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CONVEX TRACE FUNCTIONS OF SEVERAL VARIABLES ON C^*-ALGEBRAS

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Abstract. For each trace τ on a C^*-algebra A generated by mutually commuting C^*-subalgebras $A_1, A_2, \ldots, A_n$ and every convex function f of n variables we show that the function $(x_1, x_2, \ldots, x_n) \to \tau(f(x_1, x_2, \ldots, x_n))$ is convex on the space $\bigoplus\limits_{k=1}^{n}((A_k)_{\mathrm{sa}})$.

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1. INTRODUCTION

If X is a self-adjoint element in a C^*-algebra A and $\mathrm{sp}(x) \subset I$ for some interval I (we say that $x \in A^I_{\mathrm{sa}}$), then for each f in $C(I)$ we can define $f(x)$ in A by spectral theory. The behaviour of the operator-valued function $x \to f(x)$ on A^I_{sa} so obtained has always received much attention; partly because such functions show up in nearly all operator problems, and partly because the behaviour is highly non-trivial.

In 1934 K. Löwner showed that the function $x \to f(x)$ is operator monotone (increasing), i.e. $x \leqslant y$ implies $f(x) \leqslant f(y)$ when considered in the partially ordered real Banach space A_{sa}, if and only if f has an analytic extension $\widetilde{f}$ to the upper half-plane $\mathbb{C}_+$ such that $\widetilde{f}(\mathbb{C}_+) \subset \mathbb{C}_+$. In 1955 J. Bendat and S. Sherman showed that the function is operator convex, i.e. $f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y)$ in A_{sa}, if and only if $t \to f(t)t^{-1}$ is an operator monotone function (assuming that $0 \in I$ and $f(0) = 0$). A concise account of these results can be found in [11]. The main point of the story is that it is highly unusual for a function to be operator monotone or operator convex.

It is natural to try to generalize Löwner's theory to functions of several variables. A few experiments show that a satisfying spectral theory for functions of several variables can be developed only if the ingoing elements are mutually commuting. We shall therefore assume this situation. Since $\mathbb{R}^n$ really has no good order structure, monotonicity is probably not so interesting a concept. (However, see Corollary 3.3.) But convexity is still very much to the fore, and one may ask for conditions on a function f on $\mathbb{R}^n$ that will guarantee joint convexity of the operator function

$$(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$$

of the commuting variables $x_1, \ldots, x_n$. At the moment there is no completely satisfying characterization of such functions, but recent work by F. Hansen has shed much light on the problem (see [2], [8], [9]).

If we follow the operator function $x \to f(x)$ by a trace τ, to obtain the scalar function $x \to \tau(f(x))$, then general philosophy predicts that the non-commutativity of the problem disappears, since the trace is unable to distinguish xy from yx. (General philosopy does not provide any proofs, though.) Thus $x \to \tau(f(x))$ becomes an increasing function (on A^I_{sa}) whenever f is increasing (cf. [12], Theorems 2.3 and 2.4) and $x \to \tau(f(x))$ is convex as long as f itself is convex (cf. [20], Theorem 4).

In a recent paper ([10]) Hansen verified the general principle mentioned above by showing that for each convex function f of n variables the function

$$(x_1, \ldots, x_n) \to \mathrm{Tr}(f(x_1, \ldots, x_n))$$

is convex on $\bigoplus \mathcal{M}_{m_k}$. Here $f(\underline{x})$ is regarded as an element in $\bigotimes \mathcal{M}_{m_k} = \mathcal{M}_m$, where $m = m_1 m_2 \cdots m_n$, and Tr is the trace on $\mathcal{M}_m$. The proof is quite short, but builds on the general (and highly interesting) theory for operator convex functions developed in [8]. The aim of the present note is to give a self-contained proof of Hansen's result, but now extended to the infinite dimensional case and for general traces. The strategy of our proof is not original, though, but uses the applications of the Fréchet differential which Hansen and the author have developed over the years (see [2], [8], [9], [11], [12], [19]).

2. NOTATION

We consider mutually commuting C^*-algebras $A_1, \ldots, A_n$ of operators on a Hilbert space $\mathfrak{H}$ and denote by A the C^*-algebra they generate. For simplicity we assume that each A_k contains the unit 1 of $\mathcal{B}(\mathfrak{H})$. Evidently this means that A is a quotient of the maximal tensor product $\bigotimes_{\max} A_k$, but, remarkably enough, tensor product arguments will not play any rôle in the sequel.

