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IDEAL STRUCTURE IN FREE SEMIGROUPOID ALGEBRAS FROM DIRECTED GRAPHS

MICHAEL T. JURY and DAVID W. KRIBS

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ABSTRACT. A *free semigroupoid algebra* is the weak operator topology closed algebra generated by the left regular representation of a directed graph. We establish lattice isomorphisms between ideals and invariant subspaces, and this leads to a complete description of the WOT-closed ideal structure for these algebras. We prove a distance formula to ideals, and this gives an appropriate version of the Carathéodory interpolation theorem. Our analysis rests on an investigation of predual properties, specifically the $\mathbb{A}_n$ properties for linear functionals, together with a general Wold Decomposition for n-tuples of partial isometries. A number of our proofs unify proofs for subclasses appearing in the literature.

KEYWORDS: *Hilbert space, Fock space, directed graph, partial isometry, nonselfadjoint operator algebra, partly free algebra, Wold Decomposition, distance formula, Carathéodory Theorem.*

MSC (2000): 47L75, 47L55.

INTRODUCTION

In [19] and [20], the second author and Stephen Power began studying a class of operator algebras called *free semigroupoid algebras*. These are the WOT-closed (nonselfadjoint) algebras $\mathfrak{L}_G$ generated by the left regular representations of directed graphs G. Earlier work of Muhly and Solel [24], [25] considered the norm closed algebras generated by these representations in the finite graph case; they called them *quiver algebras*. In the case of single vertex graphs, the $\mathfrak{L}_G$ obtained include the classical analytic Toeplitz algebra H^∞ [13], [14], [29] and the noncommutative analytic Toeplitz algebras $\mathfrak{L}_n$ studied by Arias, Popescu, Davidson, Pitts, and others [1], [9], [12], [11], [18], [27], [28].

In this paper, we consider algebraic structure-type problems for the algebras $\mathfrak{L}_G$. In particular, we derive a complete description of the WOT-closed ideal structure; for instance, there is a lattice isomorphism between right ideals and invariant subspaces of the commutant $\mathfrak{L}_G' = \mathfrak{R}_G$ [11]. Furthermore, we prove a distance formula to ideals in these algebras [2], [10], [23]. This yields a version of the Carathéodory interpolation theorem [2], [10] for $\mathfrak{L}_G$. A valuable tool in our analysis is a general Wold Decomposition [25], [26], which we establish for n-tuples of partial isometries with initial and final projections satisfying natural conditions. This leads to information on predual properties for $\mathfrak{L}_G$. We prove $\mathfrak{L}_G$ satisfies property $\mathbb{A}_1$ [6], [11]; that is, every weak* continuous linear functional may be realized as a vector functional. A number of our proofs for general $\mathfrak{L}_G$ unify the proofs for the special cases of $\mathfrak{L}_n$ and H^∞, which were previously established by different means.

The first section contains a brief review of the notation associated with these algebras. Our attention will be focused on the cases when the directed graph G has "no sinks"; that is, every vertex in G is the initial vertex for some directed edge. We include a list of some examples generated by simple graphs. We begin the analysis proper in the second section, with a Wold Decomposition for n-tuples of partial isometries. This leads into the topic of the third section; an investigation into the basic properties of linear functionals on $\mathfrak{L}_G$. In particular, we show the ampliation algebras $\mathfrak{L}_G^{(n)}$ have the factorization property $\mathbb{A}_n$. In the subsequent section, we prove the subclass of algebras with *partly free* commutant discovered in [19], [20] are precisely those $\mathfrak{L}_G$ which satisfy the stronger factorization property $\mathbb{A}_{\aleph_0}$, when an initial restriction is made on the graph. Using property $\mathbb{A}_1$ for $\mathfrak{L}_G$ and the Beurling Theorem from [20], we establish complete lattice isomorphisms between ideals and invariant subspaces of $\mathfrak{L}_G$ in the fifth section. This allows us to describe, for example, the WOT-closure of the commutator ideal, and precisely when WOT-closed ideals are finitely generated. In the penultimate section we prove a completely isometric distance formula to ideals of $\mathfrak{L}_G$, and we apply this in special cases to obtain a Carathéodory Theorem in the final section.

1. FREE SEMIGROUPOID ALGEBRAS

Let G be a countable (finite or countably infinite) directed graph with edge set $E(G)$ and vertex set $V(G)$. Let $\mathbb{F}^+(G)$ be the free semigroupoid determined by G; that is, $\mathbb{F}^+(G)$ consists of the vertices which act as units, written as $\{k\}_{k\geqslant 1}$, and allowable finite paths in G, with the natural operations of concatenation of allowable paths. Given a path $w = e_{i_m}\cdots e_{i_1}$ in $\mathbb{F}^+(G)$, an allowable product of edges e_{i_j} in $E(G)$, we write $w = k_2wk_1$ when the initial and final vertices of w are, respectively, k_1 and k_2. Further, by $|w|$ we mean the number of directed edges which determine the path w.

ASSUMPTION 1.1. For our purposes, it is natural to restrict attention to directed graphs G with "no sinks"; that is, every vertex is the initial vertex for some directed edge.

Let $\mathcal{H}_G = \ell^2(\mathbb{F}^+(G))$ be the Hilbert space with orthonormal basis $\{\xi_w : w \in \mathbb{F}^+(G)\}$ indexed by elements of $\mathbb{F}^+(G)$. For each edge $e \in E(G)$ and vertex $k \in V(G)$, define partial isometries and projections on $\mathcal{H}_G$ by:

$$L_e\xi_w = \begin{cases} \xi_{ew} & \text{if } ew \in \mathbb{F}^+(G), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$L_k\xi_w = \begin{cases} \xi_{kw} = \xi_w & \text{if } w = kw \in \mathbb{F}^+(G), \\ 0 & \text{otherwise.} \end{cases}$$

These operators may be regarded as "partial creation operators" acting on a generalized Fock space Hilbert space. There is an equivalent tree perspective, discussed in [20], which gives an appealing visual interpretation of the actions of these operators. The vectors $\{\xi_k : k \in V(G)\}$ are called the *vacuum vectors*.

The family $\{L_e, L_k\}$ also arises through the left regular representation $\lambda : \mathbb{F}^+(G) \to \mathcal{B}(\mathcal{H}_G)$, with $\lambda(e) = L_e$ and $\lambda(k) = L_k$. The associated *free semigroupoid algebra* is the weak operator topology closed algebra generated by this family:

$$\begin{aligned} \mathfrak{L}_G &= \text{WOT-Alg}\,\{L_e, L_k : e \in E(G), k \in V(G)\} \\ &= \text{WOT-Alg}\,\{\lambda(w) : w \in \mathbb{F}^+(G)\}. \end{aligned}$$

These algebras were the focus of analysis by the second author and Power in [19], [20]. In the case of finite graphs, Muhly and Solel [24], [25] considered the norm closed algebras $\mathcal{A}_G$ generated by such a family, calling them *quiver algebras*.

There is an analogous right regular representation $\rho : \mathbb{F}^+(G) \to \mathcal{B}(\mathcal{H}_G)$, which yields partial isometries $\rho(w) \equiv R_{w'}$ for $w \in \mathbb{F}^+(G)$ acting on $\mathcal{H}_G$ by the equations $R_{w'}\xi_v = \xi_{vw}$, where w' is the word w in reverse order, with similar conventions. Observe that $R_{v'}L_w = L_wR_{v'}$ for all $v, w \in \mathbb{F}^+(G)$. In fact, the algebra

$$\begin{aligned} \mathfrak{R}_G &= \text{WOT-Alg}\,\{R_e, R_k : e \in E(G), k \in V(G)\} \\ &= \text{WOT-Alg}\,\{\rho(w) : w \in \mathbb{F}^+(G)\} \end{aligned}$$

coincides with the commutant $\mathfrak{L}_G' = \mathfrak{R}_G$. The commutant is also unitarily equivalent to the algebra $\mathfrak{L}_{G^t}$, where G^t is obtained from G simply by reversing directions of all edges. Elements of $\mathfrak{L}_G$ have Fourier expansions: If $A \in \mathfrak{L}_G$ and $k \in V(G)$, then $A\xi_k = \sum_{w=wk} a_w\xi_w$ for some scalars $a_w \in \mathbb{C}$, and the Cesaro-type sums

$$\sigma_k(A) = \sum_{|w|<k} \Big(1 - \frac{|w|}{k}\Big) a_w L_w$$

converge in the strong operator topology to A. These results are contained in [20]. We write $A \sim \sum_{w\in\mathbb{F}^+(G)} a_w L_w$ as a notational convenience. We shall also put $P_k = L_k$ and $Q_k = R_k$ for the projections determined by vertices $k \in V(G)$.

A valuable tool in our analysis is the Beurling-type invariant subspace theorem for $\mathfrak{L}_G$ proved in [20]. A non-zero subspace $\mathcal{W}$ of $\mathcal{H}_G$ is *wandering* for $\mathfrak{L}_G$ if the subspaces $L_w\mathcal{W}$ are pairwise orthogonal for distinct w in $\mathbb{F}^+(G)$. Observe that every wandering subspace $\mathcal{W}$ generates an $\mathfrak{L}_G$-invariant subspace given by

$$\mathfrak{L}_G[\mathcal{W}] = \sum_{w\in\mathbb{F}^+(G)}^{\oplus} L_w\mathcal{W}.$$

Every $\mathfrak{L}_G$-wandering vector ζ generates the cyclic invariant subspace $\mathfrak{L}_G[\zeta]$. The subspace $\mathfrak{L}_G[\zeta]$ is *minimal cyclic* if $L_k\zeta = P_k\zeta = \zeta$ for some vertex $k \in V(G)$. Notice that if ζ is a wandering vector, then each vector $P_k\zeta$ which is non-zero is wandering as well. The following result was proved in [20]. It is a generalization of Beurling's classical theorem [7] for H^∞ and a corresponding result for free semigroup algebras $\mathfrak{L}_n$ [1], [12].

THEOREM 1.2. *Every invariant subspace of $\mathfrak{L}_G$ is generated by a wandering subspace, and is the direct sum of minimal cyclic subspaces generated by wandering vectors. Every minimal cyclic invariant subspace generated by a wandering vector is the range of a partial isometry in $\mathfrak{R}_G$, and the choice of partial isometry is unique up to a scalar multiple.*

In fact, more was proved in [20]. The partial isometries in $\mathfrak{R}_G$ have a standard form, and their initial projections are sums of projections amongst $\{Q_k : k \in V(G)\}$. Given a minimal cyclic subspace $\mathfrak{L}_G[\zeta]$ with $P_k\zeta = \zeta$, a partial isometry R_ζ in $\mathfrak{R}_G$ which satisfies $\mathfrak{L}_G[\zeta] = R_\zeta\mathcal{H}_G$ is defined by $R_\zeta\xi_w = L_w\zeta$ for w in $\mathbb{F}^+(G)$, and the initial projection satisfies $R_\zeta^*R_\zeta = Q_k$. Further, any partial isometry in $\mathfrak{R}_G$ with range space $\mathfrak{L}_G[\zeta]$ is a scalar multiple of R_ζ.

We finish this section by setting aside a number of examples generated by simple graphs.

EXAMPLES 1.3. (i) The algebra generated by the graph with a single vertex and single loop edge is unitarily equivalent to the classical analytic Toeplitz algebra H^∞ [13], [14], [29].

(ii) The noncommutative analytic Toeplitz algebras $\mathfrak{L}_n$, $n \geqslant 2$ [1], [2], [9], [12], [11], [10], [18], [27], [28], arise from the graphs with a single vertex and n distinct loop edges.

(iii) The cycle algebras $\mathfrak{L}_{C_n}$ discussed in [20] are generated by the graph C_n with $2 \leqslant n < \infty$ vertices, and edges for directions $\{(2,1),\dots,(n,n-1),(1,n)\}$. These algebras may be represented as matrix function algebras.

(iv) The algebra $\mathfrak{L}_{C_\infty}$ [19], [20] generated by the infinite graph analogue of the cycle graphs is determined by the graph C_∞ with vertices $\{k\}_{k\geqslant 1}$ and directed edges $(k+1,k)$.

(v) A non-discrete example discussed in [19] is given by the graph Q consisting of vertices $\{q\}_{q\in\mathbb{Q}}$ indexed by the rational numbers, and directed edges $e_{qp} = qe_{qp}p$ whenever $p \leqslant q$. Notice that Q^{t} is graph isomorphic to Q, and thus the algebra $\mathfrak{L}_Q \simeq \mathfrak{L}_{Q^{\mathrm{t}}} \simeq \mathfrak{L}'_Q$ is unitarily equivalent to its commutant.

2. WOLD DECOMPOSITION

In this section, we establish a Wold Decomposition for n-tuples of (non-zero) operators $S = (S_1, \dots, S_n)$ which act on a common Hilbert space $\mathcal{H}$ and satisfy the following Relations denoted by $(\dagger)$:

(1) $\sum_{i=1}^{n} S_i S_i^* \leqslant I$.

