A DOUBLE COMMUTANT THEOREM
FOR OPERATOR ALGEBRAS

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ABSTRACT. Every unital nonselfadjoint operator algebra possesses canonical and functorial classes of faithful (even completely isometric) Hilbert space representations satisfying a double commutant theorem generalizing von Neumann's classical result. Examples and complementary results are given.

KEYWORDS: *Operator algebra, representation, double commutant, Hilbert module, dual algebra, generator.*

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1. INTRODUCTION

In this paper we consider possibly nonselfadjoint norm closed algebras of operators on a Hilbert space H. The *general theory* of such *operator algebras*, and of their representations, is rather sparse in contrast to the selfadjoint case, namely the C^*-algebra theory. The contrast is easily seen in the lack of certain fundamental tools which are available in the selfadjoint case, such as von Neumann's double commutant theorem or Kaplansky's density theorem. In recent years, with the help of operator space theory, the situation has changed somewhat, and the general theory of operator algebras has been growing rapidly. The theory that is emerging also has the feature that it links much more closely together the three subjects of operator algebras, C^*-algebra theory, and the theory of rings and modules. The present paper is an attempt in this spirit to establish a double commutant theorem for general operator algebras, one that would be useful at least for tackling certain problems. Our main result simultaneously resembles two classical and fundamental results: von Neumann's double commutant theorem ([17]), and Nesbitt and Thrall's purely algebraic result ([16]) to the effect that any module M over a ring R, which is a "generator for $_R$MOD", satisfies the appropriate double commutant theorem. Although for a subalgebra $A \subset B(H)$ the double commutant

A'' may not equal the weak* closure $\overline{A}^{\text{weak}^*}$ within $B(H)$, we show that important and canonical classes of completely isometric Hilbert space representations $\pi : A \to B(K)$ of A do have the *double commutant property*, by which we will mean that $\pi(A)'' = \overline{\pi(A)}^{\text{weak}^*} = \overline{\pi(A)}^{\text{WOT}}$ in $B(K)$. In fact, for a certain subclass of these representations this double commutant is also isomorphic to A^{**}. If A is a "dual operator algebra", then we can find canonical classes of completely isometric normal representations with the property $\pi(A)'' = \pi(A)$, as one is accustomed to for von Neumann algebras.

A good illustration is the very simple example $\mathcal{T}_2$, the algebra of 2×2 upper triangular matrices. The usual representation of $\mathcal{T}_2$ on $\mathbb{C}^2$ clearly does not have the double commutant property. However the direct sum ρ of the usual representation and the 1-dimensional representation consisting of evaluation at the 1-1 entry, does possess the double commutant property. Indeed this representation ρ satisfies our general "sufficient condition" for the double commutant property (Theorem 1.1), but the usual representation of $\mathcal{T}_2$ does not.

The substitution of a given embedding $A \subset B(H)$ of an operator algebra A for another in which the double commutant theorem holds, is easily justified by the trend in the recent general theory of operator algebras towards a "coordinate-free" approach, i.e. not to focus on any one fixed embedding $A \subset B(H)$. In the new perspective one is encouraged to think about all representations of A simultaneously; some may be better than others for solving certain problems.

Before we turn to specific details, we feel it is important to mention that although some of the representations we consider are quite "large", in some applications this should not matter — the important thing is often just the universal property and the functoriality, and the strong links to the C^*-algebra theory. A good illustration of this may be found in the paper [4], where a difficult problem concerning nonselfadjoint operator algebras was solved by transferring it to the C^*-algebra world using some rather large representations. We will give other such applications of our results in a sequel paper, as well as a characterization of the double commutant property which is quite different to the considerations in the present paper.

We now turn to specific details, and recall some definitions and facts (see e.g. [5] for more details, if needed). Although everything in this paper can be done within the Banach algebra context and without the matrix norms of operator space theory, for specificity we use the operator space context. Thus, following the lead suggested by Arveson ([2]), and by operator space theory, we define an *abstract operator algebra* — or simply an *operator algebra* — to be a Banach algebra A which is also an operator space, such that there exists a completely isometric homomorphism $\theta : A \to B(H)$ for some Hilbert space H. There is an abstract characterization of these operator algebras due to the first author with Ruan and Sinclair, but we shall not need this here. Except in the last section, operator algebras are assumed to have a contractive approximate identity (c.a.i.). If A has an identity of norm 1 then we say that A is *unital*.

It is helpful to use the language of Hilbert modules (see for example [18], [8], [15]). For the purposes of this paper, we define a Hilbert A-module to be a Hilbert space H which is a nondegenerate A-module via a completely contractive nondegenerate representation $\pi : A \to B(H)$. Thus $a\zeta = \pi(a)(\zeta)$, for $a \in A$, $\zeta \in H$. The theory below may be modified to include "contractive Hilbert A-modules", we

omit the easy details. Except in the final section of our paper, the word "nondegenerate" used above means that the span of such products $a\zeta$ is dense in H. This correspondence between Hilbert A-modules and completely contractive nondegenerate representations is bijective. Henceforth we use the term "representation of A" for a completely contractive nondegenerate representation. We write $_A$HMOD for the category of Hilbert A-modules, with morphisms the bounded maps which intertwine the representations (that is, the bounded A-module maps). This category is closed under direct sums and quotients by closed submodules (see e.g. [15]). If α is a cardinal then the direct sum of α copies of a Hilbert A-module H, or of its associated representation $\pi : A \to B(H)$, is called a *multiple* of H or of π; and is written as $H^{(\alpha)}$ or π^α. We say that two Hilbert A-modules are *spatially equivalent*, and write $H \cong K$, if they are isometrically A-isomorphic, that is, if there exists a unitary A-module map from H onto K. We say that a closed A-submodule K of a Hilbert A-module H is *A-complemented* if the projection of H onto K is an A-module map. This may be reformulated in several equivalent ways (see for example the discussion in [15]). We say that representations π, θ of A are *quasi-equivalent* if there is a multiple of π which is spatially equivalent to a multiple of θ. Thus two Hilbert A-modules are quasi-equivalent if and only if there are cardinals α and β (which we may clearly assume to be equal) such that $H^{(\alpha)} \cong K^{(\beta)}$. We say that a Hilbert A-module H is *A-universal*, if every $K \in {_A}$HMOD is isometrically A-isomorphic (that is, spatially equivalent) to an A-complemented submodule of a direct sum of copies of H. We say that a module $H \in {_A}$HMOD is a *generator* (respectively *cogenerator*) for $_A$HMOD if for every nonzero morphism $R : K \to L$ of $_A$HMOD, there exists a morphism $T : H \to K$ (respectively $T : L \to H$) of $_A$HMOD with $RT \neq 0$ (respectively $TR \neq 0$). We will say that H is *sub-tracing* if the definition above for generator is modified so that K ranges over the set of submodules of H. We say that H is *completely subtracing* if a countably infinite multiple of H is sub-tracing. We also use these terms when referring to the associated representation on H. Thus, for example, we will often refer to a *representation* $\pi : A \to B(H)$ as being A-universal, or sub-tracing. One would expect, just as in pure algebra, that there are many useful alternative characterizations of generators, cogenerators, and sub-tracing modules. Indeed, we will provide some in Section 2.

Note that any generator is sub-tracing. Since any multiple of a generator is also a generator, we see that any generator is completely sub-tracing. Also, any A-universal Hilbert module H is a generator. To see this, suppose that $T : K \to L$ is a nonzero A-module map. Without loss of generality, there is a cardinal α such that K is an A-complemented submodule of $H^{(\alpha)}$; let Q be the associated projection onto K from $H^{(\alpha)}$. Let ε_i be the inclusion map of H into $H^{(\alpha)}$ as its ith summand H. If every map $T \circ Q \circ \varepsilon_i$ is zero, then $T = 0$, which is a contradiction.

