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A FEW REMARKS IN NON-COMMUTATIVE ERGODIC THEORY

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ABSTRACT. Individual ergodic theorems for free group actions and Besicovitch weighted ergodic averages are proved in the context of the bilateral almost uniform convergence in the L^1-space over a semifinite von Neumann algebra. Some properties of the non-commutative counterparts of the pointwise convergence and the convergence in measure are discussed.

KEYWORDS: *Von Neumann algebra, bilateral almost uniform convergence, individual ergodic theorem, free group action, Besicovitch weights.*

MSC (2000): Primary 46L50; Secondary 47A35.

INTRODUCTION

The purpose of this article is to present several new results in the non-commutative ergodic theory. We are mainly concerned with a study of individual ergodic theorems in the non-commutative tracial L^p-spaces, even though some of the results are formulated also in a non-tracial context.

We deal with the so-called bilateral almost uniform (b.a.u.) convergence, which involves two-sided multiplication by a projection. This choice of convergence is attributed to the fact that our arguments, naturally, depend upon some kind of maximal inequality. For the space of integrable operators affiliated with a semifinite von Neumann algebra the only known maximal inequality was established in [20], and it is given through the mentioned two-sided multiplication. In Section 2 of the article we discuss some features of the b.a.u. convergence, which play an important role in the sequel.

The Banach Principle on the almost everywhere convergence of sequences of measurable functions is by far one of the finest tools of the classical ergodic theory. For a von Neumann algebra with a faithful normal semifinite trace, a non-commutative analog of the Banach Principle was established in [5]. Later it was applied in [8] to prove an individual ergodic theorem for a certain type of subsequential ergodic averages. In Section 2 we prove another ("b.a.u.") version of the non-commutative Banach Principle, which is being used extensively in the present work. Note that neither in [5] nor in [8] was it shown that the Cauchy condition obtained by means of the non-commutative Banach Principle yields the actual convergence. Section 2 provides all necessary proofs.

The individual ergodic theorem for free group actions was formulated and proved in the classical context by A. Nevo and E. Stein [11], [10]. Soon afterwards T. Walker showed [22] that it holds true in the context of a semifinite von Neumann algebra and the almost uniform convergence, substantially reducing the assumptions on the sequence of averaging maps. In Section 3 Walker's theorem is generalized in two directions. First it is shown that if we restrict ourselves to the b.a.u. convergence the theorem holds in any von Neumann algebra with a faithful normal semifinite weight. Then, using the non-commutative Banach Principle we conclude that it remains true also for integrable operators affiliated with any semifinite von Neumann algebra. This automatically guarantees that the considered sequences converge b.a.u. (and also a.u. if the von Neumann algebra is finite) over any non-commutative tracial L^p-space, $1 \leqslant p \leqslant \infty$.

In [6], for a semifinite von Neumann algebra with a faithful normal state, some of the results of [13] concerning the pointwise convergence of the Besicovitch weighted ergodic averages were generalized to the non-commutative setting. In the last Section 4 of the article, utilizing the non-commutative Banach Principle again, we prove that these results hold true in the space of integrable operators affiliated with a semifinite von Neumann algebra.

1. PRELIMINARIES

Let M be a semifinite von Neumann algebra acting on a Hilbert space H, let τ be a faithful normal semifinite trace on M, let $P(M)$ be the complete lattice of all projections in M. A densely-defined closed operator x in H is said to be *affiliated* with M if $y'x \subset xy'$ for every $y' \in M'$, where M' is the commutant of the algebra M. An operator x, affiliated with M, is said to be *τ-measurable* if for each $\epsilon > 0$ there exists $e \in P(M)$ with $\tau(e^{\perp}) \leqslant \epsilon$ such that $eH \subset D_x$, where $e^{\perp} = I - e$, I is the unit of M, D_x is the domain of definition of x. Let $S(M)$ be the set of all τ-measurable operators affiliated with M. Let $\|\cdot\|$ stand for the uniform norm in M. The *measure topology* in $S(M)$ is given by the system

$$V(\epsilon,\delta) = \{x \in S(M) : \|xe\| \leqslant \delta \text{ for some } e \in P(M) \text{ with } \tau(e^{\perp}) \leqslant \epsilon\},$$

$\epsilon > 0$, $\delta > 0$, of neighborhoods of zero. Accordingly, a sequence $\{x_n\} \subset S(M)$ converges *in measure* to $x \in S(M)$, $x_n \to x$ (m), if, given $\epsilon > 0, \delta > 0$, there is a number $N = N(\epsilon,\delta)$ such that for any $n \geqslant N$ there exists a projection $e_n \in P(M)$ satisfying the conditions $\tau(e_n^{\perp}) < \epsilon$ and $\|(x_n - x)e_n\| < \delta$.

THEOREM 1.1. ([15]; see also [9]) *Equipped with the measure topology, $S(M)$ is a complete topological $*$-algebra.*

For a positive self-adjoint operator $x = \int_0^\infty \lambda \mathrm{d}e_\lambda$ affiliated with M one can define

$$\tau(x) = \sup_n \tau\Big(\int_0^n \lambda \mathrm{d}e_\lambda\Big) = \int_0^\infty \lambda \mathrm{d}\tau(e_\lambda).$$

If $0 < p \leqslant \infty$, then

$$L^p = L^p(M,\tau) = \begin{cases} \{x \in S(M) : \|x\|_p = \tau(|x|^p)^{1/p} < \infty\} & \text{for } p \neq \infty, \\ (M, \|\cdot\|) & \text{for } p = \infty. \end{cases}$$

Here, $|x|$ is the *absolute value* of x, i.e. the square root of x^*x.

We will need the following theorem.

THEOREM 1.2. *Let $0 < p < \infty$, and let $\{x_n\} \subset L^p$ converge to $x \in S(M)$ in measure. If $s = \liminf\limits_n \|x_n\|_p < \infty$, then $x \in L^p$ and $\|x\|_p \leqslant s$.*

We provide a proof which is based on application of a very useful notion of singular number of a measurable operator. Let $t > 0$. The *t-th singular number* of an operator $x \in S(M)$ is given by the formula

$$\mu_t(x) = \inf\{\|xe\| : e \in P(M), \tau(e^\perp) \leqslant t\}.$$

THEOREM 1.3. (see Corollary 2.8 of [4]) *For every $x \in S(M)$ and $0 < p < \infty$, we have*

$$\tau(|x|^p) = \int_0^\infty \mu_t(x)^p \mathrm{d}t.$$

Proof of Theorem 1.2. Fix $\epsilon > 0$. Passing if necessary to a subsequence, we can assume that $\|x_n\|_p < s + \epsilon$. By Lemma 3.4 in [4], we have

$$\mu_t(x)^p = \lim_{n\to\infty} \mu_t(x_n)^p$$

for every t where the function $\mu_t(x)$ is continuous. Since the function $\mu_t(x)$ is monotone, the above convergence holds for almost all $t \in (0,\infty)$. Besides, due to Theorem 1.3, we have

$$\int_0^\infty \mu_t(x_n)^p \mathrm{d}t < (s+\epsilon)^p$$

for every n. Therefore, Fatou's Lemma yields that $\mu_t(x)^p$ is integrable and

$$\int_0^\infty \mu_t(x)^p \mathrm{d}t \leqslant (s+\epsilon)^p.$$

Applying Theorem 1.3 again, we conclude that $x \in L^p$ and $\|x\|_p \leqslant s + \epsilon$. Due to the arbitrariness of ϵ, we get $\|x\|_p \leqslant s$. ∎

There are several different types of convergences in $S(M)$, each of which, in the commutative case with finite measure, reduces to the almost everywhere (a.e.) convergence. We deal with the *bilateral almost uniform* (b.a.u.) (*almost uniform* (a.u.)) convergence in $S(M)$ for which $x_n \to x$ means that for every $\epsilon > 0$ there exists $e \in P(M)$ with $\tau(e^{\perp}) \leqslant \epsilon$ such that $\|e(x_n - x)e\| \to 0$ (respectively, $\|(x_n - x)e\| \to 0$). Obviously, the a.u. convergence implies b.a.u. convergence, and the converse is false (see the example in [3]). We should mention here that, in the commutative case with infinite measure, the b.a.u. convergence is stronger than the a.e. convergence. For example, if we consider the Lebesgue measure on the real line, then the sequence $x_n = \chi_{[n,\infty)}(t)$ converges a.e. to zero, while x_n does not converge b.a.u. to zero because, for every n, the measure of the interval $[n, \infty)$ (the value of the standard trace of x_n) equals ∞.