(2) For $1 \leqslant i \leqslant n$,
$$(S_i^* S_i)^2 = S_i^* S_i.$$

(3) For $1 \leqslant i, j \leqslant n$,
$$(S_i^* S_i)(S_j^* S_j) = 0 \quad \text{or} \quad S_i^* S_i = S_j^* S_j.$$

(4) For $1 \leqslant i \leqslant n$, there is a j such that
$$S_i S_i^* \leqslant S_j^* S_j.$$

(5) The distinct elements $\{P_k\}_{k \in \mathcal{S}}$ from the set $\{S_i^* S_i : 1 \leqslant i \leqslant n\}$ satisfy
$$\sum_{k \in \mathcal{S}} P_k = I.$$

NOTE 2.1. We shall behave as though n is finite in this section; there are obvious modifications which can be made in the infinite case. The first four conditions say the S_i are partial isometries with pairwise orthogonal ranges, with initial projections either orthogonal or equal, and with range projections supported on a (unique) initial projection. The last condition, which is redundant when equality is achieved in the first condition, means no non-zero vector is annihilated by all the S_i. In terms of the directed graph connection we are about to make, this means the associated directed graphs have no sinks.

DEFINITION 2.2. Given a positive integer $n \geqslant 2$, we write $\mathbb{F}_n^+$ for the (non-unital) free semigroup on n noncommuting letters. If G is a directed graph with n edges, we let $\mathbb{F}_n^+(G)$ denote the set of all finite words in the edges of G. Observe that $\mathbb{F}_n^+(G)$ contains the set of finite paths $\mathbb{F}^+(G) \setminus V(G)$ as a subset.

If $w = i_m \cdots i_1$ belongs to $\mathbb{F}_n^+$, it is convenient in this section to let $w(S)$ be the operator product $w(S) = S_{i_m} \cdots S_{i_1}$. When $S = (S_1, \dots, S_n)$ satisfies $(\dagger)$, this is a partial isometry with initial projection $w(S)^* w(S)$ equal to $S_{i_1}^* S_{i_1}$ or 0, and final projection $w(S) w(S)^*$ supported on $S_{i_m} S_{i_m}^*$. Let $\mathfrak{S}_S$ be the weak operator topology closed algebra generated by such an n-tuple and its initial projections
$$\mathfrak{S}_S = \text{WOT-Alg}\{S_1, \dots, S_n, S_1^* S_1, \dots, S_n^* S_n\}.$$

DEFINITION 2.3. A subspace $\mathcal{W}$ of $\mathcal{H}$ is *wandering* for $S = (S_1, \dots, S_n)$ satisfying $(\dagger)$ if the subspaces $w(S)\mathcal{W}$ are pairwise orthogonal for distinct words w in $\mathbb{F}_n^+$. Observe that a given partial isometry $w(S)$ may be equal to zero here. Every wandering subspace generates an invariant subspace by

$$\mathfrak{S}_S[\mathcal{W}] = \sum_{w\in\mathbb{F}_n^+}^{\oplus} w(S)\mathcal{W}.$$

The following is the pure part of the Wold Decomposition. We set it aside since it is the precise form we require in this paper.

LEMMA 2.4. *Let $S = (S_1, \dots, S_n)$ be operators on $\mathcal{H}$ satisfying (†). The subspace $\mathcal{W} = \operatorname{Ran}\left(I - \sum_{i=1}^{n} S_i S_i^*\right)$ is wandering for S. Suppose that*

$$\mathcal{H} = \sum_{w\in\mathbb{F}_n^+}^{\oplus} w(S)\mathcal{W}.$$

Let $\{P_k\}_{k\in\mathcal{S}}$ be the distinct projections from the set $\{S_i^* S_i : 1 \leqslant i \leqslant n\}$. Then

$$P_k\mathcal{W} \subseteq \mathcal{W} = \sum_{k\in\mathcal{S}}^{\oplus} P_k\mathcal{W}.$$

Let G be the directed graph (with no sinks) with vertex set $V(G) \equiv \mathcal{S}$, and n directed edges $\{e \in E(G)\}$ where the number of edges from vertex k to vertex l is given by the cardinality of the set

$$\{S_i \ : \ S_i^* S_i = P_k \text{ and } S_i S_i^* \leqslant P_l\}.$$

Given $k \in \mathcal{S}$, let $\alpha_k = \dim P_k\mathcal{W}$ and let $\{S_i^{(k)}\} = \{S_i : S_i^* S_i = P_k\}$. The sets $\{S_i^{(k)}\}$ and $\{L_e : e = ek\}$ have the same cardinality for each $k \in \mathcal{S} = V(G)$. Let $\mathcal{S}_0 = \{k \in V(G) : \alpha_k \neq 0\}$. Then there is a unitary

$$U : \mathcal{H} \longrightarrow \sum_{k\in\mathcal{S}_0}^{\oplus} (Q_k\mathcal{H}_G)^{(\alpha_k)}$$

such that for $l \in \mathcal{S}$,

$$\{US_i^{(l)}U^*\} = \Big\{ \sum_{k\in\mathcal{S}_0}^{\oplus} L_e^{(\alpha_k)}\Big|_{(Q_k\mathcal{H}_G)^{(\alpha_k)}} : e = el \Big\},$$

where $L_e^{(\alpha_k)}|_{(Q_k\mathcal{H}_G)^{(\alpha_k)}}$ is the restriction of the ampliation $L_e^{(\alpha_k)}$ to the α_k-fold direct sum of $Q_k\mathcal{H}_G$ with itself.

Proof. As $S = (S_1, \dots, S_n)$ are partial isometries with pairwise orthogonal ranges, the operator $P = I - \sum_{i=1}^{n} S_i S_i^*$ is a projection and the subspace $\mathcal{W} = P\mathcal{H}$ is wandering. Notice that

$$(S_i^* S_i)(S_j S_j^*) = (S_j S_j^*)(S_i^* S_i) = \begin{cases} S_j S_j^* & \text{if } S_j S_j^* \leqslant S_i^* S_i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the initial projections $S_i^* S_i$ commute with P, and hence $P_k\mathcal{W}$ is contained in $\mathcal{W}$ for $k \in \mathcal{S}$. Further, $\mathcal{W} = \sum_{k\in\mathcal{S}}^{\oplus} P_k\mathcal{W}$ by condition (5) in (†); the no-sink condition. The sets $\{S_i^{(k)}\}$ and $\{L_e : e = ek,\ e \in E(G)\}$ have the same cardinality from the definition of G. It is convenient to re-label $\{S_i^{(k)}\}$ as $\{S_e : e = ek, e \in E(G)\}$ for the rest of the proof.

It remains to construct a unitary U which intertwines $\{S_e : e \in E(G)\}$ with the appropriate sums of restricted ampliations of $\{L_e : e \in E(G)\}$. First note that there is a natural bijective correspondence between $\mathbb{F}_n^+(G)$ and $\mathbb{F}_n^+$ induced by the re-labeling of $\{S_1, \dots, S_n\}$ as $\{S_e : e \in E(G)\}$. The non-trivial paths in the free semigroupoid, $\mathbb{F}^+(G) \setminus V(G)$, may be regarded as a subset of $\mathbb{F}_n^+(G)$. The key point is that, under this identification, the set of all words w in $\mathbb{F}_n^+(G)$ for which the partial isometry $w(S)$ is non-zero is precisely the set $\mathbb{F}^+(G) \setminus V(G)$. Indeed, given a formal product $w = e_k \cdots e_1$ in $\mathbb{F}_n^+(G)$, the definition of G and properties (†) for $S = (S_1, \dots, S_n)$ show that

$$\begin{aligned} w(S) \neq 0 \quad & \text{if and only if} \quad S_{e_j} S_{e_j}^* \leqslant S_{e_{j+1}}^* S_{e_{j+1}} \quad \text{for } 1 \leqslant j < k \\ & \text{if and only if} \quad w = e_k \cdots e_1 \in \mathbb{F}^+(G). \end{aligned}$$

Given k in $\mathcal{S}_0$, so that $\alpha_k = \dim P_k \mathcal{W} \neq 0$, for $1 \leqslant j \leqslant \alpha_k$ let

$$\{\xi_w^{(j)} \equiv w(L) \xi_k^{(j)} : w \in \mathbb{F}^+(G), \, w = wk\}$$

be the standard orthonormal basis for the jth copy of the subspace $Q_k \mathcal{H}_G$ in the α_k-fold direct sum $(Q_k \mathcal{H}_G)^{(\alpha_k)}$. Next, for each k in $\mathcal{S}_0$ choose an orthonormal basis $\{\eta_k^{(j)} : 1 \leqslant j \leqslant \alpha_k\}$ for the (non-zero) subspace $P_k \mathcal{W}$. By hypothesis we have

$$\mathcal{H} = \operatorname{span}\{w(S)\eta_k^{(j)} : w \in \mathbb{F}_n^+(G), \, k \in \mathcal{S}_0, \, 1 \leqslant j \leqslant \alpha_k\}.$$

Moreover, as $\mathcal{W} = \sum_{k \in \mathcal{S}_0}^{\oplus} P_k \mathcal{W}$ is wandering for $\mathcal{S}$, the non-zero vectors in this spanning set form an orthonormal basis for $\mathcal{H}$. But we observed above that w in $\mathbb{F}_n^+(G) \setminus \mathbb{F}^+(G)$ implies $w(S) = 0$. Further, by the properties (†) we also have $w(S)\eta_k^{(j)} \neq 0$ precisely when $w = wk$ inside $\mathbb{F}^+(G)$. Hence the spanning set for $\mathcal{H}$ may be restricted to require w in $\mathbb{F}^+(G) \setminus V(G)$. Thus, it follows that we may define a unitary operator

$$U : \mathcal{H} \longrightarrow \sum_{k \in \mathcal{S}_0}^{\oplus} (Q_k \mathcal{H}_G)^{(\alpha_k)}$$

by intertwining these bases

$$\begin{cases} U(\eta_k^{(j)}) & = \xi_k^{(j)} \\ U(w(S)\eta_k^{(j)}) & = w(L)\xi_k^{(j)} \ (= \xi_w^{(j)}) \end{cases}$$

where $k \in \mathcal{S}_0$, $1 \leqslant j \leqslant \alpha_k$, and $w = wk \in \mathbb{F}^+(G) \setminus V(G)$. Evidently, this unitary jointly intertwines the operators $\{S_i\}$ with the ampliations $\Big\{ \sum_{k \in \mathcal{S}_0}^{\oplus} L_e^{(\alpha_k)}|_{(Q_k \mathcal{H}_G)^{(\alpha_k)}} \Big\}$ in the desired manner, and this completes the proof. ∎

It follows that the algebra $\mathfrak{S}_S$ is unitarily equivalent to a corresponding "weighted ampliation" of $\mathfrak{L}_G$, with weights α_k. We may regard the dimension $\alpha_k = \dim P_k\mathcal{W}$ as the *pure multiplicity over vertex* k in the Wold Decomposition for S. Observe that we are allowing for $\alpha_k = 0$; this means that no copy of the "tree component" subspace $Q_k\mathcal{H}_G$ appears in the range of the unitary U. This situation arises, for instance, if the operators $L = (L_e)_{e\in E(G)}$ themselves are restricted to the direct sum of a subset of these reducing subspaces $\{Q_k\mathcal{H}_G\}_{k\in V(G)}$. More generally, there is an interesting consequence of the lemma in the case that S is a joint restriction of some $L = (L_e)_{e\in E(G)}$ to a cyclic invariant subspace. A version of the following result for the ampliations of $\mathfrak{L}_G$ will play a key role in Sections 3 and 6.

COROLLARY 2.5. *Let $\mathcal{M} = \overline{\mathfrak{L}_G\xi}$ be a cyclic invariant subspace for $\mathfrak{L}_G$. Let $\mathcal{S} = \{k \in V(G) : P_k\xi \neq 0\}$. Then the operators $L_e|_{\mathcal{M}}$ are jointly unitarily equivalent to the operators $L_e|_{\sum^{\oplus}_{k\in\mathcal{S}}(Q_k\mathcal{H}_G)}$. In particular, the corresponding restrictions of $\mathfrak{L}_G$ are unitarily equivalent:*

$$\mathfrak{L}_G|_{\mathcal{M}} \simeq \mathfrak{L}_G|_{\sum^{\oplus}_{k\in\mathcal{S}}(Q_k\mathcal{H}_G)}. \tag{2.1}$$

Proof. Assume first that all the restrictions $S_e = L_e|_{\mathcal{M}}$ are non-zero. Then we may apply Lemma 2.4 to the tuple $(S_e)_{e\in E(G)}$. The wandering subspace here is

$$\mathcal{W} = \mathrm{Ran}\,\Big(I_{\mathcal{M}} - \sum_{e\in E(G)} S_eS_e^*\Big) = \dim\Big(\mathcal{M} \ominus \Big(\sum_{e\in E(G)}^{\oplus} L_e\mathcal{M}\Big)\Big).$$

But in this case $\mathcal{W}$ is spanned by $\{P_kP\xi = PP_k\xi : k \in V(G)\}$ where $P = I - \sum_e L_eL_e^*$. It follows that the subspaces $P_k\mathcal{W} \subseteq \mathcal{W}$ are at most one dimensional, and that $P_k\mathcal{W} \neq 0$ if and only if $P_k\xi \neq 0$. Hence the result follows from Lemma 2.4.

In the general case, Lemma 2.4 would produce a joint unitary equivalence between the non-zero operators amongst $\{S_e : e \in E(G)\}$ and the creation operators from a subgraph of G. (The subgraph obtained would consist of the saturations at all vertices in $\mathcal{S} = \{k \in V(G) : P_k\xi \neq 0\}$.) But it is easy to see that $S_e = 0$ if and only if e is not part of a path starting at some vertex in $\mathcal{S}$, and this also corresponds to the case that $L_e|_{\sum^{\oplus}_{k\in\mathcal{S}}(Q_k\mathcal{H}_G)^{(\alpha_k)}} = 0$. Thus, the unitary equivalence (2.1) holds in all situations. ∎

We now turn our attention to the general case. The families of partial isometries $L = (L_e)_{e\in E(G)}$ provide the models for pure partial isometries in the Wold Decomposition through their weighted ampliations as in Lemma 2.4. On the other hand, the "coisometric" component of the decomposition is characterized by determining a representation of a Cuntz-Krieger directed graph C^*-algebra [4], [21], [22].