The main results of our paper are the following:

Theorem 1.1. *Let A be an operator algebra with a contractive approximate identity. A Hilbert A-module which is a generator or cogenerator for $_A$HMOD, or which is completely sub-tracing, has the double commutant property.*

Theorem 1.2. *Let A be an operator algebra with contractive approximate identity.*

(i) *there exist A-universal representations for A;*

(ii) *any two A-universal representations for A are quasi-equivalent;*

(iii) *if π is a representation of A which is quasi-equivalent to an A-universal representation, then π is an A-universal representation;*

(iv) *if π is an A-universal representation of A on a Hilbert space H, then*

$$\pi(A)'' = \overline{\pi(A)}^{\,\text{weak}^*};$$

(v) *if π is an A-universal representation, then A^{**} is isomorphic to $\overline{\pi(A)}^{\,\text{weak}^*}$ via a completely isometric weak*-homeomorphic homomorphism $\rho : A^{**} \to \overline{\pi(A)}^{\,\text{weak}^*}$ such that $\rho(\widehat{a}) = \pi(a)$ for all $a \in A$.*

Proof. We prove only items (ii)–(iv) now, and defer the proofs of the other assertions. In fact (iii) is clear by the definitions, or is an easy exercise. The proof of (ii) is a simple application of set theory, and the well known "Eilenberg swindle". If H and K are two A-universal representations, then there exist cardinals α and β, and Hilbert A-modules M and N, such that $H \oplus M \cong K^{(\alpha)}$ and $K \oplus N \cong H^{(\beta)}$. Without loss of generality, by adding on extra multiples of H or K to the last two equations, $\alpha = \beta$. We may also assume that α is a large enough cardinal so that $\alpha \cdot \alpha$ equals α. Then

$$K^{(\alpha)} \cong K^{(\alpha)} \oplus K^{(\alpha)} \oplus \cdots \cong H \oplus M \oplus H \oplus M \oplus \cdots.$$

By associativity we get

$$K^{(\alpha)} \cong H \oplus K^{(\alpha)} \oplus K^{(\alpha)} \oplus \cdots \cong H \oplus K^{(\alpha)}.$$

Since $\alpha \cdot \alpha = \alpha$, a multiple of the last equation yields

$$K^{(\alpha)} \cong H^{(\alpha)} \oplus K^{(\alpha)}.$$

Similarly, $H^{(\alpha)} \cong H^{(\alpha)} \oplus K^{(\alpha)} \cong K^{(\alpha)}$, which proves (ii).

Item (iv) follows from Theorem 1.1 and the fact above that any A-universal Hilbert module H is a generator. ∎

It follows from (iv) and (v) that for A-universal representations, there is an automatic "Kaplansky density" result, which is really Goldstine's lemma in disguise.

The main results above are proved in the first few sections of our paper. In Sections 4–6 we give examples and complementary results. For example, we study there dual algebras; the relations between various classes of Hilbert modules; and in the final section we study operator algebras without c.a.i., showing for example that all of Theorem 1.2 with the exception of (iv) holds in complete generality.

We list now some background facts that we will make much use of (often without comment). One fact which is of great assistance when dealing with operator algebras with c.a.i. but no identity, is the following: if B is a C^*-algebra

generated (as a C^*-algebra) by a closed subalgebra A which has a c.a.i., then any $b \in B$ is a product ab' (or $b'a$) with $a \in A, b' \in B$. Equivalently:

(1.1)   Any c.a.i. for A is also one for the C^*-algebra generated by A.

See e.g. Chapter 2, [5]. We use the notation $[AK]$ for the norm closure of the span of products of a term in A with a term in K.

If $\mathcal{S} \subset B(H)$ then we define $\mathcal{S}^* = \{x^* : x \in \mathcal{S}\}$. If K is another Hilbert space (respectively if γ is a cardinal), then we write $\mathcal{S} \otimes I = \{x \otimes I : x \in \mathcal{S}\}$ for the set of appropriate "multiples" of elements in $\mathcal{S}$. This is a set of operators on $H \otimes K$ (respectively on $H^{(\gamma)}$). It is a simple computation that the following relations hold:

$$\overline{\mathcal{S} \otimes I}^{\,\mathrm{weak}^*} = \overline{\mathcal{S}}^{\,\mathrm{weak}^*} \otimes I, \qquad\qquad (\mathcal{S} \otimes I)'' = \mathcal{S}'' \otimes I,$$
$$\overline{\mathcal{S}^*}^{\,\mathrm{weak}^*} = (\overline{\mathcal{S}}^{\,\mathrm{weak}^*})^*, \qquad \text{and} \qquad (\mathcal{S}^*)'' = (\mathcal{S}'')^*.$$

Hence $\mathcal{S}$ has the double commutant property if and only if $\mathcal{S}^*$ has the double commutant property, and if and only if $\mathcal{S} \otimes I$ has the double commutant property.

2. GENERATORS AND TRACES

Following algebra texts (e.g. [1], p. 109) if H, K are Hilbert A-modules, then we define the *trace* $\mathrm{Tr}_K(H)$ to be the closure of the set of finite sums of elements taken from the ranges of bounded A-module maps from H into K. We define the *reject* $\mathrm{Rej}_K(H)$ to be the intersection of the kernels of all bounded A-module maps from K into H. Clearly $\mathrm{Tr}_K(H)$ and $\mathrm{Rej}_K(H)$ are closed submodules of K.

LEMMA 2.1. *Let H be a Hilbert A-module. Then:*
   (i) *H is a generator for $_A$HMOD if and only if $\mathrm{Tr}_K(H) = K$ for all Hilbert A-modules K;*
   (ii) *H is a cogenerator for $_A$HMOD if and only if $\mathrm{Rej}_K(H) = \{0\}$ for all Hilbert A-modules K;*
   (iii) *H is sub-tracing if and only if $\mathrm{Tr}_K(H) = K$ for closed submodules K of H.*

Proof. (i) Suppose that H is a generator. If $\mathrm{Tr}_K(H) \neq K$, then there exists a nonzero bounded A-module map $R : K \to K/\mathrm{Tr}_K(H)$ annihilating $\mathrm{Tr}_K(H)$, namely the quotient map. Since the quotient of Hilbert A-modules is a Hilbert A-module, and since H is a generator, there exists a $T \in B_A(H, K)$ with $RT \neq 0$. This contradicts the definition of the trace.

Conversely, suppose that $\mathrm{Tr}_K(H) = K$, and $R : K \to L$ is a bounded A-module map. If $R \circ T = 0$ for all $T \in B_A(H, K)$ then R is zero on $\mathrm{Tr}_K(H) = K$. So H is a generator. (iii) is proved similarly to (i).

For (ii), assume that $\mathrm{Rej}_K(H) = \{0\}$. If $R : L \to K$ is a nonzero morphism, but that $TR = 0$ for all $T \in {_A}B(K, H)$, then R maps into $\mathrm{Rej}_K(H) = \{0\}$. Hence $R = 0$. Conversely, if $\mathrm{Rej}_K(H) \neq \{0\}$, then the inclusion map $\varepsilon : \mathrm{Rej}_K(H) \to K$ is a nonzero bounded A-module map with $T\varepsilon = 0$ for all $T \in {_A}B(K, H)$. ∎

LEMMA 2.2. *Suppose that π is a generator for $_A$HMOD, that σ is a cogenerator for $_A$HMOD, and that ρ is any representation in $_A$HMOD. We have:*

(i) $\pi \oplus \rho$ *is a generator for $_A$HMOD, $\sigma \oplus \rho$ is a cogenerator for $_A$HMOD, and $\pi \oplus \sigma \oplus \rho$ is both a generator and cogenerator for $_A$HMOD;*

(ii) π *and σ are 1-1 (i.e. faithful);*

(iii) ρ *is a generator (respectively, cogenerator) for $_A$HMOD if and only if a multiple of ρ is a generator (respectively, cogenerator) for $_A$HMOD;*