A positive linear map $\alpha : L^1(M,\tau) \to L^1(M,\tau)$ will be called an *absolute contraction* if $\alpha(x) \leqslant I$ and $\tau(\alpha(x)) \leqslant \tau(x)$ for every $x \in M \cap L^1$ with $0 \leqslant x \leqslant I$. If α is a positive contraction in L^1, then, as it is shown in [20], [21], $\|\alpha(x)\|_p \leqslant \|x\|_p$ holds for each $x = x^* \in M \cap L^1$ and all $1 \leqslant p \leqslant \infty$. Besides, there exist unique continuous extensions $\alpha : L^p(M,\tau) \to L^p(M,\tau)$ for all $1 \leqslant p < \infty$ and a unique ultra-weakly continuous extension $\alpha : M \to M$. This implies that, for every $x \in L^p$ and any positive integer k, one has

$$\|\alpha^k(x)\|_p \leqslant 2\|x\|_p.$$

In [20], the following form of non-commutative individual ergodic theorem was proved.

THEOREM 1.4. *If α is an absolute contraction in $L^1 = L^1(M,\tau)$, then, for every $x \in L^1$, the averages*

$$A_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} \alpha^k(x)$$

converge b.a.u. in L^1.

The key role in the proof of Theorem 1.4 is played by the so-called maximal ergodic theorem:

THEOREM 1.5. ([20]) *Let L^1, α be as above. Then for every $x \in L^1$ and $\epsilon > 0$ there exists $e \in P(M)$ with $\tau(e^{\perp}) \leqslant 4\epsilon^{-1}\|x\|_1$ such that*

$$\|eA_n(x)e\| \leqslant 4\epsilon \quad \textit{for every } n.$$

PROPOSITION 1.6. *If the trace τ is finite and $x \in M$, then the b.a.u. convergence in Theorem* 1.4 *can be replaced with the a.u. convergence.*

Proof. Without loss of generality, one can assume that $x^* = x$. Since τ is finite, $x^2 \in L^1$. Therefore, by Theorem 1.5, for every $\epsilon > 0$ there exists $e \in P(M)$

with $\tau(e^{\perp}) \leqslant 4\epsilon^{-1}\|x^2\|_1$ such that

$$\|eA_n(x^2)e\| \leqslant 4\epsilon \quad \text{for all } n.$$

By Kadison's inequality [7], $A_n(x)^2 \leqslant A_n(x^2)$ for every n, which implies that

$$\|A_n(x)e\|^2 = \|(A_n(x)e)^*A_n(x)e\| = \|eA_n(x)^2e\| \leqslant \|eA_n(x^2)e\| \leqslant 4\epsilon.$$

As it can be easily seen, as soon as the maximal inequality in [20] is stated in terms of the one-sided multiplication, Theorem 1.4 is valid with b.a.u. replaced by a.u. Or, even better, one can use the a.u. non-commutative Banach Principle that is found in [5] to finish the proof as it is done in [8]. ∎

2. COMPLETENESS OF THE B.A.U. CONVERGENCE AND THE B.A.U. BANACH PRINCIPLE FOR $S(M)$

We say that a sequence $\{x_n\} \subset S(M)$ converges *bilaterally in measure* to $x \in S(M)$, and we will write $x_n \to x$ (b.m.) if, for any given $\epsilon > 0, \delta > 0$, it is possible to find a number $N = N(\epsilon, \delta)$ such that for every $n \geqslant N$ there exists a projection $e_n \in P(M)$ with $\tau(e_n^{\perp}) < \epsilon$ satisfying the condition $\|e_n(x_n - x)e_n\| < \delta$.

It is clear that $x_n \to x$ (m) implies $x_n \to x$ (b.m.). The next theorem asserts that, in fact, these convergences coincide on $S(M)$. The following lemma plays the key role in the proof. We denote $l(x)$ ($r(x)$) the left (respectively, right) support of a measurable operator x.

LEMMA 2.1. *Let $x \in S(M)$, and let $e \in P(M)$. If $q = I - r(e^{\perp}x)$, then $\tau(q^{\perp}) \leqslant \tau(e^{\perp})$ and $xq = exq$.*

Proof. Because the left and right supports of a measurable operator are equivalent (see 9.29 of [16]), we obtain

$$\tau(q^{\perp}) = \tau(r(e^{\perp}x)) = \tau(l(e^{\perp}x)) \leqslant \tau(e^{\perp}).$$

Since

$$xq = exq + e^{\perp}xq = exq,$$

the proof is complete. ∎

THEOREM 2.2. *If $x_n, x \in S(M)$, then $x_n \to x$* (m) *if and only if $x_n \to x$ (b.m.).*

Proof. It is enough to show that $x_n \to 0$ (b.m.) entails $x_n \to 0$ (m). To see this, let's assume that $x_n \to 0$ (b.m.) and fix $\epsilon > 0, \delta > 0$. Then we can find a number N for which $n \geqslant N$ would imply the existence of a projection $e_n \in P(M)$ with $\tau(e_n^{\perp}) < \frac{\epsilon}{2}$ such that $\|e_nx_ne_n\| < \delta$. If we define $q_n = I - r(e_n^{\perp}x_n)$ and $r_n = e_n \wedge q_n$, then it follows from Lemma 2.1 that $\tau(q_n^{\perp}) < \frac{\epsilon}{2}$, and therefore, $\tau(r_n^{\perp}) < \epsilon$. Also, we have

$$x_nr_n = x_nq_nr_n = e_nx_nq_nr_n = e_nx_nr_n = e_nx_ne_nr_n,$$

which implies that
$$\|x_n r_n\| \leqslant \|e_n x_n e_n\| < \delta.$$
Therefore, we can conclude that $x_n \to 0$ (m). ∎

THEOREM 2.3. *The algebra $S(M)$ is complete with respect to the b.a.u. convergence.*

Proof. Let $\{x_n\} \subset S(M)$ be a b.a.u. Cauchy sequence. Then it can be easily seen that $\{x_n\}$ is also a Cauchy sequence with respect to the bilateral in measure convergence. By Theorem 2.2, $\{x_n\}$ is a Cauchy sequence relative to the convergence in measure. Next, since $S(M)$ is complete with respect to the convergence in measure (Theorem 1.1), one can find $x \in S(M)$ such that $x_n \to x$ (m). Fix $\epsilon > 0$. By the assumption, there exists $e \in P(M)$ such that $\tau(e^{\perp}) < \epsilon$, and $\|e(x_m - x_n)e\| \to 0$ as $m, n \to \infty$. Therefore, there exists $x(e) \in M$ such that $ex_n e \to x(e)$ uniformly, hence $ex_n e \to x(e)$ (m). Now, by Theorem 1.1, $x_n \to x$ (m) implies $ex_n e \to exe$ (m). Therefore, $x(e) = exe$, and we get
$$\|e(x_n - x)e\| = \|ex_n e - exe\| = \|ex_n e - x(e)\| \to 0,$$
which means that $x_n \to x$ b.a.u. ∎

REMARK 2.4. Completeness of $S(M)$ with respect to the a.u. convergence follows directly from Theorem 1.1.

