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SPACES ON WHICH THE ESSENTIAL SPECTRUM OF ALL THE OPERATORS IS FINITE

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ABSTRACT. We study the Banach spaces X for which the essential spectrum $\sigma_{\mathrm{e}}(T)$ of every T in $L(X)$ is finite. We show that there exists an integer n so that $|\sigma_{\mathrm{e}}(T)| \leqslant n$ for every T. We also show that X admits an irreducible decomposition as a direct sum of indecomposable subspaces, and that the quotient algebra $L(X)/\mathrm{In}(X)$, $\mathrm{In}(X)$ the inessential operators, is isomorphic to a finite product of spaces of scalar matrices.

KEYWORDS: *Indecomposable Banach spaces, Fredholm operators, Calkin algebra.*

MSC (2000): 47A53.

1. INTRODUCTION

The essential spectrum $\sigma_{\mathrm{e}}(T)$ of the operators $T \in L(X)$ is a useful tool to obtain information on the isomorphic properties of a Banach space X; see Section 2.c of [15] and [7]. Here we study the spaces X such that $\sigma_{\mathrm{e}}(T)$ is finite for every $T \in L(X)$.

First we show that if $|\sigma_{\mathrm{e}}(T)| < \infty$ for every $T \in L(X)$, then there exists an integer n so that $|\sigma_{\mathrm{e}}(T)| \leqslant n$ for every $T \in L(X)$. Thus, denoting by Σ_{e}^n the class of the Banach spaces X such that $\max\{|\sigma_{\mathrm{e}}(T)| : T \in L(X)\} = n$, our problem is reduced to study the spaces $X \in \Sigma_{\mathrm{e}}^n$.

Let us denote by $\mathrm{In}(X, Y)$ the inessential operators in $L(X, Y)$. For definitions we refer to the end of this introduction. We obtain several characterizations of the spaces in Σ_{e}^1 in terms of the inessential operators. For example, $X \in \Sigma_{\mathrm{e}}^1$ if and only if we can write $L(X) = \mathbb{C}I_X \oplus \mathrm{In}(X)$, where I_X is the identity operator on X.

Among our results, we prove that each $X \in \Sigma_{\mathrm{e}}^n$ admits a decomposition $X = X_1 \oplus \cdots \oplus X_n$, where each X_i is a subspace of X that belongs to Σ_{e}^1. Moreover, the summands X_i can be divided into r sets $S_1, \ldots, S_r$ so that there exists a Fredholm operator in $L(X_j, X_i)$ when i and j belong to the same set and

$L(X_j, X_i) = \mathrm{In}(X_j, X_i)$ when they belong to different sets. Let us denote by n_l the number of spaces in the class S_l. Clearly $n_1 + \cdots + n_r = n$.

From the decomposition described in the previous paragraph, we show that the quotient algebra $L(X)/\mathrm{In}(X)$ is isomorphic to the product $M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$ where $M_l(\mathbb{C})$ is the algebra of complex $l \times l$ matrices. In particular,

$$\dim L(X)/\mathrm{In}(X) = n_1^2 + \cdots + n_r^2.$$

Note that the essential spectrum of an operator $T \in L(X)$ coincides with the spectrum of the image of T in $L(X)/\mathrm{In}(X)$.

From the description of $L(X)/\mathrm{In}(X)$ we derive that all the irreducible decompositions of a space $X \in \Sigma_e^n$ as a direct sum of spaces in Σ_e^1 have n summands. Even more, they have associated the same integers $n_1, \ldots, n_r$.

A Banach space X is *n-hereditarily decomposable* [5] and we write $X \in HD_n$, if n is the maximal number of the integers k such that X contains a subspace which is the direct sum of k subspaces. The space X is *n-quotient decomposable* [10] and we write $X \in QD_n$, if n is the maximal number of the integers k such that X has a quotient which is the direct sum of k (closed infinite dimensional) subspaces. It follows from the results in [5] and [10] that the spaces in HD_n or QD_n belong to Σ_e^m for some $m \leqslant n$, and there exist examples for which $m < n$.

Let SS and SC denote the strictly singular and the strictly cosingular operators, respectively. In [5] Ferenczi proves that for every operator $T \in L(X)$ on a HD_n space X, $|\sigma_e(T)| \leqslant n$ and that $\dim L(X)/SS(X) \leqslant n^2$. In [10] the authors prove for a QD_n space X that the quotient algebra $L(X)/SC(X)$ is isomorphic to a subalgebra of the $n \times n$ complex matrices $M_n(\mathbb{C})$. Thus for every $T \in L(X)$, $|\sigma_e(T)| \leqslant n$ and $\dim L(X)/SC(X) \leqslant n^2$. Our results on the structure of $L(X)/\mathrm{In}(X)$ give a better bound for $|\sigma_e(T)|$ and show that the algebra $L(X)/\mathrm{In}(X)$ is more suitable to study the essential spectrum of the spaces in Σ_e^m than $L(X)/SS(X)$ or $L(X)/SC(X)$. Note that the essential spectrum of $T \in L(X)$ coincides with the spectrum of the image of T in any of the algebras $L(X)/SS(X)$, $L(X)/SC(X)$ or $L(X)/\mathrm{In}(X)$.