(iv) *if ρ is quasi-equivalent to π (respectively to σ) then ρ is a generator (respectively cogenerator) for $_A$HMOD.*

Proof. (i) is obvious from the definitions. For (ii), suppose that $\pi(a) = 0$, and let K be a faithful A-module, with corresponding 1-1 representation σ. If $T \in {_A}B(H, K)$ then $\sigma(a)T(\zeta) = T\pi(a)\zeta = 0$. Hence $\sigma(a)$ is zero on $\mathrm{Tr}_K(H) = K$ (using Lemma 21). So $a = 0$. A similar obvious argument proves the assertion for σ. We leave (iii) and (iv) as exercises; they will not be explicitly used in the paper. ∎

PROPOSITION 2.3. *Let A be an operator algebra with c.a.i. Suppose that $\rho : A \to B(H)$ is a sub-tracing representation. Then $\rho(A)'' \subset \mathrm{alg\,lat}\,\rho(A)$.*

Proof. Fix $x \in H, x \neq 0$, and consider the closed span K of $\rho(A)x$ in H. If $\{e_\alpha\}$ is a c.a.i. for A then $\rho(e_\alpha)x \to x$. Thus $x \in K$. Suppose that $T \in \rho(A)''$. If $V \in {_A}B(H, K)$ then V (regarded as a map into H) is in $\rho(A)'$. Thus $TVH = VTH \subset K$. Consequently, T maps the trace $\mathrm{Tr}_K(H)$ into K. By Lemma 2.1 (iii), $T(x) \in T(K) \subset K$. Since x was arbitary, we are done. ∎

COROLLARY 2.4. *If A is an operator algebra with c.a.i. and if π is a completely sub-tracing representation of A on a Hilbert space H, then*

$$\pi(A)'' = \overline{\pi(A)}^{\,\mathrm{weak}^*} = \overline{\pi(A)}^{\,\mathrm{WOT}}.$$

Proof. By definition, for a separable infinite dimensional Hilbert space H_0 we have that $\pi(\cdot) \otimes I_{H_0}$ is sub-tracing. Hence by the previous proposition we have

$$(\pi(A) \otimes I_{H_0})'' \subset \mathrm{alg\,lat}\,\big(\pi(A) \otimes I_{H_0}\big) \subset \mathrm{alg\,lat}\,\big(\overline{\pi(A)}^{\,\mathrm{weak}^*} \otimes I_{H_0}\big).$$

By well known facts about "reflexive algebras" (see e.g. Lemma 15.4 in [6]),

$$\mathrm{alg\,lat}\,\big(\overline{\pi(A)}^{\,\mathrm{weak}^*} \otimes I_{H_0}\big) = \overline{\pi(A)}^{\,\mathrm{weak}^*} \otimes I_{H_0}.$$

Of course $(\pi(A) \otimes I_{H_0})'' = \pi(A)'' \otimes I_{H_0}$. Putting the facts above together yields

$$\pi(A)'' \otimes I_{H_0} \subset \overline{\pi(A)}^{\,\mathrm{weak}^*} \otimes I_{H_0}$$

so that

$$\pi(A)'' \subset \overline{\pi(A)}^{\,\mathrm{weak}^*} \subset \overline{\pi(A)}^{\,\mathrm{WOT}}.$$

The other direction is trivial since $\pi(A)''$ is weakly closed. ∎

By the last result, we are now almost done with the proof of Theorem 1.1. The final part is completed as follows. Suppose that H is a cogenerator for $_A$HMOD. Then by simple observations in the next section (before (3.1)), H is a generator for $_{A^*}$HMOD. Thus the image of A^* in $B(H)$ has the double commutant property. The facts at the end of Section 1 now complete the proof.

REMARK 2.5. It is fairly clear that the definitions of A-universal, generator, cogenerator, or sub-tracing, are *functorial*. In particular, if B is another such operator algebra, and if the categories $_A$HMOD and $_B$HMOD are equivalent as categories, then it is easy algebra to check that the equivalence functor takes A-universal representations to B-universal representations and vice versa. Similarly for generators, cogenerators, or sub-tracing Hilbert modules.

3. THE UNIVERSAL C^*-ALGEBRA AND UNIVERSAL REPRESENTATIONS

We will need to recall several simple facts (see e.g. [5] for more details, and examples, if needed). Firstly, there is a canonical functor $A \mapsto C^*(A)$ from the category of operator algebras (with c.a.i.) and completely contractive homomorphisms, to the category of C^*-algebras and $*$-homomorphisms, with the following universal property: there exists a completely isometric homomorphism $i : A \to C^*(A)$ such that $i(A)$ generates $C^*(A)$ as a C^*-algebra, and such that if $\phi : A \to D$ is any completely contractive homomorphism into a C^*-algebra D, then there exists a (necessarily unique) $*$-homomorphism $\widetilde{\phi} : C^*(A) \to D$ such that $\widetilde{\phi} \circ i = \phi$. The algebra $C^*(A)$ is called the *maximal C^*-algebra* generated by A, and is sometimes written as $C^*_{\max}(A)$. For those interested in algebra, this universal property essentially says that the functor $A \mapsto C^*(A)$ is the left adjoint to the forgetful functor from the category of C^*-algebras to the category of operator algebras.

From the universal property of $C^*(A)$, it is clear that if H is a Hilbert A-module, then the associated representation π has a unique extension $\widetilde{\pi}$ which is a nondegenerate $*$-representation of $C^*(A)$ on H. Conversely, every nondegenerate $*$-representation of $C^*(A)$ on H restricts (using the fact (1.1) from Section 1 if necessary) to a nondegenerate representation of A on H. Thus we may regard Hilbert A-modules as Hilbert $C^*(A)$-modules, and vice versa, in this canonical way. By symmetry, every Hilbert A-module is also a nondegenerate Hilbert module over the subalgebra A^* of $C^*(A)$ (one may deduce the nondegeneracy from fact (1.1) from Section 1 again). If $T : H \to K$ is a bounded A-module map, then it is easy to see that $T^* : K \to H$ is an A^*-module map; and conversely. From this we can make a few simple deductions. Firstly, it follows from the last fact that H is a generator for $_A$HMOD if and only if H is a cogenerator for $_{A^*}$HMOD. Similarly, H is a cogenerator for $_A$HMOD if and only if H is a generator for $_{A^*}$HMOD. Secondly, if we call a bounded A-module map T between Hilbert A-modules *adjointable* if T^* is also an A-module map, then we have from the above that:

(3.1) T is adjointable if and only if T is a $C^*(A)$-module map.

We shall not need adjointable maps very much, except in the special case that i is an isometric A-module map between Hilbert A-modules, such that i^* is an A-module map. It follows from (3.1) that i and i^* are $C^*(A)$-module maps. In particular we deduce that *unitary* morphisms, i.e. unitary A-module maps, are $C^*(A)$-module maps. That is, two Hilbert A-modules are spatially equivalent as A-modules if and only if they are spatially equivalent as $C^*(A)$-modules.

From the above, it also follows that the class of A-complemented submodules of a Hilbert A-module H is the same as the class of closed $C^*(A)$-submodules of H. Hence Hilbert A-module direct sums (respectively summands) of Hilbert

A-modules are the same as Hilbert $C^*(A)$-module direct sums (respectively summands). From these considerations the following result is clear. Note that part (ii) of Corollary 3.1 below establishes (i) of Theorem 1.2:

COROLLARY 3.1. *Let A be an operator algebra with* c.a.i.