Now we shall present a non-commutative variant of the Banach Principle adapted to the b.a.u. convergence. Our argument follows the general scheme suggested in [5]. Note that the main difference between the Banach Principle for the a.u. convergence [5] and its b.a.u. counterpart (Theorem 2.7 below) is the requirement that the maps a_n be positive. This difference is due to the following rather technical issue. It turns out that if one simply replaces the condition $\sup_n \|a_n(x)b\| \leqslant L$ by the obvious $\sup_n \|ba_n(x)b\| \leqslant L$ in the definition of the sets $X_{L,k}$, then the desired closedness of these sets cannot be established as it is done in [5].

LEMMA 2.5. ([5]) *Let $b \in M$, $0 \leqslant b \leqslant I$, and f be the spectral projection of b corresponding to the interval $\left[\frac{1}{2}, 1\right]$. Then:*
(i) *$\tau(f^{\perp}) \leqslant 2\tau(I - b)$;*
(ii) *there exists $b^{-} \in M$ with $\|b^{-}\| \leqslant 2$ such that $f = bb^{-}$.*

PROPOSITION 2.6. ([15], see also [5]) *If $y_n \to 0$ in $S(M)$ with respect to the measure topology, then $y_{n_k} \to 0$ a.u. for some $\{y_{n_k}\} \subset \{y_n\}$.*

Let $(X, \|\cdot\|, \geqslant)$ be an ordered real Banach space with the closed convex cone X_+, $X = X_+ - X_+$. A subset $X_0 \subset X_+$ is said to be *minorantly dense* in X_+ if for every $x \in X_+$ there is a sequence $\{x_n\}$ in X_0 such that $x_n \leqslant x$ for each n, and $\|x - x_n\| \to 0$ as $n \to \infty$. For example, if $X = L^p_{\text{sa}}(M, \tau) = \{x \in L^p(M, \tau) : x^* = x\}$ and $X_+ = L^p_+(M, \tau) = \{x \in L^p_{\text{sa}}(M, \tau) : x \geqslant 0\}$, $1 \leqslant p \leqslant \infty$, then

X is a real Banach space, X_+ is a closed convex cone in X, $X = X_+ - X_+$, and $X_0 = X_+ \cap M$ is minorantly dense in X_+. A linear map $a : X \to S(M)$ is called *positive* if $a(x) \geqslant 0$ whenever $x \in X_+$.

THEOREM 2.7. *Let X be an ordered real Banach space with the closed convex cone X_+, $X = X_+ - X_+$. Let $a_n : X \to S(M)$ be a sequence of positive continuous linear maps satisfying the condition:*

(i) *For every $x \in X_+$ and $\epsilon > 0$ there is $b \in M_+$, $0 \neq b \leqslant I$, such that $\tau(I - b) < \epsilon$, and*

$$\sup_n \|ba_n(x)b\| < \infty.$$

If, for every x from a minorantly dense subset $X_0 \subset X_+$,

(ii) *$a_m(x) - a_n(x) \to 0$ b.a.u., $m, n \to \infty$,*

then (ii) *holds on all of X.*

Proof. Since $X = X_+ - X_+$, it is enough to show that condition (ii) holds for every $x \in X_+$. Fix $x \in X_+$ and $\epsilon > 0$. For every pair L and k of positive integers define the set

$$X_{L,k} = \Big\{ x \in X_+ : \sup_n \|a_n(x)^{1/2} b\| \leqslant L \text{ for some } b \in M \\ \text{with } 0 \leqslant b \leqslant I,\ \tau(I - b) \leqslant \epsilon_k = \frac{\epsilon}{2^{k+3}} \Big\}.$$

We show that the set $X_{L,k}$ is closed in X_+. For that let $\{y_m\} \subset X_{L,k}$, and $\|y_m - x\| \to 0$. Note that, since X_+ is closed, $x \in X_+$. We have $a_1(y_m) \to a_1(x)$ in the measure topology entailing that $a_1(y_m)^{1/2} \to a_1(x)^{1/2}$ [19]. Due to Proposition 2.6, there exists a subsequence $\{y_m^{(1)}\} \subset \{y_m\}$ such that $a_1(y_m^{(1)})^{1/2} \to a_1(x)^{1/2}$ a.u. By the same reasoning, one can find a subsequence $\{y_m^{(2)}\} \subset \{y_m^{(1)}\}$ satisfying $a_2(y_m^{(2)})^{1/2} \to a_2(x)^{1/2}$ a.u. Repeating this process, for every $n \geqslant 3$ we choose a subsequence $\{y_m^{(n)}\} \subset \{y_m^{(n-1)}\}$ for which $a_n(y_m^{(n)})^{1/2} \to a_n(x)^{1/2}$ a.u. as $m \to \infty$. Define $x_m = y_m^{(m)}$. Since $\{x_m\}_{m \geqslant n}$ is a subsequence of $\{y_m^{(n)}\}$, we have

$$a_n(x_m)^{1/2} \to a_n(x)^{1/2} \text{a.u.}, \quad m \to \infty, n \geqslant 1.$$

Now, because $\{x_m\} \subset X_{L,k}$, one can choose a sequence $\{b_m\} \subset M$ such that $0 \leqslant b_m \leqslant I$, $\tau(I - b_m) \leqslant \epsilon_k$ and $\|a_n(x_m)^{1/2} b_m\| \leqslant L$ for every m, n. Since the unit ball in M is compact in the weak operator topology, for some subnet $\{b_\alpha\} \subset \{b_m\}$, there is $b \in M$ for which $b_\alpha \to b$ weakly. We clearly have $0 \leqslant b \leqslant I$. Moreover, by the well-known inequality (see, for example [1]),

$$\tau(I - b) \leqslant \liminf_\alpha \tau(I - b_\alpha) \leqslant \epsilon_k.$$

Show that $\sup\limits_n \|a_n(x)^{1/2} b\| \leqslant L$. Fix n. Since $a_n(x_m)^{1/2} \to a_n(x)^{1/2}$ a.u., given $\sigma > 0$, there exists a projection $h \in P(M)$ with $\tau(h^\perp) \leqslant \sigma$ such that

$$\|h(a_n(x_m)^{1/2} - a_n(x)^{1/2})\| = \|(a_n(x_m)^{1/2} - a_n(x)^{1/2})h\| \to 0$$

as $m \to \infty$. We show first that, with such a choice of h, we have $\|ha_n(x)^{1/2}b\| \leqslant L$. Indeed, for every $\xi, \eta \in H$ we have

$$|(h(a_n(x_m)^{1/2}b_m - a_n(x)^{1/2}b)\xi, \eta)| \leqslant |(h(a_n(x_m)^{1/2} - a_n(x)^{1/2})b_m\xi, \eta)| + |((b_m - b)\xi, a_n(x)^{1/2}h\eta)|. \tag{2.1}$$

