$$

This raises the question whether the same inequality can be proved for all $1 < p < 2$.

In Section 2 we prove a more general majorisation result from which Theorem 1.1 follows. This majorisation is the basis for several other inequalities, some of which are pointed out in Section 2. In Section 3 we prove a similar majorisation result for general Hermitian $A$ and $B$. This supplements some known results. For simplicity, we state our results for matrices. Minor modifications lead to similar results for compact operators as in [4].

## 2. THE MAIN RESULT

We will derive a majorisation relation from which Theorem 1.1 follows as corollary.

Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be sequences of real numbers whose terms are in decreasing order. If

$$\sum_{j=1}^{k} x_j \leqslant \sum_{j=1}^{k} y_j, \quad k = 1, 2, \ldots \tag{2.1}$$

we write $\{x_j\} \prec_w \{y_j\}$, and say $x$ is *weakly majorised* by $y$. If $x$ and $y$ terminate after $n$ terms, and if in addition to the inequalities (2.1) we have

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j, \tag{2.2}$$

we write $\{x_j\} \prec \{y_j\}$, and say $x$ is *majorised* by $y$. See [2] and [7] for basic properties of these relations.

Let $T$ be an $n \times n$ matrix with real and imaginary parts $A, B$. Let $\{s_j^2\}, \{\alpha_j^2\}$, and $\{\beta_j^2\}$ be the $n$-tuples whose terms are the eigenvalues of $T^*T$, $A^2$, and $B^2$, all arranged in decreasing order. Define another $n$-tuple $\{\gamma_j^2\}$ as follows. If $n$ is an even number and $n = 2m$, then

$$\gamma_j^2 = \begin{cases} \beta_{2j-1}^2 + \beta_{2j}^2 & \text{for } 1 \leqslant j \leqslant m, \\ 0 & \text{for } m+1 \leqslant j \leqslant n. \end{cases}$$

If $n$ is an odd number and $n = 2m + 1$, then

$$\gamma_j^2 = \begin{cases} \beta_{2j-1}^2 + \beta_{2j}^2 & \text{for } 1 \leqslant j \leqslant m, \\ \beta_{2m+1}^2 & \text{for } j = m+1, \\ 0 & \text{for } m+2 \leqslant j \leqslant n. \end{cases}$$

These notations will remain fixed now.

Our main result is the following.

Theorem 2.1. *Let $T = A + \mathrm{i}B$ where $A$ is a positive and $B$ a Hermitian matrix. Then we have the majorisation*

$$\{s_j^2\} \prec \{\alpha_j^2 + \gamma_j^2\}. \tag{2.3}$$

Note that the relation (2.3) is equivalent to the following three statements together:

$$\sum_{j=1}^{k} s_j^2 \leqslant \sum_{j=1}^{k} \alpha_j^2 + \sum_{j=1}^{2k} \beta_j^2 \quad \text{for } 1 \leqslant k < \frac{n}{2}, \tag{2.4}$$

$$\sum_{j=1}^{k} s_j^2 \leqslant \sum_{j=1}^{k} \alpha_j^2 + \sum_{j=1}^{n} \beta_j^2 \quad \text{for } \frac{n}{2} \leqslant k \leqslant n, \tag{2.5}$$

$$\sum_{j=1}^{n} s_j^2 = \sum_{j=1}^{n} \alpha_j^2 + \sum_{j=1}^{n} \beta_j^2. \tag{2.6}$$

Of these, the equation (2.6) is a restatement of (1.3) and is already known to be true.

The following lemma will be used in the proof of the theorem.

LEMMA 2.2. *Let $k, n$ be positive integers with $1 \leqslant k < n/2$. Let $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$, and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n-k}$ be real numbers satisfying the interlacing inequalities*

$$\mu_i \geqslant \lambda_i \geqslant \mu_{i+k} \quad \text{for } i = 1, 2, \ldots, n-k.$$

*Then we can choose $n-2k$ distinct indices $i_1, \ldots, i_{n-2k}$ out of the set $\{1, \ldots, n-k\}$ so that*

$$|\lambda_{i_s}| \geqslant |\mu_{k+s}| \quad \text{for } s = 1, \ldots, n-2k.$$

*Proof.* Three different cases can arise. We list them and the corresponding choices in each case.

