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WEIGHTED COMPOSITION OPERATORS ON SPACES OF FUNCTIONS WITH DERIVATIVE IN A HARDY SPACE

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Abstract. Let φ and ψ be two analytic functions defined on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The operator given by $f \mapsto \psi(f \circ \varphi)$ is called a weighted composition operator. For each $1 \leqslant p \leqslant \infty$, let S_p be the space of analytic functions on $\mathbb{D}$ whose derivatives belong to the Hardy space H_p. In this paper we deal with boundedness, compactness, weak compactness, and complete continuity of weighted composition operators from S_p into S_q for $1 \leqslant p, q \leqslant \infty$.

Keywords: *Weighted composition operators, S_p spaces, compact operators, weakly compact operators, completely continuous operators.*

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1. INTRODUCTION

The aim of this paper is to characterize boundedness, compactness, weak compactness and complete continuity of weighted composition operators between S_p spaces.

For $1 \leqslant p \leqslant \infty$ we denote by H_p the Hardy space on the unit disc $\mathbb{D}$ and by S_p the space of analytic functions f on $\mathbb{D}$ such that $f' \in H_p$, endowed with the norm $\|f\|_{S_p} = |f(0)| + \|f'\|_{H_p}$. It is clear that S_p is a Banach space. Denote by $\mathcal{H}(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$. A *weighted composition operator* $W_{\varphi,\psi}$ is an operator that maps $f \in \mathcal{H}(\mathbb{D})$ to the function $W_{\varphi,\psi}(f)(z) = \psi(z) f(\varphi(z))$, for all $z \in \mathbb{D}$, where φ and ψ are analytic functions defined in $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ (we will assume in what follows that ψ is not the null function).

These operators have been studied on many spaces of analytic functions. For example, on Hardy spaces ([2], [7], [8]), on the disc algebra ([15], [22]), on Bloch-type spaces ([18], [21], [23]), and on some weighted Banach spaces of analytic functions ([6], [19]).

Weighted composition operators appear in a natural way in different contexts. For example, isometries in many Banach spaces of analytic functions are weighted composition operators. For instance, de Leeuw showed that isometries in the Hardy space H_1 are weighted composition operators and Forelli obtained the same result for the Hardy spaces H_p when $1 < p < \infty$, $p \neq 2$ (see [11], [12]). Similar results for S_p spaces were obtained by Novinger and Oberlin ([20]). They proved that isometries in S_p for $1 \leqslant p < \infty$, $p \neq 2$, are $Tf(z) = \lambda_1\Big[f(0) + \int\limits_0^z W_{\varphi,\lambda_2(\varphi')^{1/p}}(f')(\xi)\,\mathrm{d}\xi\Big]$ for $f \in S_p$, where $|\lambda_1| = |\lambda_2| = 1$ and φ is a suitable analytic function such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Weighted composition operators also appear in the study of classical operators like Cesàro and Hilbert operators and they play an important role in the study of composition operators on Hardy spaces of a half-plane (see [8] for more details).

When $\psi \equiv 1$, we just have the *composition operator* C_φ defined by $C_\varphi(f) = f \circ \varphi$ and when $\varphi(z) = z$ we have the *multiplication operator* M_ψ defined by $M_\psi(f) = \psi f$. During the last century, composition operators were studied between different spaces of analytic functions. There are many papers about these operators. Among other things, they deal with boundedness and compactness of these operators. Since about 1980, this subject has undergone explosive growth (see, for example, the following recent works [4], [9], [14], and [25]).

In 1978, Roan started the study of composition operators on the space S_p ([24]). A few years later, MacCluer characterized boundedness and compactness of these operators in terms of Carleson measures showing, among other things, that for $1 \leqslant p < \infty$ the composition operator $C_\varphi : S_p \to S_p$ is compact if and only if $\|\varphi\|_{H_\infty} < 1$ (Theorem 2.3 of [16]). Recently, Arora, Mukherjee and Panigrahi found two sufficient conditions in terms of Carleson measures for boundedness and compactness of weighted composition operators on S_p ([1]). In this paper, we characterize boundedness and compactness of weighted composition operators on S_p.

In Section 2 we present the main results of this paper. In Theorems 2.1 and 2.2 we characterize boundedness, compactness, weak compactness and complete continuity of $W_{\varphi,\psi}$ from S_p into S_q, $1 \leqslant p, q \leqslant \infty$, in terms of weighted composition operators on Hardy spaces. In Section 3 we prove these results. The starting point is a study of the inclusion map from S_p into the disc algebra. In the last section we apply some known results of weighted composition operators on Hardy spaces (see [7] and [8]) to obtain other characterizations of boundedness, compactness, weak compactness and complete continuity in terms of geometric properties of φ and ψ.

In what follows, we denote by $\mathbb{T}$ the unit circle, m the normalized Lebesgue measure on $\mathbb{T}$, B_X the unit ball of the Banach space X, and by $\|T\|_{X\to Y}$ the norm of the operator $T : X \to Y$. To simplify the statements of the results, we denote by $\Phi = \{\varphi \in \mathcal{H}(\mathbb{D}) : \varphi(\mathbb{D}) \subseteq \mathbb{D}\}$. We refer the reader to [9] and [12] for the terminology and results about spaces of analytic functions.

2. MAIN RESULTS

In the following results we show that boundedness, compactness, weak compactness, and the complete continuity of $W_{\varphi,\psi} : S_p \to S_q$ can be characterized through the corresponding properties of $W_{\varphi,\psi\varphi'} : H_p \to H_q$.