Pick $\delta > 0$ and find m_0 such that

$$\|h(a_n(x_m)^{1/2} - a_n(x)^{1/2})\| < \delta \tag{2.2}$$

would hold for all $m \geqslant m_0$. Next, since $b_\alpha \to b$, one can find such an index $\alpha(\delta)$ that

$$|((b_\alpha - b)\xi, a_n(x)^{1/2}h\eta)| < \delta \tag{2.3}$$

whenever $\alpha \geqslant \alpha(\delta)$. Because $\{b_\alpha\}$ is a subnet of $\{b_m\}$, it is possible to present an index $\alpha(m_0)$ satisfying the condition $\{b_\alpha\}_{\alpha \geqslant \alpha(m_0)} \subset \{b_m\}_{m \geqslant m_0}$. In particular, if α_0 is such that $\alpha_0 \geqslant \alpha(\delta)$ and $\alpha_0 \geqslant \alpha(m_0)$, then $b_{\alpha_0} = b_{m_1}$ for some $m_1 \geqslant m_0$. It follows now from (2.1)–(2.3) that

$$\begin{aligned} |(ha_n(x)^{1/2}b\xi, \eta)| &\leqslant |(ha_n(x_{m_1})^{1/2}b_{m_1}\xi, \eta)| + |(h(a_n(x_{m_1})^{1/2} - a_n(x)^{1/2})b_{m_1}\xi, \eta)| \\ &\quad + |((b_{m_1} - b)\xi, a_n(x)^{1/2}h\eta)| \\ &\leqslant L + \|h(a_n(x_{m_1})^{1/2} - a_n(x)^{1/2})\|\|b_{m_1}\|\|\xi\|\|\eta\| + \delta \leqslant L + 2\delta \end{aligned}$$

whenever $\|\xi\| = \|\eta\| = 1$. Taking into account that $\delta > 0$ was chosen arbitrarily, we get

$$\|ha_n(x)^{1/2}b\| = \sup_{\|\xi\|=\|\eta\|=1} |(ha_n(x)^{1/2}b\xi, \eta)| \leqslant L.$$

Now, let $h_j \in P(M)$ be such that $\tau(h_j^\perp) \leqslant \frac{1}{j}$ and

$$\|h_j(a_n(x_m)^{1/2} - a_n(x)^{1/2})\| \to 0 \quad \text{as } m \to \infty.$$

We have $\|h_j a_n(x)^{1/2}b\| \leqslant L$, $j = 1, 2, \ldots$. Since $h_j \to I$ weakly, we also have $h_j a_n(x)^{1/2}b \to_w a_n(x)^{1/2}b$, and therefore,

$$\|a_n(x)^{1/2}b\| \leqslant \limsup_j \|h_j a_n(x)^{1/2}b\| \leqslant L.$$

So, the sets $X_{L,k}$ are closed in X_+ for all positive integers L, k.

Next, we shall verify that

$$X_+ = \bigcup_{L=1}^{\infty} X_{L,k}. \tag{2.4}$$

Given $x \in X_+$, condition (i) entails the existence of $b \in M$, $0 \leqslant b \leqslant I$, such that $\tau(I - b) \leqslant \epsilon_k$ and $\sup_n \|ba_n(x)b\| = \gamma < \infty$. We have $|a_n(x)^{1/2}b|^2 = ba_n(x)b \in M$, so $|a_n(x)^{1/2}b| \in M$. Since $a_n(x)^{1/2}b \in S(M)$, in its polar decomposition,

$a_n(x)^{1/2}b = u|a_n(x)^{1/2}b|$, $u \in M$. Therefore, $a_n(x)^{1/2}b \in M$ for every n, and one can write

$$\|a_n(x)^{1/2}b\|^2 = \|(a_n(x)^{1/2}b)^* a_n(x)^{1/2}b\| = \|ba_n(x)b\| \leqslant \gamma,$$

$n = 1, 2, \ldots$. Choosing L so that $L^2 \geqslant \gamma$, we get $\sup_n \|a_n(x)^{1/2}b\| \leqslant L$, i.e. $x \in X_{L,k}$, and (2.4) holds.

Since X_+ is complete, using the Baire category theorem, we find L_k, $x_k \in X_+$ and $\delta_k > 0$ such that for every $x \in X_+$ with $\|x - x_k\| < \delta_k$ there is an operator $0 \leqslant b_{x,k} \leqslant I$, which satisfies the conditions $\tau(I - b_{x,k}) \leqslant \epsilon_k$ and $\sup_n \|a_n(x)^{1/2}b_{x,k}\| \leqslant L_k$. Define $f_{x,k}$ to be the spectral projection of $b_{x,k}$ corresponding to the interval $[\frac{1}{2}, 1]$. Then, by Lemma 2.5, $\tau(I - f_{x,k}) \leqslant \frac{\epsilon}{2^{k+2}}$ and

$$\sup_n \|a_n(x)^{1/2} f_{x,k}\| \leqslant 2L_k,$$

which implies that $\sup_n \|f_{x,k}a_n(x)f_{x,k}\| \leqslant 4L_k^2$. Further, if $x \in X_+$, $\|x - x_k\| < \delta_k$, and $g_{x,k} = f_{x,k} \wedge f_{x_k,k}$, then $\tau(g_{x,k}^{\perp}) \leqslant \frac{\epsilon}{2^{k+1}}$, and

$$\sup_n \|g_{x,k}a_n(x - x_k)g_{x,k}\| \leqslant 8L_k^2.$$

This means that, if $\gamma_k = \frac{\delta_k}{8L_k^2}$, then for every $z \in X_+$ with $\|z\| \leqslant \gamma_k$ there exists such $g_{z,k} \in P(M)$ that $\tau(g_{z,k}^{\perp}) \leqslant \frac{\epsilon}{2^{k+1}}$, and

$$\sup_n \|g_{z,k}a_n(z)g_{z,k}\| \leqslant 1.$$

Let $x \in X_+$. We will show now that, for some $q \in P(M)$ with $\tau(q^{\perp}) < \epsilon$, $\|q(a_m(x) - a_n(x))q\| \to 0$ as $m, n \to \infty$, or, in other words, that $a_m(x) - a_n(x) \to 0$ b.a.u. Since X_0 is a minorantly dense set in X_+, given k, one can find $y_k \in X_0$ such that $y_k \leqslant x$, and $\|x - y_k\| \leqslant \frac{\gamma_k}{k}$. If $z_k = k(x - y_k)$, then $z_k \geqslant 0$, $\|z_k\| \leqslant \gamma_k$. Therefore, it is possible to find a projection $g_{z_k,k} \in P(M)$ enjoying the properties $\tau(g_{z_k,k}^{\perp}) \leqslant \frac{\epsilon}{2^{k+1}}$ and $\sup_n \|g_{z_k,k}a_n(z_k)g_{z_k,k}\| \leqslant 1$. Putting $g = \bigwedge_{k=1}^{\infty} g_{z_k,k}$, we have $\tau(g^{\perp}) \leqslant \frac{\epsilon}{2}$ and $\sup_n \|ga_n(z_k)g\| \leqslant 1$. Therefore,

$$\|ga_n(x - y_k)g\| = \frac{1}{k} \cdot \|ga_n(z_k)g\| \to 0, \quad k \to \infty, \text{uniformly in } n.$$

Finally, for an arbitrary $\delta > 0$ find such k_0 that

$$\|ga_n(x - y_{k_0})g\| \leqslant \frac{\delta}{3}$$

for all $n = 1, 2, \ldots$. As $y_{k_0} \in X_0$, by condition (ii), there is $p \in P(M)$, $\tau(p^{\perp}) \leqslant \frac{\epsilon}{2}$, and $N = N(p, \delta)$ such that the inequality

$$\|p(a_m(y_{k_0}) - a_n(y_{k_0}))p\| \leqslant \frac{\delta}{3}$$

holds for all $m, n \geqslant N$. Defining $q = p \wedge g$, we get $\tau(q^{\perp}) < \epsilon$ and

$$\|q(a_m(x) - a_n(x))q\|$$
$$\leqslant \|q(a_m(x - y_{k_0}) - a_n(x - y_{k_0})q\| + \|q(a_m(y_{k_0}) - a_n(y_{k_0}))q\| \leqslant \delta$$

for all $m, n \geqslant N$. ∎

REMARK 2.8. Since, as it is stated in Theorem 2.3, the space $S(M)$ is complete with respect to the b.a.u. convergence, Theorem 2.7 actually asserts the b.a.u. convergence of $a_n(x)$ in $S(M)$ for every $x \in X$.