Assume now that $I_1, \ldots, I_n$ are intervals in $\mathbb{R}$ (bounded or not) and put $\underline{I} = I_1 \times \cdots \times I_n \subset \mathbb{R}^n$. If we let $(A_k)^{I_k}_{\mathrm{sa}}$ denote the set of self-adjoint elements in A_k with spectra contained in I then for each function f in $C(\underline{I})$ and $\underline{x} = (x_1, \ldots, x_n)$ in $\bigoplus((A_k)^{I_k}_{\mathrm{sa}})$ we can define an element $f(\underline{x})$ in A. If $f = f_1 \otimes \cdots \otimes f_n$ this element is simply given by $f(\underline{x}) = f_1(x_1) \cdots f_n(x_n)$. In the general case f is a limit, uniformly on compact subsets of $\underline{I}$, of linear combinations of such pure tensor product functions, e.g. polynomials in n variables, and $f(\underline{x})$ can be defined by a limit argument. Alternatively we may assume that each A_k is a von Neumann algebra (replacing if necessary A_k by A_k''), so that $x_k = \int \lambda_k \, \mathrm{d}E_k(\lambda_k)$. Since the spectral measures $E_1, \ldots, E_n$ are mutually commuting we can define the spectral measure E on $\underline{I}$ by $E(S_1 \times \cdots \times S_n) = E_1(S_1) \cdots E_n(S_n)$, and then write

$$f(\underline{x}) = \int f(\lambda_1, \ldots, \lambda_n)\, \mathrm{d}E(\lambda_1, \ldots, \lambda_n).$$

In this paper we shall quickly reduce our considerations to functions f that can be represented in the form $f(\underline{u}) = \int \mathrm{e}^{\mathrm{i}\underline{t}\cdot\underline{u}} \widehat{f}(\underline{t})\, \mathrm{d}\underline{t}$, and in that case we can simply define $f(\underline{x}) = \int \mathrm{e}^{\mathrm{i}\underline{t}\cdot\underline{x}} \widehat{f}(\underline{t})\, \mathrm{d}\underline{t}$ as an operator integral in A over $\mathbb{R}^n$.

For fixed $\underline{x}$ the map $f \to f(\underline{x})$ is a $*$-homomorphism from $C(\underline{I})$ into A. The support of this homomorphism is a compact subset of $\underline{I}$ which we may call the joint spectrum of the n-tuple $\underline{x}$. The map can therefore be regarded as the proper generalization of the one-variable spectral theory (for self-adjoint operators), cf. [13] for an early reference. We shall here be interested in the behaviour of the continuous map $\underline{x} \to \tau(f(\underline{x}))$ from $\bigoplus((A_k)_{\mathrm{sa}}^{I_k})$ to $\mathbb{R}$ for a fixed function f and a trace τ on A.

3. MAIN RESULTS

Theorem 3.1. *If τ is a bounded trace on a unital C^*-algebra A generated by mutually commuting unital C^*-subalgebras $A_1, \ldots, A_n$ and f is a continuous, real and convex function on $\underline{I}$, then we obtain a convex operator function φ defined by*

$$\varphi(\underline{x}) = \tau(f(\underline{x})) \quad \textit{on the convex set } \bigoplus_{k=1}^{n}((A_k)_{\mathrm{sa}}^{I_k}).$$

Proof. In order to avoid unnecessary considerations about non-standard Borel spaces we notice that it suffices to prove the theorem for arbitrary separable C^*-subalgebras of the A_k's, so that we may assume that A is separable.

The set $T(A)$ of finite traces on A is a lattice cone (cf. [23] or [17], Theorem 3.1). The predual was described in Proposition 2.7 of [5]. Since A is separable the set $T_1(A)$ of tracial states of A is metrizable Choquet simplex, so there is a (unique) extreme probability measure μ on $T_1(A)$ such that $\tau = \int \sigma\, \mathrm{d}\mu(\sigma)$, cf. [4] or Theorem 9 of [21]. The extremality of μ implies that if $\partial T_1(A)$ denotes the G_δ-set of extreme points of $T_1(A)$, then μ is concentrated on $\partial T_1(A)$. In particular,

$$\tau(x) = \int_{\partial T_1(A)} \sigma(x)\, \mathrm{d}\mu(\sigma)$$

for every x in A.

Since the integral of convex functions is convex it follows that it suffices to prove the theorem for a tracial state τ in $\partial T_1(A)$. In that case, if $(\pi_\tau, \mathfrak{H}_\tau)$ denotes the GNS representation of τ, the von Neumann algebra $\pi_\tau(A)''$ is a finite factor (since every non-scalar central element z in the positive part of the unit ball of $\pi_\tau(A)''$ will produce a non-trivial trace $\tau(z\,\cdot\,)$ majorized by τ). The von Neumann subalgebras $\pi_\tau(A_k)''$ are mutually commuting and generate $\pi_\tau(A)''$, so each of these algebras is also a finite factor. If $x_k \in \pi_\tau(A_k)''_+$, the function

$x_1 \to \tau(x_1 x_2 \cdots x_n)$ is a trace on $\pi_\tau(A_1)''$, hence proportional to the restriction $\tau|\pi_\tau(A_1)''$. Consequently $\tau(x_1 x_2 \cdots x_n) = \tau(x_1)\tau(x_2 \cdots x_n)$. By iteration we find that $\tau(x_1 x_2 \cdots x_n) = \tau(x_1)\tau(x_2) \cdots \tau(x_n)$ for any n-tuple $(x_1, x_2, \ldots, x_n)$ in A such that $x_k \in \pi_\tau(A_k)''$ for every k.

