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HECKE C^*-ALGEBRAS AND AMENABILITY

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ABSTRACT. Following an idea of G. Schlichting we associate to any Hecke pair of groups a pair consisting of a totally disconnected group and a compact-open subgroup. Using this correspondence we show how the study of Hecke C^*-algebras can be reduced to the study of corners in C^*-algebras of totally disconnected groups. Afterwards this method is used to study the amenability of Hecke pairs.

KEYWORDS: *Hecke algebras, C^*-algebras, amenability, almost-normal subgroups.*

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1. INTRODUCTION

Hecke algebras were introduced in noncommutative geometry by J.-B. Bost and A. Connes in their work on spontaneous symmetry breaking of a dynamical system associated to the group pair $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{Z} \rtimes 1)$ ([3]). In that paper they show the relationship between this dynamical system and the distribution of prime numbers. The constructions rely mainly on $\mathbb{Z} \rtimes 1$ being an *almost normal subgroup* of $\mathbb{Q} \rtimes \mathbb{Q}_+^*$, in other words on $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{Z} \rtimes 1)$ being a *Hecke pair* (Definition 2.1).

In this paper we extend some ideas of [3] to define a general operator algebraic framework for the study of Hecke pairs. This extension is motivated by many interesting examples, some of them arising from the resolution of extremely simple ordinary differential equations (cf. [8]).

2. PRELIMINARIES

Let Γ be a subgroup of G. We define the functions:

(i) $L(g) = [\Gamma : \Gamma \cap g\Gamma g^{-1}] =$ "length of the Γ-orbit of $g\Gamma$ in G/Γ, where Γ acts by left multiplication" = "number of right Γ-classes in the double class $\Gamma g \Gamma$";
(ii) $R(g) = [\Gamma : \Gamma \cap g^{-1}\Gamma g] =$ "length of the Γ-orbit of Γg in $\Gamma \setminus G$, where Γ acts by right multiplication" = "number of left Γ-classes in the double class $\Gamma g \Gamma$".

These two functions are Γ-biinvariant on G with values in $\mathbb{N} \cup \{\infty\}$. They satisfy $L(g) = R(g^{-1})$. When necessary, to avoid confusion, we denote them by L_G, R_G. The equality $L(g) = 1$ holds for all $g \in G$ if and only if Γ is normal in G ($\Gamma \triangleleft G$).

Let G be a group with two subgroups Γ_1 and Γ_2. We say that Γ_1 and Γ_2 are *commensurable* (and we denote this by $\Gamma_1 \sim \Gamma_2$) if the indices $[\Gamma_1 : \Gamma_1 \cap \Gamma_2]$ and $[\Gamma_2 : \Gamma_1 \cap \Gamma_2]$ are finite.

DEFINITION 2.1. A subgroup Γ of a group G is *almost normal* (and we write $\Gamma \lessapprox G$) if $\Gamma \sim g\Gamma g^{-1}$ for any $g \in G$. In other words Γ is *almost normal* in G if any double Γ-class contains finitely many right Γ-classes (i.e. $L(g) < \infty$ for all $g \in G$).
If $\Gamma \lessapprox G$ we say that (G, Γ) is a *Hecke pair*.

Note that if Γ is normal in G, then Γ is almost normal. It is also clear that if Γ is finite or of finite index in G then $\Gamma \lessapprox G$. Further $(SL_2(\mathbb{Q}), SL_2(\mathbb{Z}))$ is a Hecke pair ([7], [3]) as well as $(\mathbb{Q} \rtimes \mathbb{Q}^*, \mathbb{Z} \rtimes 1)$ ([3]). Generally, if K is a number field, O_K is its ring of integers and $G(K)$ is an algebraic group over K, then $G(O_K) \lessapprox G(K)$ (see the discussion after Proposition 4.1).

If Γ is a compact-open subgroup of G, then (G, Γ) is a Hecke pair (for example we can take G to be the group of isometries of a locally finite tree and Γ to be a stabilizer of a vertex). In fact, as Γ is compact, all Γ-orbits in G/Γ are compact as well. But, since Γ is open, G/Γ is discrete and so Γ-orbits are finite. This case turns out to be generic as we show in Proposition 4.1.

Commensurability is a relation stable by intersection of subgroups and taking direct and inverse image by homomorphisms. As a consequence one has the following elementary properties:

(i) Let $\varphi : G_1 \to G_2$ be a group homomorphism and Γ_1 and Γ_2 two subgroups of G_1 and G_2 respectively, then

$$\Gamma_1 \lessapprox G_1 \Rightarrow \varphi(\Gamma_1) \lessapprox \varphi(G_1) \quad \text{and} \quad \Gamma_2 \lessapprox G_2 \Rightarrow \varphi^{-1}(\Gamma_2) \lessapprox G_1.$$

(ii) If $\Gamma_1 \lessapprox G$ and $\Gamma_1 \sim \Gamma_2$ then $\Gamma_2 \lessapprox G$.
(iii) If $\Gamma_1 \lessapprox G$ and $\Gamma_2 \lessapprox G$ then $\Gamma_1 \cap \Gamma_2 \lessapprox G$.
(iv) If $\Gamma_1 \lessapprox G$ and $\Gamma_2 \lessapprox G$ then the subgroup $\langle \Gamma_1, \Gamma_2 \rangle$ generated by Γ_1 and Γ_2 is not always almost normal in G, but under some additional assumptions it is [8]. For example if $\Gamma_1 \lessapprox G$ and $\Gamma_2 \triangleleft G$ then $\langle \Gamma_1, \Gamma_2 \rangle \lessapprox G$.