DEFINITION 2.6. Let $S = (S_1, \ldots, S_n)$ satisfy (†) on $\mathcal{H}$. Then

(i) S is *fully coisometric* if equality is achieved in condition (1) of (†):

$$SS^* = \begin{bmatrix} S_1 & \cdots & S_n \end{bmatrix} \begin{bmatrix} S_1^* \\ \vdots \\ S_n^* \end{bmatrix} = \sum_{i=1}^{n} S_i S_i^* = I.$$

(ii) S is *pure* if $\mathcal{H}$ is equal to the S-invariant subspace generated by the wandering subspace $\mathcal{W} = \operatorname{Ran}\Big(I - \sum\limits_{i=1}^{n} S_i S_i^*\Big)$:

$$\mathcal{H} = \sum_{w \in \mathbb{F}_n^+}^{\oplus} w(S)\mathcal{W}.$$

Every countable directed graph G with n edges can be seen to determine a fully coisometric n-tuple $S = (S_e)_{e \in E(G)}$, where the particular (†) relations between the S_e are governed by the directed graph as in the statement of Lemma 2.4. Conversely, every fully coisometric n-tuple here can be shown to be jointly unitarily equivalent to a fully coisometric n-tuple of the form $S = (S_e)_{e \in E(G)}$, where the directed graph G is again explicitly obtained as in Lemma 2.4. (See work of Brenken [8] for discussions on this topic.) Further, we have shown in Lemma 2.4 how pure n-tuples are completely determined by tuples $L = (L_e)_{e \in E(G)}$. Thus we may prove the following.

THEOREM 2.7. *Let $S = (S_1, \ldots, S_n)$ be operators on $\mathcal{H}$ satisfying* (†). *Then $S_1, \ldots, S_n$ are jointly unitarily equivalent to the direct sum of a pure n-tuple and a fully coisometric n-tuple which both satisfy* (†). *In other words, there is a directed graph G with n edges and a unitary U such that $US_1U^*, \ldots, US_nU^*$ are of the form*

$$\{US_iU^*\}_{i=1}^{n} = \Big\{ S_e \oplus \Big(\sum_{k \in \mathcal{S}_0}^{\oplus} L_e^{(\alpha_k)} \Big|_{(Q_k \mathcal{H}_G)^{(\alpha_k)}} \Big) : e \in E(G) \Big\}. \tag{2.2}$$

Let $\mathcal{H}_p = \sum\limits_{w \in \mathbb{F}_n^+}^{\oplus} w(S)\mathcal{W}$ where $\mathcal{W} = \operatorname{Ran}\Big(I - \sum\limits_{i=1}^{n} S_i S_i^*\Big)$ and let $\mathcal{H}_c = (\mathcal{H}_p)^{\perp}$. The subspaces $\mathcal{H}_c$ and $\mathcal{H}_p$ reduce $S = (S_1, \ldots, S_n)$, and the restrictions $S_i|_{\mathcal{H}_c}$ and $S_i|_{\mathcal{H}_p}$ determine the joint unitary equivalence in (2.2).

This decomposition is unique in the sense that if $\mathcal{K}$ is a subspace of $\mathcal{H}$ which reduces $S = (S_1, \ldots, S_n)$, and if the restrictions $\{S_i|_{\mathcal{K}} : 1 \leqslant i \leqslant n\}$ are pure, respectively fully coisometric, then $\mathcal{K} \subseteq \mathcal{H}_p$, respectively $\mathcal{K} \subseteq \mathcal{H}_c$.

Proof. Since each $S_i^*\mathcal{W} = \{0\}$, it is clear that $\mathcal{H}_p$ reduces $S = (S_1, \ldots, S_n)$. Thus, the restrictions $S_i|_{\mathcal{H}_p}$ form a pure n-tuple and Lemma 2.4 shows they are jointly unitarily equivalent to an n-tuple of the desired form. On the other hand, the operators $V_i = S_i|_{\mathcal{H}_c}$ satisfy

$$I_{\mathcal{H}_c} = \sum_{i=1}^{n} V_i V_i^* = \sum_{i=1}^{n} S_i S_i^* |_{\mathcal{H}_c}.$$

Indeed, if $P = I - \sum_{i=1}^{n} S_i S_i^*$, then $P\xi$ belongs to $\mathcal{W} \subseteq \mathcal{H}_p$ for all $\xi \in \mathcal{H}$. But when $\xi \in \mathcal{H}_{\mathrm{c}}$ we have $P\xi \in \mathcal{H}_{\mathrm{c}}$ since $\mathcal{H}_{\mathrm{c}}$ reduces S, and thus $P\xi \in \mathcal{H}_{\mathrm{c}} \cap \mathcal{W} \subseteq \mathcal{H}_{\mathrm{c}} \cap \mathcal{H}_p = \{0\}$, so that $P\xi = 0$ as claimed. Hence it follows that $V = (V_1, \ldots, V_n)$ satisfies (†) and is fully coisometric, and by our remarks preceding the theorem, this n-tuple is determined by the same directed graph G as the pure part of $S = (S_1, \ldots, S_n)$.

To verify the uniqueness statement we shall prove more. Let P_{c} and P_p be, respectively, the projections of $\mathcal{H}$ onto $\mathcal{H}_{\mathrm{c}}$ and $\mathcal{H}_p$. Let $Q \in \mathcal{B}(\mathcal{H})$ be a projection such that $Q\mathcal{H}$ reduces $S = (S_1, \ldots, S_n)$. We claim that $QP_{\mathrm{c}} = P_{\mathrm{c}}Q$ and $QP_p = P_pQ$; in other words, $Q\mathcal{H} = P_{\mathrm{c}}(Q\mathcal{H}) \oplus P_p(Q\mathcal{H})$ contains both subspaces $P_{\mathrm{c}}(Q\mathcal{H})$ and $P_p(Q\mathcal{H})$. To see this, first let $\{X_k : k \geqslant 1\}$ be the projections

$$X_k = \sum_{w \in \mathbb{F}_n^+;\, |w| = k} w(S)w(S)^* = \Phi^k(I),$$

where $\Phi(A) = \sum_{i=1}^{n} S_i A S_i^*$. As $\mathcal{H}_{\mathrm{c}}$ reduces S and $\Phi(P_{\mathrm{c}}) = P_{\mathrm{c}}$, we have the restrictions $X_k|_{\mathcal{H}_{\mathrm{c}}} = I_{\mathcal{H}_{\mathrm{c}}}$ for $k \geqslant 1$. Furthermore, since $\mathcal{H}_p = \sum_{w \in \mathbb{F}_n^+}^{\oplus} w(S)\mathcal{W}$ and $w(S)^*\mathcal{W} = \{0\}$ for $w \in \mathbb{F}_n^+$, it is evident that the strong operator topology limit SOT-$\lim_{k\to\infty} X_k|_{\mathcal{H}_p} = 0$.

Now let $Q\xi = \xi = P_{\mathrm{c}}\xi + P_p\xi$ belong to $Q\mathcal{H}$. It suffices to show that both $P_{\mathrm{c}}\xi$ and $P_p\xi$ belong to $Q\mathcal{H}$. But since $Q\mathcal{H}$ is reducing for S, we have

$$P_{\mathrm{c}}\xi = \lim_{k\to\infty} X_k P_{\mathrm{c}}\xi = \lim_{k\to\infty} X_k(P_{\mathrm{c}}\xi + P_p\xi) = \lim_{k\to\infty} X_k\xi \in Q\mathcal{H}.$$

This establishes the claim and finishes the proof. ∎

REMARK 2.8. This decomposition theorem plays a role in forthcoming work of the authors on dilation theory [17]. There are a number of modern generalizations of Wold's classical theorem which appear in the literature (for instance see [15], [25], [26]). A special case of Theorem 2.7 is discussed in [20]. In its general form, Theorem 2.7 is most closely related to the Wold Decomposition of Muhly and Solel [25] established for the more abstract setting of representations of C^*-correspondences. In fact, from one point of view, Theorem 2.7 can be thought of as an explicit identification, of the pure part in particular, of the components of their Wold Decomposition in some very concrete cases.

3. PREDUAL PROPERTIES

A WOT-closed algebra $\mathfrak{A}$ has property $\mathbb{A}_n$, $1 \leqslant n \leqslant \aleph_0$, if for every $n \times n$ matrix $[\varphi_{ij}]$ of weak* continuous linear functionals on $\mathfrak{A}$, there are vectors $\{\zeta_i, \eta_j : 1 \leqslant i, j \leqslant n\}$ with

$$\varphi_{ij}(A) = (A\eta_j, \zeta_i) \quad \text{for } A \in \mathfrak{A} \text{ and } 1 \leqslant i, j \leqslant n. \tag{3.1}$$

These notions are discussed in detail in [6]. Given $1 \leqslant n \leqslant \aleph_0$, recall that the nth *ampliation* of $\mathfrak{A}$ is the WOT-closed algebra $\mathfrak{A}^{(n)}$ generated by the n-fold direct sums $A^{(n)} = A \oplus \cdots \oplus A$ where $A \in \mathfrak{A}$. The infinite ampliation of $\mathfrak{A}$, which we write as $\mathfrak{A}^{(\infty)}$, satisfies property $\mathbb{A}_{\aleph_0}$ [6], and hence property $\mathbb{A}_n$ for $n \geqslant 1$. We prove the following for $\mathfrak{L}_G$ in the case that G has no sinks.

THEOREM 3.1. *$\mathfrak{L}_G^{(n)}$ has property $\mathbb{A}_n$ for $1 \leqslant n \leqslant \aleph_0$.*

Proof. Fix $1 \leqslant n \leqslant \aleph_0$ and let $\mathfrak{A} = \mathfrak{L}_G^{(n)}$ and $\mathcal{H} = \mathcal{H}_G^{(n)}$. Let $[\varphi_{ij}]$ be an $n \times n$ matrix of weak* continuous functionals on $\mathfrak{A}$. We may regard these as functionals $\widetilde{\varphi}_{ij}$ acting on $\mathfrak{A}^{(\infty)}$ by $\widetilde{\varphi}_{ij}(A^{(\infty)}) \equiv \varphi_{ij}(A)$. Since $\mathfrak{A}^{(\infty)}$ has property $\mathbb{A}_{\aleph_0}$, hence $\mathbb{A}_n$, there are vectors $\{x_i, y_j\}_{1 \leqslant i,j \leqslant n}$ in $\mathcal{H}^{(\infty)}$ such that

$$\widetilde{\varphi}_{ij}(A^{(\infty)}) = (A^{(\infty)} x_i, y_j) = (A^{(\infty)} x_i, P_{\mathcal{M}} y_j),$$

where $P_{\mathcal{M}}$ is the projection onto the $\mathfrak{A}^{(\infty)}$-invariant subspace

$$\mathcal{M} = \bigvee_{1 \leqslant i \leqslant n} \overline{\mathfrak{A}^{(\infty)} x_i}.$$

Clearly we may assume $P_{\mathcal{M}} y_j = y_j$ for $1 \leqslant j \leqslant n$.

Let $\widetilde{L}_e$ and $\widetilde{P}_k$ be the restrictions of the generators of $\mathfrak{A}^{(\infty)}$ to this invariant subspace; that is,

$$\widetilde{L}_e = (L_e^{(n)})^{(\infty)}|_{\mathcal{M}} \text{ and } \widetilde{P}_k = (P_k^{(n)})^{(\infty)}|_{\mathcal{M}}.$$

The wandering subspace here is

$$\mathcal{W} = \operatorname{Ran}\Big(I_{\mathcal{M}} - \sum_{e \in E(G)} \widetilde{L}_e \widetilde{L}_e^*\Big) = \mathcal{M} \ominus \sum_{e \in E(G)} \widetilde{L}_e \mathcal{M}.$$

Then, as in Corollary 2.5, it follows that $\alpha_k \equiv \dim \widetilde{P}_k \mathcal{W} \leqslant n$ for k in $V(G)$. Further, if we let $\mathcal{S} = \{k \in V(G) : \widetilde{P}_k \mathcal{W} \neq 0\}$, Lemma 2.4 gives a unitary

$$U : \mathcal{M} \longrightarrow \sum_{k \in \mathcal{S}}^{\oplus} (Q_k \mathcal{H}_G)^{(\alpha_k)} \hookrightarrow \mathcal{H}$$

which may be defined so that

$$U \widetilde{L}_e U^* = \sum_{k \in \mathcal{S}}^{\oplus} L_e^{(\alpha_k)}|_{(Q_k \mathcal{H}_G)^{(\alpha_k)}}.$$

Now, given $1 \leqslant i, j \leqslant n$, let $\widetilde{x}_i$ be the vector in $\mathcal{H}$ defined by

$$P_{U\mathcal{M}}(\widetilde{x}_i) = U x_i \text{ and } P_{\mathcal{H} \ominus U\mathcal{M}}(\widetilde{x}_i) = 0,$$

and similarly define $\widetilde{y}_j$ in terms of Uy_j. Moreover, for each $A^{(\infty)}$ in $\mathfrak{A}^{(\infty)}$ let $A_{\mathcal{S}} = UA^{(\infty)}U^*$. Then given A in $\mathfrak{A} = \mathfrak{L}_G^{(n)}$ and $1 \leqslant i, j \leqslant n$, we have

$$\begin{aligned}\varphi_{ij}(A) = \widetilde{\varphi}_{ij}(A^{(\infty)}) &= (A^{(\infty)}x_i, y_j) \\ &= (U^* A_{\mathcal{S}} U x_i, y_j) \\ &= (A_{\mathcal{S}} U x_i, U y_j) = (A\widetilde{x}_i, \widetilde{y}_j).\end{aligned}$$

This shows that $\mathfrak{L}_G^{(n)}$ has property $\mathbb{A}_n$. ∎

An immediate consequence of the fact that $\mathfrak{L}_G$ satisfies $\mathbb{A}_1$ is the following. This will be valuable in Section 5.