Replacing A_k by $\pi_\tau(A_k)''$ and A with the C^*-algebra generated by the von Neumann algebras $\pi_\tau(A_k)''$, but replacing also τ with its extension to $\pi_\tau(A)''$, we have an extension of the old function φ, even though we now compute the elements $f(\underline{x})$ in a new algebra. This means that it suffices to prove the theorem assuming that each A_k is a von Neumann algebra, and furthermore we may assume that

$$\tau(x_1 x_2 \cdots x_n) = \tau(x_1)\tau(x_2) \cdots \tau(x_n),$$

whenever $x_k \in A_k$ for each k.

Clearly it suffices to prove convexity of φ on bounded subcubes of $\underline{I}$. We may therefore assume that f is bounded on $\underline{I}$ and extend it to a continuous function $\widetilde{f}$ with compact support on $\mathbb{R}^n$. If now e is a positive C^∞-function on $\mathbb{R}^n$ with small support $[-\varepsilon, \varepsilon]^n$, then the function

$$g(\underline{u}) = \int_{\mathbb{R}^n} \widetilde{f}(\underline{u} - \underline{v}) e(\underline{v}) \, \mathrm{d}\underline{v}$$

is a Schwartz function on $\mathbb{R}^n$ and convex on a slightly smaller cube $\underline{I}_\varepsilon$. It follows by approximation that it suffices to prove the theorem for a Schwartz function f on $\mathbb{R}^n$ which is convex on the bounded cube $\underline{I}$. Thus we have a representation

$$f(\underline{u}) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\underline{t}\cdot\underline{u}} \widehat{f}(\underline{t}) \, \mathrm{d}\underline{t}, \quad \underline{u} \in \mathbb{R}^n.$$

For later use we note that this means that

$$f'_k(\underline{u}) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\underline{t}\cdot\underline{u}} \mathrm{i} t_k \widehat{f}(\underline{t}) \, \mathrm{d}\underline{t},$$

and similarly for the higher derivatives. These combined reductions mean that we now consider a function φ of the form

$$(*) \qquad \varphi(\underline{x}) = \int_{\mathbb{R}^n} \tau(\mathrm{e}^{\mathrm{i}t_1 x_1}) \cdots \tau(\mathrm{e}^{\mathrm{i}t_n x_n}) \widehat{f}(\underline{t}) \, \mathrm{d}\underline{t}.$$

It is well known (see e.g. [6], Exercises 3.1.8 and 3.6.4), that a Fréchet differentiable real function φ on (an open subset of) a real Banach space is convex if and only if

$$\mathrm{d}\varphi_x(h) \leqslant \varphi(x + h) - \varphi(x),$$

which again (when φ is twice differentiable) is equivalent to its second Fréchet differential being a positive (semi-)definite quadratic form. Thus we must show that

$$\mathrm{d}^2\varphi_{\underline{x}}(\underline{h}, \underline{h}) \geqslant 0$$

for each $\underline{x}$ in $\bigoplus((A_k)_{\mathrm{sa}}^{I_k})$ and every $\underline{h}$ in $\bigoplus((A_k)_{\mathrm{sa}})$. To do this we have the information that $\mathrm{d}^2 f$ is positive definite, i.e. the Hesse matrix $\big(f''_{kl}(\underline{u})\big)$ is positive definite for each $\underline{u}$ in $\underline{I}$.

Using the Dyson expansion we find that the exponential function is Fréchet differentiable on any Banach algebra with

$$\mathrm{d}(\exp)_x(h) = \int_0^1 \mathrm{e}^{sx} h\, \mathrm{e}^{(1-s)x}\, \mathrm{d}s$$

(cf. [1] or [12], Proposition 1.3). It follows that the operator function $\underline{x} \to f(\underline{x})$ has the differential

$$\mathrm{d}f_{\underline{x}}(\underline{h}) = \sum_k \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}t_1 x_1} \cdots \left(\int_0^1 \mathrm{e}^{s\mathrm{i}t_k x_k} \mathrm{i}t_k h_k \mathrm{e}^{(1-s)\mathrm{i}t_k x_k}\, \mathrm{d}s \right) \cdots \mathrm{e}^{\mathrm{i}t_n x_n} \widehat{f}(\underline{t})\, \mathrm{d}\underline{t}.$$

But then, using the multiplicative character of the trace τ,

$$\mathrm{d}\varphi_{\underline{x}}(\underline{h}) = \tau(\mathrm{d}f_{\underline{x}}(\underline{h})) = \sum_k \int_{\mathbb{R}^n} \tau(\mathrm{e}^{\mathrm{i}\underline{t}\cdot\underline{x}} h_k) \mathrm{i}t_k \widehat{f}(\underline{t})\, \mathrm{d}\underline{t} = \sum_k \tau(f'_k(\underline{x}) h_k).$$