PROPOSITION 2.2. *If $\Gamma \lessapprox G$ then the function $\Delta : G \to \mathbb{Q}^*_+$, $g \mapsto \Delta(g) = \frac{L(g)}{R(g)}$ is a group homomorphism.*

Proof. For $\Gamma_1 \sim \Gamma_2$, the relative index $[\Gamma_1 : \Gamma_2] = [\Gamma_1 : \Gamma_1 \cap \Gamma_2]/[\Gamma_2 : \Gamma_1 \cap \Gamma_2]$ has the property that if $\Gamma_1 \sim \Gamma_2 \sim \Gamma_3$ then $[\Gamma_1 : \Gamma_2][\Gamma_2 : \Gamma_3] = [\Gamma_1 : \Gamma_3]$. As

$L(g)/R(g) = [\Gamma : g\Gamma g^{-1}]$ by taking $\Gamma_1 = \Gamma$, $\Gamma_2 = g\Gamma g^{-1}$ and $\Gamma_3 = gh\Gamma h^{-1}g^{-1}$ one obtains that

$$\frac{L(g)}{R(g)}\frac{L(h)}{R(h)} = \frac{L(gh)}{R(gh)}. \qquad \blacksquare$$

REMARK 2.3. The above proposition follows from G. Schlichting's characterization of almost normality (see [5], as well Theorem 4.2) because Δ is the restriction of some modular homomorphism.

For a subgroup Γ of a group G we set $\Gamma_\triangleleft = \bigcap_{g\in G} g^{-1}\Gamma g$, the largest normal subgroup of G contained in Γ.

DEFINITION 2.4. The pair (G,Γ) is called *reduced* if $\Gamma_\triangleleft = \{e\}$. The *reduced pair* of (G,Γ) is (G',Γ') where $G' = G/\Gamma_\triangleleft$ and $\Gamma' = \Gamma/\Gamma_\triangleleft$.

If (G',Γ') is the reduced pair of (G,Γ), then it is obvious that $\Gamma \lesssim G$ if and only if $\Gamma' \lesssim G'$. In the sequel, all objects associated to a Hecke pair will be invariant under this "reduction" procedure.

3. ALGEBRAS AND DYNAMICAL SYSTEMS

In this section we recall the construction of the reduced Hecke C^*-algebra $C^*_{\mathrm{r}}(G,\Gamma)$ and the associated dynamical system $(C^*_{\mathrm{r}}(G,\Gamma),\sigma_t)$ of [3] and we give analogous definitions for $L^1(G,\Gamma)$ and $C^*(G,\Gamma)$.

For a discrete set X we denote by $\mathbb{C}(X)$ the space of complex valued functions with finite support on X. So for $\Gamma \lesssim G$ the space $\mathbb{C}(\Gamma\setminus G/\Gamma)$ is naturally identified with the subspace of Γ-invariant functions in $\mathbb{C}(G/\Gamma)$ (or in $\mathbb{C}(\Gamma\setminus G)$). In the space $\mathbb{C}(\Gamma\setminus G/\Gamma)$ there are a natural product and involution given by

$$(f_1 * f_2)(g) = \sum_{g_1\in\langle\Gamma\setminus G\rangle} f_1(gg_1^{-1})f_2(g_1) = \sum_{g_2\in\langle G/\Gamma\rangle} f_1(g_2)f_2(g_2^{-1}g) \quad \text{for } g\in G,$$

$$f^*(g) = \overline{f(g^{-1})},$$

where the functions of $\mathbb{C}(\Gamma\setminus G/\Gamma)$ are identified with Γ-biinvariant functions on G and $g_1 \in \langle\Gamma\setminus G\rangle$ (respectively $g_2 \in \langle G/\Gamma\rangle$) means that g_1 (respectively g_2) runs over a set of representatives of left (respectively right) Γ-sets. In this manner one obtains an involutive algebra $\mathbb{C}(G,\Gamma)$ called the *Hecke algebra* of (G,Γ).

We denote by $\Gamma g\Gamma$ the characteristic function in $\mathbb{C}(\Gamma\setminus G/\Gamma)$ of the corresponding double set. These functions form a basis of $\mathbb{C}(G,\Gamma)$ and for two such functions, $\Gamma g\Gamma$ and $\Gamma h\Gamma$, the product formula is

$$\Gamma g\Gamma * \Gamma h\Gamma = \sum_{\substack{g_i\Gamma\subset\Gamma g\Gamma\\ h_j\Gamma\subset\Gamma h\Gamma}} \frac{1}{L(g_ih_j)}\Gamma g_ih_j\Gamma = \sum_{h_j\Gamma\subset\Gamma h\Gamma} \frac{L(g)}{L(gh_j)}\Gamma gh_j\Gamma$$

where $g_i\Gamma \subset \Gamma g\Gamma$ means that g_i run over some $L(g)$ representatives of right Γ-classes included in $\Gamma g\Gamma$.

The algebra $\mathbb{C}(G,\Gamma)$ is unital with unit $1_{\mathbb{C}(G,\Gamma)} = \Gamma$, the characteristic function of the identity class.