For future applications we shall complement Theorem 2.7 by the following.

PROPOSITION 2.9. *Let* $0 < p < \infty$, *and let* $\{x_n\} \subset L^p$ *be such that* $\liminf_n \|x_n\|_p = s < \infty$. *If* $x_n \to x$ *b.a.u. , then* $x \in L^p$ *and* $\|x\|_p \leqslant s$.

Proof. Since $x_n \to x$ b.a.u. implies $x_n \to x$ (b.m.), by Theorem 2.2, we obtain that $x_n \to x$ (m). The rest now follows from Theorem 1.2. ∎

3. GENERALIZATION OF THE WALKER'S ERGODIC THEOREM

Let now M be an arbitrary von Neumann algebra with a faithful normal semifinite weight ϕ. The predual of M will be denoted by $M_\star$. Further,

$$\mathcal{N}_\phi = \{x \in M : \phi(x^\star x) < \infty\}, \quad M_0 = \mathcal{N}_\phi^\star \cap \mathcal{N}_\phi,$$

and let H_ϕ be the Hilbert space completion of M_0 (with respect to the scalar product $\langle x, y\rangle_\phi = \phi(y^\star x)$). The norm in H_ϕ will be denoted by $\|\cdot\|_2$. We will write Λ_ϕ for the canonical injection of M_0 into H_ϕ, and $\pi_\phi : M \to B(H_\phi)$ will be a faithful normal representation such that, for all $x \in M, y \in M_0$,

$$\pi_\phi(x)(\Lambda_\phi(y)) = \Lambda_\phi(xy)$$

holds (left regular representation). With such definitions, $\Lambda_\phi(M_0)$ becomes a full (achieved) left Hilbert algebra. We will denote M_0' the associated right Hilbert algebra with its right regular representation π_ϕ' and the involution $\eta \to \eta^\flat$. For details on Hilbert algebras see [16], [18].

Let for any $p \in (0, 1)$

$$D_p = \Big\{z \in \mathbb{C} : \Big|\sqrt{z + 4p - 4p^2} + \sqrt{z - 4p - 4p^2}\Big| \leqslant 2\sqrt{p},$$
$$\Big|\sqrt{z + 4p - 4p^2} - \sqrt{z - 4p - 4p^2}\Big| \leqslant 2\sqrt{p}\Big\}.$$

The essential of Theorem 3 of [22], formulated in the spirit of the celebrated von Neumann ergodic theorem, is the following.

THEOREM 3.1. ([22]) *Given a Hilbert space* H, x_1 *a normal operator in* $B(H)$, $x_0 = Id_H$, p_1, p_2 *real numbers such that* $p_1 + p_2 = 1$, $0 < p_1 \leqslant 1$, *and* x_n, $n \geqslant 2$, *operators satisfying the relations* $x_1 x_n = p_1 x_{n+1} + p_2 x_{n-1}$, *the sequence of partial sums*

$\frac{1}{n}\sum_{k=0}^{n-1} x_k$ converges strongly if and only if the spectrum of x_1 is contained in the set D_{p_1}. It converges to the projection P onto the set $\{\eta \in H : x_1\eta = \eta\}$.

Let us fix real numbers p_1, p_2, $p_1 + p_2 = 1$, $0 < p_1 \leqslant 1$. Let σ_1 be a normal a completely positive map acting on the algebra M, such that ϕ is σ_1-invariant and $\sigma_1(I) = I$. Further, let σ_k, $k \geqslant 2$, be positive maps acting on M satisfying the relations $\sigma_1\sigma_k = p_1\sigma_{k+1} + p_2\sigma_{k-1}$, where $\sigma_0 = I$. For each $n \in \mathbb{N}$ we define

$$s_n = \frac{1}{n}\sum_{k=0}^{n-1}\sigma_k,$$

and an operator $\widetilde{\sigma}_1$ acting on $\Lambda_\phi(M_0)$ by

$$\widetilde{\sigma}_1(\Lambda_\phi(y)) = \Lambda_\phi(\sigma_1(y)), \quad y \in M_0.$$

Complete positivity, unitality and ϕ-invariance of σ_1 imply that $\widetilde{\sigma}_1$ is contractive and as such can be continuously extended to the whole H_ϕ (the extension will be denoted by the same symbol). Using recurrency relations we define in a natural manner $\widetilde{\sigma}_0$, $\widetilde{\sigma}_k$, $k \geqslant 2$, and $\widetilde{s}_n$, $n \in \mathbb{N}$.

EXAMPLE 3.2. ([22]) Let us present an example of maps satisfying the above conditions. Let $\{a_i\}_{i=1}^r$ be a set of generators of F_r (the free group on r generators) and let $\{\alpha_i\}_{i=1}^r$ be a set of ϕ-invariant $\star$-automorphisms of the algebra M. Assume that we have a group homomorphism $\Phi : F_r \to \mathrm{Aut}(M)$ defined on the basis elements by $\Phi(a_i) = \alpha_i$, $i \in \{1, \ldots, r\}$. Let w_n, $n \in \mathbb{N}$, denote the set of all reduced words of length n belonging to F_r, and let $|w_n|$ be the cardinality of this set (e.g., $|w_1| = 2r$). The following sums were introduced in [11] and were called, respectively, *Free Group Actions* and *Free Group Partial Sums*:

$$\sigma_n = \frac{1}{|w_n|}\sum_{a\in w_n}\Phi(a), \quad s_n = \frac{1}{n}\sum_{k=0}^{n-1}\sigma_k.$$

It is clear that the corresponding recurrency relations hold true just because of the properties of free group. All the proofs will be conducted in full generality, but one may well think about this particular example, being of an independent interest.

THEOREM 3.3. *Let $x \in M_0$. If $\widetilde{\sigma}_1$ is a normal operator in $B(H_\phi)$ whose spectrum is contained in the set D_{p_1}, then the sequence $\{s_n(x)\}_{n=1}^\infty$ converges strongly to some $\widehat{x} \in M_0$. Moreover if $P \in B(H_\phi)$ is the projection onto $\{\eta \in H_\phi : \widetilde{\sigma}_1\eta = \eta\}$, then $\Lambda_\phi(\widehat{x}) = P\Lambda_\phi(x)$.*

Proof. Theorem 3.1 shows that $\widetilde{s}_n \to P$ strongly. This implies that for any $\eta_1, \eta_2 \in M_0'$, $x \in M_0$

$$\langle\pi_\phi(s_n(x))\eta_1|\eta_2\rangle = \langle\Lambda_\phi(s_n(x))|\eta_2\eta_1^\flat\rangle = \langle\widetilde{s}_n(\Lambda_\phi(x))|\eta_2\eta_1^\flat\rangle \to \langle P(\Lambda_\phi(x))|\eta_2\eta_1^\flat\rangle.$$

If we denote the right-hand side of the previous expression by $F(x,\psi)$, then, for $\psi \in M_\star$ given via $\psi(y) = \langle \pi_\phi(y)\eta_1|\eta_2\rangle$, $y \in M$, it can be easily seen that $|F(x,\psi)| \leqslant \|\psi\|\|x\|$ (all s_n's are contractive). Using the fact that the set of the above forms is dense in $M_\star$, one concludes that the functional

$$M_\star \ni \psi \to \lim_{n\to\infty} \psi(s_n(x)) \in \mathbb{C}$$

is well-defined and continuous. Therefore, there exists $\widehat{x} \in M$ such that for all $\psi \in M_\star$

$$\psi(s_n(x)) \to \psi(\widehat{x}).$$

As M_0 is σ-weakly closed, $\widehat{x} \in M_0$. It is also clear that $\|\widehat{x}\| \leqslant \|x\|$.