(i) *a Hilbert A-module H is A-universal if and only if H is $C^*(A)$-universal;*

(ii) *any $C^*(A)$-universal representation of $C^*(A)$, such as the usual "universal representation" π_{u} of $C^*(A)$, restricts to an A-universal representation of A;*

(iii) *two Hilbert A-modules are quasi-equivalent as Hilbert A-modules if and only if they are quasi-equivalent as Hilbert $C^*(A)$-modules;*

(iv) *if π and θ are quasi-equivalent representations of A, then there exists a (necessarily unique) weak*-homeomorphic completely isometric isomorphism $\rho : \overline{\pi(A)}^{\,\mathrm{weak}^*} \to \overline{\theta(A)}^{\,\mathrm{weak}^*}$ such that $\rho \circ \pi = \theta$.*

Proof. (i)–(iii) are obvious from the discussion above. Let $\mathcal{C} = C^*(A)$. If π and θ satisfy the hypothesis in (iv), then by (iii) we know that $\widetilde{\pi}$ and $\widetilde{\theta}$ are quasi-equivalent in the usual C^*-algebraic sense. Thus by 5.3.1 (ii) in [7], there is a W^*-isomorphism $\Phi : \overline{\widetilde{\pi}(\mathcal{C})}^{\,\mathrm{weak}^*} \to \overline{\widetilde{\theta}(\mathcal{C})}^{\,\mathrm{weak}^*}$ such that $\Phi \circ \pi = \theta$. The restriction of Φ to $\pi(A)$ maps onto $\theta(A)$, so that by weak*-continuity we obtain the result. ∎

Thus there is a "canonical" A-universal representation of A, namely the restriction of the universal representation of $C^*(A)$ to A. We will call this *the universal representation of A*, and we write this representation of A as π_{u}.

The following result, which shall not be used in an essential way in this paper, shows that the universal representation satisfies quite a strong form of the "sub-tracing" condition.

PROPOSITION 3.2. *Let A be an operator algebra with* c.a.i. *Suppose that $\rho : C^*(A) \to B(H)$ is a nondegenerate $*$-representation with the following property: For every state φ on $C^*(A)$ there exists a $\xi \in H$ such that $\varphi(b) = \langle \rho(b)\xi, \xi \rangle$ for all $b \in C^*(A)$. Then for every topologically singly generated A-submodule K of H, there is a partial isometry in A' with range K.*

Proof. Let $B = C^*(A)$. Fix $x \in H, \|x\| = 1$, and consider the closed span K of $\rho(A)x$ in H. As noted in the proof of Proposition 2.3, $x \in K$. Since $\rho(A)K \subset K$, by the universal property of $C^*(A)$ there exists a $*$-representation π of $C^*(A)$ on K with $\pi(a) = \rho(a)_{|K}$ for $a \in A$. Let $\varphi = \langle \pi(\cdot)x, x \rangle$. This is a state, so by hypothesis there exists a $\xi \in H$ such that $\varphi = \langle \rho(\cdot)\xi, \xi \rangle$ for all $b \in B$. Therefore

$$\|\rho(b)\xi\|^2 = \|\pi(b)x\|^2$$

for all $b \in B$, and so there is a well defined unitary V_0 from $[\rho(B)\xi]$ to $[\pi(B)x] \subset K$ taking $\rho(b)\xi$ to $\pi(b)x$. Extend V_0 to a partial isometry $V \in B(H)$ by setting it to be zero on $[\rho(B)\xi]^\perp$. It is easy to see that

$$\rho(A)[\rho(B)\xi]^\perp \subset \rho(B)[\rho(B)\xi]^\perp \subset [\rho(B)\xi]^\perp,$$

and therefore $V\rho(a)y = \rho(a)Vy = 0$ if $y \in [\rho(B)\xi]^\perp$ and $a \in A$. On the other hand, if $y = \rho(b)\xi \in [\rho(B)\xi]$, then

$$V\rho(a)y = V\rho(ab)\xi = \pi(ab)x = \rho(a)\pi(b)x = \rho(a)V\rho(b)\xi = \rho(a)Vy.$$

Hence $V \in \rho(A)'$. ∎

REMARK 3.3. If π_{u} is the universal representation of $C^*(A)$ on H_{u}, and if H_0 is any Hilbert space, then the $*$-representation $\rho = \pi_{\mathrm{u}}(\cdot) \otimes I_{H_0}$ of $C^*(A)$ on $H_{\mathrm{u}} \otimes H_0$, satisfies the conditions of the proposition.

4. DUAL OPERATOR ALGEBRAS AND NORMAL REPRESENTATIONS

In this section we turn to dual operator algebras, and we will also prove the remaining part, namely (v), of Theorem 1.2. We again begin by recalling some facts and notations (see e.g. [5] for more details, if needed). A *dual operator algebra* is an operator algebra A which has a predual such that A is completely isometrically isomorphic, via a homomorphism which is a homeomorphism for the weak* topologies, to a σ-weakly closed unital subalgebra of $B(H)$. There is an abstract characterization of dual operator algebras due to Le Merdy, with a contribution by the first author, but we shall not need this here. A *normal representation* of a dual operator algebra is a unital completely contractive weak* continuous homomorphism $\pi : A \to B(K)$. We write $_A\mathrm{NHMOD}$ for the category of the Hilbert modules corresponding to such normal representations, and call an object in $_A\mathrm{NHMOD}$ a *normal Hilbert A-module.* The morphisms are the same as in $_A\mathrm{HMOD}$. Again it is a simple exercise that $_A\mathrm{NHMOD}$ is closed under direct sums.

Let A be an operator algebra with c.a.i. It is a well known fact (that appears first in [9], and which may be deduced for example from the first part of the next proof) that A^{**} is a unital operator algebra in a canonical way. For any $H \in {}_A\mathrm{HMOD}$, with corresponding representation π, we may use the universal property of $C^*(A)$ to get a $*$-representation $\widetilde{\pi} : C^*(A) \to B(H)$. As is explained in any text on C^*-algebras, we may extend this $*$-representation in a unique fashion to a unital normal $*$-representation $\theta : C^*(A)^{**} \to B(H)$. Let $\overline{\pi}$ be θ restricted to A^{**}. Then clearly $\overline{\pi}$ is the unique normal representation $A^{**} \to B(H)$ extending π from A. Moreover, since $\overline{\pi}$ is w*-continuous, its range is contained in the dual operator algebra $\overline{\pi(A)}^{\,\mathrm{weak}^*}$.

We remark in passing that the converse is true too: any normal representation $\rho : A^{**} \to B(H)$ restricts to a nondegenerate completely contractive representation of A on H. In fact, generalizing a well known fact for C^*-algebras (see p. 53 of [18]), it is clear that the categories $_A\mathrm{HMOD}$ and $_{A^{**}}\mathrm{NHMOD}$ are equivalent for any operator algebra A with c.a.i.

Proof of (v) *of Theorem* 1.2. Applying the remark above to the universal representation π_{u} of A, we obtain a normal representation $\overline{\pi}_{\mathrm{u}} : A^{**} \to \overline{\pi_{\mathrm{u}}(A)}^{\,\mathrm{weak}^*} \subset B(H_{\mathrm{u}})$. In fact $\overline{\pi}_{\mathrm{u}}$ is completely isometric, since it is the restriction of the faithful $*$-isomorphism between $\mathcal{C}^{**}$ and $\pi_{\mathrm{u}}(\mathcal{C})''$, where $\mathcal{C} = C^*(A)$. Thus by the Krein-Smulian theorem, the image B of A^{**} under $\overline{\pi}_{\mathrm{u}}$ is weak* closed, and $\overline{\pi}_{\mathrm{u}}$ is a homeomorphism for the weak* topologies. Since B contains $\pi_{\mathrm{u}}(A)$ we have that $\overline{\pi_{\mathrm{u}}(A)}^{\,\mathrm{weak}^*} = B$. This proves the result in the special case that $\pi = \pi_{\mathrm{u}}$.

