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GENERALIZED GAUSSIAN ESTIMATES AND THE LEGENDRE TRANSFORM

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ABSTRACT. Generalized Gaussian estimates (GGEs) are the main tool for the L_p-extension of L_2-properties of elliptic operators without heat kernel, e.g. operators of higher order and operators with complex or unbounded coefficients. In this paper, we give several characterizations of GGEs which are important for the applicability of such L_p-extension results. As an application of these characterizations, we show a result on the spectral L_p-independence of operators satisfying GGEs.

KEYWORDS: *Generalized Gaussian estimates, elliptic operator, heat kernel.*

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1. INTRODUCTION AND MAIN RESULTS

A central (almost) classical tool for extending properties of non-negative selfadjoint operators $-A$ or, more generally, of generators of semigroups $(\mathrm{e}^{tA})_{t\in\mathbb{R}_+}$, from L_2 to L_p are so-called Gaussian estimates (GEs). This means that the e^{tA} have integral kernels $k_t(x,y)$ satisfying

$$|k_t(x,y)| \leqslant v_{r_t}(x)^{-1} g\Big(\frac{d(x,y)}{r_t}\Big) \quad \text{for all } x,y\in\Omega, t\in\mathbb{R}_+ \tag{1.1}$$

and some positive radii $(r_t)_{t\in\mathbb{R}_+}$. Here (Ω,d,μ) is the underlying metric measure space of homogeneous type, i.e.

$$v_{2r}(x) \leqslant C v_r(x) \quad \text{for all } x\in\Omega, r>0, \tag{1.2}$$

$v_r(x) := \mu(B(x,r))$ for the ball $B(x,r)$ around x of radius r, and $g:\mathbb{R}_{\geqslant 0}\to\mathbb{R}_+$ is some decreasing function such that $h := -\log(g)$ is convex and $\liminf\limits_{t\to\infty} \frac{h(t)}{t} > 0$. We will use the Legendre transform $h^{\#}:\mathbb{R}\to[-h(0),\infty]$ defined by

$$h^{\#}(s) := \sup_{t\geqslant 0} st - h(t), \quad s\in\mathbb{R}.$$

Indeed, numerous authors showed results of the following type: A satisfies a given operator property on L_p for all $p \in (1, \infty)$, provided A satisfies the property on L_2 and GEs of type (1.1). We want to mention the properties of generating an (analytic) semigroup [12], [21], having an H^∞ functional calculus [15], [16], maximal regularity [10], [17], and Riesz transforms [3], [9]; many other references can be found in the papers we already mentioned. The following result can be seen as the "heart" of Davies' perturbation method, a well-known and important tool for the verification of GEs. It is therefore not really new but the role of the Legendre transform has never been pointed out before.

PROPOSITION 1.1. *Let (Ω, μ, d) be a metric measure space and $\mathcal{A}$ a set of measurable functions $\phi : \Omega \to \mathbb{R}$ such that*

$$d(x, y) = \sup_{\phi \in \mathcal{A}} (\phi(x) - \phi(y)) \quad \text{for all } x, y \in \Omega.$$

Let R be a linear operator having an integral kernel $k(x, y)$ and let $r > 0$. Then the following are equivalent:

(i) $|k(x, y)| \leqslant e^{-h(d(x,y)r^{-1})}$ *for all $x, y \in \Omega$;*

(ii) $\|e^{-\rho\phi} R e^{\rho\phi}\|_{1\to\infty} \leqslant e^{h^\#(\rho r)}$ *for all $\phi \in \mathcal{A}, \rho \geqslant 0$.*

Typical examples for the set $\mathcal{A}$ are $\mathcal{A} = \{d(x, \cdot) : x \in \Omega\}$, $\mathcal{A} = \{\phi \in C_b^\infty(\Omega) : |\nabla\phi| \leqslant 1\}$ (Ω Riemannian manifold) and $\mathcal{A} = \{\langle x, \cdot\rangle : |x| = 1\}$ ($\Omega \subset \mathbb{R}^D$). Applying Proposition 1.1 for the $R_t = e^{tA}$ and $h_t(s) = \log(r_t^D g(s)^{-1})$, where $D \in \mathbb{R}_+$, is the standard method to verify estimates of the type

$$|k_t(x, y)| \leqslant r_t^{-D} g\Big(\frac{d(x, y)}{r_t}\Big) \quad \text{for all } x, y \in \Omega, t \in \mathbb{R}_+;$$