Finally we get some results about K-theory of HD_n and QD_n spaces using the algebra $L(X)/\mathrm{In}(X)$ which a priori would be more difficult to obtain starting from the algebras $L(X)/SS(X)$ or $L(X)/SC(X)$.

Throughout the paper $X, Y, Z, \ldots$ will denote complex Banach spaces. X^* will stand for the dual space of X and $L(X, Y)$ for the (continuous linear) operators from X into Y. We set $L(X) = L(X, X)$ and denote the identity map by I_X or simply I if there is no possible confusion. We denote by $K(X, Y)$ the set of all compact operators from X into Y.

An operator $T \in L(X, Y)$ is *Fredholm*, $T \in \Phi(X, Y)$, if $\mathrm{Ker}\, T$ is finite dimensional and $R(T)$ is finite codimensional (hence closed). It is *left-Atkinson*, $T \in \Phi_l(X, Y)$, if $R(T)$ is complemented and $\mathrm{Ker}\, T$ is finite dimensional, and it is *right-Atkinson*, $T \in \Phi_r(X, Y)$, if $\mathrm{Ker}\, T$ is complemented and $R(T)$ is finite codimensional.

An operator $T \in L(X,Y)$ is *inessential*, $T \in \mathrm{In}(X,Y)$, if $I - ST \in \Phi(X)$ (or equivalently $I - TS \in \Phi(Y)$) for every $S \in L(Y,X)$.

Unless the contrary is specified, all the subspaces will be closed and infinite dimensional, and the quotients will be infinite dimensional.

2. FINITELY DECOMPOSABLE SPACES

The *essential spectrum* of an operator $T \in L(X)$ is defined by

$$\sigma_{\mathrm{e}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi(X)\}.$$

Let π denote the quotient map from $L(X)$ onto the Calkin algebra $L(X)/K(X)$. It is well known that the essential spectrum of T coincides with the spectrum of $\pi(T)$ in $L(X)/K(X)$.

We are interested in the spaces X so that $\sigma_{\mathrm{e}}(T)$ is finite for every $T \in L(X)$. In order to study these spaces, the following result will be useful.

For each $n \in \mathbb{N}$, we denote by F_n the set of all $T \in L(X)$ such that $\sigma_{\mathrm{e}}(T)$ has at most n connected components.

PROPOSITION 2.1. *For each $n \in \mathbb{N}$, the set F_n is closed in $L(X)$.*

Proof. Let (T_k) be a sequence in F_n which converges to $T \in L(X)$. Suppose that $\sigma_{\mathrm{e}}(T)$ includes m connected components, $C_1, \dots, C_m$, with $m > n$. We select closed simple curves $\Gamma_1, \dots, \Gamma_m$ on $\mathbb{C} \setminus \sigma_{\mathrm{e}}(T)$ which do not intersect so that each C_i is in the interior U_i of Γ_i.

Let V be an open set containing $\sigma_{\mathrm{e}}(T)$. We claim that $\sigma_{\mathrm{e}}(T_k) \subset V$ for k large enough. If this is not the case, passing to a subsequence we can suppose that there exists $\alpha_k \in \sigma_{\mathrm{e}}(T_k)$ so that $\alpha_k \notin V$ for every k. As $|\alpha_k| \leqslant \|T_k\|$, α_k is a bounded sequence and, passing to a subsequence again, we can suppose that α_k converges to α. Since $\mathbb{C} \setminus V$ is closed, $\alpha \notin V$ and we get that $\alpha I - T = \lim(\alpha_k I - T_k)$ is Fredholm. As $\Phi(X)$ is open, $\alpha_k I - T_k$ must be Fredholm for k large, which is not the case. So the claim is proved.