(i) If $\mu_{n-k} \geqslant 0$, choose $\{i_1, \ldots, i_{n-2k}\} = \{1, 2, \ldots, n-2k\}$.

(ii) If for some $r$ with $n-k > r \geqslant k+1$, we have $\mu_r \geqslant 0 > \mu_{r+1}$, choose $\{i_1, \ldots, i_{n-2k}\} = \{1, \ldots, r-k, r+1, \ldots, n-k\}$.

(iii) If $0 > \mu_{k+1}$, choose $\{i_1, \ldots, i_{n-2k}\} = \{k+1, \ldots, n-k\}$.

In each case the assertion of the Lemma 2.2 is readily verified. ∎

*Proof of Theorem* 2.1. The proof imitates the ideas in our earlier paper ([4]) but there are a few new subtleties.

Let $k < n/2$. Because of (2.6), the inequality (2.4) is equivalent to

$$\sum_{j=k+1}^{n} s_j^2 \geqslant \sum_{j=k+1}^{n} \alpha_j^2 + \sum_{j=2k+1}^{n} \beta_j^2. \tag{2.7}$$

By a well-known minimum principle ([2], p. 24) there exists an $n \times (n-k)$ matrix $U$, with $U^*U = I$ such that

$$\sum_{j=k+1}^{n} s_j^2 = \operatorname{tr} U^*T^*TU. \tag{2.8}$$

Since $UU^* \leqslant I$,

$$\operatorname{tr} U^*T^*TU \geqslant \operatorname{tr} U^*T^*U \cdot U^*TU.$$

From this, we obtain using the relation (1.3) (with $U^*TU$ in place of $T$),

$$\operatorname{tr} U^*T^*TU \geqslant \operatorname{tr} (U^*AU)^2 + \operatorname{tr} (U^*BU)^2. \tag{2.9}$$

The operators $U^*AU$ and $U^*BU$ are compressions of $A$ and $B$ to an $(n-k)$-dimensional subspace. Hence, by Cauchy's Interlacing Theorem for eigenvalues of Hermitian matrices ([2], Corollary III. 1.5) we have

$$\lambda_j^{\downarrow}(U^*AU) \geqslant \lambda_{j+k}^{\downarrow}(A), \quad 1 \leqslant j \leqslant n-k,$$

and

$$\lambda_j^{\downarrow}(B) \geqslant \lambda_j^{\downarrow}(U^*BU) \geqslant \lambda_{j+k}^{\downarrow}(B), \quad 1 \leqslant j \leqslant n-k.$$

The first of these inequalities shows that

$$\operatorname{tr} (U^*AU)^2 = \sum_{j=1}^{n-k} (\lambda_j^{\downarrow}(U^*AU))^2 \geqslant \sum_{j=k+1}^{n} \alpha_j^2. \tag{2.10}$$

The second inequality, together with Lemma 2.2, shows that

$$\operatorname{tr}(U^*BU)^2 = \sum_{j=1}^{n-k} (\lambda_j^{\downarrow}(U^*BU))^2 \geqslant \sum_{j=2k+1}^{n} \beta_j^2. \tag{2.11}$$

Combining (2.8)–(2.11) we obtain the desired inequality (2.7).