Consider again any $\eta_1, \eta_2 \in M_0'$. We have

$$\langle \pi_\phi(\widehat{x})\eta_1|\eta_2\rangle = \langle P\Lambda_\phi(x)|\eta_2\eta_1^\flat\rangle = \langle \pi'_\phi(\eta_1)P\Lambda_\phi(x)|\eta_2\rangle,$$

where π'_ϕ denotes the right regular representation. Since η_2 was taken arbitrarily, we obtain

$$\pi_\phi(\widehat{x})\eta_1 = \pi'_\phi(\eta_1)P\Lambda_\phi(x).$$

As all s_n are $\star$-preserving, we can easily prove (using only definition of $\widehat{x}$) that $\widehat{(x^\star)} = (\widehat{x})^\star$. Applied to the above formulas, this yields

$$\pi_\phi(\widehat{x})^\star\eta_1 = \pi'_\phi(\eta_1)P\Lambda_\phi(x^\star).$$

Next, using Proposition 10.4 of [16], we infer from the above formulas and the fact that $\Lambda_\phi(M_0)$ is full that $P\Lambda_\phi(x) \in \Lambda_\phi(M_0)$, and $P\Lambda_\phi(x) = \Lambda_\phi(\widehat{x})$. Now the desired strong convergence can be obtained by taking any $\eta \in M_0'$:

$$\pi_\phi(s_n(x))\eta = \pi'_\phi(\eta)\Lambda_\phi(s_n(x)) = \pi'_\phi(\eta)\widetilde{s}_n(\Lambda_\phi(x)) \to \pi'_\phi(\eta)P\Lambda_\phi(x) = \pi_\phi(\widehat{x})\eta$$

and remembering that π_ϕ is normal. ∎

Before we prove the announced theorem on the b.a.u. convergence of the considered sequences we need to formulate a few known lemmas. The first one comes from [22] and is based on an application of combinatorial reasoning (see [11]).

LEMMA 3.4. *There exists $C_{p_1} > 0$ such that for every $n \in \mathbb{N}$ and $x \in M^+$ the inequality*

$$s_n(x) \leqslant C_{p_1}\frac{1}{3n}\sum_{k=0}^{3n-1}\sigma_1^k(x)$$

holds.

An easy consequence of the above fact is the following.

LEMMA 3.5. *For any $x \in M$ and $j \in \mathbb{N}$ we have $\|s_n(s_j(x) - x)\| \to 0$.*

Proof. Fix $j \in \mathbb{N}$. Since $s_j(x) - x = \sum\limits_{k=0}^{j-1} \Big(\frac{1}{j}(\sigma_k(x) - x)\Big)$, it is enough to prove the convergence for the sequence $\|s_n(\sigma_k(x) - x)\|$ for every fixed k. Further, using the recurrency relations among σ_k's, it can be easily verified that for each $k \in \mathbb{N}$

$$\sigma_k = \sum_{i=0}^{k} \lambda_i \sigma_1^i, \quad \sum_{i=0}^{k} \lambda_i = 1,$$

and, by linearity of s_n's, we only have to prove that $\|s_n(\sigma_1^i(x) - x)\| \to 0$ for each $i \in \mathbb{N}$. This can be obtained by using the previous lemma and dealing with the standard Cesàro averages. ∎

We will also need a maximal ergodic lemma for a series of operators and the existence of a convenient decomposition of a selfadjoint operator in M_0, both established by D. Petz in [12]. We formulate these facts in the next two lemmas. The map α in Lemma 3.6 may be considered as a non-tracial counterpart of the absolute contraction.

LEMMA 3.6. *Let $\alpha : M \to M$ be a positive linear map such that for each $x \in M, 0 \leqslant x \leqslant I$, we have $\alpha(x) \leqslant I$ and for each $x \in M^+$ we have $\phi(\alpha(x)) \leqslant \phi(x)$. If $\{x_m\}_{m=1}^{\infty}$ is a sequence of operators belonging to M^+ and $\{\varepsilon_m\}_{m=1}^{\infty}$ is a sequence of positive real numbers (estimation numbers), then there exists a projection $p \in P(M)$ such that*

$$\phi(p^{\perp}) \leqslant 2 \sum_{m=1}^{\infty} \varepsilon_m^{-1} \phi(x_m),$$

$$\Big\| p\Big(\frac{1}{r}\sum_{k=0}^{r-1} \alpha^k(x_m)\Big)p \Big\|_{\infty} \leqslant 2\varepsilon_m$$

for all $m, r \in \mathbb{N}$.

LEMMA 3.7. *Suppose that $w \in M_0$, $w = w^{\star}$. Then there exist $z \in M$, $z = z^{\star}$, $a, b \in M^+$ such that $w = z + a - b$, $\|z\| \leqslant \phi(w^2)^{1/2}$, $\phi(a) \leqslant \phi(w^2)^{1/2}$, $\phi(b) \leqslant \phi(w^2)^{1/2}$ and $\|z\|, \|a\|, \|b\| \leqslant \|w\|$.*

Now we shall prove the first main result of this section.

THEOREM 3.8. *Let $x \in M_0$. If $\widetilde{\sigma}_1$ is a normal operator in $B(H_\phi)$ whose spectrum is contained in the set D_{p_1}, then the sequence $\{s_n(x)\}_{n=1}^{\infty}$ converges b.a.u. to $\widehat{x} \in M_0$ defined in Theorem* 3.3.

Proof. Theorem 3.3 implies that, if $j \in \mathbb{N}$ and

$$\xi_j = \widetilde{s}_j(\Lambda_\phi(x)) - P\Lambda_\phi(x),$$

then $\|\xi_j\|_2 \to 0$. Moreover, $\xi_j \in \Lambda_\phi(M_0)$ and, if $\xi_j = \Lambda_\phi(y_j)$, $y_j \in M_0$, we have

$$x - \widehat{x} = y_j + x - s_j(x), \quad \phi(y_j^{\star} y_j) \to 0.$$

Decomposing y_j into its real and imaginary parts, we get

$$x - \widehat{x} = u_j + iv_j + x - s_j(x), \quad u_j = u_j^\star, v_j = v_j^\star,$$
$$\phi(u_j^2) \to 0, \quad \phi(v_j^2) \to 0.$$

From now on we can precisely follow the method from [12]. For the sake of completeness we will reproduce the proof. Let us fix $\varepsilon > 0$ and choose a subsequence $\{j_m\}_{m=1}^\infty$ such that $\phi(u_{j_m}^2)^{1/2} \leqslant 8^{-1}m^{-1}2^{-m}\varepsilon$, $\phi(v_{j_m}^2)^{1/2} \leqslant 8^{-1}m^{-1}2^{-m}\varepsilon$. For each $m \in \mathbb{N}$ we can decompose both u_{j_m} and v_{j_m} according to Lemma 3.7: $u_{j_m} = z_m + a_m - b_m$, $v_{j_m} = z'_m + a'_m - b'_m$. Now we apply Lemma 3.6 to the map σ_1 and the set of operators $\{a_m, b_m, a'_m, b'_m : m \in \mathbb{N}\}$, with the estimation numbers respectively equal to $\frac{1}{m}$, as a result finding a projection $p \in P(M)$ such that

$$\phi(p^\perp) \leqslant 4 \cdot 2 \sum_{m=1}^{\infty} m 8^{-1} m^{-1} 2^{-m} \varepsilon = \varepsilon,$$
$$\Big\| p\Big(\frac{1}{r}\sum_{k=0}^{r-1} \sigma_1^k(a_m)\Big)p\Big\| \leqslant 2m^{-1}, \quad m, r \in \mathbb{N},$$
$$\Big\| p\Big(\frac{1}{r}\sum_{k=0}^{r-1} \sigma_1^k(a'_m)\Big)p\Big\| \leqslant 2m^{-1}, \quad m, r \in \mathbb{N},$$
$$\Big\| p\Big(\frac{1}{r}\sum_{k=0}^{r-1} \sigma_1^k(b_m)\Big)p\Big\| \leqslant 2m^{-1}, \quad m, r \in \mathbb{N},$$
$$\Big\| p\Big(\frac{1}{r}\sum_{k=0}^{r-1} \sigma_1^k(b'_m)\Big)p\Big\| \leqslant 2m^{-1}, \quad m, r \in \mathbb{N}.$$