If (G', Γ') is the reduced pair of Hecke pair (G, Γ) then $\mathbb{C}(G,\Gamma) \simeq \mathbb{C}(G',\Gamma')$. When $\Gamma \lhd G$ one has the isomorphism $\mathbb{C}(G,\Gamma) \simeq \mathbb{C}(G/\Gamma)$, where $\mathbb{C}(G/\Gamma)$ is the group algebra of G/Γ.

If Γ is a finite subgroup of G, then $\Gamma \lessapprox G$ and $\mathsf{p} = \frac{1}{|\Gamma|}\sum_{\Gamma}\gamma$ is a self-adjoint projection in $\mathbb{C}(G)$. In this case one has natural isomorphism $\mathbb{C}(G,\Gamma) \simeq \mathsf{p}\mathbb{C}(G)\mathsf{p}$. We will see (Theorem 4.2) that any Hecke algebra is "almost of this kind".

Proposition 2.2 has two direct corollaries.

COROLLARY 3.1. *The formula below defines a one parameter group of automorphisms $(\sigma_t)_{t\in\mathbb{R}}$ on the Hecke algebra $\mathbb{C}(G,\Gamma)$*

$$\sigma_t(f)(g) = \Delta(g)^{-\mathrm{it}} f(g) \quad \textit{for } f \in \mathbb{C}(G,\Gamma),\ g \in \Gamma \setminus G/\Gamma \textit{ and } t \in \mathbb{R}.$$

Proof. It is obvious that σ_t is linear and that $\sigma_{t_1}\sigma_{t_2} = \sigma_{t_1+t_2}$. The compatibility with the involution is given by

$$\Delta(g^{-1})^{-\mathrm{it}} = \overline{\Delta(g)^{-\mathrm{it}}} \Rightarrow \sigma_t(f^*) = \sigma_t(f)^*.$$

One also has to check the compatibility with the product:

$$\begin{aligned}[][\sigma_t(f_1) * \sigma_t(f_2)](g) &= \sum_{g_1\in\langle\Gamma\backslash G\rangle} \Delta(gg_1^{-1})^{-\mathrm{it}} f_1(gg_1^{-1})\Delta(g_1)^{-\mathrm{it}} f_2(g_1)\\ &= \Delta(g)^{-\mathrm{it}} \sum_{g_1\in\langle\Gamma\backslash G\rangle} f_1(gg_1^{-1})f_2(g_1) = [\sigma_t(f_1 * f_2)](g). \quad \blacksquare\end{aligned}$$

COROLLARY 3.2. *There is a natural homomorphism $\varepsilon : \mathbb{C}(G,\Gamma) \to \mathbb{C}$ defined by:*

$$\varepsilon(f) = \sum_{g\in\langle\Gamma\backslash G\rangle} f(g)\sqrt{\Delta(g)} = \sum_{g\in\langle\Gamma\backslash G/\Gamma\rangle} f(g)\sqrt{L(g)R(g)} \quad \textit{for } f \in \mathbb{C}(G,\Gamma).$$

Proof. The map ε is linear and one has

$$\varepsilon(f^*) = \sum_{g\in\langle\Gamma\backslash G/\Gamma\rangle} \overline{f(g^{-1})}\sqrt{L(g)R(g)} = \sum_{g\in\langle\Gamma\backslash G/\Gamma\rangle} \overline{f(g)\sqrt{L(g)R(g)}} = \overline{\varepsilon(f)}$$

and

$$\begin{aligned}\varepsilon(f_1 * f_2) &= \sum_{g\in\langle\Gamma\backslash G\rangle} \sqrt{\Delta(g)} \sum_{g_1\in\langle\Gamma\backslash G\rangle} f_1(gg_1^{-1})f_2(g_1)\\ &= \sum_{g_1\in\langle\Gamma\backslash G\rangle} \sqrt{\Delta(g_1)}f_2(g_1) \sum_{g\in\langle\Gamma\backslash G\rangle} \sqrt{\Delta(gg_1^{-1})}f_1(gg_1^{-1}) = \varepsilon(f_1)\varepsilon(f_2). \quad \blacksquare\end{aligned}$$

The vector space $\mathbb{C}(\Gamma \setminus G)$ is a natural left $\mathbb{C}(G,\Gamma)$-module. The action λ is given by the formula:

$$[\lambda(f)\xi](g) = \sum_{g_1\in\langle\Gamma\backslash G\rangle} f(gg_1^{-1})\xi(g_1) \quad \text{for } f \in \mathbb{C}(G,\Gamma),\ \xi \in \mathbb{C}(\Gamma \setminus G) \text{ and } g \in \Gamma \setminus G.$$

It commutes with the natural right action of G on $\mathbb{C}(\Gamma\setminus G)$. In fact $\lambda : \mathbb{C}(G,\Gamma) \to \mathrm{End}_G(\mathbb{C}(\Gamma \setminus G))$ is an isomorphism. When necessary, to avoid confusions, we will denote λ by $\lambda_{\Gamma\setminus G}$. Often instead of $\lambda(f)\xi$ we use $f * \xi$. So this action written on the basis functions becomes

$$\Gamma g\Gamma * \Gamma h = \sum_{\Gamma g_i \subset \Gamma g \Gamma} \Gamma g_i h\,.$$

