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SCHUR MULTIPLIER PROJECTIONS ON THE VON NEUMANN-SCHATTEN CLASSES

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ABSTRACT. For $1 \leqslant p < \infty$ let C_p denote the usual von Neumann-Schatten ideal of compact operators on ℓ^2. The standard basis of C_p is a conditional one and so it is of interest to be able to identify the sets of coordinates for which the corresponding projection is bounded. In this paper we survey and extend the known classes of bounded projections of this type. In particular we show that some recent results from spectral theory allow one to prove boundedness of a projection by checking simple geometric conditions on the associated set of coordinates.

KEYWORDS: *Von Neumann-Schatten classes, Schur multipliers.*

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1. INTRODUCTION

Given any conditional basis $\{x_n\}$ of a Banach space X, a natural question has been to decide which multiplier operators are bounded. That is, one would like to be able to characterize those scalar sequences $\{\alpha_n\}$ such that the linear transformation $\sum a_n x_n \mapsto \sum \alpha_n a_n x_n$ is a bounded linear operator on X. Of particular importance is deciding the multiplier projections for the basis; that is the multiplier sequences that contain only zeros and ones.

For the trigonometric basis of $L^p(\mathbb{T})$ this is a classical and much studied problem. Recently much attention has been paid to analogues of this theory in non-commutative L^p spaces, and in particular, in the von Neumann-Schatten ideals $\mathcal{C}_p$.

To fix some notation, for $1 \leqslant p < \infty$, let $\mathcal{C}_p$ denote the von Neumann-Schatten ideal of compact operators on ℓ^2, with norm $\|T\|_p = (\mathrm{tr}((T^*T)^{p/2}))^{1/p}$. We take $\mathcal{C}_\infty$ to be the set of all compact operators on ℓ^2 with the usual operator norm. We shall let $\mathcal{C}_p^n$ denote the $n \times n$ matrices equipped with the corresponding norm. If $S : \mathcal{C}_p \to \mathcal{C}_p$ is a bounded linear operator, we shall denote its operator norm by $\|S\|_p$.

We shall, as usual, think of elements of $\mathcal{C}_p$ as being infinite matrices with respect to some fixed orthonormal basis. For $i, j \geqslant 1$ let E_{ij} denote the matrix with

1 in the (i, j)th entry and zero otherwise. Under many natural orderings $\{E_{ij}\}$ forms a basis for $\mathcal{C}_p$ for $1 \leqslant p \leqslant \infty$. Of course on the Hilbert space $\mathcal{C}_2$ this basis is orthonormal (and hence unconditional), but for $p \neq 2$, it has long been known that this basis is only conditional.

Let $\mathcal{Z}$ denote the set of all zero-one arrays $[a_{i,j}]_{i,j=1}^{\infty}$. (We shall usually use the term "array" for a multiplier and reserve the term "matrix" for elements of $\mathcal{C}_p$.) We are interested in the norms of projections defined by Schur multiplication by such arrays. If $A = [a_{i,j}] \in \mathcal{Z}$, define the *Schur projection* corresponding to A to be the map $P_A : T \mapsto A \circ T$, where $\circ$ denotes Schur (or Hadamard or elementwise) multiplication of matrices. Let

$$\mathcal{B}_p = \{A \in \mathcal{Z} \, : \, P_A \in B(\mathcal{C}_p)\}.$$

The set of $n \times n$ arrays with zero-one entries will be denoted by $\mathcal{Z}^n$. For $\Delta \subset \mathbb{Z}^+ \times \mathbb{Z}^+$, we shall let $A(\Delta)$ denote the characteristic array of Δ.

Some of the standard criteria for A to be in $\mathcal{B}_p$ are given in Section 2. Section 3 contains our major new tool. Using a result from spectral theory [11] we give a uniform bound on the multiplier norms of a large class of readily identifiable arrays (Corollary 3.9). In the following sections we use this and other results to provide various analogues of results from Fourier multiplier theory. In Section 6 for example, we generate several classes of Paley-Littlewood type decompositions of C_p. The final section includes two examples which illustrate how these methods could be used to increase the class of arrays known to lie in $\mathcal{B}_p$.

The spaces C_p, $1 < p < \infty$, are standard examples of UMD spaces and are therefore important "test cases" for when results valid on L^p spaces might generalize to this wider class. In order to test conjectures or provide counterexamples, it is often important to be able to construct examples in these spaces. Much of the aim of this paper is to enable one to replace some of the analysis inherent in these sorts of constructions with simpler questions concerning patterns in subsets of $\mathbb{Z}^+ \times \mathbb{Z}^+$.

Recall that if $1 \leqslant p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\mathcal{C}_p^* = \mathcal{C}_q$, under the natural pairing $\langle S, T\rangle = \mathrm{tr}(ST)$. One can check that if $A \in \mathcal{B}_p$ then

$$\langle P_A(S), T\rangle = \langle S, P_{A^t}(T)\rangle, \quad S \in \mathcal{C}_p, T \in C_q.$$

It follows easily that $\|P_A\|_p = \|P_A\|_q$ and hence $\mathcal{B}_p = \mathcal{B}_q$. Of course $\mathcal{B}_2 = \mathcal{Z}$ with $\|P_A\|_p = 1$ for all nonzero $A \in \mathcal{Z}$.

There is an extensive literature on the boundedness of Schur multipliers on $B(\ell^2)$ (which corresponds to the case $p = \infty$ here). An excellent source for the main results of that theory is [2].

2. BASIC RESULTS

In this section we record a number of facts regarding the norms of Schur multiplier projections on $\mathcal{C}_p$. Most, if not all, of these results are quite standard, but as there seems to be no general reference for them, we have included them here for completeness.

An important fact is that a randomly chosen array $A \in \mathcal{Z}$ will not give rise to a bounded Schur multiplier projection on $\mathcal{C}_p$ unless $p = 2$. (We are thinking here of $\mathcal{Z}$ as a countable product $\mathbb{Z}_2^{\mathbb{N}}$ of two point measure spaces.) Standard examples show that $\sup\{\|P_A\|_p : A \in \mathcal{Z}^n\} = \mathrm{O}(n^{|(1/2)-(1/p)|})$; indeed, as is noted in [10], $\mathbb{E}(\|P_A\|_p : A \in \mathcal{Z}^n) = \mathrm{O}(n^{|(1/2)-(1/p)|})$ too. On the other hand, the set $\mathcal{B}_p$ is not too small either. If one gives $\mathcal{Z} = \mathbb{Z}_2^{\mathbb{N}}$ the natural product topology, then the following observations show that $\mathcal{B}_p$ is an uncountable dense subset of $\mathcal{Z}$ for all p.

It is a trivial consequence of the ideal inequalities for $\mathcal{C}_p$ that if $A \in \mathcal{Z}$ is a nonzero array which is constant on each row (or on each column) then $\|P_A\|_p = 1$ for all p. This bound also holds for the projection onto the diagonal elements of the matrix. Suppose that $\{Q_i\}_{i=1}^{\infty}$ and $\{P_i\}_{i=1}^{\infty}$ are two sequences of disjoint orthogonal projections on ℓ^2. Then $P(S) = \sum_{i=1}^{\infty} Q_i S P_i$ is a projection on $\mathcal{C}_p$. If we choose $\{Q_i\}$ and $\{P_i\}$ to be projections onto the spans of disjoint blocks of standard basis vectors in ℓ^2, then P is a Schur multiplier projection. In particular, suppose that $B_1, B_2, \dots$ are rectangular subsets of $\mathbb{Z}^+ \times \mathbb{Z}^+$ of the following general form

$$\begin{pmatrix} \boxed{B_1} & & & & \\ & \boxed{B_2} & & & \\ & & \boxed{B_3} & & \cdots \\ & & & \boxed{B_4} & \\ & & \vdots & & \ddots \end{pmatrix}$$

and that S_k is the Schur projection associated to the characteristic array of B_k. Let $\emptyset \neq J \subset \mathbb{Z}^+$. Standard Hilbert space theory ensures that $\Big\|\sum_{k\in J} S_k\Big\|_\infty = 1$ and clearly $\Big\|\sum_{k\in J} S_k\Big\|_2 = 1$, so by interpolation and duality, $\Big\|\sum_{k\in J} S_k\Big\|_p = 1$ for $1 \leqslant p \leqslant \infty$.