see e.g. [1], [8], [11]. Unfortunately, this estimate is a GE of the type (1.1) (and thus helpful for the L_p-extension of L_2-properties of A) only if Ω is of polynomial volume growth (i.e. $v_r(x) \leqslant Cr^D$ for all $x \in \Omega, r > 0$) which is a strictly stronger condition than being of homogeneous type (1.2). Moreover, there are many important operators A whose semigroup e^{tA} does not satisfy GEs. This happens e.g. for elliptic operators A of order m with measurable coefficients on $\mathbb{R}^D$ if $D > m$ [2], [14] or if the coefficients are unbounded [20]. But in many of these cases A still satisfies so-called Generalized Gaussian Estimates (GGEs) of the following type [13], [22]:

$$\|\chi_{B(x,r_t)} e^{tA} \chi_{B(y,r_t)}\|_{p_0\to q_0} \leqslant v_{r_t}(x)^{1/q_0 - 1/p_0} g\Big(\frac{d(x, y)}{r_t}\Big) \tag{1.3}$$

for all $x, y \in \Omega$, $t > 0$, and for some $1 \leqslant p_0 < 2 < q_0 \leqslant \infty$. Note that the GGE (1.3) for the special case $(p_0, q_0) = (1, \infty)$ is equivalent to the GE (1.1); see Proposition 3.6 below. GGEs allow to extend L_2-properties of A to L_p for all $p \in (p_0, q_0)$ (which is in general the optimal p-interval!). We mention again the properties of generating an (analytic) semigroup [13], having an H^∞ functional calculus [6], maximal regularity [5] and Riesz transforms [7], [18]. Hence, extending $L_2(\Omega)$-properties of A to $L_p(\Omega)$ via Proposition 1.1 requires (in general) too strong restrictions on both Ω and A. This motivates the first main result of this paper which is a generalization of Proposition 1.1 for GGEs of the "right" type (1.3) on spaces of the "right" type (1.2).

THEOREM 1.2. *Let $(\Omega, \mathfrak{A}, \mu, d)$ be a space of homogeneous type and $\mathcal{A}$ a set of measurable functions $\phi : \Omega \to \mathbb{R}$ such that*

$$d(E,F) = \sup_{\phi \in \mathcal{A}} \Big(\inf_F \phi - \sup_E \phi \Big) \quad \textit{for all } E, F \in \mathcal{S} \tag{1.4}$$

and for some $\mathcal{S} \subset \mathfrak{A}$ containing all balls. Let $1 \leqslant p \leqslant q \leqslant \infty$ and $\gamma \geqslant 0$. Let R be a linear operator and $r > 0$. Then the following are equivalent:

(i) *we have for some/all $\alpha, \beta \geqslant 0$ such that $\alpha + \beta = \gamma$:*

$$\|\chi_{B(x,r)} R \chi_{B(y,r)}\|_{p\to q} \leqslant v_r(x)^{-\alpha} v_r(y)^{-\beta} \mathrm{e}^{-h(d(x,y)r^{-1})} \quad \textit{for all } x, y \in \Omega;$$

(ii) *we have for some/all $\alpha, \beta \geqslant 0$ such that $\alpha + \beta = \gamma$:*

$$\|\mathrm{e}^{-\rho\phi} v_r^{\alpha} R v_r^{\beta} \mathrm{e}^{\rho\phi}\|_{p\to q} \leqslant \mathrm{e}^{h^{\#}(\rho r)} \quad \textit{for all } \phi \in \mathcal{A}, \rho \geqslant 0;$$

(iii) *we have for some/all $\alpha, \beta \geqslant 0$ such that $\alpha + \beta = \gamma$:*

$$\|\chi_E v_r^{\alpha} R v_r^{\beta} \chi_F\|_{p\to q} \leqslant \mathrm{e}^{-h(d(E,F)r^{-1})} \quad \textit{for all } E, F \in \mathcal{S}.$$

Here the statement is written modulo identification of h (similarly for $h^{\#}$) and $\widetilde{h}$, where $\widetilde{h}(u) := ch(bu) - a$ for some constants $a, b, c > 0$ independent of R and r.

REMARK 1.3. (i) Proposition 1.1 is the part (i)⇒(ii) of Theorem 1.2 for the special case $(p,q) = (1,\infty)$ and $(\alpha, \beta, \gamma) = (0,0,0)$. Indeed, in this case, the parts (i) of Proposition 1.1 and Theorem 1.2 are equivalent; see Proposition 3.6 below.