COROLLARY 3.2. *The weak* and weak operator topologies coincide on $\mathfrak{L}_G$.*

REMARK 3.3. Applied to the case of single vertex graphs, the proof of Theorem 3.1 for $n = 1$ provides a new proof of property $\mathbb{A}_1$ for the free semigroup algebras $\mathfrak{L}_n$, $n \geqslant 2$. In fact, this proof unifies the H^∞ [6] and $\mathfrak{L}_n$ cases. The previously known proof for $\mathfrak{L}_n$ [11] of Davidson and Pitts relied on the existence of a pair of isometries with mutually orthogonal ranges in the commutant. The subclass of $\mathfrak{L}_G$ which satisfy this extra condition are discussed further below.

4. PARTLY FREE ALGEBRAS

An interesting subclass of the algebras $\mathfrak{L}_G$ discovered in [19], [20] are characterized by commutant $\mathfrak{L}_G' = \mathfrak{R}_G$ containing a pair of isometries with mutually orthogonal ranges. In the terminology of [19], [20], these are the $\mathfrak{L}_G$ with *unitally partly free* commutant, as there is a unital injection of the free semigroup algebra $\mathfrak{L}_2$ into the commutant $\mathfrak{L}_G'$ in this case.

We say that G contains a *double cycle* if there are distinct cycles $w_i = kw_ik$, $i = 1, 2$, of minimal length over some vertex k in G. By a *proper infinite (directed) path* in G, we mean an infinite path $\omega = e_{i_1}e_{i_2}e_{i_3}\cdots$ in the edges of G such that no edges are repeated and every finite segment corresponds to an allowable finite directed path in G. (Note that with our notation such an infinite path ends at the final vertex for e_{i_1}.) Say that G has the *aperiodic path property* if there exists an aperiodic infinite path; in other words, there exists a proper infinite path or a double cycle in G. Define the *attractor* at a vertex k in G to be the set consisting of k, together with all finite and infinite paths which end at k, and all vertices that are initial vertices for paths ending at k. Then G satisfies the *uniform aperiodic path entrance property* if the attractor at every vertex includes an aperiodic infinite path.

The following result is from [20] (finite case) and [19] (countably infinite case).

LEMMA 4.1. *The following assertions are equivalent for a countable directed graph G:*

(i) *G has the uniform aperiodic path entrance property;*
(ii) $\mathfrak{L}'_G$ *is unitally partly free;*
(iii) $\mathfrak{L}'_G$ *contains a pair of isometries U, V with mutually orthogonal ranges;* $U^*V = 0$.

In [19], [20] things were phrased differently. Recall that the commutant $\mathfrak{L}'_G = \mathfrak{R}_G$ is unitarily equivalent to $\mathfrak{L}_{G^{\mathrm{t}}}$, where G^{t} is the graph obtained from G by reversing directions of all directed edges. Thus, as in [19], [20], one could just as easily phrase this theorem in terms of $\mathfrak{L}_G$ being partly free, proper infinite paths defined by starting (instead of ending) at a vertex, and the graph G satisfying a corresponding exit property.

The $\mathfrak{L}_G$ satisfying the condition in Lemma 4.1 form a large class of operator algebras [19]. Indeed, the classification theorem from [20], which proved G to be a complete unitary invariant of $\mathfrak{L}_G$, shows that different graphs really do yield different algebras. Concerning the Examples from 1.3, observe that the graphs in (ii) and (v) satisfy the uniform aperiodic path entrance condition, hence $\mathfrak{L}'_G$ is unitally partly free in both cases. Further, notice that the algebra $\mathfrak{L}_{C_\infty}$ from (iv) is unitally partly free since it is unitarily equivalent to the commutant of $\mathfrak{L}_{C_\infty^{\mathrm{t}}}$.

In connection with predual properties, the importance of isometries with pairwise orthogonal ranges in the commutant of an algebra was realized in [5] and [10]. In particular, the following result was proved in [10].

LEMMA 4.2. *If $\mathfrak{A}$ is a* WOT*-closed algebra which commutes with two isometries with orthogonal ranges, then it has property $\mathbb{A}_{\aleph_0}$.*

Thus we have the following consequence of Lemma 4.1.

COROLLARY 4.3. *If G has the uniform aperiodic path entrance property, equivalently the commutant $\mathfrak{L}'_G$ is unitally partly free, then $\mathfrak{L}_G$ has property $\mathbb{A}_{\aleph_0}$.*

There is a strong partial converse of this result, to which we now turn for the rest of this section.

DEFINITION 4.4. If G is a directed graph for which the attractor at each vertex includes an infinite directed path, then we say G has the *uniform infinite path entrance property*.

The class of algebras $\mathfrak{L}_G$ generated by the graphs which satisfy this property includes, for example, all algebras satisfying the conditions of Lemma 4.1. But function algebras such as H^∞ and the cycle algebras $\mathfrak{L}_{C_n}$, $1 \leqslant n < \infty$, are also included. The following result shows that the algebras of Lemma 4.1 truly stand apart in this class. They are the only algebras which satisfy the $\mathbb{A}_{\aleph_0}$ property.

THEOREM 4.5. *Let G satisfy the uniform infinite path entrance property. Then the following assertions are equivalent:*
(i) *G has the uniform aperiodic path entrance property;*
(ii) $\mathfrak{L}'_G$ *is unitally partly free;*

(iii) $\mathfrak{L}'_G$ *contains a pair of isometries* U, V *with mutually orthogonal ranges;* $U^*V = 0$;

(iv) $\mathfrak{L}_G$ *satisfies property* $\mathbb{A}_{\aleph_0}$.

We first show that the cycle algebras do not satisfy $\mathbb{A}_{\aleph_0}$.

LEMMA 4.6. *Let* $1 \leqslant n < \infty$. *The cycle algebra* $\mathfrak{L}_{C_n}$ *does not satisfy property* $\mathbb{A}_{n+1}$.

Proof. Suppose $\mathfrak{L}_{C_n}$ satisfies $\mathbb{A}_{n+1}$. Choose a vertex k in C_n and put $\mathcal{H} = \mathcal{H}_{C_n}$. Then, since P_k belongs to $\mathfrak{L}_{C_n}$, the compression algebra $\mathfrak{A} = P_k\mathfrak{L}_{C_n}P_k|_{P_k\mathcal{H}}$ has property $\mathbb{A}_{n+1}$ as well. To see this, suppose φ is a functional on $\mathfrak{A}$. For $A \in \mathfrak{L}_{C_n}$, let $A_k = P_kAP_k|_{P_k\mathcal{H}}$ and define a functional on $\mathfrak{L}_{C_n}$ by $\widetilde{\varphi}(A) \equiv \varphi(A_k)$. If $\widetilde{\varphi}$ can be realized as a vector functional, $\widetilde{\varphi}(A) = (A\xi, \eta)$, then

$$\varphi(A_k) = \varphi(P_kA_kP_k|_{P_k\mathcal{H}}) = \widetilde{\varphi}(P_kA_kP_k) = (A_kP_k\xi, P_k\eta).$$

Let $w \in \mathbb{F}^+(G)$ be the cycle of minimal length in C_n with $w = kwk$. A consideration of Fourier expansions for elements of $\mathfrak{L}_{C_n}$ reveals that $P_k\mathfrak{L}_{C_n}P_k$ is the subalgebra of $\mathfrak{L}_{C_n}$ given by

$$P_k\mathfrak{L}_{C_n}P_k = \text{WOT–Alg}\{L_w, P_k\}.$$

Moreover, $V = P_kL_wP_k|_{P_k\mathcal{H}} = L_w|_{P_k\mathcal{H}}$ is unitarily equivalent to the canonical unilateral shift operator of multiplicity n. Indeed, P_k commutes with L_w and V is a pure isometry with

$$\text{rank}(I - VV^*) = \text{rank}(P_k(I - L_wL_w^*)P_k) = n.$$

(One multiplicity is picked up for each of the n infinite stalks in the Fock space tree.) Thus, $\mathfrak{A} = \text{WOT}-\text{Alg}\{V, I_{P_k\mathcal{H}}\}$ does not have property $\mathbb{A}_{n+1}$ since the shift of multiplicity n does not satisfy $\mathbb{A}_{n+1}$ [6]. This contradiction shows that $\mathfrak{L}_{C_n}$ does not satisfy property $\mathbb{A}_{n+1}$. ∎

Proof of Theorem 4.5. It follows from Lemma 4.1 and Corollary 4.3 that it suffices to prove (iv)⇒(i). Thus, suppose $\mathfrak{L}_G$ satisfies $\mathbb{A}_{\aleph_0}$. Given a subset $\mathcal{S} \subseteq V(G)$ of vertices in G, let $P_{\mathcal{S}}$ be the projection in $\mathfrak{L}_G$ defined by $P_{\mathcal{S}} = \sum_{k\in\mathcal{S}} P_k$. Observe that every compression algebra $P_{\mathcal{S}}\mathfrak{L}_GP_{\mathcal{S}}|_{P_{\mathcal{S}}\mathcal{H}}$ satisfies $\mathbb{A}_{\aleph_0}$ since $P_{\mathcal{S}}$ belongs to $\mathfrak{L}_G$; every functional φ on $P_{\mathcal{S}}\mathfrak{L}_GP_{\mathcal{S}}|_{P_{\mathcal{S}}\mathcal{H}}$ can be extended to $\mathfrak{L}_G$ by defining $\widetilde{\varphi}(A) \equiv \varphi(P_{\mathcal{S}}AP_{\mathcal{S}}|_{P_{\mathcal{S}}\mathcal{H}})$ as in the previous proof.

Now suppose (i) does not hold. Then there is a vertex $k \in V(G)$ such that the attractor at k contains no aperiodic infinite path. In other words, there are no paths leaving double cycles for k, and there are no infinite non-overlapping directed paths which end at k. Hence, the uniform infinite path entrance property tells us there is a path from a cycle into k. Furthermore, by moving backwards along the paths which enter k, we can find a cycle C from which there is a path into k with the following properties: The only edges that enter the vertices in C are the edges which make up the cycle, and the cardinality n of the vertex set is equal to the number of edges in the cycle. In other words, there are no multiple edges between vertices in C (using the fact that (i) fails), and there are no directed paths from vertices outside C to vertices inside C (using the uniform infinite path entrance property). Let $\mathcal{S}$ be the collection of vertices in C, so that $|\mathcal{S}| = n$. It follows from the choice of C that the algebra $P_{\mathcal{S}}\mathfrak{L}_G P_{\mathcal{S}}|_{P_{\mathcal{S}}\mathcal{H}}$ is unitarily equivalent to $\mathfrak{L}_{C_n}$. This gives a contradiction to Lemma 4.6, and hence G must in fact satisfy (i). ∎

REMARK 4.7. We wonder whether the initial restriction to G satisfying the uniform infinite path entrance property is really necessary in Theorem 4.5. It may simply be a convenient technical assumption. It seems plausible to us that Theorem 4.5 could hold without making this initial restriction on G. We would also expect that the equivalent conditions in this result could be extended to include the related factorization properties $\mathbb{A}_n(n^2)$ and $X_{0,1}$ [5], [10], which involve norm control over the vectors chosen, and perhaps even property $\mathbb{A}_2$.

5. IDEALS AND INVARIANT SUBSPACES

In this section we give a detailed description of the WOT-closed ideal structure for $\mathfrak{L}_G$. The key ingredients in our analysis are the Beurling Theorem 1.2 for $\mathfrak{L}_G$ and the $\mathbb{A}_1$ property for functionals established in Theorem 3.1. In [11], Davidson and Pitts described the ideal structure for free semigroup algebras $\mathfrak{L}_n$. For the sake of continuity in the literature, our presentation in this section will mirror their approach whenever possible.

DEFINITION 5.1. Given a directed graph G, let $\mathrm{Id}_{\mathrm{r}}(\mathfrak{L}_G)$ and $\mathrm{Id}(\mathfrak{L}_G)$ denote, respectively, the sets of all WOT-closed right and two-sided ideals. Note that $\mathrm{Id}_{\mathrm{r}}(\mathfrak{L}_G)$ and $\mathrm{Id}(\mathfrak{L}_G)$ form complete lattices under the operations of intersections and WOT-closed sums.

To streamline the presentation, we shall let ξ_ϕ be the vector given by the following weighted sum of the vacuum vectors: $\xi_\phi = \sum\limits_{k\in V(G)} \frac{1}{k}\,\xi_k$. This sum is finite precisely when G has finitely many vertices. Observe that for all words $w \in \mathbb{F}^+(G)$ we have $L_w\xi_\phi = \frac{1}{k}\xi_w$, where $w = wk$.