In other words, the operator functions $\mathrm{d}f_{\underline{x}}(\underline{h})$ and $\sum f'_k(\underline{x}) h_k$ (for fixed $\underline{h}$) have the same trace. So, therefore, have their differentials. But for the second function we can compute the Fréchet differential using the same method as above, and taking the trace afterwards we find that

$$\begin{aligned}
\mathrm{d}^2\varphi_{\underline{x}}(\underline{h}, \underline{h}) &= \tau(\mathrm{d}^2 f_{\underline{x}}(\underline{h}, \underline{h})) = \tau(\mathrm{d}(\,\mathrm{d}f_{(\cdot)}(\underline{h}))_{\underline{x}}(h)) = \tau\Big(\mathrm{d}\Big(\sum_k f'_k(\underline{\cdot}) h_k\Big)_{\underline{x}}(\underline{h})\Big) \\
&= \sum_{k,l} \int_{\mathbb{R}^n} \tau\left(\mathrm{e}^{\mathrm{i}t_1 x_1} \cdots \left(\int_0^1 \mathrm{e}^{s\mathrm{i}t_l x_l} \mathrm{i}t_l h_l \mathrm{e}^{(1-s)\mathrm{i}t_l x_l}\, \mathrm{d}s \right) \cdots \mathrm{e}^{\mathrm{i}t_n x_n} h_k \right) \widehat{f'_k}(\underline{t})\, \mathrm{d}\underline{t} \\
&= \sum_{k,l} D_{kl}. \quad \text{(To give a name for the different summands.)}
\end{aligned}$$

For $k \neq l$ the multiplicative nature of τ implies that

$$D_{kl} = \int_{\mathbb{R}^n} \tau(\mathrm{e}^{\mathrm{i}\underline{t}\cdot\underline{x}} h_l h_k) \mathrm{i}t_l \widehat{f'_k}(\underline{t})\, \mathrm{d}\underline{t} = \tau(f''_{kl}(\underline{x}) h_l h_k).$$

For $k = l$ we get the more complicated expression

$$D_{kk} = \int_{\mathbb{R}^n} \tau\left(\mathrm{e}^{\mathrm{i}t_1 x_1} \cdots \left(\int_0^1 \mathrm{e}^{s\mathrm{i}t_k x_k} h_k \mathrm{e}^{(1-s)\mathrm{i}t_k x_k} h_k\, \mathrm{d}s \right) \cdots \mathrm{e}^{\mathrm{i}t_n x_n} \right) \mathrm{i}t_k \widehat{f'_k}(\underline{t})\, \mathrm{d}\underline{t}.$$

To compute D_{kk} in more detail we need more rigid spectral decompositions. But, since each A_k is a von Neumann algebra it suffices by approximation to prove that $\mathrm{d}^2\varphi_{\underline{x}}$ is positive definite, assuming that each x_k has the form

$$x_k = \sum \lambda_{n_k} p_{n_k},$$

where $\lambda_{n_k} \in I_k$ and $\{p_{n_k}\}$ is a finite family of pairwise orthogonal projections in A_k with sum 1. Using the concept of multi-index $\alpha = (n_1, \dots, n_n)$ we let $\underline{\lambda}_\alpha = (\lambda_{n_1}, \dots, \lambda_{n_n})$ and $p_\alpha = p_{n_1} \cdots p_{n_n}$. This means that we can write

$$f(\underline{x}) = \sum_\alpha \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\underline{t}\cdot\underline{\lambda}_\alpha} p_\alpha \widehat{f}(\underline{t})\,\mathrm{d}\underline{t} = \sum_\alpha f(\underline{\lambda}_\alpha) p_\alpha.$$

Returning to our previous formula $\mathrm{d}^2\varphi_{\underline{x}}(\underline{h},\underline{h}) = \sum D_{kl}$ we find for $k \neq l$ that

$$D_{kl} = \tau(f''_{kl}(\underline{x}) h_k h_l) = \sum_\alpha f''_{kl}(\underline{\lambda}_\alpha)\tau(p_\alpha h_k h_l).$$

Note now that if $x = \sum \lambda_n p_n$ then

$$\begin{aligned}\int_0^1 \tau(\mathrm{e}^{sitx}\mathrm{i}th\,\mathrm{e}^{(1-s)\mathrm{i}tx}\mathrm{i}th)\,\mathrm{d}s &= \sum_{n,m}\int_0^1 \tau(\mathrm{e}^{sit\lambda_n}p_n\mathrm{i}th\,\mathrm{e}^{(1-s)\mathrm{i}t\lambda_m}p_m\mathrm{i}th)\,\mathrm{d}s\\ &= \sum_{n\neq m}(\mathrm{e}^{\mathrm{i}t\lambda_n} - \mathrm{e}^{\mathrm{i}t\lambda_m})(\lambda_n - \lambda_m)^{-1}\tau(p_nhp_mh)\mathrm{i}t + \sum_n \mathrm{e}^{\mathrm{i}t\lambda_n}\tau(p_nhp_nh)(\mathrm{i}t)^2.\end{aligned}$$