Now we claim that $\sigma_{\mathrm{e}}(T_k)$ meets each U_i for k large enough. If this is not the case, there exist some j and a subsequence $T_r \to T$ such that $\sigma_{\mathrm{e}}(T_r)$ does not meet U_j. Then $\pi(T_r) \to \pi(T)$ and $\sigma(\pi(T_r))$ does not meet U_j. Moreover, by the continuity of the map $S \to S^{-1}$ ([2], Theorem 3.2.3), we can assume that $(\lambda I - \pi(T_r))^{-1}$ converges to $(\lambda I - \pi(T))^{-1}$ for all λ in Γ_j. Thus by the compactness of Γ_j we have

$$0 = \int_{\Gamma_j} (\lambda I - \pi(T_r))^{-1}\, \mathrm{d}\lambda \to \int_{\Gamma_j} (\lambda I - \pi(T))^{-1}\, \mathrm{d}\lambda \neq 0$$

and we get a contradiction.

The two claims proved show that $\sigma_{\mathrm{e}}(T_k)$ has at least m connected components for k large enough, which is not the case. Hence T belongs to F_n. ∎

The following result gives a classification of the spaces X such that $|\sigma_e(T)|$ is finite for every $T \in L(X)$.

THEOREM 2.2. *Suppose that $\sigma_e(T)$ is finite for every $T \in L(X)$. Then there exists an integer n_X so that $|\sigma_e(T)| \leqslant n_X$ for every $T \in L(X)$.*

Proof. Observe that in this case $F_n = \{T \in L(X) : |\sigma_e(T)| \leqslant n\}$. By Proposition 2.1, each F_n is closed. Applying Baire's Lemma to $L(X) = \bigcup_k F_k$ we get that there exists n such that F_n has nonempty interior. We choose T_0 in the interior of F_n. Now, for every fixed $T \in L(X)$, the function $f(\lambda) = T_0 + \lambda(T - T_0)$ from $\mathbb{C}$ into $L(X)$ is analytic and $|\sigma_e(f(\lambda))| \leqslant n$ on a nonempty open set U. Since U has nonzero capacity, Theorem 3.4 of [2] applied to $\pi(f(\lambda))$ implies that $|\sigma_e(f(\lambda))| \leqslant n$ for every λ. In particular, $|\sigma_e(f(1))| = |\sigma_e(T)| \leqslant n$, so we are done. ∎

DEFINITION 2.3. We say that a Banach space X belongs to the class Σ_e^n if

$$n = \max\{|\sigma_e(T)| : T \in L(X)\}.$$

The following result follows easily from the definition of Σ_e^n.

PROPOSITION 2.4. *Let $X \in \Sigma_e^n$. Then:*

(i) *if $T \in L(X)$ is semi-Fredholm, then* $\operatorname{ind}(T) = 0$*;*
(ii) *there is no proper subspace or proper quotient of X isomorphic to X.*

Proof. (i) Observe that $\mathbb{C} \setminus \sigma_e(T)$ is connected and $\operatorname{ind}(\lambda I - T) = 0$ for $|\lambda| > \|T\|$. Since the index is a continuous map defined on the semi-Fredholm operators ([12], V.1.6 Theorem), $\operatorname{ind}(T) = 0$.

(ii) Let $Y \subseteq X$ be a subspace. Suppose that there is an isomorphism $U : X \longrightarrow Y$. Then composing U with the inclusion $i : Y \longrightarrow X$, we get an injective semi-Fredholm operator $i\,U$ on X. By the previous part $\operatorname{ind}(i\,U) = 0$, so $Y = X$.

The proof in the case of a proper quotient is analogous. ∎

Recall that a Banach space X is said to be *n-decomposable* if it admits a decomposition $X = X_1 \oplus \cdots \oplus X_n$ where X_i are (infinite dimensional) subspaces of X.

THEOREM 2.5. *Let $n \in \mathbb{N}$. For a complex Banach space X, the following assertions are equivalent:*

(i) *X is n-decomposable;*
(ii) *there exists an operator $T \in L(X)$ such that $|\sigma_e(T)| = n$;*
(iii) *there exists an operator $T \in L(X)$ such that $\sigma_e(T)$ consists of n connected components.*

Proof. Suppose that X is n-decomposable, and hence, $X = X_1 \oplus \cdots \oplus X_n$. Then $T(x_1, x_2, \ldots, x_n) := (x_1, 2x_2, \ldots, nx_n)$ defines an operator on X such that $\sigma_e(T) = \{1, 2, \ldots, n\}$. Thus, (i) implies (ii).

That (ii) implies (iii) is trivial, so let us see that (iii) implies (i). Let $T \in L(X)$ be such that $\sigma_{\mathrm{e}}(T)$ consists of n connected components, $C_1, \dots, C_n$. By Theorem V.1.8 of [12], $\lambda I - T$ is invertible on the unbounded component of $\mathbb{C} \setminus \sigma_{\mathrm{e}}(T)$ with the possible exceptions of isolated points.

We select closed simple curves $\Gamma_1, \dots, \Gamma_n$ on $\mathbb{C} \setminus \sigma(T)$ which do not intersect so that each C_i is in the interior of Γ_i.