Let us recall that an operator is compact if it maps bounded sets into relatively norm compact sets; it is weakly compact if it maps bounded sets into relatively weakly compact sets; and it is completely continuous (or Dunford-Pettis) if it maps weakly compact sets into compact sets. It is well known that if an operator is compact, then it is weakly compact and completely continuous. The other implications are not true in general.

Theorem 2.1. *Let $1 \leqslant p, q \leqslant \infty$, $\varphi \in \Phi$, and $\psi \in S_q$. $W_{\varphi,\psi}$ exists as a bounded (respectively, completely continuous) operator from S_p into S_q if and only if $W_{\varphi,\psi\varphi'}$ exists as a bounded (respectively, completely continuous) operator from H_p into H_q.*

Moreover, if $(p, q) \neq (1, \infty)$, then $W_{\varphi,\psi} : S_p \to S_q$ is compact (respectively, weakly compact) if and only if $W_{\varphi,\psi\varphi'} : H_p \to H_q$ is compact (respectively, weakly compact).

The above theorem leaves open the question of compactness and weak compactness when the domain is S_1 and the range is S_∞. In this case we obtain the following characterizations.

Theorem 2.2. *Let $\varphi \in \Phi$ and $\psi \in S_\infty$ be such that $W_{\varphi,\psi}$ exists as a bounded operator from S_1 into S_∞.*

(i) *This operator is compact if and only if $\|\varphi\|_{H_\infty} < 1$ or*

$$\lim_{|\varphi(z)|\to 1} |\psi'(z)| = \lim_{|\varphi(z)|\to 1} \frac{|\psi(z)\varphi'(z)|}{1 - |\varphi(z)|^2} = 0.$$

(ii) *It is weakly compact if and only if $\|\varphi\|_{H_\infty} < 1$ or*

$$\lim_{|\varphi(z)|\to 1} |\psi'(z)| = 0.$$

3. PROOFS

It is well known (Theorem 3.11 in [10]) that, if $f \in S_1$, then f extends continuously to $\overline{\mathbb{D}}$. Thus, the functions in S_p belong to the disc algebra A (the space of analytic functions on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ endowed with the norm $\|f\|_A = \sup_{z\in\mathbb{D}} |f(z)|$). By the closed graph theorem, the inclusion map $j_p : S_p \hookrightarrow A$ is bounded, $1 \leqslant p \leqslant \infty$. In the following result we study other properties of $j_p : S_p \hookrightarrow A$ and of the inclusion map from S_1 into H_1.

Before stating these results, we show the following lemma whose proof can be obtained by adapting the proof of Proposition 3.11 in [9].

LEMMA 3.1. *Given $1 \leqslant p, q \leqslant \infty$, let X be the space S_p or H_p and Y any of the spaces S_q, H_q or A. Let $\varphi \in \Phi$ and $\psi \in Y$ be such that $W_{\varphi,\psi}$ defines a bounded operator from X into Y. Then $W_{\varphi,\psi} : X \to Y$ is compact (respectively, weakly compact) if and only if, whenever (f_n) is a bounded sequence in X and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|W_{\varphi,\psi}(f_n)\|_Y \to 0$ (respectively, $(W_{\varphi,\psi}(f_n))$ is a weak null sequence in Y).*

PROPOSITION 3.2. (i) *The inclusion operator $j_p : S_p \hookrightarrow A$ is compact if and only if $1 < p \leqslant \infty$. Moreover, $j_1 : S_1 \hookrightarrow A$ is not weakly compact.*

(ii) *The inclusion operator $j_1 : S_1 \hookrightarrow A$ is completely continuous.*

(iii) *The inclusion operator from S_1 into H_1 is compact.*

Proof. (i) Let us assume that $1 < p \leqslant \infty$. Note that the inclusion operator from A into $C(\mathbb{T})$ (the space of continuous functions on $\mathbb{T}$) is an isometry (not onto). So, we only have to prove that $j_p : S_p \hookrightarrow C(\mathbb{T})$ is compact. We will use Ascoli's theorem. Bearing in mind that $j_p : S_p \hookrightarrow A$ is bounded, we are going to show that B_{S_p} is equicontinuous in $C(\mathbb{T})$.

We can assume that $1 < p < \infty$. Take q so that $\frac{1}{p} + \frac{1}{q} = 1$. Fix $\varepsilon > 0$ and consider $\delta = \frac{\varepsilon^q}{(2\pi)^{q/p}}$. Now, take $x = \mathrm{e}^{\mathrm{i}\theta_0}$ and $y = \mathrm{e}^{\mathrm{i}\theta_1}$ two points in $\mathbb{T}$ such that $\theta_0 < \theta_1$ and $(\theta_1 - \theta_0) < \delta$. Then, having fixed $f \in B_{S_p}$ and $0 < r < 1$, we have that

$$\begin{aligned} |f(rx) - f(ry)| &= \Big| \int_{[r\mathrm{e}^{\mathrm{i}\theta_0}, r\mathrm{e}^{\mathrm{i}\theta_1}]} f'(z)\,\mathrm{d}z \Big| = \Big| \int_{\theta_0}^{\theta_1} \mathrm{i}r\mathrm{e}^{\mathrm{i}\theta} f'(r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \Big| \\ &\leqslant r \int_{\theta_0}^{\theta_0+2\pi} |f'(r\mathrm{e}^{\mathrm{i}\theta})| \chi_{[\theta_0,\theta_1]}(\theta)\,\mathrm{d}\theta \\ &\leqslant r \Big(\int_0^{2\pi} |f'(r\mathrm{e}^{\mathrm{i}\theta})|^p\,\mathrm{d}\theta \Big)^{1/p} \Big(\int_{\theta_0}^{\theta_0+2\pi} \chi_{[\theta_0,\theta_1]}(\theta)\,\mathrm{d}\theta \Big)^{1/q} \\ &\leqslant (2\pi)^{1/p} r \|f'\|_{H_p} (\theta_1 - \theta_0)^{1/q} < (2\pi)^{1/p} \delta^{1/q} = \varepsilon. \end{aligned}$$