To prove the general case, suppose that θ is an A-universal representation of A. By (ii) of Theorem 1.2, θ is quasi-equivalent to π_{u}. By (iv) of Corollary 3.1, there exists a weak* homeomorphic completely isometric isomorphism $\rho : \overline{\pi_{\mathrm{u}}(A)}^{\,\mathrm{weak}^*} \to \overline{\theta(A)}^{\,\mathrm{weak}^*}$ such that $\rho \circ \pi = \theta$. Composing ρ with the map

$\overline{\pi}_{\mathrm{u}} : A^{**} \to \overline{\pi_{\mathrm{u}}(A)}^{\,\mathrm{weak}^*}$ of the previous paragraph, gives the desired map in (v) of Theorem 1.2. ∎

As a corollary of (iv) and (v) of Theorem 1.2, one may immediately obtain the following fact which implies some results proved in [3] and [13]:

COROLLARY 4.1. *Suppose that A is an operator algebra with* c.a.i., *which possesses an A-universal representation π with $\pi(A)'$ selfadjoint. Then A is a C^*-algebra.*

Proof. If $\pi(A)'$ is selfadjoint, then by the above $A^{**} \cong \pi(A)''$ is a W^*-algebra. The proof is completed by an appeal to the following lemma. ∎

LEMMA 4.2. *Suppose that A is an operator algebra such that A^{**} possesses an involution with respect to which A^{**} is a C^*-algebra. Then A is a C^*-algebra.*

Proof. Suppose that A is a subalgebra of a C^*-algebra B. Then we have the following closed subalgebras: $\widehat{A} \subset A^{**} \subset B^{**}$. It is well known that a contractive homomorphism between C^*-algebras is a $*$-homomorphism. Thus, if A^{**} possesses an involution as stated, then it follows that A^{**} is a $*$-subalgebra of B^{**}. We complete the proof by showing that $\widehat{A}$ is closed under the above involution. For if $a \in A$, then $\widehat{a}^* \in A^{**} \cap \widehat{B}$. By a basic fact for Banach spaces, $A^{**} \cap \widehat{B} = \widehat{A}$. ∎

A different proof of this lemma was found together with Le Merdy around year 1999.

If π is a normal completely isometric representation of a dual operator algebra, then it follows from the Krein-Smulian theorem that $\pi(A)$ is weak* closed. If further π is completely sub-tracing, then it follows immediately from Corollary 2.4 that

$$\pi(A)'' = \pi(A).$$

One may define *normal generators* and *normal A-universal representations* for the category $_A$NHMOD, in an obvious way. It is clear as before that every normal A-universal is a normal generator, and that every normal generator is completely sub-tracing.

In order to see that for any dual operator algebra there do exist normal A-universal representations, and for its own intrinsic interest, we will define a *maximal W^*-algebra $W^*(A)$* of a dual operator algebra A. This is a W^*-algebra, together with a weak* continuous completely isometric homomorphism $j : A \to W^*(A)$ whose range generates $W^*(A)$ as a W^*-algebra, and which possesses the following universal property: given any normal representation $\pi : A \to B(H)$, there exists a normal $*$-representation $W^*(A) \to B(H)$ extending π. It is elementary to define this if $A = B^{**}$ for an operator algebra B, in this case simply let $W^*(A) = C^*(B)^{**}$, and one may easily check that this has the desired universal property. But if A is a general dual operator algebra a little more care is needed in order to show the existence of $W^*(A)$. Although such "existence proofs" are standard fare, we include most of the details below for the readers convenience.

Let A be a dual operator algebra. We suppose that the cardinality of A is less than or equal to a cardinal $I = 2^J$, where J is infinite, and define $\mathcal{F}$ to be the set of normal completely contractive representations $\pi : A \to B(\ell^2(J))$ where J varies over the cardinals corresponding to subsets of I. We write $H_\pi = \ell^2(J)$.

Define $j = \bigoplus \{\pi : \pi \in \mathcal{F}\}$, that is, $j(a) = \bigoplus_{\pi \in \mathcal{F}} \pi(a)$ for all $a \in A$. This is a normal completely contractive representation of A on a Hilbert space $H^{\mathrm{w}} = \bigoplus_{\pi \in \mathcal{F}} H_\pi$. In fact j is also completely isometric, as may be seen by the standard arguments (Sketch: take any one normal completely isometric representation σ on a Hilbert space H. If H is of dimension $\leqslant I$, we are done. If not, then for each finite subset F of H set H_F to be the Hilbert space generated by $\sigma(A)F$, and set π_F to be $\sigma(\cdot)_{|_{H_F}}$. Each π_F is unitarily equivalent to a representation on $\mathcal{F}$. Also, for each $x = [x_{ij}] \in M_n(A)$ we may clearly find such a finite set F such that the norm of $[\pi_F(x_{ij})]$ is close to that of $[\sigma(x_{ij})]$.) Thus, by the Krein-Smulian theorem, j is a homeomorphism for the weak* topologies, with weak* closed range. The projection of H^{w} onto its πth coordinate will be written as P_π. We define $W^*(A)$ to be the von Neumann algebra inside $B(H^{\mathrm{w}})$ generated by $\{j(a) : a \in A\}$. If $\theta : A \to B(H)$ is any normal completely contractive representation of A, with dimension $H \leqslant I$, then there is a unitary U such that $\rho = U^*\theta(\cdot)U \in \mathcal{F}$. Define $\widetilde{\rho} : W^*(A) \to B(H_\rho)$ to be $\widetilde{\rho}(T) = P_\rho T_{|_{H_\rho}}$. Then $\widetilde{\rho}$ is a weak* continuous $*$-homomorphism on $W^*(A)$, and $\widetilde{\rho} \circ j = \rho$. Then $\widetilde{\theta} = U\widetilde{\rho}(\cdot)U^*$ is a weak* continuous $*$-homomorphism $W^*(A) \to B(H)$, and $\widetilde{\theta} \circ j = \theta$. Clearly $\widetilde{\theta}$ is the unique such $*$-homomorphism.

Thus we have shown that $W^*(A)$ has the desired universal property at least for representations on Hilbert spaces of dimension $\leqslant I$. From this fact and a routine Zorn's lemma argument it is not hard to show that $W^*(A)$ has the desired universal property for arbitrary dimensions of normal representations.

We next prove a variant on Proposition 2.3:

PROPOSITION 4.3. *Suppose that $\rho : W^*(A) \to B(H)$ is a normal $*$-representation with the property that for every normal state ϕ on $W^*(A)$, there is an $x \in H$ such that $\phi = \langle \rho(\cdot)x, x \rangle$ on $W^*(A)$. Then $\rho(A)'' \subset \mathrm{alg\,lat}\,\rho(A)$.*

Proof. Note $K = [\rho(A)x]$ has cardinality $\leqslant I^{\aleph_0} = I$, for I as above. Thus $\dim(K) \leqslant I$, and so there is a normal $*$-representation π of $W^*(A)$ on K extending $\rho(a)_{|_K}$. The rest of the proof is the same as that of Proposition 3.2 combined with Proposition 2.3. ∎

REMARK 4.4. If π is any faithful normal $*$-representation of $W^*(A)$ on a Hilbert space K, then $\pi(\cdot) \otimes I_\infty$ satisfies the hypothesis of Proposition 4.3.

COROLLARY 4.5. *For a dual operator algebra A, and for any faithful normal $*$-representation π of $W^*(A)$, we have $\pi(A)'' = \pi(A)$.*

Proof. This is almost identical to the proof of Corollary 2.4. ∎

One may show using standard facts (see e.g. 1.3 in [18]) that any faithful normal representation of $W^*(A)$ restricts to a normal A-universal representation of A. One may also prove "normal" analogues of parts (i)–(iii) of Theorem 1.2, for a dual operator algebra A and normal A-universal representations. Indeed this follows exactly the proof of Theorem 1.2.