The action λ extends naturally to an involutive representation of $\mathbb{C}(G,\Gamma)$ in the Hilbert space of square summable functions $\ell^2(\Gamma \setminus G)$. We will also denote this representation by λ and we call it *the left regular representation* of $\mathbb{C}(G,\Gamma)$. In the case of $\Gamma \triangleleft G$, this representation coincides with the left regular representation of the group algebra $\mathbb{C}(G/\Gamma)$. The *reduced Hecke C^*-algebra* of (G,Γ) is the C^*-algebra generated by $\lambda(\mathbb{C}(G,\Gamma))$ in $B(\ell^2(\Gamma \setminus G))$. It is denoted by $C^*_{\mathrm{r}}(G,\Gamma)$. The *von Neumann algebra* of (G,Γ) is the bi-commutant $\lambda(\mathbb{C}(G,\Gamma))''$ and we denote it by $L(G,\Gamma)$.

Let $\rho_{\Gamma\setminus G}$ be the right quasi-regular representation of G on $\ell^2(\Gamma \setminus G)$. As the characteristic function $\Gamma \in \ell^2(\Gamma \setminus G)$ is a cyclic vector for $\rho_{\Gamma\setminus G}$, any element of the commutant $\rho_{\Gamma\setminus G}(G)'$ is characterized by the image of this vector in $\ell^2(\Gamma \setminus G)$. In consequence, the algebra $\mathbb{C}(G,\Gamma)$ is weakly dense in $\rho_{\Gamma\setminus G}(G)'$, so $\rho_{\Gamma\setminus G}(G)' = L(G,\Gamma)$ ([3], [2]).

We consider the vector state $\omega_{\Gamma,\Gamma}$ on $C^*_{\mathrm{r}}(G,\Gamma)$ defined by $\omega_{\Gamma,\Gamma}(f) = \langle \Gamma, f*\Gamma\rangle$, where $\Gamma \in \ell^2(\Gamma \setminus G)$. In the case of $\Gamma \triangleleft G$, this state corresponds to the standard trace on $C^*_{\mathrm{r}}(G/\Gamma)$. But, in general, it is not a trace for $\Gamma \ntriangleleft G$.

PROPOSITION 3.3. ([3]) *Let $\Gamma \ntriangleleft G$. There exists a unique one parameter group of automorphisms $(\sigma_t)_{t\in\mathbb{R}} \in \mathrm{Aut}\,(C^*_{\mathrm{r}}(G,\Gamma))$ such that*

$$\sigma_t(f)(g) = (\Delta(g))^{-\mathrm{i}t} f(g) \quad \textit{for } f \in \mathbb{C}(G,\Gamma),\ g \in \Gamma \setminus G/\Gamma \textit{ and } t \in \mathbb{R}\,.$$

Moreover $(\sigma_{-t})_{t\in\mathbb{R}}$ is the modular group of the faithful state $\omega_{\Gamma,\Gamma}$.

Thus the dynamical system associated to a Hecke pair is $(C^*_{\mathrm{r}}(G,\Gamma),\sigma_t)$ ([3]). Let us recall that when $\Gamma \triangleleft G$ the dynamical system is trivial because $\sigma_t \equiv 1$. This is also the case for any finite dimensional Hecke C^*-algebra.

We endow the involutive algebra $\mathbb{C}(G,\Gamma)$ with the following Banach norm:

$$\|f\|_{L^1} = \varepsilon(|f|) = \sum_{g\in\Gamma\setminus G} |f(g)|\sqrt{\Delta(g)} = \sum_{h\in\Gamma\setminus G/\Gamma} |f(h)|\sqrt{L(h)R(h)}\,.$$

The completion $L^1(G,\Gamma)$ of $\mathbb{C}(G,\Gamma)$ with respect to this norm is the L^1-*Banach algebra* of the Hecke pair. The enveloping C^*-algebra of $L^1(G,\Gamma)$ is $C^*(G,\Gamma)$, the *(maximal) Hecke C^*-algebra* of the pair (G,Γ).

The one parameter group of automorphisms $(\sigma_t)_{t\in\mathbb{R}}$ of $\mathbb{C}(G,\Gamma)$ (defined in Corollary 3.1) extends naturally to $L^1(G,\Gamma)$, therefore to $C^*(G,\Gamma)$, because $\|\sigma_t(f)\|_{L^1} = \|f\|_{L^1}$ for any f in $\mathbb{C}(G,\Gamma)$.

We remark here that the algebra $C^*(G,\Gamma)$ is not so "universal" as the group C^*-algebra is. Actually it can happen that some involutive representation of $\mathbb{C}(G,\Gamma)$ on a Hilbert space does not extend to $C^*(G,\Gamma)$.

EXAMPLE 3.4. Consider the Hecke pair $(D_\infty, \mathbb{Z}/2\mathbb{Z})$, where D_∞ is the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. In this case a basis of $\mathbb{C}(G,\Gamma)$ is given by the characteristic functions $\{\delta_n\}_{\mathbb{N}}$ of $\mathbb{Z}/2\mathbb{Z}$-double classes $\{(\pm n, \pm 1)\}$ in $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. The product on this basis is $\delta_n * \delta_m = \delta_{|n-m|} + \delta_{n+m}$ for all $n, m \in \mathbb{N}^*$, and the involution is trivial ($\delta_n^* = \delta_n$). Let $\pi : \mathbb{C}(G,\Gamma) \to \mathbb{C}$ be the involutive representation given by $\pi(\delta_n) = \mathrm{e}^n + \mathrm{e}^{-n}$ for $n \in \mathbb{N}^*$ and $\pi(\delta_0) = 1$. It is obvious that this representation is not continuous for the L^1 norm as $\|\delta_n\|_{L^1} \leqslant 2$ and $\|\pi(\delta_n)\|$ is unbounded.