If $\mathcal{J}$ belongs to $\mathrm{Id}_{\mathrm{r}}(\mathfrak{L}_G)$, then the subspace

$$\overline{\mathcal{J}\xi_\phi} = \overline{\mathcal{J}\mathfrak{L}_G\xi_\phi} = \overline{\mathcal{J}\mathcal{H}_G} = \overline{\mathcal{J}\mathfrak{R}_G\xi_\phi} = \mathfrak{R}_G\overline{\mathcal{J}\xi_\phi}.$$

Thus, the range subspace of $\mathcal{J}$ satisfies $\overline{\mathcal{J}\mathcal{H}_G} = \overline{\mathcal{J}\xi_\phi}$ and is $\mathfrak{R}_G$-invariant. If, in addition, $\mathcal{J}$ belongs to $\mathrm{Id}(\mathfrak{L}_G)$, then $\mathfrak{L}_G\mathcal{J}\xi_\phi = \mathcal{J}\xi_\phi$, and $\overline{\mathcal{J}\xi_\phi}$ is $\mathfrak{L}_G$-invariant. Hence $\overline{\mathcal{J}\xi_\phi}$ belongs to both $\mathrm{Lat}(\mathfrak{L}_G)$ and $\mathrm{Lat}(\mathfrak{R}_G)$ when $\mathcal{J}$ is a two-sided ideal.

On the other hand, if $\mathcal{M}$ belongs to $\mathrm{Lat}(\mathfrak{R}_G)$, it follows that the set $\{A \in \mathfrak{L}_G : A\xi_\phi \in \mathcal{M}\}$ is contained in $\mathrm{Id}_{\mathrm{r}}(\mathfrak{L}_G)$. Indeed, this set is clearly WOT-closed and for $X \in \mathfrak{L}_G$,

$$AX\xi_\phi \in \overline{A\mathcal{H}_G} = \overline{A\mathfrak{R}_G\xi_\phi} = \overline{\mathfrak{R}_G A\xi_\phi} \subseteq \mathcal{M}.$$

Furthermore, if $\mathcal{M}$ is also $\mathfrak{L}_G$-invariant, then this set evidently forms a two-sided ideal.

The following theorem shows that ideals in $\mathrm{Id_r}(\mathfrak{L}_G)$ and $\mathrm{Id}(\mathfrak{L}_G)$ are fully determined by their ranges.

THEOREM 5.2. *Let* $\mu : \mathrm{Id_r}(\mathfrak{L}_G) \to \mathrm{Lat}(\mathfrak{R}_G)$ *be defined by* $\mu(\mathcal{J}) = \overline{\mathcal{J}\xi_\phi}$*. Then* μ *is a complete lattice isomorphism. The restriction of* μ *to the set* $\mathrm{Id}(\mathfrak{L}_G)$ *is a complete lattice isomorphism onto* $\mathrm{Lat}(\mathfrak{L}_G) \cap \mathrm{Lat}(\mathfrak{R}_G)$*. The inverse map* ι *sends a subspace* $\mathcal{M}$ *to*

$$\iota(\mathcal{M}) = \{A \in \mathfrak{L}_G \ : \ A\xi_\phi \in \mathcal{M}\}.$$

Proof. We have observed above that the maps μ and ι map into the correct subspace lattice and ideal lattice respectively.

We show first that $\mu\iota$ is the identity map. Let $\mathcal{M}$ belong to $\mathrm{Lat}(\mathfrak{R}_G)$. It is clear that $\mu\iota(\mathcal{M})$ is contained in $\mathcal{M}$. Conversely, let $\{\xi_{k,j}\}_{k,j}$ be an orthonormal basis for the $\mathfrak{R}_G$-wandering subspace $\mathcal{W} = \mathcal{M} \ominus \sum_e^{\oplus} R_e\mathcal{M}$, with $Q_k\xi_{k,j} = \xi_{k,j}$. Then from the Beurling Theorem (the $\mathfrak{R}_G$ version) we have

$$\mathcal{M} = \sum_{j,k}^{\oplus} \mathfrak{R}_G[\xi_{k,j}] = \sum_{j,k}^{\oplus} \mathrm{Ran}(L_{\xi_{k,j}}).$$

As $L^*_{\xi_{k,j}} L_{\xi_{k,j}} = P_k$, it follows that $L_{\xi_{k,j}}\xi_\phi = \frac{1}{k}\,\xi_{k,j}$ is in $\mathcal{M}$ and $L_{\xi_{k,j}}$ belongs to $\iota(\mathcal{M})$. Hence

$$\mathcal{M} = \sum_{j,k}^{\oplus} \mathrm{Ran}(L_{\xi_{k,j}}) \subseteq \overline{\iota(\mathcal{M})\mathcal{H}_G} = \overline{\iota(\mathcal{M})\xi_\phi} = \mu(\iota(\mathcal{M})).$$

Thus, $\mu\iota(\mathcal{M}) = \mathcal{M}$, as required.

To see that $\iota\mu$ is the identity, fix $\mathcal{J}$ in $\mathrm{Id_r}(\mathfrak{L}_G)$ and let $\mathcal{M} = \mu(\mathcal{J})$. It is clear from the definitions that $\mathcal{J}$ is contained in $\iota\mu(\mathcal{J})$. We first show that for every ξ in $\mathcal{H}_G$,

$$\overline{\mathcal{J}\xi} = \overline{\iota\mu(\mathcal{J})\xi}. \tag{5.1}$$

From the Beurling Theorem, the cyclic $\mathfrak{L}_G$-invariant subspace $\mathfrak{L}_G[\xi]$ decomposes as $\mathfrak{L}_G[\xi] = \sum_{k\in\mathcal{S}}^{\oplus} R_{\zeta_k}\mathcal{H}_G$, where $\mathcal{S} = \{k \in V(G) : P_k\xi \neq 0\}$, the vectors $\zeta_k = P_k\zeta_k$ are $\mathfrak{L}_G$-wandering, and the R_{ζ_k} are partial isometries in $\mathfrak{R}_G$ with pairwise orthogonal ranges. It follows that

$$\overline{\mathcal{J}\xi} = \overline{\mathcal{J}\mathfrak{L}_G\xi} = \overline{\mathcal{J}\sum_{k\in\mathcal{S}}^{\oplus} R_{\zeta_k}\mathcal{H}_G} = \overline{\sum_{k\in\mathcal{S}}^{\oplus} R_{\zeta_k}\mathcal{J}\mathcal{H}_G} = \sum_{k\in\mathcal{S}}^{\oplus} R_{\zeta_k}\mathcal{M}.$$

But since $\mu\iota$ is the identity we have $\mu(\iota\mu(\mathcal{J})) = \mu(\mathcal{J}) = \mathcal{M}$, hence the same computation for $\overline{\iota\mu(\mathcal{J})\xi}$ yields the same result. This yields (5.1).

Next suppose that φ is a WOT-continuous linear functional on $\mathfrak{L}_G$ which annihilates the ideal $\mathcal{J}$. By Theorem 3.1, there are vectors ξ, η in $\mathcal{H}_G$ with $\varphi(A) =$

$(A\xi, \eta)$ for A in $\mathfrak{L}_G$. Since $\varphi(\mathcal{J}) = 0$, the vector η is orthogonal to $\overline{\mathcal{J}\xi} = \overline{\iota\mu(\mathcal{J})\xi}$. Hence φ also annihilates $\iota\mu(\mathcal{J})$. Thus, since the weak* and WOT topologies on $\mathfrak{L}_G$ coincide, we have $\mathcal{J} = \iota\mu(\mathcal{J})$ by the Hahn-Banach Theorem.

Thus we have shown that μ establishes a bijective correspondence between elements of $\mathrm{Id}_{\mathrm{r}}(\mathfrak{L}_G)$ and $\mathrm{Lat}(\mathfrak{R}_G)$ which maps $\mathrm{Id}(\mathfrak{L}_G)$ onto $\mathrm{Lat}(\mathfrak{L}_G) \cap \mathrm{Lat}(\mathfrak{R}_G)$, and that $\iota = \mu^{-1}$. It is elementary to verify that μ and ι are complete lattice isomorphisms, and, since this may be accomplished in exactly the same way as for the free semigroup algebras $\mathfrak{L}_n$ [11], we leave these remaining details to the interested reader. ∎

Let us apply the theorem to the case of singly generated ideals.

COROLLARY 5.3. *Let A belong to $\mathfrak{L}_G$. Then the* WOT*-closed two-sided ideal generated by A is given by* $\{X \in \mathfrak{L}_G : X\xi_\phi \in \overline{\mathfrak{L}_G A \mathcal{H}_G}\}$.

An interesting special case occurs when $A = L_w$ is a partial isometry coming from a word w in $\mathbb{F}^+(G)$. The corresponding ideal is easily described in terms of Fourier expansions.

COROLLARY 5.4. *For w in $\mathbb{F}^+(G)$, the* WOT*-closed two-sided ideal generated by L_w is given by*

$$\{X \in \mathfrak{L}_G : (X\xi_\phi, \xi_v) \neq 0 \textit{ only if } v = u_1 w u_2;\ u_1, u_2 \in \mathbb{F}^+(G)\}.$$

The theorem also leads to a simple characterization of the WOT-closure of the commutator ideal of $\mathfrak{L}_G$. Define the *G-symmetric Fock space* to be the subspace of $\mathcal{H}_G$ spanned by the vectors $\sum_{\sigma \in S_r} \xi_{\sigma(w)}$, where $w \in \mathbb{F}^+(G)$ with $|w| = r$, S_r is the symmetric group on r letters, and $\sigma(w)$ is the word with letters in w permuted by σ. We put $\xi_{\sigma(w)} = 0$ and $L_{\sigma(w)} = 0$ when $\sigma(w)$ is not an allowable finite path in G. The terminology here is motivated by the case of a trivial graph with one vertex and a number of loop edges. For a general graph though, there may be very little "symmetry" associated with $\mathcal{H}_G$ (see Example 5.6 below).

By considering Fourier expansions it is not hard to see that the linear span of the commutators of the form $[L_v, L_w] = L_v L_w - L_w L_v$ for v, w in $\mathbb{F}^+(G)$ is WOT-dense in the WOT-closure of the commutator ideal of $\mathfrak{L}_G$. But a simple exercise in combinatorics shows that each of these commutators is determined by elementary commutators. For instance, given edges e, f, g in $E(G)$ observe that

$$[L_{ef}, L_g] = L_e[L_f, L_g] + [L_e, L_g]L_f.$$

It follows that the WOT-closure of the commutator ideal and its range subspace have the following form.

COROLLARY 5.5. *The* WOT*-closure of the commutator ideal of $\mathfrak{L}_G$ is the* WOT*-closed two-sided ideal given by*

$$\begin{aligned}\overline{\mathfrak{C}} &= \langle [L_v, L_w] : v, w \in \mathbb{F}^+(G)\rangle \\ &= \langle [L_e, L_f], [L_e, P_k] : e, f \in E(G), k \in V(G)\rangle.\end{aligned}$$

The corresponding range subspace in $\mathrm{Lat}(\mathfrak{L}_G) \cap \mathrm{Lat}(\mathfrak{R}_G)$ *is*

$$\mu(\overline{\mathfrak{C}}) = (\mathcal{H}_G^{\mathrm{s}})^{\perp} = \mathrm{span}\{\xi_{uefv} - \xi_{ufev} : e \neq f,\, u, v \in \mathbb{F}^+(G)\}.$$

Notice that an elementary commutator will typically collapse; for instance, $L_eL_f = 0$ if the final vertex of f is different than the initial vertex of e. In fact, L_eL_f and L_fL_e are both non-zero precisely when ef (and fe) forms a cycle in G. The following class of examples differ greatly from the algebras $\mathfrak{L}_n$ and H^∞.

EXAMPLE 5.6. Let G be a directed graph with no cycles; that is, no paths with the same initial and final vertices. Then every commutator $[L_e, L_f] = L_eL_f - L_fL_e$ with $e, f \in E(G)$ is equal to L_eL_f, $-L_fL_e$, or 0. Further, since there are no loop edges,

$$[L_e, P_k] = \begin{cases} L_e & \text{if } e = ek, \\ -L_e & \text{if } e = ke, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by Corollary 5.5, the WOT-closure of the commutator ideal is equal to the WOT-closed ideal generated by the L_e,

$$\overline{\mathfrak{C}} = \langle L_e : e \in E(G)\rangle,$$

and the range is given by,

$$\mu(\overline{\mathfrak{C}}) = (\mathcal{H}_G^{\mathrm{s}})^{\perp} = \mathrm{span}\{\xi_w : w \in \mathbb{F}^+(G) \setminus V(G)\} = \mathcal{H}_G \ominus \mathrm{span}\{\xi_k : k \in V(G)\}.$$

It also follows that $\mathfrak{L}_G/\overline{\mathfrak{C}}$ is completely isometrically isomorphic to the algebra $\mathrm{span}\{P_k : k \in V(G)\}$ in this case (see Section 6).

We finish this section with an investigation into factorization in right ideals.

LEMMA 5.7. *Let $\{L_{\zeta_j} : 1 \leqslant j \leqslant s\}$ be a finite set of partial isometries in $\mathfrak{L}_G$ with pairwise orthogonal ranges $\mathcal{M}_j$. Let $\mathcal{M} = \sum_{j=1}^{s}{}^{\oplus}\mathcal{M}_j$ and $\mathcal{J} = \iota(\mathcal{M})$. Then $\mathcal{J} = \{A \in \mathfrak{L}_G : \mathrm{Ran}(A) \subseteq \mathcal{M}\}$ and every element A of $\mathcal{J}$ factors as*

$$A = \sum_{j=1}^{s} L_{\zeta_j} A_j \quad \textit{with } A_j \in \mathfrak{L}_G.$$

In particular, the algebraic right ideal generated by the finite set $\{L_{\zeta_j} : 1 \leqslant j \leqslant s\}$ coincides with $\mathcal{J}$.