Going back to the previous formula and using Lemma 3.4 we obtain (for any $m, n \in \mathbb{N}$)

$$\begin{aligned}\|ps_n(x - \widehat{x})p\| &= \|ps_n(u_{j_m} + iv_{j_m} + x - s_{j_m}(x))p\| \\ &\leqslant \|p(s_n u_{j_m})p\| + \|p(s_n v_{j_m})p\| + \|ps_n(x - s_{j_m}(x))p\| \\ &\leqslant \phi(u_{j_m}^2)^{1/2} + 4C_{p_1}\frac{1}{m} + \phi(v_{j_m}^2)^{1/2} + 4C_{p_1}\frac{1}{m} + \|s_n(x - s_{j_m}(x))\|,\end{aligned}$$

and an application of Lemma 3.5 ends the proof. ∎

Using the results of Section 2 we can conclude the following.

THEOREM 3.9. *Let M be a von Neumann algebra with a normal semifinite faithful trace τ, and let $x \in L^1(M, \tau)$. If $\{\sigma_n\}_{n=0}^\infty$ is a sequence of maps acting on M and satisfying the conditions described after Theorem* 3.1, *then the sequence $\{s_n(x)\}_{n=1}^\infty$ converges b.a.u. to some $\widehat{x} \in L^1$.*

Proof. Assume first that $x \geqslant 0$. As it is clear that σ_1 is an absolute contraction, we can use Lemma 1.5 and Lemma 3.4 to deduce that for every $\varepsilon > 0$ there exists $p \in P(M)$ such that $\tau(p^{\perp}) < \varepsilon$ and for all $n \in \mathbb{N}$ we have

$$\|ps_n(x)p\| \leqslant \varepsilon^{-1}\|x\|_1.$$

Each s_n is continuous and positive as a map from L^1_{sa} to $S(M)$. Theorem 3.8 implies that for any $x \in M \cap L^2(M)_+$ the sequence $\{s_n(x)\}_{n=1}^{\infty}$ is b.a.u. convergent. Since $M \cap L^2(M)_+$ is a minorantly dense subset of L^1_+, we are in a position to apply the non-commutative Banach principle (Theorem 2.7) to infer that for any $x \in L^1_{\mathrm{sa}}$ the sequence $\{s_n(x)\}_{n=1}^{\infty}$ is b.a.u. convergent in $S(M)$. Theorem 2.9 allows us to conclude that the limit belongs to L^1 and the standard decomposition of $x \in L^1$ into its real and imaginary part ends the proof. ∎

REMARK 3.10. If τ is finite, then the averages $s_n(x)$ converge also a.u. , which is a consequence of the original Walker's theorem and proof of Proposition 1.6.

4. BOUNDED BESICOVITCH SEQUENCES AND B.A.U. CONVERGENCE OF BB-WEIGHTED AVERAGES

Let now M be a semifinite von Neumann algebra with separable predual, and let τ be a faithful normal semifinite trace on M.

We start with definition of a bounded Besicovitch sequence [13]. Let $\mathbb{K}$ denote the unit circle in $\mathbb{C}$. A function $P_s : \mathbb{Z} \to \mathbb{C}$ will be called a *trigonometric polynomial* if $P_s(k) = \sum\limits_{j=1}^{s} r_j \lambda_j^k, k \in \mathbb{Z}$, for some $\{r_j\}_1^s \subset \mathbb{C}$ and $\{\lambda_j\}_1^s \subset \mathbb{K}$. A sequence $\{\beta_k\}$ of complex numbers is called a *bounded Besicovitch sequence* (BB-sequence) if

(i) $|\beta_k| \leqslant B < \infty$ for every k, and

(ii) given $\epsilon > 0$, there exists a trigonometric polynomial P_s such that

$$\limsup_n \frac{1}{n} \sum_{k=0}^{n-1} |\beta_k - P_s(k)| < \epsilon. \tag{4.1}$$

Let us denote by μ the normalized Lebesgue measure on $\mathbb{K}$. Let $\widetilde{M}$ be the von Neumann algebra of all essentially bounded ultra-weakly measurable functions $h : (\mathbb{K}, \mu) \to M$ equipped with the trace

$$\widetilde{\tau}(h) = \int\limits_{\mathbb{K}} \tau(h(z))\mathrm{d}\mu(z),$$

$h \geqslant 0$, and let $\widetilde{L}^1$ be the Banach space of all Bochner μ-integrable functions $f : (\mathbb{K}, \mu) \to L^1(M, \tau)$ (see p. 68 of [14]). Being the predual to $\widetilde{M}$, the space $\widetilde{L}^1$ is isomorphic to $L^1(\widetilde{M}, \widetilde{\tau})$.

For the sake of completeness, we provide the proof for the next lemma, although it differs from Lemma 2 in [2] only by the underlying space.

LEMMA 4.1. *If* $\widetilde{L}^1 \ni f_n \to f \in \widetilde{L}^1$ b.a.u. *(a.u.), then* $f_n(z) \to f(z)$ *b.a.u. (a.u.) for almost every* $z \in \mathbb{K}$.

Proof. For every positive integer k there exists a projection $e_k \in \widetilde{M}$ such that $\widetilde{\tau}(e_k^\perp) < \frac{1}{k^4}$ and

$$\|e_k(f_n - f)e_k\| \to 0.$$

Then $e_k(z) \in P(M)$ and

$$\|e_k(z)(f_n(z) - f(z))e_k(z)\| \to 0$$

for almost all $z \in \mathbb{K}$. Let

$$Z_k = \Big\{z \in \mathbb{K} : \tau(e_k(z)^\perp) < \frac{1}{k^2}\Big\}, \quad Z_k^c = \mathbb{K} \setminus Z_k.$$

We have

$$\frac{1}{k^4} > \widetilde{\tau}(e_k^\perp) = \int_{\mathbb{K}} \tau(e_k(z)^\perp) \mathrm{d}\mu(z) \geqslant \mu(Z_k^c)\frac{1}{k^2},$$

which implies that

$$\mu(Z_k^c) < \frac{1}{k^2}.$$

Putting $Z' = \bigcup_{n\geqslant 1} \bigcap_{k\geqslant n} Z_k$, we get

$$\mu(Z'^c) = \mu\Big(\bigcap_{n\geqslant 1}\bigcup_{k\geqslant n} Z_k^c\Big) \leqslant \mu\Big(\bigcup_{k\geqslant n} Z_k^c\Big)$$

for every n. Since

$$\mu\Big(\bigcup_{k\geqslant n} Z_k^c\Big) \leqslant \sum_{k\geqslant n}\frac{1}{k^2} \to 0$$

as $n \to \infty$, we conclude that $\mu(Z'^c) = 0$. Now, if $z \in Z'$, then there exists such n that $z \in Z_k$ whenever $k \geqslant n$. Therefore, given $\epsilon > 0$, one can find $e \in P(M)$ with $\tau(e^\perp) < \epsilon$ and such that

$$\|e(f_n(z) - f(z))e\| \to 0,$$

i.e. $f_n(z) \to f(z)$ b.a.u. on Z'. ∎

LEMMA 4.2. *Let* α *be an absolute contraction in* $L^1(M, \tau)$*. Then, for any trigonometric polynomial* P_s *and every* $x \in L^1$*, the averages*

$$\widetilde{a}_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} P_s(k)\alpha^k(x) \tag{4.2}$$

converge b.a.u. If $x \in M$ *and* τ *is finite, then the averages* (4.2) *converge also a.u.*