If U and V are any permutation matrices, then $P_{UAV}(S) = UP_A(U^*SV^*)V$. Now pre or post multiplying by a unitary is an isometry on $\mathcal{C}_p$, and so $\|P_{UAV}\|_p = \|P_A\|_p$. This implies that $\mathcal{B}_p$ is stable under permutations of the rows and/or the columns of the zero-one arrays.

We shall say that $A_1 \in \mathcal{Z}^n$ is a subarray of $A \in \mathcal{Z}$ if A_1 is formed by deleting all but finitely many of the rows and all but finitely many of the columns of A. If

A_1 is a subarray of A then $\|P_{A_1}\|_p \leqslant \|P_A\|_p$. Indeed,

$$\|P_A\|_p = \sup\{\|P_{A_1}\|_p \,:\, A_1 \text{ is a subarray of } A\}$$

and so for many purposes, it suffices to consider the finite dimensional situation.

DEFINITION 2.1. An array $A \subset \mathcal{Z}$ is *periodic of period* $(I, J) \subset \mathbb{Z}^+ \times \mathbb{Z}^+$ if for all i, j, $a_{i,j} = a_{i+I,j} = a_{i,j+J}$. We shall say that A is periodic of period (I, ∞) if for all i, j, $a_{i,j} = a_{i+I,j}$, and periodic of period (∞, J) if for all i, j, $a_{i,j} = a_{i,j+J}$.

PROPOSITION 2.2. *If A is periodic of period (I, J) then $\|P_A\|_p \leqslant \min\{I, J\}$ for $1 \leqslant p \leqslant \infty$.*

Proof. If all but one of the first I rows (or J columns) of A are zero, then $P_A(S) = QSP$ for orthogonal projections Q and P onto the spans of suitable collections of the standard basis vectors in ℓ^2, and hence $\|P_A\|_p = 1$. The result now follows from the triangle inequality. ∎

Proving boundedness beyond these most obvious ones has been significantly more difficult. The first major result was due to Macaev (see [14]) who showed that the upper triangular truncation map is bounded on $\mathcal{C}_p$ for $1 < p < \infty$ (but not for $p = 1$ or $p = \infty$). This has often been seen as an analogue of the statement that the Riesz projection from $L^p(\mathbb{T})$ to $H^p(\mathbb{T})$ is bounded. To fix some notation, throughout we shall let $\mathbb{U} = P_{A(\Delta)}$ denote the upper triangular truncation projection corresponding to the set $\Delta = \{(i, j) \,:\, i \leqslant j\}$.

In the 1980s Bourgain [8] showed boundedness results for a class of "Toeplitz" arrays which are analogous to the Marcinkiewicz multiplier results from Fourier analysis. In particular, for $1 < p < \infty$ there is a constant K_p such that if $A \in \mathcal{Z}$ is any array which is constant on diadic blocks of (long) diagonals, then $\|P_A\|_p \leqslant K_p$. In Section 4 below we shall show how these results follow from the transference techniques of Berkson and Gillespie.

A significant difficulty that faces anyone who wishes to know whether a particular array does produce a bounded Schur multiplier on $\mathcal{C}_p$ is that many of the main results show that arrays generated by some procedure are bounded, but do not provide easy criteria for checking just which arrays are of that form. Our aim in this present paper is to give results where the criteria are more easily verifiable.

Interpolation theory and standard examples show that if $2 < p < r < \infty$ then $\mathcal{B}_\infty \subsetneq \mathcal{B}_r \subset \mathcal{B}_p \subsetneq \mathcal{B}_2 = \mathcal{Z}$. It has long been an open question as to whether the middle inclusion is always strict. A recent major step in this direction was made by A. Harcharras [15] who showed that if p is an even integer then there exist arrays $A \in \mathcal{Z}$ which are $\mathcal{C}_p$ Schur multipliers but not $\mathcal{C}_r$ multipliers.

After this work was completed we became aware of some recent work of Clément, de Pagter, Sukochev and Witvliet [19], [9], [20]. Many of the results that we obtain here can also be deduced from their work.

3. OBTAINABLE ARRAYS

Some bounds on norms of Schur multipliers on $\mathcal{C}_p$ spaces can be obtained from some recent results from spectral theory. Well-bounded and scalar-type spectral operators may be characterized (at least on reflexive Banach spaces) by the fact that they admit integral representations with respect to suitable families of projections. They are distinguished by the fact that for scalar-type spectral operators, the corresponding decomposition of the Banach space is of an unconditional nature, whilst for well-bounded operators one gets a decomposition of a conditional nature. The appropriate background about well-bounded and scalar-type spectral operators may be found in [12]. We shall, however, briefly recall some definitions and results from [11] and [6].

Let X denote a complex reflexive Banach space. Given any well-bounded operator $S \in B(X)$ there exists a unique spectral family of projections $\{E(s)\}_{s\in\mathbb{R}}$ such that $S = \int\limits_{\sigma(S)}^{\oplus} s\,\mathrm{d}E(s)$, where the integral converges as a strong operator topology limit of Riemann-Stieltjes sums. It has been more traditional to write this as $\int\limits_{[a,b]}^{\oplus} s\,\mathrm{d}E(s)$ where $[a,b]$ is any interval containing $\sigma(S)$. It is however more natural to stress explicitly the way in which the domain of integration depends on S. That one can make this precise was shown in [1].

DEFINITION 3.1. ([6], Definition 2.4) Let $\{E_\gamma\}_{\gamma\in\Gamma}$ be a family of operators on a Banach space X. We shall say that $\{E_\gamma\}$ is *R-bounded* (or has the *R-property*) if there is a constant C such that for all $N \in \mathbb{N}$, all $\gamma_1, \dots, \gamma_N \in \Gamma$ and all $x_1, \dots, x_N \in X$,

$$\int_0^1 \Big\| \sum_{j=1}^{N} r_j(t) E_{\gamma_j} x_j \Big\| \,\mathrm{d}t \leqslant C \int_0^1 \Big\| \sum_{j=1}^{N} r_j(t) x_j \Big\| \,\mathrm{d}t, \tag{3.1}$$

where $\{r_j\}_{j=1}^{\infty}$ denote the Rademacher functions on $[0,1]$.

The main result of [11] is the following.

THEOREM 3.2. ([11], Theorem 3.1) *Suppose that:*

(i) $T \in B(X)$ *is a well-bounded operator of type* (B) *whose spectral family has the R-property;*

(ii) $S \in B(X)$ *is a real scalar-type spectral operator which commutes with* T*;*

(iii) p *is a real polynomial of two variables.*

Then $p(T,S)$ *is well-bounded.*

Perhaps just as importantly, in the case that X is reflexive, the proof of this theorem gives bounds on the norms of the projections that make up the spectral family for $p(T,S)$, and it is those bounds which will be important here.

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ be bounded sequences of scalars. Let $D_\lambda \in B(\ell^2)$ denote the diagonal operator

$$D_\lambda(x_1, x_2, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \dots).$$

Define operators C_λ and R_μ on $\mathcal{C}_p$ by

$$C_\lambda(S) = SD_\lambda, \quad R_\mu(S) = D_\mu S.$$

Thus C_λ (respectively R_μ) is just the Schur multiplier which corresponds to an array with λ_j in the jth column (respectively μ_i in the ith row).