Inserting this in the expression for D_{kk} we get

$$\begin{aligned}D_{kk} &= \sum_\Delta (f'_k(\underline{\lambda}_\alpha) - f'_k(\underline{\lambda}_\beta))(\underline{\lambda}_{\alpha(k)} - \underline{\lambda}_{\beta(k)})^{-1}\tau(p_\alpha h_k p_\beta h_k)\\ &\quad + \sum_\alpha f''_{kk}(\underline{\lambda}_\alpha)\tau(p_\alpha h_k p_\alpha h_k).\end{aligned}$$

Here Δ is the set of pairs of multi-indices (α,β), so that $\alpha(l) = \beta(l)$ for all $l \neq k$, but $\alpha(k) \neq \beta(k)$.

Since f is convex, each of its partial derivatives is monotone increasing. Consequently

$$(f'_k(\underline{\lambda}_\alpha) - f'_k(\underline{\lambda}_\beta))(\underline{\lambda}_{\alpha(k)} - \underline{\lambda}_{\beta(k)})^{-1} \geqslant 0$$

for all α, β in Δ. Moreover, since $h_k = h_k^*$,

$$\tau(p_\alpha h_k p_\beta h_k) = \tau(p_\alpha h_k p_\beta h_k p_\alpha) \geqslant 0.$$

Deleting the positive Δ-terms from D_{kk} we therefore have

$$(**)\qquad \mathrm{d}^2\varphi_{\underline{x}}(\underline{h},\underline{h}) \geqslant \sum_\alpha\sum_{k\neq l} f''_{kl}(\underline{\lambda}_\alpha)\tau(p_\alpha h_k h_l) + \sum_\alpha\sum_k f''_{kk}(\underline{\lambda}_\alpha)\tau(p_\alpha h_k p_\alpha h_k).$$

For a fixed $\alpha = (n_1, \dots, n_n)$ we put $p_k = p_{n_k}$, $p_l = p_{n_l}$ and $q = \prod p_{n_j}$, $j \neq k, l$. Then with $\gamma_k = \tau(p_k h_k)\tau(p_k)^{-1}$ we can write

$$\tau(p_\alpha h_k h_l) = \tau(q)\tau(p_k h_k)\tau(p_l h_l) = \tau(p_\alpha)\gamma_k\gamma_l.$$

If $r = \prod p_{n_j}$, $j \neq k$, then by the Cauchy-Schwarz inequality we get

$$\begin{aligned}\tau(p_\alpha)\gamma_k^2 &= \tau(p_\alpha)\tau(p_k)^{-2}\tau(p_k h_k)^2 = \tau(p_\alpha)^{-1}\tau(r)^2\tau(p_k h_k)^2\\ &= \tau(p_\alpha)^{-1}\tau(p_\alpha h_k)^2 = \tau(p_\alpha)^{-1}\tau(p_\alpha p_\alpha h_k p_\alpha)^2\\ &\leqslant \tau(p_\alpha)^{-1}(\tau(p_\alpha)\tau(p_\alpha h_k p_\alpha h_k p_\alpha)) = \tau(p_\alpha h_k p_\alpha h_k).\end{aligned}$$

Inserting these estimates in $(**)$ we see that for each α we have

$$\sum_{k\neq l} f''_{kl}(\underline{\lambda}_\alpha)\tau(p_\alpha h_k h_l) + \sum_k f''_{kk}(\underline{\lambda}_\alpha)\tau(p_\alpha h_k p_\alpha h_k)$$
$$\geqslant \sum_{k\neq l} f''_{kl}(\underline{\lambda}_\alpha)\tau(p_\alpha)\gamma_k\gamma_l + \sum_k f''_{kk}(\underline{\lambda}_\alpha)\tau(p_\alpha)\gamma_k^2 = \tau(p_\alpha)\sum_{k,l} f''_{kl}(\underline{\lambda}_\alpha)\gamma_k\gamma_l \geqslant 0,$$

because the Hessian $(f''_{kl}(\underline{\lambda}_\alpha))$ is positive definite. Thus $\mathrm{d}^2\varphi_{\underline{x}}(\underline{h},\underline{h})$ majorizes a sum (over all multi-indices) of positive terms, hence is positive. ∎