The analytic operational calculus ([18], Section V.8) allows us to define

$$P_i := \int_{\Gamma_i} (\lambda I - T)^{-1} \mathrm{d}\lambda, \quad i = 1, \dots, n.$$

Then each P_i is a projection. Moreover,

$$\pi(P_i) = \int_{\Gamma_i} (\lambda I - \pi(T))^{-1} \mathrm{d}\lambda \neq 0,$$

where π is the quotient map onto the Calkin algebra $L(X)/K(X)$. Thus $R(P_i)$ is infinite dimensional. By Theorem V.9.1 of [18], $X = R(P_1) \oplus \cdots \oplus R(P_n)$. Thus, X is n-decomposable. ∎

REMARK 2.6. It is not true in general that the operators acting on finitely decomposable spaces have finite essential spectrum, or index equal to 0 when they are semi-Fredholm. In Section 4.2 of [11] we can find an indecomposable space X and an operator $T \in L(X)$ such that $\sigma_{\mathrm{e}}(T) = \{\alpha \in \mathbb{C} : |\alpha| = 1\}$ and $\mathrm{ind}(T) = -1$.

Recall that $X \in HD_n$ if n is the maximal integer such that X has a n-decomposable subspace; and $X \in QD_n$ if n is the maximal integer such that X has a n-decomposable quotient.

REMARK 2.7. The spaces $X \in HD_n$ or $X \in QD_n$ belong to Σ_{e}^m for some $m \leqslant n$, but $m < n$ in some cases. We refer to Section 3 of [6] for examples of spaces $X \in QD_2$ which are hereditarily indecomposable.

From now on we restrict ourselves to consider spaces in Σ_{e}^n.

LEMMA 2.8. *Suppose that $X \in \Sigma_{\mathrm{e}}^n$. Then each complemented subspace Y of X belongs to Σ_{e}^m for some $m \leqslant n$. Moreover, if Y is finite codimensional, then $Y \in \Sigma_{\mathrm{e}}^n$.*

Proof. Let $Y \subseteq X$ be a complemented subspace with $X = Y \oplus Z$. Given $S \in L(Y)$ we denote by T the extension of S to X defined by $T(y+z) = S(y)$. It is clear that

$$\sigma_{\mathrm{e}}(S) \subset \sigma_{\mathrm{e}}(T) \subset \sigma_{\mathrm{e}}(S) \cup \{0\}.$$

Moreover, if Z is finite dimensional and for an operator $T \in L(X)$ we denote by S the operator on Y given by the matricial representation of T associated to $X = Y \oplus Z$, then it is easy to see that $\sigma_{\mathrm{e}}(S) = \sigma_{\mathrm{e}}(T)$. ∎

PROPOSITION 2.9. *Let X be a Banach space. The following conditions are equivalent:*

(i) $X \in \Sigma_e^1$;
(ii) $L(X,Y) = \Phi_l(X,Y) \cup \mathrm{In}(X,Y)$, *for each* Y;
(iii) $L(Y,X) = \Phi_r(Y,X) \cup \mathrm{In}(Y,X)$, *for each* Y;
(iv) $L(X) = \Phi(X) \cup \mathrm{In}(X)$.

Proof. (i) $\Rightarrow$ (ii). Let $T \in L(X,Y)$, $T \notin \mathrm{In}(X,Y)$. Then there exists $A \in L(Y,X)$ such that $I_X - AT \notin \Phi(X)$. Since $X \in \Sigma_e^1$, $\sigma_e(AT) = \{1\}$. In particular, AT is Fredholm; hence $T \in \Phi_l(X,Y)$ ([18], Exercise 4.3).

(ii) $\Rightarrow$ (iv). Taking $X = Y$ we have $L(X) = \Phi_l(X) \cup \mathrm{In}(X)$, so it will be enough to prove that $\Phi_l(X) = \Phi(X)$. In fact, let $T \in \Phi_l(X)$. Then we have decompositions $X = M \oplus F = N \oplus G$ so that $T = \mathrm{diag}(U,0)$ where $U : M \longrightarrow N$ is an isomorphism and $\dim F < \infty$. The operator $\widetilde{T}$ with matrix $\mathrm{diag}(U^{-1},0)$ with respect to the previous decomposition is not inessential because $R(I - \widetilde{T}T) = F$. Thus $\widetilde{T} \in \Phi_l(X)$ and $\mathrm{Ker}\, \widetilde{T} = G$ has finite dimension, i.e., T is Fredholm.