LEMMA 3.3. *For all bounded real sequences λ and μ, $C_\lambda + R_\mu$ is a well-bounded operator of type* (B) *on $\mathcal{C}_p$ for $1 < p < \infty$.*

Proof. It is clear from the definition that C_λ and R_μ commute. It is also easy to check that both operators are real scalar-type spectral. One can either check this directly, or else note that for any polynomial g,

$$g(C_\lambda)(S) = Sg(D_\lambda) = SD_{g(\lambda)},$$

where $g(\lambda) = (g(\lambda_1), g(\lambda_2), \dots)$ and so $\|g(C_\lambda)\| \leqslant \sup_{z \in \sigma(C_\lambda)} |g(z)|$. The fact that C_λ (and R_μ) are real scalar-type spectral now follows immediately from Theorem 6.10 of [12].

Since $\mathcal{C}_p$ is a UMD space, it now follows from Theorem 3.5 of [11] that $C_\lambda + R_\mu$ is a well-bounded operator of type (B). ∎

In the context of Lemma 3.3, since $C_\lambda + R_\mu$ is a well-bounded operator of type (B) there exists a unique spectral family of projections $\{E_{\lambda\mu}(s)\} \subset B(\mathcal{C}_p)$ such that $C_\lambda + R_\mu = \int^{\oplus}_{\sigma(C_\lambda + R_\mu)} s \, \mathrm{d}E_{\lambda\mu}(s)$.

LEMMA 3.4. *For all $1 < p < \infty$ there exists a constant K_p such that for all bounded real sequences λ and μ and all $s \in \mathbb{R}$, $\|E_{\lambda\mu}(s)\|_p \leqslant K_p$.*

Proof. As above, for any bounded Borel measurable function f,

$$\|f(C_\lambda)\| \leqslant \sup_{z \in \sigma(C_\lambda)} |f| \quad \text{and} \quad \|f(R_\mu)\| \leqslant \sup_{z \in \sigma(R_\mu)} |f|.$$

The result then follows from Corollary 3.7 of [11]. ∎

It is not too hard to see that every projection $E_{\lambda\mu}(s)$ is actually of the form P_A for some zero-one array A. Indeed, for all $s \in \mathbb{R}$, $E_{\lambda\mu}(s) = P_{A(\Delta)}$ where $\Delta = \{(i, j) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : \mu_i + \lambda_j \leqslant s\}$. The next step then is to characterize those arrays that can occur in this way.

DEFINITION 3.5. A zero-one array A will be said to be *obtainable* if there exist bounded real sequences λ and μ such that (with the above notation) $P_A = E_{\lambda\mu}(0)$.

DEFINITION 3.6. A zero-one array $A = [a_{i,j}]$ has property (β) if it does not contain either

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

as a subarray. That is, A has property (β) if there do not exist indices i_1, i_2, j_1, j_2 such that

$$a_{i_1,j_1} = a_{i_2,j_2} = 1 \quad \text{and} \quad a_{i_2,j_1} = a_{i_1,j_2} = 0.$$

Let A be a zero-one array. We shall use A to define some relations on $\mathbb{Z}^+$. If $j_1, j_2 \in \mathbb{Z}^+$ then:

(i) $j_1 =_A j_2$ if column j_1 (of A) is identical to column j_2;

(ii) $j_1 \prec_A j_2$ if there exists $i \in \mathbb{Z}^+$ such that $a_{\ell,j_1} = a_{\ell,j_2}$ for all $\ell < i$, $a_{i,j_1} = 1$ and $a_{i,j_2} = 0$;

(iii) $j_1 \preceq_A j_2$ if $j_1 =_A j_2$ or $j_1 \prec_A j_2$. It is straightforward to check that $\preceq_A$ is a partial order on $\mathbb{Z}^+$.

LEMMA 3.7. *Suppose that A has property (β). Then:*

(i) *if $a_{i,j_1} = 1$, then $a_{i,j_2} = 1$ for all j_2 with $j_2 \preceq_A j_1$;*
(ii) *if $a_{i,j_1} = 0$, then $a_{i,j_2} = 0$ for all j_2 with $j_1 \preceq_A j_2$;*

Proof. (i) Suppose that $a_{i,j_1} = 1$ and that $j_2 \preceq_A j_1$. If $a_{i,j_2} = 0$, then (as $j_2 \prec_A j_1$) there exists $i_1 < i$ such that $a_{i_1,j_1} = 0$ and $a_{i_1,j_2} = 1$, contradicting property (β). Thus $a_{i_1,j_2} = 1$. Proving (ii) is similar. ∎

THEOREM 3.8. *A zero-one array $A = [a_{i,j}]$ is obtainable if and only if it has property (β).*

Proof. Suppose first that A does not have property (β), and that

$$\begin{pmatrix} a_{i_1,j_1} & a_{i_1,j_2} \\ a_{i_2,j_1} & a_{i_2,j_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose that A is obtainable. Thus there exist real bounded sequences λ and μ such that $E_{\lambda\mu}(0) = P_A$. This corresponds to the inequalities

$$\mu_{i_1} + \lambda_{j_1} \leqslant 0, \quad \mu_{i_1} + \lambda_{j_2} > 0, \quad \text{and} \quad \mu_{i_2} + \lambda_{j_1} > 0, \quad \mu_{i_2} + \lambda_{j_2} \leqslant 0.$$

Thus

$$\mu_{i_1} + \lambda_{j_1} - \mu_{i_1} - \lambda_{j_2} < 0 \quad \text{and} \quad \mu_{i_2} + \lambda_{j_1} - \mu_{i_2} - \lambda_{j_2} > 0$$

which is clearly a contradiction. Thus A is not obtainable.

Suppose now that A has property (β). We first construct a map $\rho : \mathbb{Z}^+ \to (-1,1)$ for which:

(1) $j_1 \prec_A j_2$ if and only if $\rho(j_1) < \rho(j_2)$, and

(2) for all $j \in \mathbb{Z}^+$, there is no strictly increasing sequence $\{\rho(j_k)\}_{k=1}^{\infty}$ such that $\rho(j) = \lim_k \rho(j_k)$.

Such a map can be constructed recursively as follows. Let $\rho(1) = 0$. Now suppose that the algorithm has fixed the values of $\rho(1), \dots, \rho(k-1)$, and that $j_1, \dots, j_{k-1}$ is a permutation of $\{1, \dots, k-1\}$ such that

$$\rho(j_1) \leqslant \rho(j_2) \leqslant \cdots \leqslant \rho(j_{k-1}).$$

If $k =_A j_\ell$ for some $\ell < k$ then set $\rho(k) = \rho(j_\ell)$. If there exists $\ell < k$ such that $j_{\ell-1} \prec_A k \prec_A j_\ell$, then set

$$\rho(k) = \frac{2^k - 1}{2^k} \rho(j_{\ell-1}) + \frac{1}{2^k} \rho(j_\ell).$$

If $k \prec_A j_1$ then set $\rho(k) = \frac{2^k-1}{2^k}(-1) + \frac{1}{2^k}\rho(j_1)$. The only remaining possibility is that $j_{k-1} \prec_A k$. In this case set $\rho(k) = \frac{2^k-1}{2^k}\rho(j_{k-1}) + \frac{1}{2^k}$. The resulting map clearly has property (1) above.

A simple induction proof shows that for all j, $\{2^{n(j)}\rho(k)\}_{k=1}^{j} \subset \mathbb{Z}$, where $n(j) = \frac{(j+2)(j-1)}{2}$. The construction of ρ ensures that for $\ell > j$, either $\rho(\ell) \geqslant \rho(j)$, or else

$$\begin{aligned}\rho(\ell) &< \left(\rho(j) - \frac{1}{2^{n(j)}}\right) + \frac{1}{2^{n(j)}}\left(\frac{1}{2^{j+1}} + \frac{1}{2^{j+2}} + \cdots + \frac{1}{2^\ell}\right) \\ &< \rho(j) - \frac{1}{2^{n(j)}} + \frac{1}{2^{n(j)+1}} = \rho(j) - \frac{1}{2^{n(j)+1}}\end{aligned}$$

which gives property (2).