Now let $n/2 \leqslant k \leqslant n$. Again, because of (2.6) the inequality (2.5) is equivalent to

$$\sum_{j=k+1}^{n} s_j^2 \geqslant \sum_{j=k+1}^{n} \alpha_j^2.$$

The proof of this is easier. Just drop the last term in (2.9) and use (2.10). ∎

To derive Theorem 1.1 from Theorem 2.1 we need to establish a relation between the norms of the sequences $\{\gamma_j\}$ and $\{\beta_j\}$. Given an $n$-tuple $\delta = \{\delta_1, \ldots, \delta_n\}$ and an $m$-tuple $\sigma = \{\sigma_1, \ldots, \sigma_m\}$ we write $\delta \vee \sigma$ for the $(n+m)$-tuple obtained by combining them. We identify $\delta \vee 0$ with $\delta$. We write $\delta^2$ for the tuple $\{\delta_1^2, \ldots, \delta_n^2\}$.

Note that from the definition of $\{\gamma_j\}$ it follows that

$$\gamma^2 \vee \gamma^2 \prec 2\beta^2. \tag{2.12}$$

LEMMA 2.3. *We have the inequalities*

$$\|\gamma\|_p^2 \leqslant 2^{1-2/p}\|\beta\|_p^2 \quad \textit{for } 2 \leqslant p \leqslant \infty,$$
$$\|\gamma\|_p^2 \geqslant 2^{1-2/p}\|\beta\|_p^2 \quad \textit{for } 1 \leqslant p \leqslant 2.$$

*Proof.* For all values of $p > 0$

$$\|\gamma\|_p^2 = 2^{-2/p}\|\gamma \vee \gamma\|_p^2 = 2^{-2/p}\|\gamma^2 \vee \gamma^2\|_{p/2}.$$

Using familiar properties of majorisation, one has from (2.12)

$$\|\gamma^2 \vee \gamma^2\|_{p/2} \leqslant 2\|\beta^2\|_{p/2} \quad \text{for } 2 \leqslant p \leqslant \infty,$$

and the opposite inequality for $1 \leqslant p \leqslant 2$. Hence, for $2 \leqslant p \leqslant \infty$, we have

$$\|\gamma\|_p^2 \leqslant 2^{1-2/p}\|\beta^2\|_{p/2} = 2^{1-2/p}\|\beta\|_p^2,$$

and for $1 \leqslant p \leqslant 2$, the inequality is reversed. ∎

*Proof of Theorem* 1.1. Let $\Gamma$ be the positive diagonal operator with $\gamma_1, \ldots, \gamma_n$ down its diagonal. Let $p \geqslant 2$. Using the convexity of the function $f(t) = t^{p/2}$ on the positive half-line, and the Minkowski inequality one obtains from (2.3) the inequality

$$\|T\|_p^2 \leqslant \|A\|_p^2 + \|\Gamma\|_p^2. \tag{2.13}$$

(This argument is used in [4].) Now use Lemma 2.3 to obtain (1.8).

All the inequalities involved in this argument are reversed for $1 \leqslant p \leqslant 2$. This leads to (1.9). ∎

We now discuss the sharpness of the bounds (1.8) and (1.9).

When $p \geqslant 2$, the factor $2^{1-2/p}$ occuring in (1.8) can not be replaced by any smaller number. Borrowing from [5] the example

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix},$$

we see that in this case the right hand side of (1.8) is equal to $1 + 2t^2$ for all values of $p$. If for some $2 < p < \infty$, we could replace the factor $2^{1-2/p}$ in (1.8) by a smaller number, then we would have in this case

$$\|T\|^2 \leqslant \|T\|_p^2 < 1 + (2-\varepsilon)t^2.$$

But by choosing $t$ small we can bring $\|T\|^2$ arbitrarily close to $1 + 2t^2$.

Note that for the operator norm, the inequality (2.13) improves upon (1.5) replacing the term $2\|B\|^2 = 2s_1^2$ by $\|\Gamma\|^2 = s_1^2 + s_2^2$.

Because of symmetry considerations one would expect the inequality (1.9) to be sharp as well. But, this is not the case as we will now see.