(iv) $\Rightarrow$ (i). Let $T \in L(X)$. As $\sigma_e(T)$ is nonempty we can find $\lambda_0 \in \mathbb{C}$ so that $\lambda_0 I - T = S \notin \Phi(X)$. Then $S \in \mathrm{In}(X)$ and $\sigma_e(T) = \sigma_e(\lambda_0 I - S) = \{\lambda_0\}$, so $X \in \Sigma_e^1$.

The proofs of (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are analogous to those of (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iv), respectively. ∎

REMARK 2.10. It is worth to compare Proposition 2.9 with the following results from [20]:

$X \in HD_1$ if and only if $L(X,Y) = \Phi_+(X,Y) \cup SS(X,Y)$ for every Y;
$X \in QD_1$ if and only if $L(Y,X) = \Phi_-(Y,X) \cup SC(Y,X)$ for every Y.

Here Φ_+, SS, Φ_- and SC are the upper semi-Fredholm, strictly singular, lower semi-Fredholm and strictly cosingular operators, respectively.

PROPOSITION 2.11. *We have $X \in \Sigma_e^1$ if and only if $L(X) = \mathbb{C}I_X \oplus \mathrm{In}(X)$.*

Proof. Let us suppose $L(X) = \mathbb{C}I_X \oplus \mathrm{In}(X)$ and let $T \in L(X)$. Then $T = \lambda I + S$ for some $\lambda \in \mathbb{C}$, $S \in \mathrm{In}(X)$ and $\sigma_e(T) = \sigma_e(\lambda I) = \{\lambda\}$.

Conversely, given $T \in L(X)$ with $\sigma_e(T) = \{\lambda\}$, we have $\lambda I - T \notin \Phi(X)$ and Proposition 2.9 implies that $\lambda I - T \in \mathrm{In}(X)$, so we are done. ∎

PROPOSITION 2.12. *Suppose that $X \in \Sigma_e^1$ and let Y be a Banach space.*

(i) *If there exists $T \in \Phi_l(X,Y)$, then $L(X,Y) = \mathbb{C}\,T \oplus \mathrm{In}(X,Y)$.*
(ii) *If there exists $T \in \Phi_r(Y,X)$, then $L(Y,X) = \mathbb{C}\,T \oplus \mathrm{In}(Y,X)$.*
(iii) *If $Y \in \Sigma_e^1$, and there exists $T \in L(X,Y) \setminus \mathrm{In}(X,Y)$, then $T \in \Phi(X,Y)$ and $L(X,Y) = \mathbb{C}\,T \oplus \mathrm{In}(X,Y)$.*

Proof. (i) Let $T \in L(X,Y)$, $T \notin \mathrm{In}(X,Y)$. Since $X \in \Sigma_e^1$, by Proposition 2.9, $T \in \Phi_l(X,Y)$. Therefore, there exists $U \in L(Y,X)$ such that $TU = I_Y - K$, K a compact operator ([18], Exercise IV.2).

Given $S \in L(X,Y)$, we have $\sigma_e(US) = \{\lambda_0\}$ for some $\lambda_0 \in \mathbb{C}$. Thus $\lambda_0 I_X - US \in \mathrm{In}(X)$, by Proposition 2.9, and $T(\lambda_0 I_X - US) = \lambda_0 T - TUS = \lambda_0 T - S + KS \in \mathrm{In}(X,Y)$. Hence $S = \lambda_0 T + K_0$, with $K_0 \in \mathrm{In}(X,Y)$ and we are done.

Analogous proof for part (ii). Part (iii) follows from the fact that if $Y \in \Sigma_e^1$, and $T \notin \mathrm{In}(X,Y)$, then $T \in \Phi_l(X,Y) \cap \Phi_r(X,Y) = \Phi(X,Y)$ by Proposition 2.9. ∎

DEFINITION 2.13. Let X and Y be Banach spaces. We say that X and Y are *essentially isomorphic* if $\Phi(X,Y) \neq \emptyset$.

We say that X and Y are *essentially incomparable* if $L(X,Y) = \mathrm{In}(X,Y)$.

COROLLARY 2.14. *Let X and Y be spaces in Σ_e^1. Then either X and Y are essentially isomorphic, or they are essentially incomparable.*

Proof. This follows from part (iii) of Proposition 2.12. ∎

REMARK 2.15. It follows easily from the definition that X and Y are essentially isomorphic if and only if X has a finite codimensional subspace which is isomorphic to a finite codimensional subspace of Y. Moreover, since the composition of Fredholm operators is a Fredholm operator, the property of being essentially isomorphic is transitive.

REMARK 2.16. The essentially incomparable spaces were introduced and studied in [7]. It was proved there that for X, Y arbitrary Banach spaces it holds that

$$L(X,Y) = \mathrm{In}(X,Y) \text{ if and only if } L(Y,X) = \mathrm{In}(Y,X).$$

DEFINITION 2.17. Let X be a Banach space. A decomposition $X = X_1 \oplus \cdots \oplus X_m$ is said to be *irrefinable* if the summands X_i are indecomposable.