The maximal C^*-algebra of this pair is $C^*(G,\Gamma) = C[0,1]$, the algebra of continuous functions on $[0,1]$. This is a consequence of the next proposition.

PROPOSITION 3.5. *Let Γ be a finite group acting by automorphisms on a commutative group N. Then $C^*(N \rtimes \Gamma, \Gamma) \simeq C(\widehat{N}/\Gamma)$.*

Proof. As Γ is finite, $\mathbb{C}(N \rtimes \Gamma, \Gamma) = \mathsf{p}\mathbb{C}(N \rtimes \Gamma)\mathsf{p}$ just as $L^1(N \rtimes \Gamma, \Gamma) = \mathsf{p}L^1(N \rtimes \Gamma)\mathsf{p}$ and $C^*(N \rtimes \Gamma, \Gamma) = \mathsf{p}C^*(N \rtimes \Gamma)\mathsf{p}$. But $C^*(N \rtimes \Gamma) \simeq C(\widehat{N}) \rtimes \Gamma$, so one obtains $\mathsf{p}(C(\widehat{N}) \rtimes \Gamma)\mathsf{p} = C(\widehat{N}/\Gamma)$, giving the isomorphism $C^*(N \rtimes \Gamma, \Gamma) \simeq C(\widehat{N}/\Gamma)$. ∎

The involutive homomorphism $\varepsilon : \mathbb{C}(G,\Gamma) \to \mathbb{C}$ defined in Corollary 3.2 can be extended to a C^*-algebra homomorphism $\varepsilon : C^*(G,\Gamma) \to \mathbb{C}$ and plays the role of the trivial representation.

PROPOSITION 3.6. *There is a natural projection of $C^*(G,\Gamma)$ onto $C^*_{\mathrm{r}}(G,\Gamma)$.*

The proof of this proposition is a consequence of Theorem 4.2 below. Thus a natural question arises: under which conditions on (G,Γ) do these two C^*-algebras coincide? The answer is given in the last section of this paper.

4. UNDERLYING TOPOLOGY

At the end of the 70's G. Schlichting, studying the periodic actions of groups, introduced topological groups associated to Hecke pairs ([6], [7]). This idea is quite natural and has a major contribution to the understanding of almost normal subgroups and their algebras.

Schlichting's idea is the following: we endow the set $\mathrm{App}\,(G/\Gamma)$ of maps from G/Γ to G/Γ with the pointwise convergence topology. It follows from Ascoli's theorem that $\Gamma \lesssim\!\!\!\!\!\triangleleft\; G$ if and only if Γ is relatively compact in $\mathrm{App}\,(G/\Gamma)$. But, under this condition, the completion $\overline{\Gamma}$ of the image of Γ in $\mathrm{App}\,(G/\Gamma)$ is a compact group. Furthermore the completion of the image of G in $\mathrm{App}\,(G/\Gamma)$ is a totally disconnected group $\overline{G}$ containing $\overline{\Gamma}$ as a compact-open subgroup. Actually $\overline{\Gamma}$ is the stabilizer in $\overline{G}$ of the identity class $\Gamma \in G/\Gamma$. We will see that from our point of view the study of a pair (G,Γ) can be reduced to that of $(\overline{G},\overline{\Gamma})$.

PROPOSITION 4.1. *(G,Γ) is a Hecke pair if and only if there is a reduced pair (G',Γ') such that:*

(i) *G' is a (totally disconnected locally compact) topological group.*

(ii) *Γ' is a compact-open subgroup of G'.*

(iii) *There exists a group homomorphism $\varphi : G \to G'$ such that:*

(a) $\overline{\varphi(G)} = G'$;

(b) $\varphi^{-1}(\varphi(G) \cap \Gamma') = \Gamma$.

The pair (G',Γ') is unique (unique in the sense that if (G'',Γ'') is another such pair with $\phi : G \to G''$, then there is a natural isomorphism $I : G'' \xrightarrow{\simeq} G'$ such that $I(\Gamma'') = \Gamma'$ and $I \circ \phi = \varphi$) and we call it the totally disconnected pair associated to (G,Γ).

Proof. The existence follows from Schlichting's construction of $(\overline{G},\overline{\Gamma})$ given above. One has to prove the uniqueness part. As Γ' is open and $\varphi(G)$ is dense in G', one has $G' = \varphi(G)\Gamma'$ and using condition (iii) (b) one obtains that the map φ induces isomorphisms $G'/\Gamma' = \varphi(G)/\varphi(\Gamma) = G/\Gamma$. Thus Γ' injects into $\mathrm{App}\,(G/\Gamma)$ and stabilizes the base point $\Gamma \in G/\Gamma$. Thus on the one hand $\Gamma' \subset \overline{\Gamma}$ as a compact subgroup, on the other hand Γ' contains the dense subgroup $\varphi(\Gamma)$ of $\overline{\Gamma}$. As a consequence Γ' is naturally isomorphic to $\overline{\Gamma}$. In conclusion, as $G' = \varphi(G)\Gamma'$ and $\overline{G} = \varphi(G)\overline{\Gamma}$, the natural action of G' on G/Γ induces an isomorphism $G' \simeq \overline{G}$. ∎

The uniqueness condition implies that if $(\overline{G},\overline{\Gamma})$ is the totally disconnected pair associated to (G,Γ) then $(\overline{G},\overline{\Gamma})$ is its own associated totally disconnected pair.