5. COMPLEMENTS AND EXAMPLES

If A is a C^*-algebra then the A-universal representations are quite well understood. We will recap some facts almost all of which may be found in Section 1 of [18], and then we will contrast these with the situation for nonselfadjoint algebras. Recall, from Section 1 above, that a module $H \in {}_A\mathrm{HMOD}$ is a *generator* (respectively *cogenerator*) for ${}_A\mathrm{HMOD}$ if for every nonzero morphism $R : K \to L$ of ${}_A\mathrm{HMOD}$, there exists a morphism $T : H \to K$ (respectively $T : L \to H$) of ${}_A\mathrm{HMOD}$ with $RT \neq 0$ (respectively $TR \neq 0$). We shall say that H is a *semigenerator* (respectively *semicogenerator*) if the condition in the last sentence is valid in the case that R is the identity map on a nonzero Hilbert module. Thus, for example, H is a semigenerator if for every nonzero Hilbert module K there is a nonzero bounded A-module map $T : H \to K$. We shall say that H is a *$*$-generator* (respectively an *$*$-semigenerator*) if the definition above for generator (respectively semigenerator) is modified so that the map T considered there is required to be adjointable (that is, T^* is also an A-module map).

If A is a C^*-algebra then generators, cogenerators, and $*$-generators, are the same thing, due to the fact that T is a bounded A-module map if and only if T^* is one also. Similarly, for C^*-algebras semigenerators, $*$-semigenerators, and semicogenerators, coincide. Indeed one has:

THEOREM 5.1. ([18], Section 1) *Let A be a C^*-algebra, and let $\pi : A \to B(H)$ be a nondegenerate $*$-representation. View H as a Hilbert A-module in the usual way. The following are equivalent:*

(i) *the canonical (and unique) weak* continuous map $\overline{\pi} : A^{**} \to B(H)$ extending π, is 1-1;*

(ii) *H is a semigenerator for ${}_A\mathrm{HMOD}$;*

(iii) *H is a generator for ${}_A\mathrm{HMOD}$;*

(iv) *H is A-universal.*

The above is quite useful. For example it follows immediately from (i) that for a finite dimensional C^*-algebra , the A-universal representations are exactly the faithful unital $*$-representations.

Next, let A be a nonselfadjoint operator algebra, and let H be a Hilbert A-module, and let $\pi : A \to B(H)$ be the associated representation. We consider the following properties that H may or may not have:

(DCP) π has the double commutant property; that is $\pi(A)'' = \overline{\pi(A)}^{\text{weak}^*}$.

    (I) π has the double commutant property and the canonical (and unique) weak* continuous map $\overline{\pi} : A^{**} \to B(H)$ extending π, is completely isometric.

    (II) H is a semigenerator for $_A$HMOD.

    (II)$'$ H is a semicogenerator for $_A$HMOD.

    (II)$''$ H is a *-semigenerator for $_A$HMOD.

    (III) H is a generator for $_A$HMOD.

    (III)$'$ H is a cogenerator for $_A$HMOD.

    (III)$''$ H is a *-generator for $_A$HMOD.

    (IV) H is A-universal.

The following table summarizes several earlier observations. We leave omitted details to the reader.

$$(\text{IV}) \Rightarrow (\text{III}) \Rightarrow (\text{II})$$

$$(\text{IV}) \Rightarrow (\text{III})' \Rightarrow (\text{II})'$$

$$(\text{IV}) \Rightarrow (\text{I}) \Rightarrow (\text{DCP})$$

$$(\text{IV}) \Leftrightarrow (\text{III})'' \Leftrightarrow (\text{II})''$$

$$((\text{III}) \text{ or } (\text{III})') \Rightarrow (\text{DCP}).$$

EXAMPLE 5.2. Let $A = \mathcal{T}_2$ be the algebra of 2×2 upper triangular matrices. In this case it is possible to precisely characterize the representations with the double commutant property. First notice that the representations π of $\mathcal{T}_2$ on a Hilbert space H are of one of the following types:

    (a) $H = H_1 \oplus H_2$ with $H_1 \neq \{0\}$, $H_2 \neq \{0\}$ and $H_1 \perp H_2$; and there exists a contraction $T : H_2 \to H_1$ such that $\pi(A)(\zeta + \eta) = a_{11}\zeta + a_{12}T(\eta) + a_{22}\eta$, for all $\zeta \in H_1, \eta \in H_2$, and $A = [a_{ij}] \in \mathcal{T}_2$.

    (b) $H \neq \{0\}$ and $\pi(A)(\zeta) = a_{11}\zeta$, for all $\zeta \in H$ and $A = [a_{ij}] \in \mathcal{T}_2$.

    (c) $H \neq \{0\}$ and $\pi(A)(\zeta) = a_{22}\zeta$, for all $\zeta \in H$ and $A = [a_{ij}] \in \mathcal{T}_2$.

    (d) $H = \{0\}$.

We will not discuss the trivial case (d) below. Clearly types (b) and (c) possess the double commutant property. We write a representation π as in (a) above as a 3-tuple (H_1, H_2, T).

PROPOSITION 5.3. *Let π be a type* (a) *representation of $\mathcal{T}_2$, with associated 3-tuple (H_1, H_2, T) as above.*

    (i) π possesses the double commutant property if and only if $T : H_2 \to H_1$ is not invertible;

    (ii) π is a semigenerator if and only if $T(H_2)$ is not dense in H_1;

    (iii) π is a semicogenerator if and only if T is not 1-1;

    (iv) π is a generator if and only if $T(H_2)$ is not dense in H_1 and $T \neq 0$;

    (v) π is a cogenerator if and only if T is not 1-1 and $T \neq 0$;

    (vi) π is (completely) sub-tracing if and only if $T(H_2)$ is not dense in H_1.

Proof. (i) An elementary computation shows that $\pi(A)'$ consists of all operators $A \oplus D$, where $A \in B(H_1), D \in B(H_2)$, such that $AT = TD$. One observation which will be useful later is that if $\zeta \in H_2, \eta \in H_1$ then $A = T\zeta \otimes \eta$ and $D = \zeta \otimes T^*\eta$ satisfies $AT = TD$.

If T is invertible then the set of solutions (A, D) to the equation $AT = TD$ is $\{(A, T^{-1}AT) : A \in B(H_1)\}$. In this case the operator z defined to be T^{-1} on

H_1 and zero on H_2, is easily seen to be in $\pi(A)''$, since $z(A \oplus T^{-1}AT)(\zeta + \eta) = T^{-1}A\zeta = (A \oplus T^{-1}AT)z(\zeta + \eta)$ for $\zeta \in H_1, \eta \in H_2$. However z is clearly not in $\overline{\pi(A)}^{\,\text{weak}^*} = \pi(A)$.

On the other hand, suppose that T is not invertible. If T is the zero operator, then any $A \in B(H_1), D \in B(H_2)$ satisfies $AT = TD$, from which it is easily seen that $\pi(A)'' = \pi(A)$. Thus we may suppose that $T \neq 0$. An operator R in $\pi(A)''$ may be written as a 2×2 operator matrix with respect to the decomposition $H_1 \oplus H_2$. The 1-1 entry x of this matrix must therefore commute with any A satisfying $AT = TD$ as above. Picking $A = T\zeta \otimes \eta$ and $D = \zeta \otimes T^*\eta$ as above, we have that $xT\zeta \otimes \eta = T\zeta \otimes x^*\eta$, so that $\eta \otimes xT\zeta = x^*\eta \otimes T\zeta$. It follows from this that $x^*\eta$ is a scalar multiple of η for every vector η, which implies that $x \in \mathbb{C}I_{H_1}$. A similar argument shows that the 2-2 entry of R is in $\mathbb{C}I_{H_2}$. A similar argument shows that the 1-2 entry y of R is a scalar multiple of T. To complete the proof, we need to show that the 2-1 entry z of R is zero. The fact that $Dz = zA$ yields as above that $\zeta \otimes z^*T^*\eta = zT\zeta \otimes \eta$ for all $\zeta \in H_2, \eta \in H_1$ as above. It follows that either $zT = Tz = 0$, or that T is both left and right invertible. The latter is impossible, by hypothesis. Thus if $T(H_2)$ is dense in H_1 then $z = 0$. On the other hand, if $T(H_2)$ is not dense in H_1, then set $D = 0$ and let $A = \xi \otimes \sigma$, where $\sigma \in T(H_2)^\perp$. Clearly $AT = 0 = TD$, so that $zA = z\xi \otimes \sigma = 0$. Since ξ is arbitrary we must have $z = 0$.