*Proof of Theorem* 1.2. First consider the case when $\mathcal{H}$ is 2-dimensional. Because of (1.3) the inequality (1.15) is equivalent to the statement

$$|\det T| \geqslant \det A + |\det B|. \tag{2.14}$$

By applying unitary conjugations, we may assume

$$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \quad B = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix},$$

where $a, b$ are positive and $c, s, t$ are real numbers. The inequality (2.14) then says

$$|(ab - c^2 - st) + \mathrm{i}(at + bs)| \geqslant ab - c^2 + |st|.$$

If $st$ is negative, this is obvious. If $st$ is positive, this inequality follows from

$$(ab - c^2 + st)^2 - (ab - c^2 - st)^2 = 4(ab - c^2)st \leqslant 4abst \leqslant (at + bs)^2.$$

Now let $\mathcal{H}$ be $n$-dimensional. Again, because of (1.3) the inequality (1.15) reduces to the statement

$$\sum_{i<j} s_i s_j \geqslant \sum_{i<j} \alpha_i \alpha_j + \sum_{i<j} |\beta_i \beta_j|,$$

where $s_i, \alpha_i$ and $|\beta_i|$, $1 \leqslant i \leqslant n$, are the singular values of $T, A$, and $B$, respectively. Let $\Lambda^2 T$ denote the second antisymmetric tensor power of $T$. Then, this inequality can be restated as

$$\|\Lambda^2 T\|_1 \geqslant \|\Lambda^2 A\|_1 + \|\Lambda^2 B\|_1. \tag{2.15}$$

For each $i < j$, let $C_{ij}(T)$ be the determinant of the $2 \times 2$ principal submatrix of $T$ defined as $\begin{pmatrix} t_{ii} & t_{ij} \\ t_{ji} & t_{jj} \end{pmatrix}$. These are the diagonal entries of the matrix $\Lambda^2 T$. Since every unitarily invariant norm is diminished by a pinching ([2], p. 97),

$$\|\Lambda^2 T\|_1 \geqslant \sum_{i,j} |C_{ij}(T)|.$$

By (2.14) $|C_{ij}(T)| \geqslant C_{ij}(A) + |C_{ij}(B)|$, for all $i, j$. By applying a unitary conjugation, we may assume that $B$ is diagonal. Then, it is easy to see from the above considerations, that

$$\|\Lambda^2 T\|_1 \geqslant \operatorname{tr} \Lambda^2 A + \operatorname{tr} |\Lambda^2 B|.$$

This proves (2.15). ∎

REMARK 2.4. The arguments we have given lead to an inequality for all unitarily invariant norms. We have from (2.3)

$$\left|\!\left|\!\left| \begin{pmatrix} T^*T & 0 \\ 0 & T^*T \end{pmatrix} \right|\!\right|\!\right| \leqslant \left|\!\left|\!\left| \begin{pmatrix} A^2 & 0 \\ 0 & A^2 \end{pmatrix} \right|\!\right|\!\right| + \left|\!\left|\!\left| \begin{pmatrix} \Gamma^2 & 0 \\ 0 & \Gamma^2 \end{pmatrix} \right|\!\right|\!\right|$$

for every unitarily invariant norm. From this we get using (2.12)

$$\left|\!\left|\!\left| \begin{pmatrix} T^*T & 0 \\ 0 & T^*T \end{pmatrix} \right|\!\right|\!\right| \leqslant \left|\!\left|\!\left| \begin{pmatrix} A^2 & 0 \\ 0 & A^2 \end{pmatrix} \right|\!\right|\!\right| + \left|\!\left|\!\left| \begin{pmatrix} 2B^2 & 0 \\ 0 & 0 \end{pmatrix} \right|\!\right|\!\right|.$$

This implies that for all $Q$-norms ([2]), we have

$$\left\| \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \right\|_Q^2 \leqslant \left\| \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \right\|_Q^2 + 2\left\| \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \right\|_Q^2. \tag{2.16}$$

Since $\|\cdot\|_p$ is a $Q$-norm for $p \geqslant 2$, and

$$\left\| \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \right\|_p = 2^{1/p} \|T\|_p,$$

the inequality (1.8) is included in (2.16).