In the trivial case $[\Gamma : \Gamma_{\triangleleft}] < \infty$, the totally disconnected pair associated to (G,Γ) is just the reduced pair.

The above proposition shows that a general construction of Hecke pairs is obtained by taking a (totally disconnected) group with two subgroups, one of them compact-open (the other being not necessarily dense). For example, let K be a number field and O_K be its ring of integers. Let $\widehat{K}$ and $\widehat{O}_K$ be their completions with respect to all non-Archimedean valuations of K. Let $G(K)$ be an algebraic group on K. We take the "big" group to be $G(\widehat{K})$ with the two subgroups $G(\widehat{O}_K)$ and $G(K)$. As $G(\widehat{O}_K)$ is a compact-open subgroup of $G(\widehat{K})$ one has that $G(\widehat{O}_K) \trianglelefteq G(\widehat{K})$. But as $G(\widehat{O}_K) \cap G(K) = G(O_K)$ we obtain that $G(O_K) \trianglelefteq G(K)$. We remark here that the totally disconnected pair associated to $(G(K), G(O_K))$ is not necessarily the pair $(G(\widehat{K}), G(\widehat{O}_K))$. Actually it is the reduced pair of $(\overline{G(K)}, \overline{G(O_K)})$ where the closures are taken in $G(\widehat{K})$. For example, given a ring A we set $G(A) = P_A = A \rtimes A^*$. Let $K = \mathbb{Q}$ and $O_K = \mathbb{Z}$, thus $\widehat{K} = \mathcal{A}$ and $\widehat{O}_K = \mathcal{R}$, where $\mathcal{A}$ is the ring of finite adeles and $\mathcal{R}$ is its maximal compact subring. So $G(K) = \mathbb{Q} \rtimes \mathbb{Q}^*$, $G(O_K) = \mathbb{Z} \rtimes \mathbb{F}_2$, where $\mathbb{F}_2 = \{1,-1\}$, $G(\widehat{K}) = \mathcal{A} \rtimes \mathcal{A}^*$ and $G(\widehat{O}_K) = \mathcal{R} \rtimes \mathcal{R}^*$, but the associated pair of $(G(K), G(O_K))$ is $(\mathcal{A} \rtimes \mathbb{Q}^*, \mathcal{R} \rtimes \mathbb{F}_2)$.

To simplify notations, we assume for the remainder of this section that (G,Γ) is a reduced pair.

Let μ be the left invariant Haar measure of $\overline{G}$ such that $\mu(\overline{\Gamma}) = 1$ and $\Delta : \overline{G} \to \mathbb{R}_+^*$ be the modular function of $\overline{G}$. Then $\mathrm{d}\mu(ts) = \Delta(s)\,\mathrm{d}\mu(t)$. So one has $\mu(\overline{\Gamma} g \overline{\Gamma}) = L(g)$. Hence $\mu(\overline{\Gamma} g^{-1} \overline{\Gamma}) = \frac{R(g)}{L(g)} \mu(\overline{\Gamma} g \overline{\Gamma})$ and as a consequence $\Delta(g) = \frac{L(g)}{R(g)}$. This justifies the use of the Δ notation for $\overline{G}$ in Proposition 2.2.

Let $L^1(\overline{G})$ be the Banach algebra $L^1(\overline{G}, \nu)$ where $\nu = \Delta^{-1/2}\mu$ is the symmetric Haar measure such that $\nu(\overline{\Gamma}) = 1$. The product of $L^1(\overline{G})$ is the standard convolution of functions and the involution is $f^*(g) = \overline{f(g^{-1})}$. The algebra $C_{\mathrm{c}}(\overline{G})$ is the involutive subalgebra of $L^1(\overline{G})$ of continuous functions with compact support. As $\overline{\Gamma}$ is a compact-open subgroup of $\overline{G}$ with $\nu(\overline{\Gamma}) = 1$, the characteristic function of $\overline{\Gamma}$ is a self-adjoint projection in $C_{\mathrm{c}}(\overline{G})$.

THEOREM 4.2. *Let (G, Γ) be a Hecke pair and $(\overline{G}, \overline{\Gamma})$ be its associated totally disconnected pair. Let p be the self-adjoint projection of $C_{\mathrm{c}}(\overline{G})$ corresponding to the characteristic function of $\overline{\Gamma}$. Then one has the following isomorphisms:*

(i) $\mathbb{C}(G, \Gamma) \simeq \mathbb{C}(\overline{G}, \overline{\Gamma}) \simeq \mathsf{p}C_{\mathrm{c}}(\overline{G})\mathsf{p}$;
(ii) $L^1(G, \Gamma) \simeq L^1(\overline{G}, \overline{\Gamma}) \simeq \mathsf{p}L^1(\overline{G})\mathsf{p}$;
(iii) $C^*(G, \Gamma) \simeq C^*(\overline{G}, \overline{\Gamma}) \simeq \mathsf{p}C^*(\overline{G})\mathsf{p}$;
(iv) $C^*_{\mathrm{r}}(G, \Gamma) \simeq C^*_{\mathrm{r}}(\overline{G}, \overline{\Gamma}) \simeq \mathsf{p}C^*_{\mathrm{r}}(\overline{G})\mathsf{p}$;
(v) $L(G, \Gamma) \simeq L(\overline{G}, \overline{\Gamma}) \simeq \mathsf{p}L(\overline{G})\mathsf{p}$.