For each $j \in \mathbb{Z}^+$, let $\lambda_j = \rho(j)$. For $i \in \mathbb{Z}^+$, let

$$\mu_i = \begin{cases} 1 & \text{if } a_{i,j} = 0 \text{ for all } j, \\ -\sup\{\rho(j) \,:\, a_{i,j} = 1\} & \text{otherwise.} \end{cases}$$

Note that if $a_{i,j_1} = 1$, then clearly $\rho(j_1) \leqslant \sup\{\rho(j) \,:\, a_{i,j} = 1\}$ and so $\lambda_{j_1} + \mu_i \leqslant 0$.

Suppose now that $a_{i,j_1} = 0$. Then one of the three following (mutually exclusive) cases holds:

(i) $a_{i,j} = 0$ for all j; or
(ii) there exists j_2 such that $a_{i,j_2} = 1$ and $-\mu_i = \rho(j_2)$; or
(iii) $-\mu_i > \rho(j)$ for all j such that $a_{i,j} = 1$.

If case (i) holds then $\mu_i = 1$ and so $\lambda_{j_1} + \mu_i > 0$. If case (ii) holds, then, by Lemma 3.7(i), $j_2 \prec_A j_1$. Thus, by property (1) of ρ, $-\mu_i = \lambda_{j_2} < \lambda_{j_1}$ and hence $\lambda_{j_1} + \mu_i > 0$.

If case (iii) holds then there exists an increasing sequence $\{\rho(j_k)\}_{k=2}^\infty$ such that $a_{i,j_k} = 1$ and $-\mu_i = \lim_{k\to\infty} \rho(j_k)$. As before, we must have $j_k \prec_A j_1$ for all $k \geqslant 2$ and so $\lambda_{j_k} < \lambda_{j_1}$. Thus $0 = \lim_{k\to\infty} \lambda_{j_k} + \mu_i \leqslant \lambda_{j_1} + \mu_i$. But $\lambda_{j_1} + \mu_i$ can not be zero since this would imply that $\rho(j_1) = \lim_k \rho(j_k)$ contradicting property (2) of ρ. Thus again we have $\lambda_{j_1} + \mu_i > 0$.

Thus $\lambda_j + \mu_i \leqslant 0$ if and only if $a_{i,j} = 1$. It follows $E_{\lambda\mu}(0) = P_A$ and so A is obtainable. ∎

COROLLARY 3.9. *Suppose that* $1 < p < \infty$. *Then there exists a constant* K_p *such that if* $A \in \mathcal{Z}$ *has property* (β), *then* $\|P_A\|_p \leqslant K_p$.

Since the upper triangular truncation map is not bounded if $p = 1$ or ∞, this theorem does not extend to the extreme values of p. The constant K_p, which is related to the norm of the vector-valued Hilbert transform on $L^2(\mathbb{R}; \mathcal{C}_p)$, may be taken to be 1 for $p = 2$, and grows unboundedly as p approaches 1 and ∞.

Of course condition (β) is rather special. In the following sections we shall use Corollary 3.9 to obtain boundedness results for a much wider class of projections. We give here just two simple applications which will be useful later in the paper.

COROLLARY 3.10. *Suppose that* $1 < p < \infty$ *and that* $A = [a_{i,j}] \in \mathcal{Z}$ *is such that for every* $j \in \mathbb{Z}^+$ *there exists* $i_0(j) \in \mathbb{Z}^+ \cup \{\infty\}$ *such that*

$$a_{i,j} = \begin{cases} 1 & \text{if } i < i_0(j), \\ 0 & \text{if } i \geqslant i_0(j). \end{cases}$$

Then $\|P_A\|_p \leqslant K_p$.

Proof. Such an array clearly has property (β). ∎

COROLLARY 3.11. *Suppose* $1 < p < \infty$ *and that* $A = [a_{i,j}] \in \mathcal{Z}$ *is such that* $\max\limits_j \sum\limits_{i=1}^{\infty} |a_{i+1,j} - a_{i,j}| = n$. *Then* $\|P_A\|_p \leqslant (n+1)K_p$.

Proof. In this case we can write $P_A = \sum\limits_{k=0}^{n} (-1)^k P_{A_k}$ where each array A_k satisfies the hypothesis of Corollary 3.10. ∎

Clearly we could swap rows and columns in Corollaries 3.10 and 3.11. One might also note that if each column of A is eventually zero, then the conclusion of Corollary 3.11 can be strengthened slightly to the estimate that $\|P_A\|_p \leqslant nK_p$.

4. TOEPLITZ AND HANKEL PROJECTIONS

In this section we shall give some results on the boundedness or otherwise of Schur multiplier projections where the multiplier pattern is constant on one or other of the sets of diagonals.

Let $b = b_1, b_2, b_3, \ldots$ be a string of binary digits. We shall say that the string is *rational* if the real number $\sum\limits_j b_j 2^{-j}$ is rational and *irrational* otherwise. Thus b is rational if and only if the string is eventually repeating. (One might complain that the strings $1000\ldots$ and $0111\ldots$ correspond to the same real number: this will not be a matter for concern in what follows.) As before, the set of all binary strings can be thought of as the probability space $\mathbb{Z}_2^{\mathbb{N}}$.

Let $n(b,i)$ denote the position of the ith 1 in the string b. Define $n(b,0)$ to be 0. We shall say that b has *increasing gaps* if $n(b,i+2)-n(b,i+1) \geqslant n(b,i+1)-n(b,i)$ for all $i \geqslant 0$, and we shall say that b is *lacunary* (of ratio λ) if there exists $\lambda > 1$ such that $n(b,i+1)/n(b,i) \geqslant \lambda$ for all i. For a string b, define the Toeplitz projection T_b to be Schur multiplication by the array

$$A_b^T = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & \ldots \\ 0 & b_1 & b_2 & b_3 & \ldots \\ 0 & 0 & b_1 & b_2 & \ldots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}$$

and the Hankel projection H_b to be the Schur multiplication by the array

$$A_b^H = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & \ldots \\ b_2 & b_3 & b_4 & b_5 & \ldots \\ b_3 & b_4 & b_5 & b_6 & \ldots \\ \vdots & \vdots & & \ddots & \end{pmatrix}.$$

The following proposition lists some easy consequences of the results in the last section.

PROPOSITION 4.1. *Suppose that* $1 < p < \infty$, $p \neq 2$. *Then:*

(i) $\|H_b\|_p \leqslant \|T_b\|_p$ *and so, if* T_b *is bounded, then so is* H_b*;*
(ii) *for almost all binary strings* b, T_b *and* H_b *are unbounded on* $\mathcal{C}_p$*;*
(iii) *if* b *is rational,* T_b *and* H_b *are bounded on* $\mathcal{C}_p$*;*
(iv) *if* b *is lacunary then* H_b *is bounded.*

Proof. (i) This follows from the fact that the $n \times n$ truncate of A_b^H is (flipped upside down) the top right corner of the $(2n-1)\times(2n-1)$ truncate of A_b^T.

(ii) We shall show that almost every Toeplitz projection contains every finite array as a subarray. Fix $m \geqslant 1$ and let $B_m \in \mathcal{Z}^m$.

Almost all irrational strings contain all finite strings as substrings. Suppose that b has this property. Then at some point in the string, the string of digits that make up the first column of B_m appears in reverse order as $b_k, b_{k+1}, \ldots, b_\ell$. Clearly then the first column of B_m appears as the start of column ℓ of A_b^T. The same reasoning shows that each column of B_m appears as the start of a column of A_b^T (and we can choose these to occur in the same order as they appear in B_m). Thus B_m is a subarray of A_b^T. It follows from the remarks in Section 2 that T_b is not bounded when $p \neq 2$.