In the case of a countably infinite set of partial isometries $\{L_{\zeta_j} : j \geqslant 1\}$ in $\mathfrak{L}_G$ with pairwise orthogonal ranges, every element A of $\mathcal{J}$ (which is not a finitely generated *algebraic ideal in this case) factors as a* WOT*-convergent sum*

$$A = \text{WOT-}\sum_{j\geqslant 1} L_{\zeta_j} A_j \quad \textit{with } A_j \in \mathfrak{L}_G.$$

Proof. We shall focus on the case where the L_{ζ_j} form a finite set. The countable case is easily obtained from this analysis.

Each $\mathcal{M}_j$ is $\mathfrak{R}_G$-invariant, hence so is $\mathcal{M}$. Thus, if A in $\mathfrak{L}_G$ satisfies $A\xi_\phi \in \mathcal{M}$, then $\operatorname{Ran}(A) = \overline{A\mathcal{H}_G}$ is contained in $\mathcal{M}$, and we have

$$\mathcal{J} = \{A \in \mathfrak{L}_G : \operatorname{Ran}(A) \subseteq \mathcal{M}\}.$$

This shows that $\mathcal{J}$ is a WOT-closed right ideal containing $\{L_{\zeta_j} : 1 \leqslant j \leqslant s\}$.

On the other hand, since the projection onto $\mathcal{M}$ is given by $P_{\mathcal{M}} = \sum_{j=1}^{s} L_{\zeta_j} L_{\zeta_j}^*$, for A in $\mathcal{J}$ we have

$$A = \Big(\sum_{j=1}^{s} L_{\zeta_j} L_{\zeta_j}^*\Big) A = \sum_{j=1}^{s} L_{\zeta_j} A_j,$$

where $A_j = L_{\zeta_j}^* A$. We finish the proof by showing that each A_j is in $\mathfrak{L}_G$. First note that A_j clearly commutes with the projections $Q_k \in \mathfrak{R}_G = \mathfrak{L}_G'$. Further, as $\mathfrak{R}_G$-wandering vectors for $\mathcal{M}$, each ζ_j is orthogonal to $\sum_e^{\oplus} R_e(\operatorname{Ran}(A)) \subseteq \sum_e^{\oplus} R_e\mathcal{M}$. Thus for $w \in \mathbb{F}^+(G)$, we have $(R_e^* A^* \zeta_j, \xi_w) = (\zeta_j, R_e A \xi_w) = 0$, and hence $R_e^* A^* \zeta_j = 0$. Whence, using Lemma 11.1 of [20], given $e = ek$ in $E(G)$ we have

$$\begin{aligned} A_j R_e - R_e A_j &= L_{\zeta_j}^* A R_e - R_e L_{\zeta_j}^* A = (L_{\zeta_j}^* R_e - R_e L_{\zeta_j}^*) A \\ &= k^2 (\xi_\phi (R_e^* L_{\zeta_j} \xi_\phi)^*) A = k(\xi_\phi (A^* R_e^* \zeta_j)^*) = k(\xi_\phi (R_e^* A^* \zeta_j)^*) = 0. \end{aligned}$$

Therefore A_j belongs to $\mathfrak{R}_G' = \mathfrak{L}_G$, as required. ∎

A special case of the lemma concerns the two-sided ideals in $\mathfrak{L}_G$ generated by the partial isometries from paths of a given length, $\{L_w : w \in \mathbb{F}^+(G), |w| = s\}$.

COROLLARY 5.8. *For $s \geqslant 1$, every A in $\mathfrak{L}_G$ can be written as a sum*

$$A = \sum_{|w|<s} a_w L_w + \sum_{|w|=s} L_w A_w,$$

where $a_w \in \mathbb{C}$ and $A_w \in \mathfrak{L}_G$ for $w \in \mathbb{F}^+(G)$. In the case that there are infinitely many paths of a given length, $\{a_w\}_{|w|<s}$ belongs to ℓ^2 and the sums are WOT*-convergent.*

We mention that the previous two results applied to the $\mathfrak{L}_n$ case include a uniqueness of factorization. For the general $\mathfrak{L}_G$ case this uniqueness does not hold; ostensibly because the generators here are partial isometries instead of isometries. However, the elements A_j in $\mathfrak{L}_G$ from Lemma 5.7 can be chosen uniquely under the extra constraint $L_{\zeta_j}^* L_{\zeta_j} A_j = A_j$.

Together with Theorem 5.2, the previous lemma may be used to describe precisely when a right ideal is finitely generated. The following result generalizes Theorem 2.10 of [11].

THEOREM 5.9. *Let $\mathcal{J}$ be a* WOT*-closed right ideal in $\mathfrak{L}_G$. Let $\mathcal{M} = \mu(\mathcal{J})$ in* $\mathrm{Lat}(\mathfrak{R}_G)$ *and let $\mathcal{W}$ be the $\mathfrak{R}_G$-wandering subspace for $\mathcal{M}$. If the sum of the dimensions of the wandering subspaces $\{Q_k\mathcal{W} : k \in V(G)\}$ is finite, $s < \infty$, then $\mathcal{J}$ is generated by s partial isometries with pairwise orthogonal ranges as an algebraic right ideal. When this wandering dimension is infinite, $s = \infty$, $\mathcal{J}$ is not finitely generated as a* WOT*-closed right ideal, but it is generated by countably many partial isometries as a* WOT*-closed right ideal.*

REMARK 5.10. There is no analogous structure for the WOT-closed left ideals of $\mathfrak{L}_G$. While some partial results go through, there are factorization pathologies in left ideals. Indeed, this was discovered in [18] for the case of free semigroup algebras $\mathfrak{L}_n$. For instance, the algebraic left ideal determined by a generator of $\mathfrak{L}_n$ is not even norm closed. The basic point is that these generators have proper factorizations inside the algebra, a property which is exclusive to the noncommutative setting.

We conclude this section with a comment on related work in the literature.

REMARK 5.11. The results on ideals of $\mathfrak{L}_G$ presented in this section generalize the characterization of ideals in H^∞ [13], [14], as well as work of Davidson and Pitts [12] in the case of free semigroup algebras $\mathfrak{L}_n$. In their work on quiver algebras, Muhly and Solel [25] briefly considered the ideal structure for $\mathfrak{L}_G$, obtaining a version of Theorem 5.2 in the special case that G is a finite graph and satisfies a certain *entrance condition*. We mention that their condition can be seen to be equivalent to the finite graph case of the *uniform aperiodic path entrance property* from Theorem 4.1. Thus, by Theorem 4.1, Muhly and Solel actually established the lattice isomorphism theorem in the case that G is finite and $\mathfrak{L}_G$ has unitally partly free commutant, and hence our result is an extension of theirs to the general $\mathfrak{L}_G$ case when G has no sinks.

6. DISTANCE FORMULA TO IDEALS

Let $\mathcal{M}_n(\mathfrak{L}_G)$ denote the algebra of $n \times n$ matrices with entries in $\mathfrak{L}_G$, equipped with the operator norm in $\mathcal{B}(\mathcal{H}_G)^{(n)}$. We prove the following distance formula to ideals in $\mathcal{M}_n(\mathfrak{L}_G)$.

THEOREM 6.1. *Let $\mathcal{J}$ be a* WOT*-closed right ideal of $\mathfrak{L}_G$, and let $\mathcal{M} = \overline{\mathcal{J}\mathcal{H}_G} = \overline{\mathcal{J}\xi_\phi}$ be the range subspace in* $\mathrm{Lat}(\mathfrak{R}_G)$. *Then for every A in $\mathcal{M}_n(\mathfrak{L}_G)$, we have*

$$\mathrm{dist}(A, \mathcal{M}_n(\mathcal{J})) = \|(P_{\mathcal{M}}^{\perp} \otimes I_n)A\|.$$

REMARK 6.2. Our proof is based on McCullough's distance formula for ideals in dual algebras [23]. We shall proceed by establishing a variant of this result (Lemma 6.3), then combine it with the Wold Decomposition Lemma 2.4 to obtain the distance formula. Note that Lemma 6.3 does not use the fact that $\mathfrak{L}_G$ is WOT-closed, only that it is weak* closed. One could prove the distance formula using the results of Section 3 (thus using the Wold decomposition implicitly, and avoiding Lemma 6.3), but we have arranged the proof this way to emphasize the role of the Wold decomposition. The proof we give reduces to that given by McCullough for H^∞, and is similar in flavour to the Arias-Popescu proof in the case of free semigroup algebras $\mathfrak{L}_n$ [2].

We first fix some notation. Let $1 \leqslant n \leqslant \aleph_0$. Throughout this section, Z will denote a positive trace class operator in $\mathcal{B}(\mathcal{H}_G^{(n)})$, factored as $Z = \sum_{i\geqslant 1} z_i z_i^*$, with each z_i in $\mathcal{H}_G^{(n)}$. Let $\widetilde{z} = [z_1\, z_2\, \cdots]^{\mathrm{t}}$ denote the corresponding vector in the infinite direct sum $\mathcal{H}_G^{(\infty)}$. Note that $\|\widetilde{z}\|^2 = \sum_{i\geqslant 1} \|z_i\|^2 = \mathrm{tr}(Z)$. We will let $\mathfrak{A} = \mathcal{M}_n(\mathfrak{L}_G)$, and for a WOT-closed right ideal $\mathcal{J}$ in $\mathfrak{L}_G$, we let $\mathfrak{A}_\mathcal{J} = \mathcal{M}_n(\mathcal{J})$.

For a given positive trace class operator Z on $\mathcal{H}_G^{(n)}$, let

$$\mathcal{M}(Z) = \overline{\mathfrak{A}^{(\infty)}\widetilde{z}} \quad \text{and} \quad \mathcal{N}(Z) = \overline{\mathfrak{A}_\mathcal{J}^{(\infty)}\widetilde{z}}\,,$$

and let $P_{\mathcal{M}(Z)}$ and $P_{\mathcal{N}(Z)}$ be the projections onto $\mathcal{M}(Z)$ and $\mathcal{N}(Z)$ respectively.

LEMMA 6.3. *For $A \in \mathfrak{A}$,*

$$\mathrm{dist}(A, \mathfrak{A}_\mathcal{J}) = \sup_Z \|P_{\mathcal{N}(Z)}^\perp A^{(\infty)} P_{\mathcal{M}(Z)}\|$$

where the supremum is taken over all positive trace class operators Z in $\mathcal{B}(\mathcal{H}_G^{(n)})$.

Proof. We must show that

$$\inf_{B-A\in\mathfrak{A}_\mathcal{J}} \|B\| = \sup_Z \|P_{\mathcal{N}(Z)}^\perp A^{(\infty)} P_{\mathcal{M}(Z)}\|;$$

the left-hand side being the definition of $\mathrm{dist}(A, \mathfrak{A}_\mathcal{J})$. We first prove the distance is bounded above by this supremum.

Since $\mathfrak{A}$ is weak* closed, it is isometrically isomorphic to the dual of a Banach space; namely

$$\mathfrak{A}_* = \mathcal{C}_1(\mathcal{H}_G^{(n)}) \ / \ \mathfrak{A}_\perp$$

where $\mathfrak{A}_\perp$ consists of the trace-class operators $T \in \mathcal{C}_1(\mathcal{H}_G^n)$ such that $\mathrm{tr}(AT) = 0$ for every $A \in \mathfrak{A}$. Let π denote the quotient map

$$\pi : \mathcal{C}_1(\mathcal{H}_G^{(n)}) \longrightarrow \mathfrak{A}_*\,.$$

Then the action of an element $A \in \mathfrak{A}$ on the predual $\mathfrak{A}_*$ is given by

$$L_A(F) \equiv \mathrm{tr}(A\widetilde{F})$$

where $\widetilde{F}$ is any representative of the coset $F \in \mathfrak{A}_*$. Let $\mathfrak{A}_{\mathcal{J}\perp}$ denote the preannihilator of $\mathfrak{A}_{\mathcal{J}}$ in the predual $\mathfrak{A}_*$; that is,

$$\mathfrak{A}_{\mathcal{J}\perp} = \{F \in \mathfrak{A}_* : L_A(F) = 0 \text{ for all } A \in \mathfrak{A}_{\mathcal{J}}\}.$$

Given $F \in \mathfrak{A}_*$ and $\varepsilon > 0$, there exists a trace class operator Y on $\mathcal{H}_G^{(n)}$ such that $\pi(Y) = F$, and $\|Y\|_1 < \|F\| + \varepsilon$. Let $Y^* = VZ$ be the polar decomposition of Y^*, with V a partial isometry and $Z = (YY^*)^{1/2}$. For this Z, let $\widetilde{z}$ be as above. Then given $A \in \mathfrak{A}$,

$$(A^{(\infty)}\widetilde{z}, V^{*(\infty)}\widetilde{z}) = \mathrm{tr}(AZV^*) = \mathrm{tr}(AY) = L_A(F).$$

Thus, when $A \in \mathfrak{A}_{\mathcal{J}}$, we have $(A^{(\infty)}\widetilde{z}, V^{*(\infty)}\widetilde{z}) = 0$.