REMARK 2.5. The majorisation (2.3) can be used to derive several other inequalities. Using the convexity of the function $f(t) = (-1/2)\log t$ on the positive half-line we get from it the family of inequalities

$$\prod_{j=n-k+1}^{n} s_j^2 \geqslant \prod_{j=n-k+1}^{n} (\alpha_j^2 + \gamma_j^2) \quad \text{for } 1 \leqslant k \leqslant n. \tag{2.17}$$

The special case $k = n$, gives

$$|\det T| \geqslant \prod_{j=1}^{n} (\alpha_j^2 + \gamma_j^2)^{1/2}. \tag{2.18}$$

This is a considerable strengthening of a well-known inequality of Ostrowski and Taussky which says

$$|\det T| \geqslant \det A = \prod_{j=1}^{n} \alpha_j$$

whenever $T = A + \mathrm{i}B$ with $A$ positive.

## 3. MORE RESULTS AND REMARKS

In this section we remove the restriction that $A$ be positive, and consider $T = A + \mathrm{i}B$ with $A, B$ Hermitian.

As before, let $\{s_j^2\}, \{\alpha_j^2\}$ and $\{\beta_j^2\}$ be the eigenvalues of $T^*T, A^2$, and $B^2$, all arranged in decreasing order. We associated with $\{\beta_j^2\}$ another $n$-tuple $\{\gamma_j^2\}$; see the discussion preceding Theorem 2.1. Exactly in the same way, construct another $n$-tuple $\{\delta_j^2\}$ out of $\{\alpha_j^2\}$. Examining the proof of Theorem 2.1, one can easily prove the following.

THEOREM 3.1. *Let $T = A + \mathrm{i}B$, where $A$ and $B$ are Hermitian. Then we have the majorisation*

$$\{s_j^2\} \prec \{\delta_j^2 + \gamma_j^2\}. \tag{3.1}$$

If $\Delta$ and $\Gamma$ are the positive diagonal operators, with $\delta_1, \ldots, \delta_n$ and $\gamma_1, \ldots, \gamma_n$ down their diagonals, we can derive from (3.1) the inequality

$$\|T\|_p^2 \leqslant \|\Delta\|_p^2 + \|\Gamma\|_p^2 \quad \text{for } 2 \leqslant p \leqslant \infty, \tag{3.2}$$

and the reverse inequality for $1 \leqslant p \leqslant 2$. These two inequalities, in turn, lead to the pair of inequalities (1.13) and (1.14).

The majorisation (3.1) supplements some known results. Let $\tau_j^2$ be the sequence defined by $\tau_j^2 = (1/2)(s_j^2 + s_{n-j+1}^2)$. Ando and Bhatia ([1]) have shown that

$$\{\tau_j^2\} \prec \{\alpha_j^2 + \beta_j^2\}. \tag{3.3}$$

Bhatia and Kittaneh ([3]) have observed that

$$s^2 \vee s^2 \prec 2\lambda(A^2 + B^2) \vee 0. \tag{3.4}$$

Here, on the left hand side we have the sequence obtained by taking two copies of the eigenvalues of $T^*T$, and on the right hand side the sequence obtained by taking the eigenvalues of $2(A^2 + B^2)$ together with zeros.

Each of the majorisations (3.1), (3.3) and (3.4) could be used to obtain the inequalities (1.13) and (1.14). However, these three relations are independent of each other. More of their consequences and related results may be found in [1], [2], [3] and [10].

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*RAJENDRA BHATIA*
*Indian Statistical Institute*
*New Delhi–110 016*
*INDIA*

*E-mail:* rbh@isid.ac.in

*XINGZHI ZHAN*
*Department of Mathematics*
*East China Normal University*
*Shanghai 200062*
*CHINA*

*E-mail:* zhan@math.ecnu.edu.cn


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