(iii) Suppose first that b is a periodic sequence with period k. The boundedness of T_b follows from the boundedness of the projection associated to the corresponding periodic array, and the boundedness of the upper triangular projection. Indeed, by Proposition 2.2, $\|T_b\|_p \leqslant kK_p$.

If b is not periodic, then write $T_b = T_{b_1} + T_{b_2} - T_{b_3}$ where b_1 is periodic, and b_1 and b_2 have only finitely many nonzero terms. Since T_{b_2} and T_{b_3} are bounded, so is T_b.

The proof for Hankel matrices is essentially the same.

(iv) Suppose that b is lacunary with ratio λ. Write $H_b = U + D + L$ where U is the Schur projection corresponding to the upper triangular part of A_b^H, D corresponds to the diagonal part, and L corresponds to the lower triangular part. Now $\|D\|_p = 1$, and $\|U\|_p = \|L\|_p$ so it suffices to show that U is bounded. Let UA denote the upper triangular part of A_b^H. The result follows from Corollary 3.11 and the following claim.

Claim. There are at most $(\log 2/\log\lambda)$ 1's in any column of UA.

The ith 1 in b generates a truncated diagonal in UA which runs from column $\lfloor n(b,i)/2\rfloor + 1$ to column $n(b,i)$. In order that a column contain $(k+1)$ 1's it would be necessary that for some i, $\lfloor n(b,i+k)/2\rfloor + 1 \leqslant n(b,i)$. However, if $k \geqslant \log 2/\log\lambda$, then for all i,

$$n(b,i+k) \geqslant \frac{\lambda^k}{2} + 1 > n(b,i). \quad \blacksquare$$

With more machinery, one can obtain more sophisticated bounds for these types of projections. Let us recall some results from [4] and [6].

Let s_n denote the diadic elements of $\mathbb{Z}$. That is $s_n = 2^{n-1}$ for $n > 0$ and $s_n = -2^{-n}$ for $n \leqslant 0$. We shall denote by $\mathfrak{M}(\mathbb{Z})$ the unital Banach algebra of Marcinkiewicz multipliers comprising those complex sequences $\phi = \{\phi_n\}_{n\in\mathbb{Z}}$ for which

$$\|\phi\|_{\mathfrak{M}(\mathbb{Z})} = \sup_{n\in\mathbb{Z}} |\phi_n| + \sup_{n\in\mathbb{Z}} \sum_{j=s_n+1}^{s_{n+1}} |\phi_j - \phi_{j-1}|$$

is finite. The unit circle will be denoted by $\mathbb{T}$.

THEOREM 4.2. ([4], Theorem 4.2; [6], Theorem 1.3) *Let $\omega \mapsto R_\omega$ be a strongly continuous representation of $\mathbb{T}$ on a* UMD *space X. Then:*

(i) *For $n \in \mathbb{Z}$, the formula*

$$P_n x = \frac{1}{2\pi}\int_{\mathbb{T}} \omega^{-n} R_\omega x \,\mathrm{d}\omega, \quad x \in X, \tag{4.1}$$

defines a sequence of (disjoint) projections on X.

(ii) *For each $\phi \in \mathfrak{M}(\mathbb{Z})$, the series $\sum_{n=1}^{\infty} \phi_{-n}P_{-n}$ and $\sum_{n=0}^{\infty} \phi_n P_n$ converge in the strong operator topology.*

(iii) *There is a constant K_X such that for all $\phi \in \mathfrak{M}(\mathbb{Z})$, letting $c = \sup\{\|R_\omega\| : \omega \in \mathbb{T}\}$ we have $\Big\|\sum_{n=-\infty}^{\infty} \phi_n P_n\Big\| \leqslant c^2 K_X \|\phi\|_{\mathfrak{M}(\mathbb{Z})}$.*

When applied to the regular representation of $\mathbb{T}$ by translation operators on $L^p(\mathbb{T})$, $1 < p < \infty$, the projection P_n defined by Equation (4.1) is just the projection onto the nth term of the Fourier series, and part (iii) of the theorem just gives the Marcinkiewicz Multiplier Theorem. By choosing an appropriate group action on $\mathcal{C}_p$ we shall get some analogous theorems for Toeplitz and Hankel type Schur multipliers on these spaces.

Since these techniques rely on having an underlying group structure, for the remainder of this section it will be more appropriate to take as our underlying Hilbert space $L^2(\mathbb{T})$ and consider the elements of $\mathcal{C}_p$ to be acting on this space, with bi-infinite matrices with respect to the basis $\{e_k\}_{k\in\mathbb{Z}}$ where $e_k(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}kt}$.

For $\omega \in \mathbb{T}$ define U_ω on $L^2(\mathbb{T})$ by $(U_\omega f)(\mathrm{e}^{\mathrm{i}t}) = f(\omega \mathrm{e}^{\mathrm{i}t})$. Clearly $\|U_\omega\| = 1$ for all ω. Note that $(U_\omega e_k)(\mathrm{e}^{\mathrm{i}t}) = e_k(\omega \mathrm{e}^{\mathrm{i}t}) = \omega^k e_k(\mathrm{e}^{\mathrm{i}t})$.

For $\omega \in \mathbb{T}$ define $R_\omega \in B(\mathcal{C}_p)$ by $R_\omega(A) = U_{\omega^{-1}} A U_\omega$. This defines a group action on $\mathcal{C}_p$.

LEMMA 4.3. *The map $\omega \mapsto R_\omega$ is strongly continuous on $\mathcal{C}_p$.*

Proof. Fix $S \in \mathcal{C}_p$. First note that it suffices to show that this action is continuous at $\omega_0 = 1$. That is, we need to show that

$$\lim_{\omega\to 1} \|R_\omega(S) - R_1(S)\|_p = \lim_{\omega\to 1} \|U_{\omega^{-1}} S U_\omega - S\|_p = \lim_{\omega\to 1} \|S U_\omega - U_\omega S\|_p = 0.$$

Fix $\varepsilon > 0$. Now choose a finite rank operator $S_0 = \sum_{k=1}^{n} \alpha_k f_k \otimes g_k$ such that $\|S - S_0\|_p < \frac{\varepsilon}{4}$. Then

$$\|S U_\omega - U_\omega S\|_p = \|(S - S_0)U_\omega - U_\omega(S - S_0) + (S_0 U_\omega - U_\omega S_0)\|_p \tag{4.2}$$

$$\leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \|S_0 U_\omega - S\|_p + \|U_\omega S_0 - S\|_p. \tag{4.3}$$

Note that for $f, g \in L^2(\mathbb{T})$, $U_\omega(f \otimes g) = (U_\omega f) \otimes g$. Thus

$$\|U_\omega S_0 - S\|_p = \Big\| \sum_{k=1}^{n} \alpha_k ((U_\omega - I) f_k) \otimes g_k \Big\|_p \leqslant \sum_{k=1}^{n} |\alpha_k| \|(U_\omega - I) f_k\|_{L^2} \|g_k\|_{L^2} < \frac{\varepsilon}{4}$$

for ω close enough to 1. The final term in (4.3) above is dealt with similarly, and hence for ω close enough to 1, $\|R_\omega(S) - R_1(S)\|_p < \varepsilon$. ∎

For $n \in \mathbb{Z}$, define the projection P_n on $\mathcal{C}_p$ by

$$P_n(S) = \frac{1}{2\pi} \int_{\mathbb{T}} \omega^{-n} R_\omega(S)\, \mathrm{d}\omega.$$

Suppose $k \in \mathbb{Z}$ and $S = [s_{i,j}] \in \mathcal{C}_p$. Then

$$P_n(S)e_k = \frac{1}{2\pi}\int\limits_{\mathbb{T}} \omega^{-n} U_{\omega^{-1}} S U_\omega e_k \,\mathrm{d}\omega = \frac{1}{2\pi}\int\limits_{\mathbb{T}} \omega^{-n}\omega^k U_{\omega^{-1}}\Big(\sum_\ell s_{\ell k} e_\ell\Big)\,\mathrm{d}\omega$$
$$= \frac{1}{2\pi}\int\limits_{\mathbb{T}} \omega^{k-n}\sum_\ell s_{\ell k}\omega^{-\ell} e_\ell \,\mathrm{d}\omega = \sum_\ell s_{\ell k}\Big(\frac{1}{2\pi}\int\limits_{\mathbb{T}} \omega^{k-n-\ell}\,\mathrm{d}\omega\Big) e_\ell$$
$$\text{(as the sum converges uniformly on } \mathbb{T} \text{ in } \ell^2 \text{ norm)}$$
$$= s_{k-n,k} e_{k-n}\,.$$

Thus P_n is the Schur projection which picks out the nth diagonal of the matrix.