Now, since $\mathfrak{A}_{\mathcal{J}}^{(\infty)}\widetilde{z}$ is dense in $\mathcal{N}(Z)$, it follows that

$$\mathrm{Ran}(P_{\mathcal{M}(Z)}V^{*(\infty)}P_{\mathcal{M}(Z)}) \subseteq \mathcal{M}(Z) \ominus \mathcal{N}(Z).$$

We now compute, for any $A \in \mathfrak{A}_{\mathcal{J}}$,

$$\begin{aligned} L_A(F) &= (A^{(\infty)}\widetilde{z}, V^{*(\infty)}\widetilde{z}) \\ &= (A^{(\infty)}\widetilde{z}, P_{\mathcal{M}(Z)}V^{*(\infty)}P_{\mathcal{M}(Z)}\widetilde{z}) \\ &= (A^{(\infty)}\widetilde{z}, P_{\mathcal{N}(Z)}^{\perp}P_{\mathcal{M}(Z)}V^{*(\infty)}P_{\mathcal{M}(Z)}\widetilde{z}) \\ &= (P_{\mathcal{N}(Z)}^{\perp}A^{(\infty)}\widetilde{z}, P_{\mathcal{M}(Z)}V^{*(\infty)}P_{\mathcal{M}(Z)}\widetilde{z}). \end{aligned}$$

Thus by the Cauchy-Schwarz inequality,

$$|L_A(F)| \leqslant \|P_{\mathcal{N}(Z)}^{\perp}A^{(\infty)}P_{\mathcal{M}(Z)}\|\|V^{*(\infty)}\widetilde{z}\|\|\widetilde{z}\|.$$

As V is a partial isometry, $\|V^{*(\infty)}\widetilde{z}\| \leqslant \|\widetilde{z}\|$ and thus

$$\|\widetilde{z}\|^2 = \mathrm{tr}(Z) = \|Y\| < \|F\| + \varepsilon,$$

implies (since ε was arbitrary) that

$$\begin{aligned} |L_A(F)| &\leqslant \|P_{\mathcal{N}(Z)}^{\perp}A^{(\infty)}P_{\mathcal{M}(Z)}\|\|\widetilde{z}\|^2 \\ &\leqslant \Big(\sup_Z \|P_{\mathcal{N}(Z)}^{\perp}A^{(\infty)}P_{\mathcal{M}(Z)}\|\Big)\|F\|. \end{aligned}$$

By the Hahn-Banach theorem, the functional L_A on $\mathfrak{A}_{\mathcal{J}\perp}$ extends to a functional L on all of $\mathfrak{A}_*$ with

$$\|L\| \leqslant \sup_Z \|P_{\mathcal{N}(Z)}^{\perp}A^{(\infty)}P_{\mathcal{M}(Z)}\|$$

and $L(F) = L_A(F)$ for $F \in \mathfrak{A}_{\mathcal{J}\perp}$.

Since $(\mathfrak{A}_*)^* = \mathfrak{A}$, there exists an operator B in $\mathfrak{A}$ such that $\|B\| = \|A\|$ and

$$\begin{aligned} L(F) &= L_B(F) \quad \text{for } F \in \mathfrak{A}_*, \\ L_A(F) &= L_B(F) \quad \text{for } F \in \mathfrak{A}_{\mathcal{J}\perp}. \end{aligned}$$

In particular, this means that $B - A$ belongs to $(\mathfrak{A}_{\mathcal{J}\perp})^{\perp}$, and hence to $\mathfrak{A}_{\mathcal{J}}$.

Thus, we have shown that there exists $B \in \mathfrak{A}$ such that $B - A \in \mathfrak{A}_{\mathcal{J}}$ and

$$\|B\| \leqslant \sup_{Z} \|P^{\perp}_{\mathcal{N}(Z)} A^{(\infty)} P_{\mathcal{M}(Z)}\|,$$

and so

$$\operatorname{dist}(A, \mathfrak{A}_{\mathcal{J}}) = \inf_{B-A \in \mathfrak{A}_{\mathcal{J}}} \|B\| \leqslant \sup_{Z} \|P^{\perp}_{\mathcal{N}(Z)} A^{(\infty)} P_{\mathcal{M}(Z)}\|.$$

To prove the reverse inequality, observe that if $B - A$ belongs to $\mathfrak{A}_{\mathcal{J}}$, then

$$P^{\perp}_{\mathcal{N}(Z)} (B - A)^{(\infty)} P_{\mathcal{M}(Z)} = 0,$$

as $\mathcal{N}(Z)$ is the range of $\mathfrak{A}_{\mathcal{J}}^{(\infty)}$ restricted to $\mathcal{M}(Z)$. Thus

$$P^{\perp}_{\mathcal{N}(Z)} B^{(\infty)} P_{\mathcal{M}(Z)} = P^{\perp}_{\mathcal{N}(Z)} A^{(\infty)} P_{\mathcal{M}(Z)}.$$

Hence, given B with $B - A$ in $\mathfrak{A}_{\mathcal{J}}$, we have

$$\begin{aligned} \|B\| = \|B^{(\infty)}\| &\geqslant \|B^{(\infty)} P_{\mathcal{M}(Z)}\| \\ &\geqslant \|P^{\perp}_{\mathcal{N}(Z)} B^{(\infty)} P_{\mathcal{M}(Z)}\| \\ &= \|P^{\perp}_{\mathcal{N}(Z)} A^{(\infty)} P_{\mathcal{M}(Z)}\|. \end{aligned}$$

Therefore,

$$\inf_{B-A \in \mathfrak{A}_{\mathcal{J}}} \|B\| \geqslant \sup_{Z} \|P^{\perp}_{\mathcal{N}(Z)} A^{(\infty)} P_{\mathcal{M}(Z)}\|.$$

This concludes the proof of the lemma. ∎

Proof of Theorem 6.1. Suppose $A, B \in \mathfrak{A}$ and $A - B \in \mathfrak{A}_{\mathcal{J}}$. Then, since $P^{\perp}_{\mathcal{M}} \otimes I_n$ annihilates $\mathfrak{A}_{\mathcal{J}}$,

$$(P^{\perp}_{\mathcal{M}} \otimes I_n) B = (P^{\perp}_{\mathcal{M}} \otimes I_n) A.$$

Hence $\|B\| \geqslant \|(P^{\perp}_{\mathcal{M}} \otimes I_n) B\| = \|(P^{\perp}_{\mathcal{M}} \otimes I_n) A\|$, and we have

$$\operatorname{dist}(A, \mathfrak{A}_{\mathcal{J}}) = \inf_{B-A \in \mathfrak{A}_{\mathcal{J}}} \|B\| \geqslant \|(P^{\perp}_{\mathcal{M}} \otimes I_n) A\|.$$

To prove the reverse inequality, we employ the previous lemma and the Wold Decomposition. By Lemma 6.3, it will suffice to show that

$$\|P^{\perp}_{\mathcal{N}(Z)} A^{(\infty)} P_{\mathcal{M}(Z)}\| \leqslant \|(P^{\perp}_{\mathcal{M}} \otimes I_n) A\|$$

for each positive trace class operator Z.

For $T_1, \ldots, T_n$ in $\mathfrak{L}_G$, let $\widetilde{T}$ denote the n-tuple $\widetilde{T} = (T_1, \ldots, T_n)$, treated as a column vector. We begin by constructing an auxilliary Hilbert space $\mathcal{K}$ in the following manner: Given a positive trace class operator Z, define a Hermitian form on the set of n-tuples of elements of $\mathfrak{L}_G$ via

$$(\widetilde{T}, \widetilde{S}) = \sum_{i \geqslant 1} (\widetilde{T}^{\mathrm{t}} z_i, \widetilde{S}^{\mathrm{t}} z_i)_{(\mathcal{H}_G)^{(n)}} = \sum_{i,k,l} (T_k z_i^k, S_l z_i^l)_{\mathcal{H}_G},$$

where z_i^k is the kth coordinate from $z_i \in \mathcal{H}_G^{(n)}$. The collection of n-tuples $\widetilde{T}$ equipped with this Hermitian form gives a pre-Hilbert space; taking the quotient modulo null vectors and closing, we obtain a Hilbert space which we denote $\mathcal{H}_G(Z)$. We let $\mathcal{K} = (\mathcal{H}_G(Z))^{(n)}$ denote the direct sum of n copies of $\mathcal{H}_G(Z)$.

If $A = (A_{jk})_{j,k=1}^n$ belongs to $\mathcal{M}_n(\mathfrak{L}_G)$, we may define a bounded operator $M_{A,Z}$ on $\mathcal{K}$ by $M_{A,Z} \equiv (A_{jk}^{(n)})_{j,k=1}^n$. The effect of $M_{A,Z}$ on a vector $(\widetilde{T}_1, \ldots, \widetilde{T}_n)$ in $\mathcal{K}$ will be

$$M_{A,Z}(\widetilde{T}_1, \ldots, \widetilde{T}_n) = \Big(\sum_{k=1}^n A_{1k}(\widetilde{T}_k)_k, \ldots, \sum_{k=1}^n A_{nk}(\widetilde{T}_k)_k \Big).$$

Now, let $\widetilde{B}_j$ denote the transpose of the jth row of $B \in M_n(\mathfrak{L}_G)$, and define a map Φ by

$$\Phi(B^{(\infty)}\widetilde{z}) = (\widetilde{B}_1, \ldots, \widetilde{B}_n)$$

between dense subspaces of $\mathcal{M}(Z)$ and $\mathcal{K}$. An elementary calculation shows that Φ is a surjective isometry between these dense subspaces, and so extends to a unitary map from $\mathcal{M}(Z)$ onto $\mathcal{K}$. It is clear that, if we let $\mathcal{K}_{\mathcal{J}}$ denote the closure in $\mathcal{K}$ of vectors for which each entry of each summand is an operator in $\mathcal{J}$, then Φ carries $\mathcal{N}(Z)$ (the range of the restriction of $\mathfrak{A}_{\mathcal{J}}^{(\infty)}$ to $\mathcal{M}(Z)$) onto $\mathcal{K}_{\mathcal{J}}$. Moreover, for each $A \in \mathfrak{A}$, this map intertwines $A^{(\infty)}|_{\mathcal{M}(Z)}$ and $M_{A,Z}$. Hence these maps are unitarily equivalent,

$$A^{(\infty)}|_{\mathcal{M}(Z)} \simeq M_{A,Z}.$$

Consider now the operators Ψ_e on $\mathcal{H}_G(Z)$, defined by

$$\Psi_e(T_1, \ldots, T_n) \equiv (L_e T_1, \ldots, L_e T_n).$$

With each $\dim \Psi_e^* \Psi_e \mathcal{W} \leqslant n$, the set $\{\Psi_e : e \in E(G)\}$ satisfies the hypotheses of Lemma 2.4, and so there exists a unitary operator

$$U : \mathcal{H}_G(Z) \longrightarrow \mathcal{H} \equiv \sum_{k \in \mathcal{S}}{}^{\oplus} (Q_k \mathcal{H}_G)^{(\alpha_k)}$$

such that

$$U \Psi_e U^* = \sum_{k \in \mathcal{S}}{}^{\oplus} L_e^{(\alpha_k)}|_{(Q_k \mathcal{H}_G)^{(\alpha_k)}} \quad \text{for } e \in E(G),$$

where $\mathcal{S}$ is the subset of vertices $V(G)$ determined as in Lemma 2.4 by the non-zero subspaces amongst $\{\Psi_e^* \Psi_e \mathcal{W} : e \in E(G)\}$.

The ampliation $U^{(n)}$ is a unitary map from $\mathcal{K}$ onto $\mathcal{H}^{(n)}$. Under U, the (j,k) block $A_{jk}^{(n)}$ of $M_{A,Z}$ is taken to $A_{jk}^{(n)}|_{\mathcal{H}}$. We now let M_A denote the operator on $\mathcal{H}_G^{(n^2)}$ which has the form of an $n \times n$ block matrix whose (j,k) block is $A_{jk}^{(n)}$. With these definitions, $U^{(n)} M_{A,Z} (U^{(n)})^* = M_A|_{\mathcal{H}^{(n)}}$. Moreover, recalling that $\mathcal{M} = \mathrm{Ran}(\mathcal{J}) = \overline{\mathcal{J}\mathcal{H}_G} \subseteq \mathcal{H}_G$, we see that $U^{(n)}$ maps $\mathcal{K}_{\mathcal{J}}$ onto $\mathcal{M}^{(n^2)} \cap \mathcal{H}^{(n)} = (\mathcal{M}^{(n)} \cap \mathcal{H})^{(n)}$. Thus, it follows that

$$P_{\mathcal{N}(Z)}^{\perp} A^{(\infty)} P_{\mathcal{M}(Z)} \simeq P_{\mathcal{K}_{\mathcal{J}}}^{\perp} M_{A,Z} \simeq (P_{\mathcal{M}}^{\perp} \otimes I_{n^2}) M_A|_{\mathcal{H}^{(n)}}.$$

Finally, let P denote the projection of $\mathcal{H}_G^{(n)}$ onto $\mathcal{H}$. By reordering the summands of $\mathcal{H}_G^{(n^2)}$ with a canonical shuffle, M_A is seen to be unitarily equivalent to the direct sum of n copies of A with itself. This reordering fixes $P_{\mathcal{M}}^{\perp} \otimes I_{n^2}$, and takes P onto another projection, say P'. Thus, for every $A \in \mathcal{M}_n(\mathfrak{L}_G)$,

$$\begin{aligned}\|P_{\mathcal{N}(Z)}^{\perp} A^{(\infty)} P_{\mathcal{M}(Z)}\| &= \|(P_{\mathcal{M}}^{\perp} \otimes I_{n^2}) M_A|_{\mathcal{H}^{(n)}}\| \\ &= \|(P_{\mathcal{M}}^{\perp} \otimes I_n)^{(n)} A^{(n)} P'\| \\ &\leqslant \|(P_{\mathcal{M}}^{\perp} \otimes I_n) A\|,\end{aligned}$$

and this completes the proof. ∎

The conclusion of Theorem 6.1 for two-sided ideals follows from the result for right-sided ideals in precisely the same way as the $\mathfrak{L}_n$ case [10]. In particular, the following result may be applied to the WOT-closure of the commutator ideal of $\mathfrak{L}_G$, as well as the radical of $\mathfrak{L}_G$ [20].