Finally, since $f \in C(\overline{\mathbb{D}})$, we have that $|f(x) - f(y)| = \lim_{r \to 1} |f(rx) - f(ry)| \leqslant \varepsilon$.

Next, we are going to show that the inclusion operator $j_1 : S_1 \hookrightarrow A$ is not weakly compact. Let (a_n) be any real sequence in $(0,1)$ such that $a_n \to 1$ and take $g_n(z) = (1 - a_n^2)\frac{z}{1 - a_n z}$. It is easy to obtain that g_n belongs to the unit ball of S_1 and (g_n) converges to zero uniformly on compact sets of $\mathbb{D}$. In addition, $g_n(1) = 1 + a_n \to 2$. So, (g_n) cannot be a weak null sequence in A. Thus, by Lemma 3.1, $j_1 : S_1 \hookrightarrow A$ is not weakly compact.

(ii) The operator $T : H_1 \to S_1$ defined by $T(f)(z) = \int_0^z f(\xi)\mathrm{d}\xi$ for $f \in H_1$ and $z \in \mathbb{D}$ is an isometry into a subspace of S_1 with codimension 1. So, $j_1 : S_1 \hookrightarrow A$ is completely continuous if and only if $S : H_1 \to A$ given by $S(f)(z) = \int_0^z f(\xi)\mathrm{d}\xi$, for each $z \in \mathbb{D}$, is completely continuous. By Theorem 2 of [13], S is completely continuous if and only if $U := S \mid H_\infty : H_\infty \to A$ is compact. But, U is compact since $j_\infty : S_\infty \hookrightarrow A$ is compact (by (i)), $\widetilde{S} : H_\infty \to S_\infty$ given by $\widetilde{S}(f)(z) = \int_0^z f(\xi)\,\mathrm{d}\xi$, for $z \in \mathbb{D}$, is well defined, and $U = j_\infty \circ \widetilde{S}$.

(iii) Again we will use Lemma 3.1. Let (f_n) be a sequence in B_{S_1} that goes to zero uniformly on compact subsets of $\mathbb{D}$. Since $f_n(0) \to 0$, we can suppose that $f_n(0) = 0$ for all n.

Since $f_n' \in H_1$, by Fejér-Riesz's inequality (Theorem 3.13 of [10]), $\int_0^1 |f_n'(t\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}t$ converges for each $\theta \in [0, 2\pi]$. Thus, since $f_n \in C(\overline{\mathbb{D}})$, we have that

$$\begin{aligned}
\|f_n\|_{H_1} &= \frac{1}{2\pi}\int_0^{2\pi} |f_n(\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^{\mathrm{e}^{\mathrm{i}\theta}} f_n'(\xi)\,\mathrm{d}\xi\right|\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^1 f_n'(t\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}t\right|\mathrm{d}\theta \leqslant \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 |f_n'(t\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}t\,\mathrm{d}\theta \\
&= \int_0^1 \frac{1}{2\pi}\int_0^{2\pi} |f_n'(t\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta\,\mathrm{d}t = \int_0^1 M_1(t, f_n')\,\mathrm{d}t.
\end{aligned}$$

On the other hand, $M_1(t, f_n') \leqslant \|f_n'\|_{H_1} = \|f_n\|_{S_1} \leqslant 1$ and, for each $t \in (0,1)$, we have that $M_1(t, f_n') \to 0$, when n tends to infinity. Then $\lim_{n\to\infty}\int_0^1 M_1(t, f_n')\,\mathrm{d}t = 0$. Thus, $\lim_{n\to\infty}\|f_n\|_{H_1} = 0$ and, by Lemma 3.1, the inclusion operator from S_1 into H_1 is compact. ∎

Since $(W_{\varphi,\psi}(f))' = \psi' f \circ \varphi + \psi\varphi' f' \circ \varphi$, the study of $W_{\varphi,\psi} : S_p \to S_q$ is "equivalent" to the study of any of these operators: $W_{\varphi,\psi'} : S_p \to H_q$ and $W_{\varphi,\psi\varphi'} : H_p \to H_q$. Before proving Theorem 2.1, we have to study some properties of weighted composition operators from S_p in H_q.