Proof. Pick $\lambda \in K$. If $f \in \widetilde{L}^1$, we define $\widetilde{\alpha} : \widetilde{L}^1 \to \widetilde{L}^1$ by the formula

$$\widetilde{\alpha}(f)(z) = \alpha(f(\lambda z))$$

where $z \in \mathbb{K}$. One can easily verify that the system $(\widetilde{M}, \widetilde{\tau}, \widetilde{L}^1, \widetilde{\alpha})$ satisfies all the conditions of Theorem 1.4. Therefore, for every $f \in \widetilde{L}^1$, the averages

$$\frac{1}{n}\sum_{k=0}^{n-1} \widetilde{\alpha}^k(f)$$

converge b.a.u. in $\widetilde{L}^1$. By Lemma 4.1, we arrive at the b.a.u. convergence of the averages

$$\frac{1}{n}\sum_{k=0}^{n-1} \widetilde{\alpha}^k(f)(z) = \frac{1}{n}\sum_{k=0}^{n-1} \alpha^k(f(\lambda^k z))$$

for almost all $z \in \mathbb{K}$. Applying this to the function

$$f(z) = zx,$$

which apparently is Bochner μ-integrable, i.e. $f \in \widetilde{L}^1$, we obtain the b.a.u. convergence of the averages

$$z \cdot \frac{1}{n}\sum_{k=0}^{n-1} \lambda^k \alpha^k(x),$$

for almost all $z \in \mathbb{K}$. Consequently, the averages

$$\frac{1}{n}\sum_{k=0}^{n-1} \lambda^k \alpha^k(x)$$

converge b.a.u. for every $\lambda \in \mathbb{K}$, and we obtain the convergence of (4.2). The a.u. convergence when τ is finite follows from Proposition 1.6. ∎

A proof of the next technical lemma can be found in [8].

LEMMA 4.3. *If a sequence $\{x_n\}$ in M is such that for every $\epsilon > 0$ there are a b.a.u. (a.u.) convergent sequence $\{\widetilde{x}_n\} \subset M$ and a positive integer n_0 satisfying $\|x_n - \widetilde{x}_n\| < \epsilon$ for all $n \geqslant n_0$, then $\{x_n\}$ converges b.a.u. (a.u.).*

THEOREM 4.4. *With α and $\{\beta_k\}$ as above, the averages*

$$a_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} \beta_k \alpha^k(x) \tag{4.3}$$

converge b.a.u. for every $x \in L^1 \cap M$.

Proof. Pick $\epsilon > 0$ and find a trigonometric polynomial P_s such that the condition (4.1) is satisfied. By Lemma 4.2, the b.a.u. convergence of the averages $\widetilde{a}_n(x)$ holds. We have

$$\|a_n(x) - \widetilde{a}_n(x)\| \leqslant \Big(\frac{1}{n}\sum_{k=0}^{n-1} |\beta_k - P_s(k)|\Big) \cdot 2\|x\|.$$

Therefore,

$$\|a_n(x) - \widetilde{a}_n(x)\| < 2\epsilon\|x\|$$

for sufficiently large n. The b.a.u. convergence of $a_n(x)$ now follows from Lemma 4.3. ∎

REMARK 4.5. If τ is finite, then the averages (4.3) converge also a.u.

Using the b.a.u. Banach Principle given in Theorem 2.7, we can now extend the convergence stated in Theorem 4.4 to the space $L^1(M,\tau)$.

THEOREM 4.6. *Let $\{\beta_k\}$ be a* BB-*sequence. For α an absolute contraction in L^1, the averages* (4.3) *converge b.a.u. in L^1 for every $x \in L^1$.*

Proof. Note first that, since $\{\beta_k\}$ is bounded, we can assume that $|\beta_k| \leqslant 1$ for every k. By Theorem 4.4, the averages (4.3) converge b.a.u. if $x \in X_0 = L^1_+ \cap M$. The b.a.u. convergence of the sequence $(a_n(x))^* = \frac{1}{n}\sum\limits_{k=0}^{n-1}\overline{\beta}_k\alpha^k(x)$ implies that both

$$a_n^{(\mathrm{r})}(x) = \frac{1}{n}\sum_{k=o}^{n-1}\mathrm{Re}(\beta_k)\alpha^k(x) \quad \text{and} \quad a_n^{(\mathrm{i})}(x) = \frac{1}{n}\sum_{k=0}^{n-1}\mathrm{Im}(\beta_k)\alpha^k(x)$$

converge b.a.u., $x \in X_0$. Define

$$a_n^{(\mathrm{R})}(x) = a_n^{(\mathrm{r})}(x) + A_n(x), \quad a_n^{(\mathrm{I})}(x) = a_n^{(\mathrm{i})}(x) + A_n(x), x \in L^1,$$

where $A_n(x) = \frac{1}{n}\sum\limits_{k=0}^{n-1}\alpha^k(x)$. Let us show that, given $\epsilon > 0$ and $x \in L^1_+$, there exists a projection $e \in P(M)$ with $\tau(e^\perp) \leqslant \epsilon$ such that

$$\sup_n \|ea_n^{(\mathrm{R})}(x)e\| < \infty. \tag{4.4}$$

By Theorem 1.5, there is $e \in P(M)$ such that $\tau(e^\perp) \leqslant \epsilon$ and

$$\sup_n \|eA_n(x)e\| < \infty.$$

Since $0 \leqslant \mathrm{Re}(\beta_k) + 1 \leqslant 2$, we have

$$ea_n^{(\mathrm{R})}(x)e \leqslant 2eA_n(x)e$$

for every n, which immediately yields (4.4). Next, because the continuous linear maps $a_n^{(\mathrm{R})} : X = L^1_{\mathrm{sa}} \to S(M)$ are positive, and the set X_0 is minorantly dense in $X_+ = L^1_+$, by Theorem 2.7, we obtain the b.a.u. convergence of $a_n^{(\mathrm{R})}(x)$ for all $x \in L^1_{\mathrm{sa}}$. Remembering that the averages $A_n(x)$ also converge b.a.u. (Theorem 1.4), we get the convergence of $a_n^{(\mathrm{r})}(x)$, $x \in L^1_{\mathrm{sa}}$. Analogously, $a_n^{(\mathrm{i})}(x)$ converge b.a.u. for all $x \in L^1_{\mathrm{sa}}$. Therefore, the averages $a_n(x) = a_n^{(\mathrm{r})}(x) + ia_n^{(\mathrm{i})}(x)$ converge b.a.u. for every $x \in L^1_{\mathrm{sa}}$, hence for every $x \in L^1$.

It remains to show that the limits of these averages belong to L^1. Taking into account that $\|\alpha^k(x)\|_1 \leqslant 2\|x\|_1$ for every k, we get $\|a_n(x)\|_1 \leqslant 2\|x\|_1$ for all $x \in L^1$. This finishes proof due to Proposition 2.9. ∎

A linear map $\alpha : L^1 \to L^1$ is said to be *weakly mixing* (see [6]) if the following two conditions hold:

(i) $\alpha x = x$ implies $x = cI$ with c a constant (ergodicity of α);

(ii) α has no eigenvalues different from 1.

REMARK 4.7. Like it was carried out in [13] and [6], it is possible to show that, if the contraction α is weakly mixing, then the averages (4.3) converge to scalar operators for all $x \in L^1$.

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