The following result follows immediately. The constant K_p which occurs is the same as that in Corollary 3.9.

COROLLARY 4.4. *Let $b = \{b_n\}_{n\in\mathbb{Z}}$ be a binary string in $\mathfrak{M}(\mathbb{Z})$ and let P_b be the corresponding Toeplitz array with (i,j)th entry b_{j-i}. Let T_b denote the corresponding Schur projection. Then for $1 < p < \infty$, T_b is bounded and*

$$\|T_b\|_p \leqslant K_p \|b\|_{\mathfrak{M}(\mathbb{Z})}.$$

COROLLARY 4.5. *If b is lacunary, then T_b is bounded.*

Proof. Suppose that b is lacunary with factor λ. It is easy to see that in any diadic block, the number of 1's must be less than $1 + \log 2/\log\lambda$. Thus $\|b\|_{\mathfrak{M}(\mathbb{Z})} \leqslant 2 + 2\log 2/\log\lambda$ and so $b \in \mathfrak{M}(\mathbb{Z})$. ∎

The same method proves the following, which appears as Corollary 20 of [8].

COROLLARY 4.6. *If b is constant on diadic blocks (i.e. from s_n to $s_{n+1} - 1$), then T_b is bounded.*

The corresponding results for "bilateral" Hankel projections follows by permuting the columns of the Toeplitz array. The results for one sided Toeplitz and Hankel Schur projections follows from taking the appropriate corner of the bilateral Toeplitz array.

5. ORDERINGS ON $\mathbb{Z}^+ \times \mathbb{Z}^+$

Roughly speaking randomness in the way the multiplier array is filled out leads to unboundedness of the corresponding projection, whilst having some degree of pattern leads to boundedness. In this section we look at some particular examples of this.

One could obtain arrays from binary strings by endowing $\mathbb{Z}^+ \times \mathbb{Z}^+$ with an ordering and then placing the ith element of the string in the ith position according to that ordering. There are, of course, any number of orderings one might use. Four relatively natural ones are described below. Throughout, let $b = (b_0, b_1, b_2, \dots)$ denote a binary string.

Tensor ordering. The *tensor ordering projection* associated with b, denoted TO_b is the Schur multiplication by the array

$$A^{T,b} = [a^{T,b}_{i,j}] = \begin{pmatrix} b_0 & b_1 & b_4 & b_5 & b_{16} & \dots \\ b_2 & b_3 & b_6 & b_7 & \dots & \\ b_8 & b_9 & b_{12} & b_{13} & \dots & \\ b_{10} & b_{11} & b_{14} & b_{15} & \dots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}.$$

In this ordering, once you have ordered a $2^n \times 2^n$ block, you fill out the other three corners of the $2^{n+1} \times 2^{n+1}$ following the same pattern.

Hankel ordering. The *Hankel ordering projection* associated with b, denoted HO_b is the Schur multiplication by the array

$$A^{H,b} = [a^{H,b}_{i,j}] = \begin{pmatrix} b_0 & b_1 & b_3 & b_6 & b_{10} & \dots \\ b_2 & b_4 & b_7 & b_{11} & \dots & \\ b_5 & b_8 & b_{12} & b_{17} & \dots & \\ b_9 & b_{13} & b_{18} & b_{24} & \dots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}.$$

In other words, this ordering works its way down the short diagonals.

Maximum ordering. The *maximum ordering projection* associated with b, denoted MO_b is the Schur multiplication by the array

$$A^{M,b} = [a^{M,b}_{i,j}] = \begin{pmatrix} b_0 & b_1 & b_4 & b_9 & \dots \\ b_3 & b_2 & b_5 & b_{10} & \dots \\ b_8 & b_7 & b_6 & b_{11} & \dots \\ b_{15} & b_{14} & b_{13} & b_{12} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Radial ordering. The radial ordering of $\mathbb{Z}^+ \times \mathbb{Z}^+$ orders points (i,j) according to the size of $i^2 + j^2$. If $i_0^2 + j_0^2 = i_1^2 + j_1^2$, we shall order these so that the point with smaller row index is listed first. Thus the *radial ordering projection* associated with b, denoted RO_b is the Schur multiplication by the array

$$A^{R,b} = [a^{R,b}_{i,j}] = \begin{pmatrix} b_0 & b_1 & b_4 & b_8 & \dots & \\ b_2 & b_3 & b_6 & b_{11} & \dots & \\ b_5 & b_7 & b_{10} & b_{13} & \dots & \\ b_9 & b_{12} & b_{14} & b_{19} & \dots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}.$$

PROPOSITION 5.1. *If b is a rational string, then TO_b, HO_b and MO_b are bounded on $\mathcal{C}_p$ for $1 < p < \infty$.*

Proof. As before, it suffices to consider the case when b is periodic. We shall write $\mathrm{ind}_n^T(i,j) = k$ if, given any sequence $b = (b_0, b_1, \dots, b_{n-1}, b_0, b_1, \dots)$ of period n, $a_{i,j}^{T,b} = b_k$. Clearly if b has period n and $\mathrm{ind}_n^T(i_1, j) = \mathrm{ind}_n^T(i_2, j)$ for all j then row i_1 and row i_2 of $A^{T,b}$ must be identical. We define $\mathrm{ind}_n^H(i,j)$, $\mathrm{ind}_n^M(i,j)$ and $\mathrm{ind}_n^R(i,j)$ analogously.

Consider first TO_b. Note that for all $i \in \mathbb{Z}^+$, row i of $A^{T,b}$ is completely determined by $\mathrm{ind}_n^T(i,1) \in \{0, 1, \dots, n-1\}$ since

$$\mathrm{ind}_n^T(i,2) = \mathrm{ind}_n^T(i,1) + 1 \mod n,$$

$$\mathrm{ind}_n^T(i,3) = \mathrm{ind}_n^T(i,1) + 4 \mod n$$

and so forth. It follows that there are at most n distinct row sequences. As in the proof of Proposition 2.2, we can therefore write TO_b as the sum of n norm 1 projections and hence $\|TO_b\|_p \leqslant n$.

The situation for HO_b is similar, but a little more complicated. A simple induction proof shows that if $\mathrm{ind}_n^H(i,1) = k$ then

$$\mathrm{ind}_n^H(i,j) = k + (j-1)i + \frac{(j-1)(j-2)}{2} \mod n, \quad j \in \mathbb{Z}^+.$$

Thus the row is determined not just by where in the repeating pattern the first element of the row appears, but also by the index of the row modulo n. There are therefore at most n^2 different row patterns and so $\|HO_b\|_p \leqslant n^2$.