Proposition 3.3. *Let $1 \leqslant p, q \leqslant \infty$ and $\varphi \in \Phi$.*

(i) *If $\psi \in H_q$, then $W_{\varphi,\psi}$ defines a bounded operator from S_p into H_q.*
(ii) *If $(p,q) \neq (1,\infty)$ and $\psi \in H_q$, then $W_{\varphi,\psi} : S_p \to H_q$ is compact.*
(iii) *Let $\psi \in H_\infty$. The following assertions are equivalent:*
   (a) *The operator $W_{\varphi,\psi} : S_1 \to H_\infty$ is compact.*
   (b) *The operator $W_{\varphi,\psi} : S_1 \to H_\infty$ is weakly compact.*
   (c) $\|\varphi\|_{H_\infty} < 1$ *or* $\lim_{|\varphi(z)|\to 1} |\psi(z)| = 0$.
(iv) *Let $\psi \in H_\infty$. Then $W_{\varphi,\psi} : S_1 \to H_\infty$ is completely continuous.*

Proof. (i) Let $f \in S_p$. Since the inclusion map $j_p : S_p \hookrightarrow A$ is bounded, we have that $\|W_{\varphi,\psi}(f)\|_{H_q} = \|\psi f \circ \varphi\|_{H_q} \leqslant \|f\|_A \|\psi\|_{H_q} \leqslant \|j_p\|_{S_p \to A}\|\psi\|_{H_q}\|f\|_{S_p}$ for all $f \in S_p$.

(ii) We will apply Lemma 3.1. Let (f_n) be a sequence in B_{S_p} such that $(f_n) \to 0$ uniformly on compact subsets of $\mathbb{D}$. Now, to prove that $\|\psi f_n \circ \varphi\|_{H_q} \to 0$ we have to distinguish between $p > 1$ and $p = 1$.

Assume that $p > 1$. By Proposition 3.2(i) and Lemma 3.1, we have that $\|f_n\|_A \to 0$. Therefore, $\|\psi f_n \circ \varphi\|_{H_q} \leqslant \|f_n\|_A \|\psi\|_{H_q} \to 0$.

Now, suppose that $p = 1$ and $1 \leqslant q < \infty$. On the one hand, by Proposition 3.2(ii) and Lemma 3.1, $\|f_n\|_{H_1} \to 0$. So, there exists a subsequence (we denote it also by f_n) such that $f_n(z) \to 0$ almost everywhere in $\mathbb{T}$. In particular, $\psi(z) f_n(\varphi(z)) \to 0$ almost everywhere in $\mathbb{T}$. On the other hand,

$$|\psi(z) f_n(\varphi(z))| \leqslant \|f_n\|_A |\psi(z)| \leqslant \|j_1\|_{S_1 \to A} \|f_n\|_{S_1} |\psi(z)| \leqslant \|j_1\|_{S_1 \to A} |\psi(z)|,$$

for each $z \in \mathbb{T}$. So, by the dominated convergence theorem, we have that $\|\psi f_n \circ \varphi\|_{H_q} \to 0$.

(iii) It is obvious that (a) implies (b).

(b) implies (c). Assume that $\|\varphi\|_{H_\infty} = 1$ and $\lim\limits_{|\varphi(z)| \to 1} |\psi(z)| \neq 0$. Then there exist a sequence (z_n) in $\mathbb{D}$ and a constant $c > 0$ such that $|\varphi(z_n)| \to 1$ and $|\psi(z_n)| \geqslant c$ for all n. We can assume that $\varphi(z_n) \to b \in \mathbb{T}$ and $\psi(z_n) \to a \in \mathbb{C}$ with $a \neq 0$. Now, consider the function

$$f_n(z) = (1 - |\varphi(z_n)|^2) \frac{\bar{b} z}{1 - |\varphi(z_n)| \bar{b} z}.$$

The sequence (f_n) belongs to B_{S_1} and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. So, by hypothesis, $(W_{\varphi,\psi}(f_n))$ is a weak null sequence in H_∞. However,

$$\begin{aligned} \lim_n \lim_m W_{\varphi,\psi}(f_n)(z_m) &= \lim_n (1 - |\varphi(z_n)|^2) \lim_m \frac{\psi(z_m) \bar{b} \varphi(z_m)}{1 - |\varphi(z_n)| \bar{b} \varphi(z_m)} \\ &= \lim_n (1 - |\varphi(z_n)|^2) \frac{a \bar{b} b}{1 - |\varphi(z_n)| \bar{b} b} \\ &= \lim_n (1 + |\varphi(z_n)|) a = 2a \neq 0. \end{aligned}$$

So, by Lemma 4.1 of [8], $(W_{\varphi,\psi}(f_n))$ is not a weak null sequence in H_∞, a contradiction.

(c) implies (a). If we assume that $\|\varphi\|_{H_\infty} < 1$ or $\lim\limits_{|\varphi(z)| \to 1} |\psi(z)| = 0$, then, by Proposition 2.3 of [5], $W_{\varphi,\psi} : H_\infty \to H_\infty$ is compact. Since the inclusion from S_1 into H_∞ is bounded, we have that $W_{\varphi,\psi} : S_1 \to H_\infty$ is compact.

(iv) Let (f_n) be a weak null sequence in S_1. By Proposition 3.2(iii), $\lim\limits_n \|f_n\|_{H_\infty} = 0$. Since $\|W_{\varphi,\psi}(f_n)\|_{H_\infty} = \|\psi f_n \circ \varphi\|_{H_\infty} \leqslant \|\psi\|_{H_\infty} \|f_n\|_{H_\infty}$, we have that $\|W_{\varphi,\psi}(f_n)\|_{H_\infty} \to 0$. ∎