THEOREM 2.18. *Let $X \in \Sigma_e^n$. Then there exist integers $n_1, \dots, n_r$ so that $n_1 + \cdots + n_r = n$ and*

$$L(X)/\mathrm{In}(X) \simeq M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C}).$$

In particular, $\dim L(X)/\mathrm{In}(X) = n_1^2 + \cdots + n_r^2$.

Proof. By Theorem 2.5 there exists a decomposition $X = X_1 \oplus \cdots \oplus X_n$ with n summands. Clearly this decomposition is irrefinable.

Applying Corollary 2.14, we can divide the set $\{1, \dots, n\}$ into r subsets $S_1, \dots, S_r$ so that there exists a Fredholm operator in $L(X_j, X_i)$ when i and j belong to the same set and $L(X_j, X_i) = \mathrm{In}(X_j, X_i)$ when they belong to different sets. Let us denote by n_l the number of spaces in the class S_l, so that $n_1 + \cdots + n_r = n$.

Clearly, for every S_l there exists a space Y which is isomorphic to a finite codimensional subspace of each X_i with $i \in S_l$. Let U_i denote an isomorphism from Y into a finite codimensional subspace of X_i. Since $R(U_i)$ is complemented, there exists $V_i \in L(X_i, Y)$ such that $V_i U_i = I_Y$.

For every $i, j \in S_l$ we define $E_{ij} := U_i V_j \in L(X_j, X_i)$. Then, for $i, j, k \in S_l$ we have $E_{ij} E_{jk} = E_{ik}$.

Let $T \in L(X)$ and let (T_{ij}) be the associated matrix with respect to the given decomposition. By Proposition 2.12, there exist $\lambda_{ij} \in \mathbb{C}$ and $S_{ij} \in \mathrm{In}(X_j, X_i)$ so that $T_{ij} = \lambda_{ij}E_{ij} + S_{ij}$ when i and j belong to the same set, and $T_{ij} \in \mathrm{In}(X_j, X_i)$ when i and j belong to different subsets. In the latter case we set $\lambda_{ij} = 0$.

The identities $E_{ij}E_{jk} = E_{ik}$ imply that the map

$$\Theta : T \in L(X) \longrightarrow (\lambda_{ij}) \in M_n(\mathbb{C})$$

induces an algebra isomorphism from $L(X)/\mathrm{In}(X)$ into a subalgebra of $M_n(\mathbb{C})$ isomorphic to $M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$. ∎

It is worth to observe that for any ideal J of $L(X)$ such that $K(X) \subseteq J \subseteq \mathrm{In}(X)$ we have $\mathrm{In}(X) = \pi^{-1}(\mathrm{rad}(L(X)/J))$ where π is the quotient map from $L(X)$ onto $L(X)/J$. Therefore $\mathrm{rad}(L(X)/J) = \mathrm{In}(X)/J$. In particular, $L(X)/\mathrm{In}(X)$ is semisimple.

REMARK 2.19. Note that $L(X)/\mathrm{In}(X)$ is finite dimensional when $X \in \Sigma_e^n$. Therefore, since $L(X)/\mathrm{In}(X)$ is semisimple, the existence of the isomorphism

$$L(X)/\mathrm{In}(X) \simeq M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$$

can be obtained also by applying the Theorem of Wedderburn on finite semisimple algebras over an algebraically closed field ([2], Theorem 2.1.2).

The following example, that can be found in the proof of Proposition 4.5 of [13], shows that for spaces $X \in \Sigma_e^n$, the quotient algebra $L(X)/SS(X)$ is not always semisimple.

EXAMPLE 2.20. Let X be a hereditarily indecomposable Banach space and let us consider an ascending chain of subspaces $X_1 \subset \cdots \subset X_n = X$ such that X_i has infinite codimension in X_{i+1} for $i = 1, \ldots, n-1$. If we denote by J_{ik} the inclusion map of X_i into X_k for $i \leqslant k$, then we have:

– For $i \leqslant k$, $L(X_i, X_k) = \mathbb{C}J_{ik} \oplus SS(X_i, X_k)$ and for $i > k$, $L(X_i, X_k) = SS(X_i, X_k)$. Both facts follow from Proposition 1 in [4].

– As $SS(X_i, X_k) \subseteq \mathrm{In}(X_i, X_k)$, then $L(X_i, X_k) = \mathrm{In}(X_i, X_k)$ for $i > k$ and, by symmetry (see Remark 2.16), $L(X_i, X_k) = \mathrm{In}(X_i, X_k)$ for $i \neq k$.