3.2. REMARK. In the course of the proof of Theorem 3.1 we found a complete formula for the second Fréchet differential of the function $\varphi(\underline{x}) = \tau(f(\underline{x}))$ defined on n-tuples in $\bigoplus_{k=1}^{n}((A_k)_{\mathrm{sa}}^{I_k})$, regardless of the convexity of f, but assuming that τ is multiplicative on products from the A_k's. Assuming also that each x_k has the form $x_k = \sum \lambda_{n_k} p_{n_k}$, where $\lambda_{n_k} \in I_k$ and $\{p_{n_k}\}$ is a finite family of pairwise orthogonal projections in A_k with sum 1, and using the concept of multi-index $\alpha = (n_1, \ldots, n_n)$, so that $\underline{\lambda}_\alpha = (\lambda_{n_1}, \ldots, \lambda_{n_n})$ and $p_\alpha = p_{n_1}\cdots p_{n_n}$, we can write

$$\mathrm{d}^2\varphi_{\underline{x}}(\underline{h},\underline{h}) = \sum_{k,l}\sum_\alpha (f''_{kl}(\underline{\lambda}_\alpha)\tau(p_\alpha h_k p_\alpha h_l)$$
$$+ \sum_k \sum_{\Delta_k} (f'_k(\underline{\lambda}_\alpha) - f'_k(\underline{\lambda}_\beta))(\underline{\lambda}_{\alpha(k)} - \underline{\lambda}_{\beta(k)})^{-1}\tau(p_\alpha h_k p_\beta h_k),$$

where Δ_k denotes the set of pairs of multi-indices (α,β), so that $\alpha(l) = \beta(l)$ for all $l \neq k$, but $\alpha(k) \neq \beta(k)$. The first part of this expression is very nearly a sum of quadratic forms (and becomes one if we replace everywhere in the diagonals $\tau(p_\alpha h_k p_\alpha h_k)$ by the smaller number $\tau(p_\alpha)(\tau(p_{\alpha(k)}h_k)^2(\tau(p_{\alpha(k)})^{-2})$. The second part is a sum of difference quotients, not unlike those encountered in Löwner's theory for matrix monotonicity. We see that $\mathrm{d}^2\varphi_{\underline{x}}$ is positive if the Hesse matrix $\mathrm{d}^2 f(\underline{\lambda}_\alpha)$ is positive for all $\underline{\lambda}_\alpha$, and if moreover $f'_k(\underline{\lambda}_\beta) \leqslant f'_k(\underline{\lambda}_\alpha)$ for all α,β in Δ_k such that $\underline{\lambda}_{\beta(k)} < \underline{\lambda}_{\alpha(k)}$.

3.3. COROLLARY. *If τ is a bounded trace on A and f is a continuous real function on $\underline{I}$ which is increasing in the sense that $f(\underline{u}) \leqslant f(\underline{u}+\underline{v})$ whenever $\underline{v} \in \mathbb{R}^n_+$, then the function $\varphi(\underline{x}) = \tau(f(\underline{x}))$ is increasing on $\bigoplus((A_k)_{\mathrm{sa}}^{I_k})$ in the same sense.*

Proof. We may assume that τ is multiplicative on $A = A_1A_2\cdots A_n$ and that f is a Schwartz function on $\mathbb{R}^n$ and monotone increasing on $\underline{I}$. But then in the proof of Theorem 3.1 we showed that

$$\mathrm{d}\varphi_{\underline{x}}(\underline{h}) = \sum_k \tau(f'_k(\underline{x})h_k).$$

By assumption $f'_k \geqslant 0$ for every k, so if $\underline{h} \in \bigoplus((A_k)_+)$ we see that

$$\tau(f'_k(\underline{x})h_k) = \tau(h_k^{1/2}f'_k(\underline{x})h_k^{1/2}) \geqslant 0,$$

whence $\mathrm{d}\varphi_{\underline{x}}(\underline{h}) \geqslant 0$.

Thus, by Theorem 2.1 of [12] or Theorem 2.7 of [19],

$$\varphi(\underline{x} + \underline{h}) - \varphi(\underline{x}) = \int_0^1 \mathrm{d}\varphi_{\underline{x}+s\underline{h}}(\underline{h})\,\mathrm{d}s \geqslant 0$$

if $\underline{h} \geqslant 0$, as desired. ∎

4. UNBOUNDED TRACES

We wish to consider unbounded traces on a C^*-algebra A generated by mutually commuting C^*-subalgebras $A_1, \dots, A_n$. Here the unit is just in the way, and we shall assume instead that τ is a densely defined, lower semi-continuous trace on A.

Let $K(A)_+$ denote the cone in A_+ hereditarily generated by sums of elements x in A_+, such that $xy = x$ for some y in A_+. Then $K(A) = \operatorname{span} K(A)_+$ is the minimal dense ideal of A (cf. [18], 5.6). It follows that the set $T(A)$ of densely defined, lower semi-continuous traces on A can be identified with the space of positive tracial functionals on $K(A)$ (cf. [18], 5.6.7). In our situation, where A is generated by products $A_1 A_2 \cdots A_n$ we note that $K(A_1)K(A_2)\cdots K(A_n) \subset K(A)$. In fact, $K(A)_+$ is hereditarily generated by sums of such products. To see this, consider x_k, y_k in $(A_k)_+$ for $1 \leqslant k \leqslant n$, such that $x_k y_k = x_k$. Then with $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_n$ we have $xy = x$, whence $x \in K(A)$.