Proof of Theorem 2.1. We start with boundedness. Assume that $W_{\varphi,\psi} : S_p \to S_q$ is bounded. Then there exists a constant $C_1 > 0$ such that $\|W_{\varphi,\psi}(f)\|_{S_q} \leqslant C_1 \|f\|_{S_p}$ for all $f \in S_p$. Moreover, by Proposition 3.3(i), there exists a constant $C_2 > 0$ such that $\|W_{\varphi,\psi'}(f)\|_{H_q} \leqslant C_2 \|f\|_{S_p}$ for all $f \in S_p$. Thus, take $f \in H_p$ and consider the function $g \in S_p$ such that $g' = f$ and $g(0) = 0$. We have that

$$\begin{aligned} \|W_{\varphi,\psi\varphi'}(f)\|_{H_q} &= \|\psi \varphi' f \circ \varphi\|_{H_q} = \|\psi \varphi' f \circ \varphi + \psi' g \circ \varphi - \psi' g \circ \varphi\|_{H_q} \\ &\leqslant \|(\psi g \circ \varphi)'\|_{H_q} + \|\psi' g \circ \varphi\|_{H_q} \leqslant \|W_{\varphi,\psi}(g)\|_{S_q} + \|W_{\varphi,\psi'}(g)\|_{H_q} \\ &\leqslant (C_1 + C_2) \|g\|_{S_p} = (C_1 + C_2) \|f\|_{H_p}. \end{aligned}$$

The converse is analogous using again Proposition 3.3(i).

Next, we deal with complete continuity. Assume that $W_{\varphi,\psi} : S_p \to S_q$ is completely continuous. Take a weak null sequence (f_n) in H_p. For each n, let us consider the function $g_n \in S_p$ such that $g_n' = f_n$ and $g_n(0) = 0$. By hypothesis and bearing in mind that (g_n) is also a weak null sequence in S_p, we have that $\|W_{\varphi,\psi}(g_n)\|_{S_q} \to 0$. Moreover, by Proposition 3.3(ii) and (iv), $\|W_{\varphi,\psi'}(g_n)\|_{H_q}$ also converges to zero. Now, since

$$\begin{aligned}\|W_{\varphi,\psi\varphi'}(f_n)\|_{H_q} &= \|\psi\varphi' f_n \circ \varphi\|_{H_q} \leqslant \|\psi\varphi' f_n \circ \varphi + \psi' g_n \circ \varphi\|_{H_q} + \|\psi' g_n \circ \varphi\|_{H_q} \\ &= \|(\psi g_n \circ \varphi)'\|_{H_q} + \|W_{\varphi,\psi'}(g_n)\|_{H_q} \\ &\leqslant \|W_{\varphi,\psi}(g_n)\|_{S_q} + \|W_{\varphi,\psi'}(g_n)\|_{H_q},\end{aligned}$$

we have that $\|W_{\varphi,\psi\varphi'}(f_n)\|_{H_q} \to 0$. The converse is proved in an analogous manner.

(ii) When $(p, q) \neq (1, \infty)$, if one uses Lemma 3.1 and Proposition 3.3(ii), the proofs of the characterizations of compactness and weak compactness are similar to the proof of complete continuity and so we omit them. ∎

We know that the continuity of $W_{\varphi,\psi} : S_p \to S_q$ implies that $\psi \in S_q$, so ψ extends continuously to $\overline{\mathbb{D}}$. Moreover, $\psi\varphi \in S_q$. Then φ extends continuously almost everywhere to $\overline{\mathbb{D}}$. Now we present an example of functions $\varphi \notin A$ and $\psi \in S_q$ for which $W_{\varphi,\psi}$ still defines a bounded operator from S_p in S_q.

Example 3.4. Let $\varphi(z) = \exp\left(\frac{z+1}{z-1}\right)$ and $\psi(z) = (1 - z)^2$. Observe that $\psi \in S_1$ (in fact, $\psi \in S_\infty$) and $\varphi \in \Phi$ because φ is analytic in $\mathbb{D}$ and

$$|\varphi(z)| = \exp\left(\mathrm{Re}\left(\frac{z+1}{z-1}\right)\right) = \exp\left(\frac{|z|^2 - 1}{|1 - z|^2}\right) \leqslant 1.$$

Moreover, φ is not continuous in $z = 1$ because of

$$\varphi(\mathrm{e}^{\mathrm{i}\theta}) = \cos\left(\frac{\sin\theta}{\cos\theta - 1}\right) + \mathrm{i}\sin\left(\frac{\sin\theta}{\cos\theta - 1}\right)$$

and

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\cos\theta - 1} = -\infty.$$

On the other hand, $W_{\varphi,\psi} : S_1 \to S_1$ is bounded. In fact, if $f \in S_1$ and $z \in \mathbb{D}$ we have that

$$\begin{aligned}(\psi f \circ \varphi)'(z) &= \psi'(z) f(\varphi(z)) + \psi(z)\varphi'(z) f'(\varphi(z)) \\ &= 2(1 - z) f(\varphi(z)) - 2\varphi(z) f'(\varphi(z)),\end{aligned}$$

and so $(\psi f \circ \varphi)'$ is the sum of a function in H_∞ and another in H_1. That is, $\psi f \circ \varphi \in S_1$.