Let us define $Z = X_1 \oplus \cdots \oplus X_n$. If we set every element $T \in L(X)$ in matrix form $T = (T_{ik})$ with $T_{ik} \in L(X_k, X_i)$, then we get that $L(Z)/SS(Z)$ is isomorphic to the algebra of all lower triangular $n \times n$ matrices and that $L(Z)/\mathrm{In}(Z)$ is isomorphic to the algebra of all diagonal $n \times n$ matrices. Thus, $\mathrm{rad}(L(Z)/SS(Z)) = \mathrm{In}(Z)/SS(Z) \neq 0$. Indeed, $\dim \mathrm{rad}(L(Z)/SS(Z)) = n(n-1)/2$.

COROLLARY 2.21. *Let X be a Σ_e^n space. Then every irrefinable decomposition of X has n summands.*

Proof. Let $X = X_1 \oplus \cdots \oplus X_m$ be an irrefinable decomposition. Obviously, $m \leqslant n$. Moreover, if we proceed as in the proof of Theorem 2.18, we get that $|\sigma_e(T)| \leqslant m$, for every $T \in L(X)$. Thus $m = n$. ∎

COROLLARY 2.22. *Let* $X \in \Sigma_e^n$ *and* $Y \in \Sigma_e^m$. *Then* $X \oplus Y \in \Sigma_e^{n+m}$.

Proof. Let $X = X_1 \oplus \cdots \oplus X_n$ and $Y = Y_1 \oplus \cdots \oplus Y_m$ be irrefinable decompositions. Then $X_1 \oplus \cdots \oplus X_n \oplus Y_1 \oplus \cdots \oplus Y_m$ is an irrefinable decomposition of $X \oplus Y$, and the result follows from Corollary 2.21. ∎

Let $X \in \Sigma_e^n$ and let $X = X_1 \oplus \cdots \oplus X_n$ be an irrefinable decomposition. As in the proof of Theorem 2.18, we divide the set $\{1, \ldots, n\}$ into r subsets $S_1, \ldots, S_r$ so that X_i and X_j are essentially isomorphic when i and j belong to the same set, and they are essentially incomparable when they belong to different sets. We denote by n_l the number of spaces in the class S_l. We also assume that $n_1 \leqslant \cdots \leqslant n_r$.

We set

$$\tau(X_1 \oplus \cdots \oplus X_n) := (n_1, \ldots, n_r).$$

THEOREM 2.23. *Let* $X \in \Sigma_e^n$ *and let* $Y_1 \oplus \cdots \oplus Y_n$ *and* $Z_1 \oplus \cdots \oplus Z_n$ *be irrefinable decompositions of* X. *Then*

$$\tau(Y_1 \oplus \cdots \oplus Y_n) = \tau(Z_1 \oplus \cdots \oplus Z_n).$$

Moreover, each summand Y_i *is essentially isomorphic to some* Z_j, *and vice versa.*

Proof. Let q_i be the projection of X onto Z_i. Clearly, for each index i there exists an index j so that $(q_i)_{|Y_j}$ is not inessential. Hence, by Proposition 2.14, $(q_i)_{|Y_j}$ is Fredholm.

After a reordering of the summands we can suppose that $i = j = 1$ and set $Y =: Y_1 \oplus \cdots \oplus Y_k$, $Z =: Z_1 \oplus \cdots \oplus Z_l$, for $Y_1, \ldots, Y_k$ the summands essentially isomorphic to Y_1, and $Z_1, \ldots, Z_l$ the summands essentially isomorphic to Z_1. By Corollary 2.22, $Y \in \Sigma_e^k$ and $Z \in \Sigma_e^l$.

Since the sum of a Fredholm operator and an inessential operator is Fredholm [16], the operator from Y into Z induced by both decompositions is Fredholm. Thus, there exist isomorphic finite codimensional subspaces $Y_0 \subseteq Y$ and $Z_0 \subseteq Z$. By Lemma 2.8 we have $Y_0 \in \Sigma_e^k$ and $Z_0 \in \Sigma_e^l$, so $k = l$. ∎

REMARK 2.24. The first part of Theorem 2.23 can be proved in a more algebraic way by observing that the numbers n_i are uniquely determined by X. In fact, they are the dimensions of the irreducible representations of the semisimple algebra $L(X)/\mathrm{In}(X) \simeq M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$.

3. K-THEORY FOR Σ_e^n SPACES

The basic concepts, definitions and results used in this section can be found in [3], [17] or [19]. Let K_0 and K_1 denote the usual K-functors in K-theory. Given a projection $P \in L(X)$, $[P]_0$ stands for the class of P in $K_0(L(X))$.