To see that MO_b is bounded, consider the upper and lower triangular parts separately. For the upper triangular part, the column sequence is completely determined by $\mathrm{ind}_n^M(1,j)$. Indeed, let $U = \mathbb{U}MO_b$. Then $U = P_A$ where A is the array whose (i,j)th element is b_k with $k = \mathrm{ind}_n^M(1,j) + (i-1) \mod n$. It follows that A has at most n distinct column sequences and hence that $\|U\|_p \leqslant nK_p$. The lower triangular part has the same bound, so $\|MO_b\|_p \leqslant 2nK_p$. ∎

Another way of avoiding randomness is to make the array sparse in an appropriate sense.

PROPOSITION 5.2. *Let* $1 < p < \infty$.

(i) *If b has increasing gaps then MO_b is bounded on $\mathcal{C}_p$.*
(ii) *If b is lacunary then TO_b, HO_b and RO_b are bounded on $\mathcal{C}_p$.*

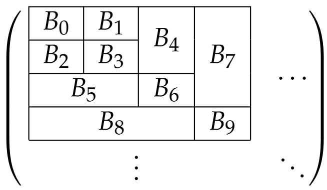
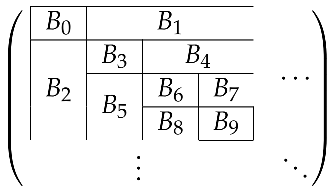
Proof. (i) If b has increasing gaps then each column in the upper triangular part of $A^{M,b}$ can contain at most one 1. Similarly, each row in the lower triangular part can contain at most one 1. It follows that $\|MO_b\|_p \leqslant 4K_p$.

(ii) Similar, but slightly more sophisticated reasoning shows that there is an upper bound to the number of 1's in each column of the upper triangular part, and the number of 1's in each row of the lower triangular part. ∎

6. UNCONDITIONAL DECOMPOSITIONS OF $\mathcal{C}_p$

One consequence of the characterization of obtainable arrays in Section 3 is that one obtains some less obvious bounded Boolean algebras of projections on $\mathcal{C}_p$ for $1 < p < \infty$.

Split $\mathbb{Z}^+ \times \mathbb{Z}^+$ into rectangular subarrays as in either of the diagrams below. Let $m(k) = k - 1 \bmod 3$. We shall call B_k a *superdiagonal* subarray if $m(k) = 0$; a *subdiagonal* subarray if $m(k) = 1$ and a *diagonal* subarray if $m(k) = 2$.

(i)  (ii)  .

The actual dimensions of these subarrays is not important. One could even choose some of the dimensions to be zero.

Let S_k denote the projection given by Schur multiplication by the characteristic function of B_k.

THEOREM 6.1. *Suppose that* $1 < p < \infty$ *and that* $\emptyset \neq J \subset \mathbb{N}$. *Then* $\sum_{k\in J} S_k$ *converges in the strong operator topology (in* $B(\mathcal{C}_p)$*) and*

$$\Big\| \sum_{k\in J} S_k \Big\|_p \leqslant 2K_p + 1.$$

Proof. It suffices to assume that J is finite and then to give the above bound on the norm of $S = \sum_{k\in J} S_k$. Write $S = U + L + D$ where

$$U = \sum_{\substack{k\in J \\ m(k)=0}} S_k, \quad L = \sum_{\substack{k\in J \\ m(k)=1}} S_k, \quad D = \sum_{\substack{k\in J \\ m(k)=2}} S_k.$$

Clearly U and L both correspond to obtainable arrays, whereas, as we remarked in Section 2, $\|D\|_p = 1$. ∎

One can also take diadic decompositions based on circles (or indeed certain other families of convex curves centred at the origin). For $k \geqslant 1$ let

$$\Delta_k = \Big\{ (i,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \; : \; 2^{k-1} \leqslant \sqrt{i^2 + j^2} < 2^k \Big\}.$$

Let $S_k = P_{A(\Delta_k)}$ be the projection which corresponds to Schur multiplication by the characteristic function of Δ_k.

THEOREM 6.2. *Suppose that* $1 < p < \infty$ *and that* $\emptyset \neq J \subset \mathbb{N}$. *Then* $\sum_{k\in J} S_k$ *converges in the strong operator topology and*

$$\Big\| \sum_{k\in J} S_k \Big\|_p \leqslant 4K_p.$$

Proof. Again, it suffices to assume that J is finite. As before, let $\mathbb{U}$ be the upper triangular truncation operator. Write $S = \sum_{k\in J} S_k = U + L$ where $U = \mathbb{U}S$ and $L = (I - \mathbb{U})S$.

For $k \in \mathbb{Z}^+$, let $\Delta_k^+ = \{(i,j) \in \Delta_k : i \leqslant j\}$ and let

$$\ell(k) = \min\{j : (i,j) \in \Delta_k^+ \text{ for some } i\},$$
$$r(k) = \max\{j : (i,j) \in \Delta_k^+ \text{ for some } i\}.$$

It is clear that $r(k) = 2^k - 1$. With a little more work one can show that $\ell(k) = \lfloor 2^{k-3/2} \rfloor + 1$. It follows immediately that for all k, $r(k) < \ell(k+2)$.

It follows that if $A = [a_{i,j}]$ is the array corresponding to U, then for each $j \in \mathbb{Z}^+$, $\sum_{i=1}^{\infty} |a_{i+1,j} - a_{i,j}| \leqslant 2$. By the comments following Corollary 3.11 then, $\|U\|_p \leqslant 2K_p$.

Essentially the same proof shows that $\|L\|_p \leqslant 2K_p$ and hence $\|S\|_p \leqslant 4K_p$. ∎

The blocks that make up the above decompositions all have zero density in $\mathbb{Z}^+ \times \mathbb{Z}^+$, where for $\Delta \subset \mathbb{Z}^+ \times \mathbb{Z}^+$ we define the density to be

$$\mathrm{d}(\Delta) = \limsup_{n\to\infty} \frac{\{|(i,j) \in \Delta : 1 \leqslant i,j \leqslant n|\}}{n^2}.$$

The following example is made up of blocks all of which have positive density.

For $k = 0, 1, 2, \ldots$ let

$$\begin{aligned}\Delta_k = \{(i,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : &(i \bmod 2^{k+1} = 2^k \text{ and } j \bmod 2^{k+1} \neq 0) \\ &\text{or } (j \bmod 2^{k+1} = 2^k \text{ and } i \bmod 2^{k+1} \neq 0)\}.\end{aligned} \tag{6.1}$$

This is perhaps better explained by the following matrix in which the (i,j)th entry is k if $(i,j) \in \Delta_k$.

$$\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 & 0 & 3 & 0 & \ldots \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 3 & 1 & \ldots \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 3 & 0 & \ldots \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & \ldots \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 3 & 0 & \ldots \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 3 & 1 & \ldots \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 3 & 0 & \ldots \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \ldots \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to check that $\mathrm{d}(\Delta_k) = \dfrac{2^{k+2}-3}{2^{2k+2}}$. Let S_k denote the Schur projection corresponding to Δ_k.

THEOREM 6.3. *Suppose that* $1 < p < \infty$ *and that* $\emptyset \neq J \subset \mathbb{N}$. *Then* $\sum_{k\in J} S_k$ *converges in the strong operator topology and*

$$\Big\| \sum_{k\in J} S_k \Big\|_p \leqslant K_p.$$

Proof. Let $\Delta = \bigcup_{k\in J} \Delta_k$. We shall show that $A(\Delta)$ is an obtainable array, from which the result will follow immediately.