Proof of Theorem 2.2. (i) Suppose that $W_{\varphi,\psi} : S_1 \to S_\infty$ is compact and $\|\varphi\|_{H_\infty} = 1$. Let (z_n) be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$. Consider, for each n, the function

$$f_n(z) = 2\frac{1 - |\varphi(z_n)|^2}{1 - \overline{\varphi(z_n)}z} - \left(\frac{1 - |\varphi(z_n)|^2}{1 - \overline{\varphi(z_n)}z}\right)^2.$$

Since the sequence (f_n) is bounded in S_1 and converges to zero uniformly on compact subsets of $\mathbb{D}$, by Lemma 3.1, $\|W_{\varphi,\psi}(f_n)\|_{S_\infty} \to 0$. From $f_n(\varphi(z_n)) = 1$ and $f_n'(\varphi(z_n)) = 0$ for all n, it follows that

$$\|W_{\varphi,\psi}(f_n)\|_{S_\infty} \geqslant |\psi'(z_n)f_n(\varphi(z_n)) + \psi(z_n)\varphi'(z_n)f_n'(\varphi(z_n))| = |\psi'(z_n)|.$$

So, $|\psi'(z_n)| \to 0$. This proves that $\lim_{|\varphi(z)|\to 1} |\psi'(z)| = 0$.

Now, to show that $\lim_{|\varphi(z)|\to 1} \frac{|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} = 0$ we use the above arguments with the sequence (g_n) defined by

$$g_n(z) = \Big(\frac{1-|\varphi(z_n)|^2}{1-\overline{\varphi(z_n)}z}\Big)^2 - \frac{1-|\varphi(z_n)|^2}{1-\overline{\varphi(z_n)}z}$$

for all n. Note that $g_n(\varphi(z_n)) = 0$ and $g_n'(\varphi(z_n)) = \frac{\overline{\varphi(z_n)}}{1-|\varphi(z_n)|^2}$ for all n.

Conversely, by Proposition 3.2 of [8] and Proposition 3.3(iii), $W_{\varphi,\psi\varphi'} : H_1 \to H_\infty$ and $W_{\varphi,\psi'} : S_1 \to H_\infty$ are compact. Hence, it is easy to show that $W_{\varphi,\psi} : S_1 \to S_\infty$ is compact.

(ii) First of all, note that, by hypothesis and Theorem 2.1(i), the operator $W_{\varphi,\psi\varphi'} : H_1 \to H_\infty$ is bounded. Thus, by Theorem 4.2 of [8], $W_{\varphi,\psi\varphi'} : H_1 \to H_\infty$ is weakly compact. So, weak compactness of $W_{\varphi,\psi} : S_1 \to S_\infty$ is equivalent to weak compactness of $W_{\varphi,\psi'} : S_1 \to H_\infty$ (we leave the details to the reader). We finish the proof by applying Proposition 3.3(iii). ∎

We finish this section with an example which shows that compactness of $W_{\varphi,\psi} : S_p \to S_q$ does not imply that $\varphi \in A$.

Example 3.5. Let $\varphi(z) = \frac{1}{2}\exp\left(\frac{1+z}{1-z}\right)$ and $\psi(z) = (1-z)^2$. Arguing as in Example 3.4, we have that $W_{\varphi,\psi} : S_1 \to S_1$ is bounded and $\varphi \notin A$. Moreover, since $\|\varphi\|_{H_\infty} < 1$, the operator $W_{\varphi,\psi} : S_1 \to S_1$ is compact.

4. COROLLARIES

In this section we apply recent results about weighted composition operators on Hardy spaces got in [7] and [8] to obtain characterizations of boundedness, compactness, weak compactness and the complete continuity of weighted composition operators on S_p spaces.

We begin with boundedness and compactness. When $1 \leqslant p \leqslant q < \infty$, we obtain a characterization of boundedness and compactness of $W_{\varphi,\psi} : S_p \to S_q$ in terms of Carleson measures. Such criteria have been used to characterize boundedness and compactness of composition operators in different papers (see, for example, [7], [8], [16], [17]).

The measure we are going to use is the following: Given $f \in H_p$, it is well known from Fatou's theorem that the radial limits $\lim_{r\to 1^-} f(re^{i\theta})$ exist almost everywhere in $\mathbb{T}$ and this radial limit, that we denote also by f, belongs to $L_p(\mathbb{T}, m)$.

Thus, taking $\varphi \in \Phi$ and $\psi \in \mathcal{H}(\mathbb{D})$ such that $\psi\varphi' \in H_q$, we define the measure $\mu_{\varphi,\psi\varphi',q}$ on $\overline{\mathbb{D}}$ by

$$\mu_{\varphi,\psi\varphi',q}(E) := \int\limits_{\varphi^{-1}(E)\cap\mathbb{T}} |\psi\varphi'|^q \,\mathrm{d}m,$$

where E is a measurable subset of the closed unit disc $\overline{\mathbb{D}}$.

COROLLARY 4.1. *Assume that* $\varphi \in \Phi$.

(i) *Let* $1 \leqslant p \leqslant q < \infty$ *and* $\psi \in S_q$. $W_{\varphi,\psi}$ *defines a bounded (respectively, compact) operator from* S_p *into* S_q *if and only if* $\mu_{\varphi,\psi\varphi',q}$ *is a* q/p*-Carleson (respectively, compact) measure on* $\overline{\mathbb{D}}$.