Let X be a Σ_e^n space (for instance a HD_n space or a QD_n space) and let $X = X_1 \oplus \cdots \oplus X_n$ be an irrefinable decomposition with $\tau(X_1 \oplus \cdots \oplus X_n) = (n_1, \ldots, n_r)$. We select summands $X_{i_1}, \ldots, X_{i_r}$ pairwise essentially incomparable and denote by P_j the natural projection on X with $R(P_j) = X_{i_j}$. Let P_0 denote a projection on X with one-dimensional range.

It is not difficult to see that two projections on a Banach space X define the same class in $K_0(L(X))$ if and only if their ranges are linearly homeomorphic (see Proposition 2.1 of [13]). Therefore, for X as in the previous paragraph, it is reasonable to hope that the K_0 group of $L(X)$ can be described in terms of the classes $[P_0]_0, \dots, [P_r]_0 \in K_0(L(X))$. We will show that this is the case.

In the particular case that X is indecomposable, the algebra $\mathcal{A} = L(X)/J$ for $J = SC(X)$ or $J = SS(X)$, is a subalgebra of the algebra $\mathcal{U}_n$ of all the upper triangular $n \times n$ complex matrices with constant diagonal ([10], Theorem 5.7). For example, if $\mathcal{A} = \mathcal{U}_n$, $\mathcal{A}$ is homotopy equivalent (as an algebra) to its diagonal, i.e., to $\mathbb{C}$. Therefore, $K_0(\mathcal{A}) = \mathbb{Z}$, $K_1(\mathcal{A}) = 0$. Since $K_0(J) = \mathbb{Z}[P_0]_0$ and $K_1(J) = 0$ (see Theorem 5.2 of [14]) it follows easily from the cyclic six-term exact sequence ([17], Theorem 12.1.2) that $K_0(L(X)) = \mathbb{Z}^2 \simeq \mathbb{Z}[P_0]_0 \oplus \mathbb{Z}[I_X]_0$, ($P_0$ as above).

Now we can obtain a more general result using the algebra $L(X)/\mathrm{In}(X)$. Observe that a priori it does not seems straightforward to prove this result directly for the algebras $L(X)/SS(X)$ or $L(X)/SC(X)$.

THEOREM 3.1. *With the previous notations, let J be a non-zero, closed ideal of inessential operators on X and let $\mathcal{B} = L(X)/J$. Then $K_1(L(X)) = K_1(\mathcal{B}) = 0$ and*

$$K_0(L(X)) = \bigoplus_{i=0}^{r} \mathbb{Z}[P_i]_0; \qquad K_0(\mathcal{B}) = \bigoplus_{i=1}^{r} \mathbb{Z}[P_i]_0.$$

Proof. Let us consider the exact sequence $0 \longrightarrow J \longrightarrow L(X) \longrightarrow \mathcal{B} \longrightarrow 0$. By Theorem 5.2 of [14], the associated cyclic six-term exact sequence looks as follows

$$0 \longrightarrow K_1(L(X)) \longrightarrow K_1(\mathcal{B}) \xrightarrow{\delta_1} \mathbb{Z} = K_0(J) \xrightarrow{\omega} K_0(L(X)) \longrightarrow K_0(\mathcal{B}) \longrightarrow 0$$

where $\omega(k) = k[P_0]_0$ and δ_1 is called index map because it fits into the commutative diagram (see Proposition 4.1 of [13])

$$\begin{array}{ccccc} \Phi(X^m) & \xrightarrow{\text{onto}} & \mathrm{Inv}_m(\mathcal{B}) & \longrightarrow & K_1(\mathcal{B}) \\ \downarrow{\scriptstyle \mathrm{ind}} & & & & \downarrow{\scriptstyle \delta_1} \\ \mathbb{Z} & & \xrightarrow[\omega]{\sim} & & K_0(J) \end{array}$$

where $\mathrm{Inv}_m(\mathcal{B})$ are the invertible elements of $M_m(\mathcal{B})$.

By Corollary 2.22, X^m is in Σ_{e}^{nm}. Therefore $\delta_1 = 0$ by Proposition 2.4. It follows from the exact sequence that $K_0(\mathcal{B})$ and $K_1(\mathcal{B})$ do not depend on J. Therefore, we can calculate them taking $J = \mathrm{In}(X)$. In that case we have, by Theorem 2.18, that $\mathcal{B} = M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$, so $K_1(\mathcal{B}) = 0$. We conclude the proof by observing that $[P_i]_0$ generates the factor $K_0(M_{n_i}(\mathbb{C}))$ in $K_0(\mathcal{B})$ for $i = 1, \dots, n$, and $[P_0]_0$ generates $K_0(J)$. ∎

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