We leave (ii)–(vi) as simple but tedious exercises. For example, to check (iv) one assumes that $T(H_2)$ is not dense in H_1, and then one considers the various cases that can arise (corresponding to the types (a)–(c) of $\mathcal{T}_2$-modules) for nonzero maps R between Hilbert $\mathcal{T}_2$-modules. ∎

From the above it is easy to find very simple finite dimensional completely isometric representations of $\mathcal{T}_2$ satisfying (I), but not (II), (II)$'$, (III), (III)$'$, or (IV). Similarly (II), or even (II) together with (II)$'$, does not imply (IV). And (III), or even (III) together with (III)$'$, does not imply (IV). Indeed by the last result one can easily find finite dimensional completely isometric representations satisfying (I), (III), and (III)$'$; however no finite dimensional representation of $\mathcal{T}_2$ can be A-universal. To see this note that by (i) of Corollary 3.1 any A-universal representation on a Hilbert space H is $C^*(A)$-universal, and therefore extends to a faithful representation of $C^*(A)$ on H. Indeed the A-universal representations are, by (i) of Corollary 3.1 and Theorem 5.1, in 1-1 correspondence with the normal faithful $*$-representations of $C^*(A)^{**}$. However in [3] Section 2 it is shown that $C^*(\mathcal{T}_2)$ is infinite dimensional.

EXAMPLE 5.4. We consider a generalization of Example 5.2, which will show for example that (II) does not imply the double commutant property, and also that even for completely isometric representations (II) and (III) may differ.

Let X be an operator space, and let $\mathcal{U}(X)$ be the canonical "upper triangular" algebra consisting of upper triangular 2×2 matrices with scalars on the diagonal and X in the 1-2 corner. Then $\mathcal{U}(X)$ has a canonical operator space structure making it a unital operator algebra. See the last section in [3] for example. As is spelled out there, the nontrivial representations π of $\mathcal{U}(X)$ are in 1-1 correspondence with completely contractive maps $\alpha : X \to B(H_2, H_1)$. Also, π is completely isometric if and only if α is completely isometric. We may thus associate with π the tuple (H_1, H_2, α). If $X = \mathbb{C}$ then this is simply the (H_1, H_2, T) notation we

met in Example 5.2. If π is such a representation, with $H_1 \neq \{0\}$ and $H_2 \neq \{0\}$, then it is easy to compute the commutant $\pi(A)'$. Analoguously to Example 5.2, this commutant consists of the operators $A \oplus D$ with $A \in B(H_1), D \in B(H_2)$, such that $A\alpha(x) = \alpha(x)D$ for all $x \in X$. The second commutant $\pi(A)''$ therefore is the set of 2×2 operator matrices

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

satisfying the equations $Ax = xA, Dw = wD, Ay = yD$ and $zA = Dz$, whenever $A \in B(H_1), D \in B(H_2)$ satisfy $A\alpha(x) = \alpha(x)D$ for all $x \in X$. From this and Theorem 1.2 (respectively, Corollary 4.4) we can deduce "double commutant theorems" for X. For example it follows that:

COROLLARY 5.5. *For any operator space (respectively dual operator space) X, there exists a completely isometric (respectively and weak* homeomorphic) linear $\alpha : X \to B(H_2, H_1)$ such that the weak* closure of $\alpha(X)$ (respectively such that $\alpha(X)$) coincides with the set of operators $S \in B(H_2, H_1)$ such that $AS = SD$, whenever $A \in B(H_1), D \in B(H_2)$ satisfies $AT = TD$ for all $T \in \alpha(X)$.*

It is elementary to check that a representation π of $\mathcal{U}(X)$, associated with a tuple (H_1, H_2, α) as above (with H_1 and H_2 nonzero), is a semigenerator if and only if the span of the ranges of the operators $\alpha(x)$, for all $x \in X$, is not dense in H_1. Also, the representation is a semicogenerator if and only if $\bigcap_{x \in X} \text{Ker } \alpha(x) \neq \{0\}$.

From this it is quite easy to find semigenerators or semicogenerators which do not satisfy (DCP). For example, choose α with the span of the ranges of the operators $\alpha(x)$ not dense in H_1, but for which there exist operators $w \in B(H_2)$ which are not scalar multiples of the identity such that $Dw = wD$ for all $A \in B(H_1), D \in B(H_2)$ satisfying $A\alpha(x) = \alpha(x)D$ for all $x \in X$. (For a concrete such example let $X = \ell^\infty$ and $\alpha(x) = S \, \text{diag}\{x\}$, where S is the forward shift.) Then the 2×2 operator matrix with w in the 2-2 entry and other entries zero, is in $\pi(A)''$ but not in the weak* closure of $\pi(A)$. Thus (II) does not imply the (DCP). Similar considerations show that (II)$'$ does not imply the (DCP).

We now exhibit an example of a completely isometric representation π of the type considered in Example 5.4, which satisfies (II) and (II)$'$, but which is not a generator (i.e. does not have (III)). One such is given by $\alpha : \ell^\infty \to B(\ell_2)$ of the form $\alpha(x) = S \, \text{diag}\{0, x\}$, where S is the forward shift again. It is clear that this has property (II) and (II)$'$. To see that (III) fails we appeal to the following

CLAIM. Let X be a non-reflexive dual operator space. Then any representation $\pi : \mathcal{U}(X) \to B(H)$ associated with a 3-tuple (H_1, H_2, α) with α weak* continuous, is a generator.

To prove this Claim, consider a fixed non weak* continuous contractive linear functional β on X, and consider the representation of $\mathcal{U}(X)$ on $K = \mathbb{C}^2$ associated with the tuple $(\mathbb{C}, \mathbb{C}, \beta)$. Let $L = \mathbb{C}$, with the "type (c) action" of $\mathcal{U}(X)$ described in Example 5.2, and let $R \in B(K, L)$ be the projection onto the second coordinate. This is clearly a nonzero A-module map on K. A nonzero A-module map $T : H \to K$ is easily seen to be necessarily of the form $T_1 \oplus T_2$, where $T_1 \in B(H_1, K_1), T_2 \in B(H_2, K_2)$ may be any pair satisfying $T_1\alpha(x) = \beta(x)T_2$ for all $x \in X$. In fact this is true for any $\mathcal{U}(X)$-modules H, K. If $T_2 \neq 0$ this implies that $\beta(x)$ is a constant

multiplied by $\langle \alpha(x)\xi, \sigma \rangle$ for some vectors ξ, σ. This implies the contradiction that β is weak* continuous. Thus $T_2 = 0$, so that $RT = 0$.

The examples above rule out most of the variants for nonselfadjoint algebras of the remaining implications of Theorem 5.1. Some questions which we have not taken the time to settle, are the following: Does a completely isometric representation satisfying ((II) and (II)′) automatically possess the double commutant property? Also, for faithful representations of unital operator algebras A, how close is the condition (DCP) to the condition ((III) or (III)′)? To the condition ((II) or (II)′)?

Finally, we remark that there are other variants on the definition of "generator", which are situated between (III) and (IV). In particular, the class of Hilbert A-modules H with the following property: For any other Hilbert A-module K there is a cardinal α and a bounded module map $T : H^{(\alpha)} \to K$ which is surjective (respectively, has dense range, is a 1-quotient map). We will not say anything further about these three classes except that they contain (but are not equal to) the class (IV), and are contained in class (III), and hence are faithful and satisfy the double commutant property.