For $k \geqslant 0$ let

$$\begin{aligned} P_k &= \{ j \in \mathbb{Z}^+ \,:\, (i,j) \in \Delta_k \text{ for some } i \in \mathbb{Z}^+ \}, \\ H_k &= \{ (i,j) \in \Delta_k \,:\, i \bmod 2^{k+1} = 2^k \}, \\ V_k &= \{ (i,j) \in \Delta_k \,:\, j \bmod 2^{k+1} = 2^k \}. \end{aligned}$$

(Pictorially, H_k contains the points in the horizontal parts of the crosses that make up Δ_k and V_k contains that points in the vertical parts.) Note that

$$P_0 \subset P_1 \subset P_2 \cdots \tag{6.2}$$

and that $\Delta_k = H_k \cup V_k$ for all k. Note also that

$$\text{if } (i,j) \in H_k \text{ and } j' \in P_k, \text{ then } (i,j') \in H_k \subset \Delta_k. \tag{6.3}$$

Let $A(\Delta) = [a_{i,j}]$. Suppose that $A(\Delta)$ is not obtainable. That is, one can choose $i_1, i_2, j_1, j_2 \in \mathbb{Z}^+$ such that $a_{i_1,j_1} = a_{i_2,j_2} = 1$ and $a_{i_1,j_2} = a_{i_2,j_1} = 0$. Thus, there exist $k_1 \leqslant k_2$ such that $(i_1, j_1) \in \Delta_{k_1}$ and $(i_2, j_2) \in \Delta_{k_2}$.

Suppose first that $(i_2, j_2) \in H_{k_2}$. Now $j_1 \in P_{k_1}$ so by (6.2), $j_1 \in P_{k_2}$. It follows from (6.3) then, that $(i_2, j_1) \in \Delta_{k_2} \subset \Delta$ and so $a_{i_2,j_1} = 1$, contradicting the fact that $A(\Delta)$ is not obtainable. Supposing instead that $(i_2, j_2) \in V_{k_2}$ leads to a similar contradiction, and so $A(\Delta)$ must be obtainable. ∎

Of course with a little imagination, one might write down many more such decompositions.

In the following corollary one may let $\{S_k\}$ be any of the decompositions defined in this section.

COROLLARY 6.4. *Suppose that* $1 < p < \infty$. *Then there exists a constant* α_p *such that given any* $T \in \mathcal{C}_p$ *and any sequence of signs* $\{\varepsilon_k\}_{k\in\mathbb{N}}$,

$$\alpha_p^{-1} \|T\|_p \leqslant \Big\| \sum_{k=0}^{\infty} \varepsilon_k S_k(T) \Big\|_p \leqslant \alpha_p \|T\|_p.$$

In particular, $\{S_k\}_{k=0}^{\infty}$ *generates a bounded Boolean algebra of projections on* $\mathcal{C}_p$.

7. CONCLUSION

It is possible to combine many of the results in the previous sections to produce bounds for wider classes of Schur multiplier projections. Unfortunately we know of no useful characterization of any of these wider classes, so in this section we shall just give a few examples which illustrate what can be done.

EXAMPLE 7.1. Let

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & \ddots \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ddots \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \ddots \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & \ddots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then for $1 < p < \infty$, $A \in \mathcal{B}_p$. To see this note that $\mathcal{B}_p$ is a lattice and that $A = A_1 \vee A_2$ where

$$A_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & \\ 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 1 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Of course $\|P_{A_1}\|_p \leqslant 2K_p$ by Proposition 4.1, and $\|P_{A_2}\|_p \leqslant K_p$ by Corollary 3.9, so $\|P_A\|_p = \|P_{A_1} + P_{A_2} - P_{A_1}P_{A_2}\|_p \leqslant 3K_p + 2K_p^2$.

EXAMPLE 7.2. The arguments used to prove uniform bounds on the Schur multiplier projections that come from obtainable arrays can in fact be iterated to obtain bounds for a larger class of projections.

The unconditional decompositions examined in Section 6 enable one to construct a large variety of real scalar-type spectral operators on $\mathcal{C}_p$ for $1 < p < \infty$. Let $\{S_k\}$ denote any of the unconditional families of projections discussed in Section 6 and let $\{\lambda_k\}$ be a bounded sequence of real numbers. Then $S = \sum_{k=0}^{\infty} \lambda_k S_k$ is a real scalar-type spectral operator. Of course S is also well-bounded, and, as is noted in Remark 2.5 of [11], the associated spectral family for S is R-bounded.

Let P_n be the Schur projection onto the nth diagonal of a matrix in $\mathcal{C}_p$. For $m \geqslant 0$, let $E_m = \sum_{n=-m}^{m} P_m$. Theorem 4.2(iii) implies that the family of projections

$\{E_m\}$ is uniformly bounded. One can therefore easily form a spectral family from this family of projections and then the associated well-bounded operator. For example, for any decreasing sequence of positive reals $\{\mu_m\}$ which converges to 0, $T = \sum_{m=0}^{\infty} \mu_m P_m + \sum_{m=1}^{\infty} \mu_{-m} P_{-m}$ defines a well-bounded operator. Furthermore, the spectral family for this T has the R-property (see Corollary 5.24 of [6]).

It follows from Theorem 3.2 that if T is a well-bounded operator whose spectral family has the R-property, and S is a commuting real scalar-type spectral operator, then $S+T$ is well-bounded. The projections in the spectral family for $S+T$ are again Schur multiplier projections. Using the results of [11] one gains bounds on the norms of these projections.

We shall just give one example to illustrate the type of zero-one arrays that one can show give bounded Schur multipliers. For $k \geqslant 0$, let Δ_k be the subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$ defined in Equation (6.1). Define $S \in B(\mathcal{C}_p)$ by

$$S = \sum_{k=0}^{\infty} \cos(2k+1) P_{A(\Delta_k)}$$

(where the sum converges in the strong operator topology). For $i, j \geqslant 1$, let

$$t_{ij} = \begin{cases} \sin(j) & \text{if } i < j, \\ \cos(i) & \text{if } i > j, \\ 0 & \text{if } i = j, \end{cases}$$

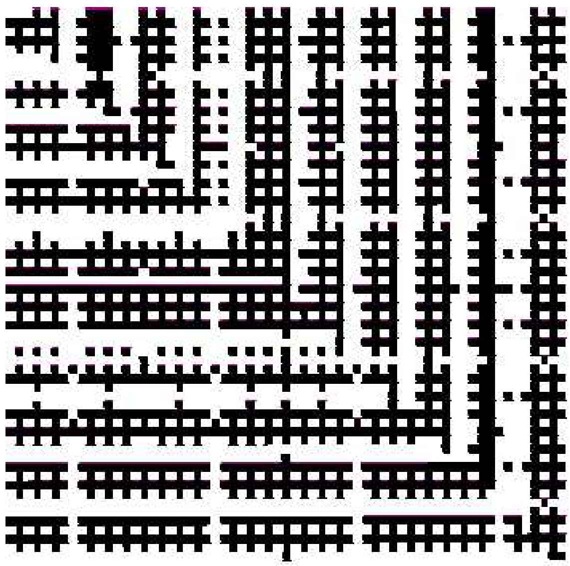


FIGURE 1. $A(\Gamma_0)$

and let $T \in B(\mathcal{C}_p)$ be defined as Schur multiplication by the array $[t_{ij}]$. (There is of course nothing special about the particular eigenvalues of S and T chosen here. We have been specific only in order to ensure that Figure 1 is reproducible by the reader.)

The results of Section 6 imply that T and S are commuting real scalar-type spectral operators. Let $\{E(s)\}$ denote the spectral family for the well-bounded operator $T+S$. Then for all $s \in \mathbb{R}$, $E(s) = P_{A(\Gamma_s)}$ where $\Gamma_s = \{(i,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : t_{ij} + s_{ij} \leqslant s\}$. It follows from the above and Corollary 3.7 of [11] that there is a constant k_p that depends only on p such that for all $s \in \mathbb{R}$, $\|E(s)\|_p \leqslant k_p$.

The arrays that appear from this construction are certainly of a more varied form than those we have considered earlier in this paper. As an example, Figure 1 shows the first 64 rows and columns of the array $A(\Gamma_0)$, where a black square indicates a 1 and a white square indicates a 0.

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