4.1. THEOREM. *With notations as above, if τ is a densely defined, lower semi-continuous trace on A then for every positive, continuous and convex function f on $\underline{I} = I_1 \times \cdots \times I_n$, where $0 \in I_k$ for each k and $f(\underline{0}) = 0$, the operator function*

$$\varphi(\underline{x}) = \tau(f(\underline{x})) \quad \text{on the convex set } \bigoplus_{k=1}^{n} ((A_k)_{\mathrm{sa}}^{I_k})$$

is convex, possibly with infinite values.

Proof. As in the proof of Theorem 3.1 we may assume that A is separable. By assumption τ belongs to the simplicial cone $T(A)$ of densely defined, lower semi-continuous traces on A, cf. [17], Theorems 3.1 and 3.2. By Theorem 5.8.3 and its Corollaries 5.8.4 and 5.8.5 of [18] each τ in $T(A)$ has the form $\int \tau_t \,\mathrm{d}\mu(t)$, where $t \to \tau_t$ is a Borel map into the set of characters in $T(A)$, i.e. points on the extreme rays of $T(J)$ (cf. [21], Proposition 11.1).

As in the proof of Theorem 3.3 it follows that it suffices to prove the result, assuming that τ is a character in $T(A)$. In that case, if $(\pi_\tau, \mathfrak{H}_\tau)$ denotes the GNS representation of τ, the von Neumann algebra $\pi_\tau(A)''$ is a semi-finite factor (since every non-scalar central element z in the positive part of the unit ball of $\pi_\tau(A)''$ will produce a non-trivial trace $\tau(z\,\cdot)$ majorized by τ). The von Neumann subalgebras $\pi_\tau(A_k)''$ are mutually commuting and generate $\pi_\tau(A)''$, so each of these algebras is also a factor. For each $(n-1)$-tuple $x_2, \cdots, x_n$ where $x_k \in K(A_k)_+$ the map $x_1 \to \tau(x_1 x_2 \cdots x_n)$ on $K(A_1)$ extends to a semi-finite normal trace τ_1 on $\pi_\tau(A_1)''$. It follows by iteration that each of the factors $\pi_\tau(A_k)''$ is semi-finite with a trace τ_k and that the function $x \to \tau_1(x_1)\tau_2(x_2)\cdots\tau_n(x_n)$, defined for any element $x = x_1 x_2 \cdots x_n$ in $\pi_\tau(A)''$ such that $x_k \in K(A_k)$ for every k, extends to a semi-finite trace on $\pi_\tau(A)''$, hence must be proportional to

τ. Consequently we may assume that $\tau(x_1x_2\cdots x_n)=\tau_1(x_1)\tau_2(x_2)\cdots\tau_n(x_n)$ for any such n-tuple. Moreover, if J_k denotes the ideal of definition for τ_k we have $K(J_1)K(J_2)\cdots K(J_n)\subset K(J_\tau(\pi_\tau(A)''))$.

Replacing A_k with J_k and A with the C^*-algebra generated by the ideals J_k in the von Neumann algebras $\pi_\tau(A_k)''$, but replacing also τ with its extension to $\pi_\tau(A)''$, we have an extension of the old function φ, even though we now compute the elements $f(\underline{x})$ in a new algebra. This means that it suffices to prove the theorem assuming that each A_k is a norm-closed ideal in some von Neumann algebra and has a lower semi-continuous, densely defined trace τ_k such that

$$\tau(x_1x_2\cdots x_n)=\tau_1(x_1)\tau_2(x_2)\cdots\tau_n(x_n)$$

whenever $x_k\in K(A_k)$ for each k.

From now on the proof proceeds much as in the finite case. We reduce to the case where f is a Schwartz function on $\mathbb{R}^n$, convex on some bounded cube $\underline{I}=I_1\times\cdots\times I_n$ in $\mathbb{R}^n$, so that we can define

$$f(\underline{x})=\int\limits_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\underline{t}\cdot\underline{x}}\widehat{f}(\underline{t})\,\mathrm{d}\underline{t}.$$

Since A generated by the commuting algebras A_k, each of which is a closed ideal in some von Neumann algebra, we see by approximation that it suffices to prove convexity of the function $\varphi(\underline{x})=\tau(f(\underline{x}))$, assuming that $\underline{x}=(x_1,\ldots,x_n)$ where for each k we have $x_k=\sum\lambda_{n_k}p_{n_k}$. Here $\lambda_{n_k}\in I_k$ (and are pairwise distinct), and $\{p_{n_k}\}$ is a finite family of pairwise orthogonal projections with sum 1 such that $p_{n_k}\in K(A_k)$ unless $\lambda_{n_k}=0$. Thus, adopting the multi-index notation we have

$$f(\underline{x})=\sum_\alpha\int\limits_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\underline{t}\cdot\underline{\lambda}_\alpha}p_\alpha\widehat{f}(\underline{t})\,\mathrm{d}\underline{t}=\sum_\alpha f(\underline{\lambda}_\alpha)p_\alpha,$$

and $p_\alpha\in K(A)$ unless $\underline{\lambda}_\alpha=\underline{0}$.