6. NONUNITAL OPERATOR ALGEBRAS

In this section we verify that all of Theorem 1.2, with the exception of part (iv), is valid more generally for operator algebras A which do not have a c.a.i. We shall see that if part (iv) was valid too then A must have a c.a.i.

We say that a homomorphism $\pi : A \to B(H)$ is *nondegenerate* if the span of terms of the form $c_1 c_2 \cdots c_n \zeta$, for $\zeta \in H$ and $c_i \in \pi(A) \cup \pi(A)^*$, is dense in H. Perhaps a better name for this is *∗-nondegeneracy*, but for simplicity we will use the other name here. We will not use this fact, but any contractive homomorphism $\pi : A \to B(K)$ can be replaced by a nondegenerate one, by restricting each $\pi(a)$ to the closed subspace of K densely spanned by the products $c_1 c_2 \cdots c_n \zeta$ mentioned above. We remark that if A has a c.a.i. then this new definition of nondegeneracy of representations coincides with the old. To see this, suppose that $\{e_\alpha\}$ is a c.a.i. for A, and that π is a contractive homomorphism which is nondegenerate in the new sense above. By fact (1.1) from Section 1 we know that $\{\pi(e_\alpha)\}$ is a c.a.i. for the C^*-subalgebra of $B(H)$ generated by $\pi(A)$. Thus $\pi(e_\alpha) \to \mathrm{Id}$ strongly on H. The converse is easier.

If A is any operator algebra then a recent paper ([14]) proves the remarkable results that (a) there are unique matrix norms on $A^+ = A \oplus \mathbb{C}$ such that A^+ is a unital abstract operator algebra (with identity $1_+ = (0, 1)$) containing A completely isometrically, and (b) given a contractive (respectively completely contractive, isometric, completely isometric) homomorphism $\varphi : A \to B$ between operator algebras, the extension $\varphi^+ : A^+ \to B^+$ given by $\varphi^+(a + \lambda 1_+) = \varphi(a) + \lambda 1_+$, for $a \in A$, $\lambda \in \mathbb{C}$, is also a contractive (respectively completely contractive, isometric, completely isometric) homomorphism. From this it is easy to define a C^*-envelope (in the spirit of Arveson and Hamana ([12], [2]) and a maximal universal C^*-algebra of operator algebras without a c.a.i. Again for specificity will do this in the operator space framework, as opposed to the Banach algebra version.

If A is any operator algebra then we define $C_e^*(A)$ (respectively $C^*(A)$) to be the C^*-subalgebra of $C_e^*(A^+)$ (respectively $C^*(A^+)$) generated by the copy

of A. See [12] and [2] for the basic properties of the C^*-envelope $C_e^*(A^+)$. We claim that $C_e^*(A)$ (respectively $C^*(A)$) has the appropriate universal properties, analogous to the well known properties they have in the case that A has a c.a.i. We first treat $C_e^*(A)$. If $\pi : A \to B$ is a completely isometric homomorphism into a C^*-algebra B such that $\pi(A)$ generates B as a C^*-algebra, then $\pi^+ : A^+ \to B^+$ is a completely isometric homomorphism into a C^*-algebra, whose range generates B^+ as a C^*-algebra. Thus by the universal property of $C_e^*(A^+)$, there is a surjective $*$-homomorphism $\rho : B^+ \to C_e^*(A^+)$ such that $\rho \circ \pi^+ = j$, where $j : A^+ \to C_e^*(A^+)$ is the canonical embedding. Let θ be ρ restricted to B, then θ is a $*$-homomorphism with

$$\theta(\pi(a)) = \rho(\pi^+(a)) = j(a) \in C_e^*(A)$$

for all $a \in A$. Thus θ maps B into $C_e^*(A)$, and the above shows that $(C_e^*(A), j)$ has the universal property which one would desire for a "C^*-envelope of A".

We now check that $C^*(A)$ has the universal property which one would desire. Suppose that π is a completely contractive homomorphism from A into a C^*-algebra B. By the universal property of $C^*(A^+)$, there is a $*$-homomorphism $\rho : C^*(A^+) \to B^+$ such that $\rho \circ \kappa = \pi^+$, where $\kappa : A^+ \to C^*(A^+)$ is the canonical embedding. Let θ be ρ restricted to $C^*(A)$; then θ is a $*$-homomorphism with

$$\theta(\kappa(a)) = \rho(\kappa(a)) = \pi^+(a) = \pi(a) \in B,$$

for all $a \in A$. Thus θ maps $C^*(A)$ into B.

If A is a nonunital operator algebra then we let $_A$HMOD be the category of nondegenerate Hilbert A-modules, using the definition of "nondegenerate" given at the beginning of this section. By the universal property of $C^*(A)$, the objects in $_A$HMOD are "the same as" the objects in $_{C^*(A)}$HMOD. In particular, a completely contractive representation of A is nondegenerate in the new sense if and only if the associated representation of $C^*(A)$ is nondegenerate in the usual sense. We may define direct sums in $_A$HMOD by associating them with the corresponding direct sums in $_{C^*(A)}$HMOD. Thus, a direct sum of Hilbert A-modules is nondegenerate if and only if every one of the individual summand Hilbert A-modules is nondegenerate. The fact from Section 3 labelled (3.1) is still valid with the same proof, and so we may treat A-complemented submodules and direct summands in $_A$HMOD just as we did before. Indeed Corollary 3.1 also carries over verbatim, as does (i)–(iii) of Theorem 1.2. We define the *universal representation π_u* of A to be the restriction to A of the universal representation π_u of $C^*(A)$. The facts in the second paragraph of Section 4 also transfer immediately, the only difference being that A^{**} and θ there need not be unital. Now we see that (v) of Theorem 1.2 carries over verbatim too. Thus all of Theorem 1.2, with the exception of part (iv), is valid when A is an operator algebra with no c.a.i.

Indeed, it is clear that if (iv) and (v) of Theorem 1.2 both hold, then A^{**} is unital, and from the theory of Banach algebras it follows that A has a c.a.i.

COROLLARY 6.1. *An operator algebra A possesses a c.a.i. if and only if for every nondegenerate contractive homomorphism $\pi : A \rightarrow B(H)$, we have $x \in [\pi(A)x]$ whenever $x \in H$.*

Proof. The one direction is easy, using the facts noted at the beginning of this section. The other direction may be proved by noting that if the hypothesis holds then the rest of the proof of Theorem 1.2 (iv) is easily amended to yield the nonunital case. Hence $A^{**} \cong A''$ is unital, and so A has a c.a.i. as mentioned above Corollary 6.1. ∎

REMARKS 6.2. A direct "reflexivity" proof of Corollary 6.1 may also be given. Also, we point out that the qualification "for all" in Corollary 6.1 may not be replaced by "for some completely isometric nondegenerate homomorphism $\pi : A \rightarrow B(H)$". To see this consider the unitary operator U on $H = L^2[0, 2\pi]$ given by an irrational rotation. If $A \subset B(H)$ is the uniform closure of the span of $U, U^2, U^3, \ldots$, then it is fairly clear that $f \in [Af]$ for all $f \in L^2[0, 2\pi]$. In particular A acts nondegenerately on H. If A contained a c.a.i. $\{E_\alpha\}$ then $E_\alpha U \rightarrow U$, and so $E_\alpha \rightarrow \mathrm{Id}_H$. Hence $\mathrm{Id}_H \in A$. On the other hand, A is contained in the closure of the C^*-algebra generated by I and U, and this latter C^*-algebra is isomorphic to the set $C(\mathbb{T})$ of continuous functions on the circle, by basic spectral theory. Under this isomorphism A corresponds to the *nonunital* ideal $zA(\mathbb{D})$ in the disk algebra $A(\mathbb{D}) \subset C(\mathbb{T})$. Here z represents the function $e^{i\theta} \mapsto e^{i\theta}$ on $\mathbb{T}$. This contradiction shows that A does not have a c.a.i.

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