(ii) *Let* $1 < p < \infty$ *and* $\psi \in S_\infty$. $W_{\varphi,\psi}$ *defines a bounded (respectively, compact) operator from* S_p *into* S_∞ *if and only if*

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)\varphi'(z)|^p}{1-|\varphi(z)|^2} < \infty \quad (\textit{respectively, } \|\varphi\|_{H_\infty} < 1 \textit{ or } \lim_{|\varphi(z)|\to 1} \frac{|\psi(z)\varphi'(z)|^p}{1-|\varphi(z)|^2} = 0).$$

(iii) *Let* $\psi \in S_\infty$. $W_{\varphi,\psi}$ *defines a bounded (respectively, compact) operator from* S_∞ *into* S_∞ *if and only if*

$$\psi\varphi' \in H_\infty \quad (\textit{respectively, } \|\varphi\|_{H_\infty} < 1 \textit{ or } \lim_{|\varphi(z)|\to 1} |\psi(z)\varphi'(z)| = 0).$$

(iv) *Let* $1 \leqslant q < \infty$ *and* $\psi \in S_q$. $W_{\varphi,\psi}$ *defines a bounded (respectively, compact) operator from* S_∞ *into* S_q *if and only if*

$$\psi\varphi' \in H_q \quad (\textit{respectively, } m(\{z \in \mathbb{T} : |\varphi(z)| = 1\}) = 0).$$

(v) *Let* $\psi \in S_\infty$. $W_{\varphi,\psi}$ *defines a bounded operator from* S_1 *into* S_∞ *if and only if*

$$\sup_{z\in\mathbb{D}} \frac{|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} < \infty.$$

Proof. For boundedness we apply Theorem 2.1 above and Proposition 2.6 and Theorem 2.10 of [8]. Compactness follows from Theorem 2.1 above and Proposition 3.2, Proposition 3.4, Theorem 3.13 of [8], and Proposition 2.4 of [5]. ∎

With regard to weak compactness and complete continuity we have the following corollaries. Since S_p is reflexive when $1 < p < \infty$, it suffices to look at S_1 and S_∞.

COROLLARY 4.2. *Assume that* $\varphi \in \Phi$.

(i) *Let* $\psi \in S_1$. *The operator* $W_{\varphi,\psi} : S_1 \to S_1$ *is weakly compact if and only if it is compact.*

(ii) *Let* $\psi \in S_\infty$. *The operator* $W_{\varphi,\psi} : S_\infty \to S_\infty$ *is weakly compact if and only if it is compact.*

(iii) *Let* $\psi \in S_1$. *The operator* $W_{\varphi,\psi} : S_\infty \to S_1$ *is weakly compact.*

Proof. (i) and (ii) follow from Theorem 2.1(ii), Theorem 3.4 of [7], and Proposition 2.3 of [5].

(iii) Bourgain proved that every operator defined on H_∞ and with range a separable Banach space is weakly compact ([3]). So, by Theorem 2.1(ii), $W_{\varphi,\psi} : S_\infty \to S_1$ is weakly compact. ∎

Corollary 4.3. *Assume that* $\varphi \in \Phi$.

(i) *Let* $1 \leqslant q < \infty$ *and* $\psi \in S_q$ *be such that* $W_{\varphi,\psi}$ *defines a bounded operator from* S_∞ *into* S_q. *Then this operator is completely continuous.*

(ii) *Let* $\psi \in S_\infty$ *be such that* $W_{\varphi,\psi}$ *defines a bounded operator from* S_∞ *into* S_∞. *Then this operator is completely continuous if and only if it is compact.*

(iii) *Let* $1 < q \leqslant \infty$ *and* $\psi \in S_q$ *be such that* $W_{\varphi,\psi}$ *defines a bounded operator from* S_1 *into* S_q. *Then this operator is completely continuous.*

(iv) *Let* $\psi \in S_1$ *be such that* $W_{\varphi,\psi}$ *defines a bounded operator from* S_1 *into* S_1. *Then this operator is completely continuous if and only if* $m(\{z \in \mathbb{T} : |\varphi(z)| = 1\}) = 0$.

Proof. Bourgain proved that an operator defined on H_∞ is completely continuous if and only if it is weakly compact (see [3] or [5]). Thus, by Theorem 2.1(i), $W_{\varphi,\psi} : S_\infty \to S_q$ is completely continuous if and only if it is weakly compact. So, (i) and (ii) follow from this fact and Corollary 4.3(ii) and (iii).

(iii) and (iv) follow from Theorem 2.1, Proposition 5.1 of [8], and Theorem 4.1 of [7]. ∎

MacCluer got that if $C_\varphi : S_p \to S_p$ is compact for some $p > 1$ then $C_\varphi : S_1 \to S_1$ is compact (Lemma 2.2 of [16]). The next result generalizes this fact to weighted composition operators. We omit its proof because it is obtained by MacCluer's only by minor modifications.

Proposition 4.4. *Let* $1 < p \leqslant q \leqslant \infty$, $\varphi \in \Phi$, *and* $\psi \in S_q$ *be such that* $W_{\varphi,\psi} : S_p \to S_q$ *is compact. Then* $W_{\varphi,\psi} : S_1 \to S_t$ *is compact where* t *is given by* $1 + \frac{1}{q} = \frac{1}{t} + \frac{1}{p}$.

Now, we give a necessary condition to have compactness of $W_{\varphi,\psi} : S_p \to S_p$.

Lemma 4.5. *Let* $\varphi \in \Phi$ *and* $\psi \in S_1$ *be such that* $\varphi(0) = 0$ *and* $\|\psi\|_A = 1$. *Let us denote by* $Y_0 = \{f \in S_1 : f(0) = 0\}$. *If* $W_{\varphi,\psi} : S_1 \to S_1$ *is compact, then*

$$\|W^n_{\varphi,\psi} \mid Y_0\| \to 0$$

where $W^1_{\varphi,\psi} = W_{\varphi,\psi}$ *and* $W^{n+1}_{\varphi,\psi} = W_{\varphi,\psi} \circ W^n_{\varphi,\psi}$, *for all* $n \in \mathbb{N}$.