To prove positivity of the second Fréchet differential of φ we further observe that by approximation it suffices to show that

$$\mathrm{d}^2\varphi_{\underline{x}}(\underline{h},\underline{h})\geqslant 0$$

when $\underline{h}=(h_1,\ldots,h_n)$ and $h_k\in K(A_k)_{\mathrm{sa}}$ for all k. By computation we get

$$\begin{aligned}\mathrm{d}f_{\underline{x}}(\underline{h})&=\sum_k\int\limits_{\mathbb{R}^n}\int\limits_0^1\mathrm{e}^{s\mathrm{i}\underline{t}\cdot\underline{x}}\mathrm{i}t_kh_k\,\mathrm{e}^{(1-s)\mathrm{i}\underline{t}\cdot\underline{x}}\,\mathrm{d}s\widehat{f}(\underline{t})\,\mathrm{d}\underline{t}\\&=\sum_k\sum_{\alpha,\beta}\int\limits_{\mathbb{R}^n}\int\limits_0^1\mathrm{e}^{s\mathrm{i}\underline{t}\cdot\underline{\lambda}_\alpha}p_\alpha h_k\mathrm{e}^{(1-s)\mathrm{i}\underline{t}\cdot\underline{\lambda}_\beta}p_\beta\,\mathrm{d}s\mathrm{i}t_k\widehat{f}(\underline{t})\,\mathrm{d}\underline{t}.\end{aligned}$$

Consequently, since $p_\alpha p_\beta=0$ if $\alpha\neq\beta$,

$$\begin{aligned}\mathrm{d}\varphi_{\underline{x}}(\underline{h})=\tau(\mathrm{d}f_{\underline{x}}(\underline{h}))&=\sum_k\sum_\alpha\int\limits_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\underline{t}\cdot\underline{\lambda}_\alpha}\tau(p_\alpha h_k)\mathrm{i}t_k\widehat{f}(\underline{t})\,\mathrm{d}\underline{t}\\&=\sum_k\sum_\alpha f_k'(\underline{\lambda}_\alpha)\tau(p_\alpha h_k)=\sum_k\tau(f_k'(\underline{x})h_k).\end{aligned}$$

Thus $\mathrm{d}f_{\underline{x}}(\underline{h})$ and $\sum f'_k(\underline{x})h_k$ have the same trace. Computing the Fréchet differential of the second function (with respect to $\underline{x}$ and the increment $\underline{h}$) we find the expression

$$\sum_{k,l}\sum_{\alpha,\beta}\int_{\mathbb{R}^n}\int_0^1 \mathrm{e}^{s\mathrm{i}\underline{t}\cdot\underline{\lambda}_\alpha}p_\alpha h_l\,\mathrm{e}^{(1-s)\mathrm{i}\underline{t}\cdot\underline{\lambda}_\beta}p_\beta\,\mathrm{d}s h_k \mathrm{i}t_l\widehat{f}'_k(\underline{t})\,\mathrm{d}\underline{t}.$$

Taking the trace this means, as in the bounded case, that $\mathrm{d}^2\varphi_{\underline{x}}(\underline{h},\underline{h}) = \sum\limits_{kl} D_{kl}$. For $k \neq l$ we get

$$D_{kl} = \sum_\alpha \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\underline{t}\cdot\underline{\lambda}_\alpha}\tau(p_\alpha h_k h_l)\mathrm{i}t_l\widehat{f}'_k(\underline{t})\,\mathrm{d}\underline{t} = \sum_\alpha f''_{kl}(\underline{\lambda}_\alpha)\tau(p_\alpha h_k h_l) = \tau(f''_{kl}(\underline{x})h_k h_l),$$

using the multiplicative form of τ. For $k = l$ we find that

$$D_{kk} = \sum_\Delta (f'_k(\underline{\lambda}_\alpha) - f'_k(\underline{\lambda}_\beta))(\lambda_{\alpha(k)} - \lambda_{\beta(k)})^{-1}\tau(p_\alpha h_k p_\alpha h_k) + \sum_\alpha f''_{kk}(\underline{\lambda}_\alpha)\tau(p_\alpha h_k p_\alpha h_k).$$

The expressions for the D_{kl} are thus exactly as in the bounded case, and the proof is completed as in Theorem 3.1. ∎

4.2. Remark. The condition above that τ be densely defined can be relaxed. All that is needed for the argument to go through is that τ restricted to its ideal of definition J_τ has a desintegration $\tau = \int \sigma \mathrm{d}\mu(\sigma)$, such that each σ splits as a product of non-trivial traces on the subalgebras A_k. Clearly, the closer A is to a tensor product $A = \bigotimes A_k$ the more likely this is to happen.

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