Furthermore the dynamical systems of (G, Γ) are the corner restrictions of the corresponding modular dynamical system of $\overline{G}$.

Proof. From the construction of $\overline{G}$ it follows that the inclusion $G \hookrightarrow \overline{G}$ induces the canonical isomorphisms $G/\Gamma \simeq \overline{G}/\overline{\Gamma}$ and $\Gamma \setminus G/\Gamma \simeq \overline{\Gamma} \setminus \overline{G}/\overline{\Gamma}$. As a consequence one has the isomorphisms $\mathbb{C}(G, \Gamma) \simeq \mathbb{C}(\overline{G}, \overline{\Gamma})$ and $C^*_{\mathrm{r}}(G, \Gamma) \simeq C^*_{\mathrm{r}}(\overline{G}, \overline{\Gamma})$. At the level of von Neumann algebras one obtains $L(G, \Gamma) \simeq L(\overline{G}, \overline{\Gamma})$. Also one has $L_{\overline{G}} \equiv L_G$ and $R_{\overline{G}} \equiv R_G$ on $G \subset \overline{G}$. So the norms $\| \cdot \|_1$ coincide on $\mathbb{C}(G, \Gamma) \simeq \mathbb{C}(\overline{G}, \overline{\Gamma})$, which implies that $L^1(G, \Gamma) \simeq L^1(\overline{G}, \overline{\Gamma})$ and $C^*(G, \Gamma) \simeq C^*(\overline{G}, \overline{\Gamma})$. The dynamical systems constructed in the preceding section are also isomorphic because the restriction of $\Delta_{\overline{G}}$ on G coincides with Δ in Proposition 2.2.

A $\overline{\Gamma}$-biinvariant function on $\overline{G}$ has compact support if and only if its support as a function on $\overline{\Gamma} \setminus \overline{G}/\overline{\Gamma}$ is finite. Thus $\mathsf{p}C_{\mathrm{c}}(\overline{G})\mathsf{p}$ is canonically isomorphic to $\mathbb{C}(\overline{G}, \overline{\Gamma})$ as a vector space. It is easy to verify that the product and the involution are also the same. So $\mathsf{p}C_{\mathrm{c}}(\overline{G})\mathsf{p} \simeq \mathbb{C}(\overline{G}, \overline{\Gamma})$. In this algebra the norms induced by $L^1(\overline{G})$ and $L^1(\overline{G}, \overline{\Gamma})$ coincide, which implies that $\mathsf{p}L^1(\overline{G})\mathsf{p} \simeq L^1(\overline{G}, \overline{\Gamma})$. And $\mathsf{p}C^*(\overline{G})\mathsf{p} \simeq C^*(\overline{G}, \overline{\Gamma})$, because $C^*(\mathsf{p}L^1(\overline{G})\mathsf{p}) = \mathsf{p}C^*(L^1(\overline{G}))\mathsf{p}$. As $\mathsf{p}C^*_{\mathrm{r}}(\overline{G})\mathsf{p}$ is the C^*-algebra generated by $\mathsf{p}C_{\mathrm{c}}(\overline{G})\mathsf{p}$ in $\mathsf{p}L^2(\overline{G}, \nu)$, to see that $C^*_{\mathrm{r}}(\overline{G}) \simeq C^*_{\mathrm{r}}(\overline{G}, \overline{\Gamma})$ it is enough to remark that $\ell^2(\overline{\Gamma} \setminus \overline{G}) \simeq \mathsf{p}L^2(\overline{G}, \nu)$ and that the left regular representation of $\mathbb{C}(\overline{G}, \overline{\Gamma})$ on this Hilbert space coincides with that of $\mathsf{p}C_{\mathrm{c}}(\overline{G})\mathsf{p} \simeq \mathbb{C}(\overline{G}, \overline{\Gamma})$ obtained by restriction of the left regular representation of $\overline{G}$. It also follows from this fact that $L(\overline{G}, \overline{\Gamma}) \simeq \mathsf{p}L(\overline{G})\mathsf{p}$. We finish this proof by remarking that all the dynamical systems constructed in the preceding section coincide with the corner restrictions of the corresponding modular dynamical system of $\overline{G}$ because $\Delta_{\overline{G}}^{-\mathrm{i}t}(\mathsf{p}) = \mathsf{p}$ and $\Delta_{\overline{G}}$ coincides with Δ on $\Gamma \setminus G/\Gamma \simeq \overline{\Gamma} \setminus \overline{G}/\overline{\Gamma}$. ∎

5. AMENABILITY OF HECKE PAIRS

In [3] the authors prove that any involutive representation of $\mathbb{C}(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}}^{+})$ on a Hilbert space extends to a representation of the C^*-algebra $C_{\mathrm{r}}^*(P_{\mathbb{Q}}^{+}, P_{\mathbb{Z}}^{+})$. To obtain this result they use the amenability of the groupoid $\mathcal{G} = \{(a,b) \in \mathcal{R}\times\mathbb{Q}_{+}^{*} : ab \in \mathcal{R}\} \subset \mathcal{A} \rtimes \mathbb{Q}_{+}^{*}$ where $\mathcal{R}$ is the maximal compact subring of the ring $\mathcal{A}$ of finite adeles. We now prove a generalization of this result that gives a new characterization of Eymard's notion of amenability of G/Γ in the particular case when (G,Γ) is a Hecke pair.