COROLLARY 6.4. *Let $\mathcal{J}$ be a* WOT*-closed two-sided ideal in $\mathfrak{L}_G$, with range subspace $\mathcal{M} = \mu(\mathcal{J})$. Then $\mathfrak{L}_G/\mathcal{J}$ is completely isometrically isomorphic to the compression algebra $P_{\mathcal{M}}^{\perp}\mathfrak{L}_G P_{\mathcal{M}}^{\perp} = P_{\mathcal{M}}^{\perp}\mathfrak{L}_G$.*

7. CARATHÉODORY THEOREM

As an application of the distance formula, in this section we prove the analogue of the Carathéodory Theorem [2], [10] for the algebras $\mathfrak{L}_G$. The Carathéodory problem specifies an initial segment of the general Fourier series for elements of $\mathfrak{L}_G$, then asks when the segment can be completed to an operator in $\mathfrak{L}_G$ of norm at most one.

We say that a subset Λ of $\mathbb{F}^+(G)$ is a *left lower set* if u belongs to Λ whenever $w = uv$ does. In other words, Λ is closed under taking left subpaths; these are the subpaths of w obtained by moving from a vertex in w along the rest of $w = yw$ to its final vertex y. Let P_Λ be the orthogonal projection onto the subspace $\mathcal{M}_\Lambda = \operatorname{span}\{\xi_w : w \in \Lambda\}$. Observe that $\mathcal{M}_\Lambda$ is invariant for $\mathfrak{R}_G^*$, and hence $\mathcal{M}_\Lambda^{\perp}$ belongs to $\operatorname{Lat}(\mathfrak{R}_G)$. Recall that elements A of $\mathfrak{L}_G$ have Fourier expansions $A \sim \sum_{w\in\mathbb{F}^+(G)} a_w L_w$, and hence it follows that elements of the matrix algebras $\mathcal{M}_k(\mathfrak{L}_G)$ have natural Fourier expansions as well.

THEOREM 7.1. *Let G be a countable directed graph and suppose Λ is a left lower set of $\mathbb{F}^+(G)$. Given $k \geqslant 1$, let $\{C_w : w \in \Lambda\}$ be matrices in $\mathcal{M}_k(\mathbb{C})$. Then there is an element A in the unit ball of $\mathcal{M}_k(\mathfrak{L}_G)$ with Fourier coefficients $A_w = C_w$ for $w \in \Lambda$ if and only if*

$$\Big\|(P_\Lambda \otimes I_k)\Big(\sum_{w\in\Lambda} L_w \otimes C_w\Big)\Big\| \leqslant 1.$$

When Λ is an infinite set this sum is understood to converge WOT *in the Cesaro sense.*

Proof. This result can be proved as an immediate consequence of the distance formula. The set of elements of $\mathfrak{L}_G$ which interpolate the zero data $C_w \equiv 0$ for $w \in \Lambda$ is equal to the WOT-closed right ideal $\mathcal{J}$ with range $\mu(\mathcal{J}) = P_\Lambda^\perp \mathcal{H}_G$. Thus, if we are given data C_w, the desired element exists precisely when the distance from $\sum_{w\in\Lambda} L_w \otimes C_w$ to $\mathcal{M}_k(\mathcal{J})$ is at most one, and the result follows from Theorem 6.1. ∎

The simplest example of a lower set is the set $\mathbb{F}^+(G)_k$ of all paths in G of length at most k. The (two-sided) ideal in this case is given by

$$\iota(\mathcal{M}^\perp_{\mathbb{F}^+(G)_k}) \equiv \mathfrak{L}_G^{0,k} = \Big\{ A \in \mathfrak{L}_G : A = \sum_{|w|>k} a_w L_w \Big\}.$$

Thus we obtain the following corollary. Let P_{Λ_k} be the projection onto the subspace $\mathcal{M}_{\mathbb{F}^+(G)_k}$.

COROLLARY 7.2. *Given a formal power series* $\sum_{|w|\leqslant k} a_w w$ *in the semigroupoid algebra* $\mathbb{C}\mathbb{F}^+(G)$, *there is an element* A *in* $\mathfrak{L}_G$ *with* $\|A\| \leqslant 1$ *and* $A - \sum_{w\in\mathbb{F}^+(G)_k} a_w L_w$ *in* $\mathfrak{L}_G$ *if and only if*

$$\Big\| P_{\Lambda_k}\Big(\sum_{w\in\mathbb{F}^+(G)_k} a_w L_w \Big) \Big\| \leqslant 1.$$

REMARK 7.3. There is a different, more self-contained proof of the Carathéodory Theorem for $\mathfrak{L}_G$ which is worth discussing since it yields general information on elements of $\mathfrak{L}_G$. Furthermore, this alternative approach gives a new proof for free semigroup algebras $\mathfrak{L}_n$ which generalizes the Parrot's Lemma cum Toeplitz matrix approach for H^∞.

We first establish some notation and make some simple observations. Given a lower set Λ in $\mathbb{F}^+(G)$, let $\Lambda_k = \{w \in \Lambda : |w| \leqslant k\}$ for $k \geqslant 0$ and let E_k be the projection of $\mathcal{H}_G$ onto $\operatorname{span}\{\xi_w : w \in \Lambda_k\}$. Then each E_k is a subprojection of P_Λ and $P_\Lambda = \sum_{k\geqslant 0}^{\oplus} F_k$, where $F_{k+1} = E_{k+1} - E_k$ and $F_0 = E_0$. The lower set property yields the identity

$$E_{k+1} R_e E_k = E_{k+1} R_e \quad \text{for } e \in E(G). \tag{7.1}$$

Also note that $Q_y = \sum_{w=wy}^{\oplus} \xi_w \xi_w^*$ for every vertex $y \in V(G)$, where $\xi_w\xi_w^*$ is the rank one projection onto the span of ξ_w, and that

$$R_e(\xi_w\xi_w^*)R_e^* = (R_e\xi_w)(R_e\xi_w)^* = \xi_{we}\xi_{we}^*.$$

Further, it is clear that the projections $\{Q_y, P_\Lambda, E_k\}$ are mutually commuting; for instance, $Q_y P_\Lambda E_k$ is the projection onto the subspace

$$Q_y P_\Lambda E_k \mathcal{H}_G = \operatorname{span}\{\xi_w : w \in \Lambda_k, w = wy\}.$$

We need the following estimates for elements of $\mathfrak{L}_G$ to apply Parrot's Lemma.

PROPOSITION 7.4. *Let G be a countable directed graph and let Λ be a lower set in $\mathbb{F}^+(G)$. Given $X \in \mathfrak{L}_G$, define*

$$A_k = E_k X E_k \text{ and } B_k = E_{k+1} X (E_{k+1} - E_0) \quad \text{for } k \geqslant 0.$$

Then

$$\|B_k\| \leqslant \|A_k\| \leqslant \|X\| \quad \text{for } k \geqslant 0.$$

Proof. Fix $k \geqslant 0$. As Q_y commutes with X and each E_k, we have $A_k = \sum_y^{\oplus} A_k Q_y$. Thus,

$$\|A_k\| = \sup_y \|E_k X Q_y E_k\|.$$

On the other hand,

$$B_k = E_{k+1} X (E_{k+1} - E_0) E_{k+1} = \sum_{w \in \Lambda_{k+1} \setminus \Lambda_0} E_{k+1} X (\xi_w \xi_w^*) E_{k+1}.$$

But this sum may be written as

$$B_k = \sum_{e \in E(G)} \sum_{u \in \Lambda_k} E_{k+1} X (\xi_{ue} \xi_{ue}^*) E_{k+1}.$$

Indeed, since Λ is a lower set, every $w \in \Lambda_{k+1} \setminus \Lambda_0$ is of the form $w = ue$ for some $u \in \Lambda_k$ and edge e. On the other hand, if $w = ue$, with $u \in \Lambda_k$, is not in Λ_{k+1}, then $(\xi_w \xi_w^*) E_{k+1} = \xi_w (E_{k+1} \xi_w)^* = 0$, so the corresponding term in this sum vanishes. Thus, we have

$$\begin{aligned}
B_k &= \sum_{e \in E(G)} \sum_{w \in \Lambda_k} E_{k+1} X R_e (\xi_w \xi_w^*) R_e^* E_{k+1} \\
&= \sum_{e \in E(G)} \sum_{w \in \Lambda_k} E_{k+1} R_e (E_k X (\xi_w \xi_w^*) E_k) R_e^* E_{k+1} \\
&= \sum_{y \in V(G)} \sum_{e = ye}^{\oplus} E_{k+1} R_e \Big(\sum_{w \in \Lambda_k;\, w = wy} E_k X (\xi_w \xi_w^*) E_k \Big) R_e^* E_{k+1} \\
&= \sum_y \sum_{e = ye}^{\oplus} E_{k+1} R_e (E_k X Q_y E_k) R_e^* E_{k+1}.
\end{aligned}$$

Therefore, as the ranges of $\{E_{k+1} R_e : e \in E(G)\}$ are pairwise orthogonal for fixed $k \geqslant 0$, it follows that

$$\begin{aligned}
\|B_k\| &= \sup_y \sup_{e = ye} \|E_{k+1} R_e (E_k X Q_y E_k) R_e^* E_{k+1}\| \\
&\leqslant \sup_y \|E_k X Q_y E_k\| = \|A_k\|,
\end{aligned}$$

for $k \geqslant 0$, and this completes the proof. ∎

These are precisely the estimates required to apply Parrot's Lemma in the "bottom left corner" argument for the proof of the Carathéodory Theorem. In the H^∞ case, equality is achieved with $\|B_k\| = \|A_k\|$. This is easily seen from the Toeplitz matrix perspective for elements of H^∞. This is also the case for free semigroup algebras $\mathfrak{L}_n$. Let us discuss the special case of $\mathfrak{L}_2$ to illustrate this generalized Toeplitz matrix structure.

EXAMPLE 7.5. Let $\{1,2\}$ be the (noncommuting) generators of $\mathbb{F}_2^+$. The full Fock space has an orthonormal basis given by $\{\xi_\phi, \xi_w : w \in \mathbb{F}_2^+\}$. Elements X of $\mathfrak{L}_2$ have Fourier expansions given by their action on the vacuum vector; $X\xi_\phi = \sum\limits_{w\in\mathbb{F}_2^+\cup\{\phi\}} a_w \xi_w$. Let $\Lambda = \mathbb{F}_2^+ \cup \{\phi\}$ be the trivial lower set, so that $P_\Lambda = I$. Consider the ordering for $\Lambda_2 = \{w : |w| \leqslant 2\}$ given by

$$\Lambda_2 = \{\phi, \{1,2\}, \{1^2, 12, 21, 2^2\}\}.$$

The projection E_2 has range space span$\{\xi_w : w \in \Lambda_2\}$. Hence the compression of $X \sim \sum\limits_w a_w L_w$ in $\mathfrak{L}_2$ to E_2 is unitarily equivalent to

$$A_2 = E_2 X E_2 \simeq \left[\begin{array}{c|cc|cccc} a_\phi & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a_1 & a_\phi & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & a_\phi & 0 & 0 & 0 & 0 \\ \hline a_{1^2} & a_1 & 0 & a_\phi & 0 & 0 & 0 \\ a_{12} & 0 & a_1 & 0 & a_\phi & 0 & 0 \\ a_{21} & a_2 & 0 & 0 & 0 & a_\phi & 0 \\ a_{2^2} & 0 & a_2 & 0 & 0 & 0 & a_\phi \end{array}\right].$$

On the other hand, $B_2 = E_3 X(E_3 - E_0)$ can be seen to be unitarily equivalent to $A_2 \otimes I_2$ where I_2 is the scalar 2×2 identity matrix. Thus, in this case $\|B_2\| = \|A_2\|$. More generally, if $\Lambda = \mathbb{F}_n^+ \cup \{\phi\}$ and X belongs to $\mathfrak{L}_n$, the operators B_k are unitarily equivalent to $A_k \otimes I_n$. The "n-branching" in $\mathbb{F}_n^+$ creates n-ampliations in the generalized Toeplitz matrices.

REMARK 7.6. We mention that when G has loop edges, it is also possible to derive a version of Pick's interpolation theorem [2], [10], [29] for $\mathfrak{L}_G$. Indeed, in [20] it was shown how loop edges over vertices explicitly define eigenvectors for $\mathfrak{L}_G^*$, with corresponding eigenvalues making up unit balls in the complex spaces $\mathbb{C}^k$. But the notation is rather cumbersome, and thus, as it is a direct generalization of the Pick theorem from [2], [10] for $\mathfrak{L}_n$, we shall not present it here.

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MICHAEL T. JURY, DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907, USA
E-mail address: jury@math.purdue.edu

DAVID W. KRIBS, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF GUELPH, GUELPH, ONTARIO N1G 2W1, CANADA
E-mail address: dkribs@uoguelph.ca

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