Proof. Since $W_{\varphi,\psi\varphi'} : H_1 \to H_1$ is compact, by Proposition 3.3 of [8], the Lebesgue measure of the set $\{z \in \mathbb{T} : |\varphi(z)| = 1\}$ is zero. So, φ is not an automorphism of the unit disc. Now, the Denjoy-Wolff's theorem (p. 79 in [25]) shows that the sequence (φ_n) converges to zero uniformly on compact subsets of $\mathbb{D}$ where $\varphi_1 = \varphi$ and $\varphi_{n+1} = \varphi \circ \varphi_n$, for all n. Let us define the operator $T := W_{\varphi,\psi} \mid Y_0$ and take λ an eigenvalue of T. There exists $f \neq 0$ in Y_0 such that $W_{\varphi,\psi}(f)(z) = \psi(z) f(\varphi(z)) = \lambda f(z)$ for all z de $\mathbb{D}$. We are going to show that $|\lambda| < 1$. Let $z \in \mathbb{D}$ be such that $f(z) \neq 0$. Since $\|\psi\|_A = 1$, we have

$$\begin{aligned} |\lambda^n f(z)| &= |W^n_{\varphi,\psi}(f)(z)| = |\psi(z)(\psi(\varphi(z))) \cdots (\psi(\varphi_{n-1}(z)))(f(\varphi_n(z)))| \\ &\leqslant \|\psi\|^n_A |f(\varphi_n(z))| = |f(\varphi_n(z))|. \end{aligned}$$

But $|f(\varphi_n(z))| \to |f(0)| = 0$, so that $|\lambda^n f(z)| \to 0$, which implies $|\lambda| < 1$. That is, the spectral radius of T, $r(T)$, satisfies that $r(T) < 1$. In particular, $\lim_{n\to\infty} \|T^n\| = 0$. ∎

Proposition 4.6. *Let* $1 \leqslant p \leqslant \infty$, $\varphi \in \Phi$, *and* $\psi \in S_p$. *If* $W_{\varphi,\psi} : S_p \to S_p$ *is compact, then*

$$\{b \in \mathbb{T} : |\varphi(b)| = 1\} \subseteq \{b \in \mathbb{T} : |\psi(b)| < \|\psi\|_A\}.$$

Proof. Note that we may assume that $\|\psi\|_A = 1$ and, by Proposition 4.4, that $p = 1$. We start proving the result when $\varphi(0) = 0$. Take $b \in \mathbb{T}$ such that $|\varphi(b)| = 1$. We can suppose that $\psi(b) \neq 0$. So, there exist a sequence (z_n) in $\mathbb{D}$ and $a \in \mathbb{T}$ such that $z_n \to b$ and $\varphi(z_n) \to a$. Moreover, since the function $\varphi\psi \in A$ and $\psi(b) \neq 0$, φ is continuous in b. In particular, $\varphi(b) = a$. In addition, we can assume that $\varphi(b) = b$. Write $\chi_1(z) = z$. By Lemma 4.5, $\|W^n_{\varphi,\psi}(\chi_1)\|_{S_1} \to 0$. Now $|\psi(b)| < 1$ follows from

$$|\psi(b)|^n = |\psi(b)\varphi(b)|^n = |W^n_{\varphi,\psi}(\chi_1)(b)| \leqslant \|W^n_{\varphi,\psi}(\chi_1)\|_A \leqslant \|j_1\|_{S_1 \hookrightarrow A} \|W^n_{\varphi,\psi}(\chi_1)\|_{S_1}.$$

Suppose now that $\varphi(0) = \alpha \neq 0$. The Möbius transformation $\varphi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$ satisfies $\varphi_\alpha \circ \varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\varphi_\alpha(\varphi(0)) = 0$. Moreover, if $b \in \mathbb{T}$, then it is satisfied that $|\varphi_\alpha(\varphi(b))| = 1$ if and only if $|\varphi(b)| = 1$. Moreover, $W_{\varphi_\alpha \circ \varphi, \psi} = W_{\varphi,\psi} \circ C_{\varphi_\alpha}$ is compact if $W_{\varphi,\psi}$ is. So, by the first part of the proof, we have that

$$\{b \in \mathbb{T} : |\varphi(b)| = 1\} = \{b \in \mathbb{T} : |\varphi_\alpha(\varphi(b))| = 1\} \subseteq \{b \in \mathbb{T} : |\psi(b)| < \|\psi\|_A\}. \qquad \blacksquare$$

If we apply last theorem to composition operators C_φ we obtain the following result due to MacCluer: If $1 \leqslant p \leqslant \infty$ and $\varphi \in \Phi \cap S_p$, then $C_\varphi : S_p \to S_p$ is compact if and only if $\|\varphi\|_{H_\infty} < 1$ (Theorem 2.3 of [16]).

In the next example we note that the converse of the above proposition is not true.

Example 4.7. Let $\varphi(z) = \frac{z+1}{2}$ and $\psi(z) = z + \mathrm{i}$. It is easy to see that the operator $W_{\varphi,\psi} : S_\infty \to S_\infty$ is bounded and not compact (by Corollary 4.1). On the other hand, $\{b \in \mathbb{T} : |\varphi(b)| = 1\} = \{1\}$ and $|\psi(1)| = |1 + \mathrm{i}| = \sqrt{2} < \|\psi\|_A = 2$.

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