Recall there are many different equivalent definitions for amenability of groups and that almost all of them extend to the more general case of homogeneous spaces G/Γ where Γ is a closed subgroup of G ([4]). For example G/Γ is amenable in the sense of Eymard if one of the following equivalent conditions is satisfied (here μ is a quasi-invariant measure on G/Γ) :

(M) There exists a G invariant mean on $L^\infty(G/\Gamma,\mu)$.

(P_p^*) For any finite subset F of G and any $\varepsilon > 0$, there is a function $f \in L^p(G/\Gamma,\mu)$, with norm $\|f\|_p = 1$, such that for any $g \in F$, one has

$$\|g * f - f\|_p < \varepsilon;$$

(PF) For any compact convex subset Q of a locally convex Hausdorff space, if G acts continuously by affine automorphisms on Q in such a way that there is a Γ-fixed point, then there is a G-fixed point in Q;

(F) The trivial representation 1_G is weakly contained in the quasi-regular one $\lambda_{G/\Gamma} : G \to B(L^2(G/\Gamma))$.

We emphasize that what is important here is not the space G/Γ, but the pair (G,Γ). Indeed there are pairs such that $G_1/\Gamma_1 \simeq G_2/\Gamma_2$ with G_1/Γ_1 amenable and G_2/Γ_2 nonamenable (actually the notion of amenability of G/Γ is a particular one of that of groupoids; [1]).

PROPOSITION 5.1. *For $\Gamma \lesssim G$, the following conditions are equivalent:*

(i) G/Γ *is amenable in the sense of Eymard;*

(ii) $C_{\mathrm{r}}^*(G,\Gamma) \simeq C^*(G,\Gamma)$*;*

(iii) *The representation ε (Corollary 3.2) of $L^1(G,\Gamma)$ is weakly contained in the left regular representation $\lambda_{\Gamma\backslash G}$;*

(iv) $\overline{G}$ *is amenable.*

Proof. We replace (G,Γ) by its associated totally disconnected pair $(\overline{G},\overline{\Gamma})$. Indeed the conditions (ii), (iii) and (iv) are trivially invariant by the substitution $(G,\Gamma) \rightsquigarrow (\overline{G},\overline{\Gamma})$. Moreover G/Γ is amenable if and only if $\overline{G}/\overline{\Gamma}$ is. The forward direction follows from condition (PF) above, and the converse is a direct consequence of the (P_1^*) characterization of amenability.

We now prove the proposition for $(\overline{G},\overline{\Gamma})$.

(i) $\Leftrightarrow$ (iv) Follows directly from Section 3.1° of [4] because $\overline{\Gamma}$ is compact and so amenable.

(iv) $\Rightarrow$ (ii) $\overline{G}$ being amenable one has $C^*(\overline{G}) \simeq C_{\mathrm{r}}^*(\overline{G}) \Rightarrow \mathsf{p}C^*(\overline{G})\mathsf{p} \simeq \mathsf{p}C_{\mathrm{r}}^*(\overline{G})\mathsf{p}$.

(ii) $\Rightarrow$ (iii) Is trivial.

(iii) ⇒ (iv) We shall denote ε by $\varepsilon_{(\overline{G},\overline{\Gamma})}$ and the trivial representation of $\overline{G}$ by $\varepsilon_{\overline{G}}$, so $\varepsilon_{\overline{G}}(f) = \int f \,\mathrm{d}\nu$. Let p be the self-adjoint projection of $C_{\mathrm{c}}(\overline{G})$ corresponding to the characteristic function of $\overline{\Gamma}$ (like in Theorem 4.2). If $\overline{G}$ is not amenable then the trivial representation of $\overline{G}$ is not weakly contained in the regular one, so there is a function $f \in C_{\mathrm{c}}(\overline{G})$ such that $\|f\|_{C^*_{\mathrm{r}}(\overline{G})} < \varepsilon_{\overline{G}}(f)$. On the one hand $\|\mathsf{p}f\mathsf{p}\|_{C^*_{\mathrm{r}}(\overline{G},\overline{\Gamma})} = \|\mathsf{p}f\mathsf{p}\|_{C^*_{\mathrm{r}}(\overline{G})} \leqslant \|f\|_{C^*_{\mathrm{r}}(\overline{G})}$ and on the other hand $\varepsilon_{\overline{G}}(f) = \varepsilon_{\overline{G}}(\mathsf{p}f\mathsf{p}) = \varepsilon_{(\overline{G},\overline{\Gamma})}(\mathsf{p}f\mathsf{p})$. Thus $\varepsilon_{(\overline{G},\overline{\Gamma})}$ is not weakly contained in $\lambda_{\Gamma\backslash G}$. ∎

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