(ii) The implication (ii)⇒(iii) of Theorem 1.2 for $\gamma = \frac{1}{p} - \frac{1}{q}$, $\mathcal{A} = \{\phi \in C_{\mathrm{b}}^{\infty}(\Omega) : \|\nabla^j \phi\|_\infty \leqslant 1 \text{ for } j = 1, \ldots, m\}$ and the simple case $\Omega = \mathbb{R}^D$ is implicitly used in [13] where Davies verifies GGEs for elliptic operators A of order $2m$ on $\mathbb{R}^D$. Note that our hypothesis (1.4) appears in p.147 of [13].

(iii) The same implication (ii)⇒(iii) for $\gamma = \frac{1}{p} - \frac{1}{q}$, $\mathcal{A} = \{\langle x, \cdot\rangle : |x| = 1\}$ and $\Omega = \mathbb{R}^D$ is implicitly used (for E, F =cubes in $\mathbb{R}^D$) in [22] where Schreieck and Voigt verify GGEs for Schrödinger operators with singular potentials on $\mathbb{R}^D$.

(iv) Theorem 1.2 (in its detailed version Proposition 2.1 below) is the central tool for the optimal extension of GGEs for real times t as in (1.3) to GGEs for complex times z of the type

$$\|\chi_{B(x,r_z)} \mathrm{e}^{zA} \chi_{B(y,r_z)}\|_{p_0 \to q_0} \leqslant v_{r_z}(x)^{1/q_0 - 1/p_0} C(z) g\Big(\frac{d(x,y)}{r_z}\Big),$$

for all $x, y \in \Omega$ and $z \in \mathbb{C}_+$; see the proof of Theorem 2.1 in [4]. This extension is of great importance for many applications of GGEs. For example, the last estimate implies directly by Proposition 2.1(ii)

$$\|e^{zA}\|_{p_0\to p_0} \leqslant C_0 C(z) \quad \text{for all } z \in \mathbb{C}_+.$$

This $\|e^{zA}\|_{p_0\to p_0}$-estimate is optimal for the class of operators A satisfying (1.3) (take $A = \Delta_{\mathbb{R}^D}$) and crucial for the L_p-boundedness of Riesz means of the Schrödinger group $(e^{itA})_{t\in\mathbb{R}}$; see Theorems 1.1 and 1.3 in [4].

Another application of Theorem 1.2 is the L_p-independence of the spectrum of operators R satisfying one of the equivalent conditions (i),(ii),(iii) (for $\gamma = \frac{1}{p} - \frac{1}{q}$). For $(p,q) \neq (1,\infty)$, the first result in this direction is in [22], further contributions may be found in [13], [12], [19], [20] and the literature cited in these works. We take the opportunity to give the following result which extends several of those we just mentioned. Moreover, the proof shows a nice interaction between the different parts of Theorem 1.2 (and Proposition 2.1 below).

PROPOSITION 1.4. *Let (Ω,μ,d) be a space of homogeneous type and $1 \leqslant p \leqslant q \leqslant \infty$. Let R be a linear operator and $r > 0$ such that*

$$\|\chi_{B(x,r)} R \chi_{B(y,r)}\|_{p\to q} \leqslant v_r(x)^{1/q-1/p} e^{-h(d(x,y)r^{-1})} \quad \textit{for all } x,y \in \Omega. \tag{1.5}$$

Then $R \in \mathfrak{L}(L_u(\Omega))$ for all $u \in [p,q]$, and the spectrum of R on $L_u(\Omega)$ is independent of $u \in [p,q]$.

As usual, on $L_\infty(\Omega)$ one considers the unique w^*-continuous extension of R. By the spectral mapping theorem, Proposition 1.4 yields spectral L_p-independence also for generators A of analytic semigroups $(e^{tA})_{r\in\mathbb{R}_+}$ satisfying GGEs.

COROLLARY 1.5. *Let (Ω,μ,d) be a space of homogeneous type and $1 \leqslant p \leqslant p_0 \leqslant q \leqslant \infty$. Let $(e^{tA})_{t\in\mathbb{R}_+}$ be a bounded analytic semigroup on $L_{p_0}(\Omega)$ such that*

$$\|\chi_{B(x,r_t)} e^{tA} \chi_{B(y,r_t)}\|_{p_0\to q_0} \leqslant v_{r_t}(x)^{1/q-1/p} g\Big(\frac{d(x,y)}{r_t}\Big) \quad \textit{for all } x,y \in \Omega, t \in \mathbb{R}_+.$$

Then, for all $u \in [p,q], u \neq \infty$, the semigroup $(e^{tA})_{t\in\mathbb{R}_+}$ is bounded analytic on $L_u(\Omega)$, and the spectrum of A on $L_u(\Omega)$ is independent of $u \in [p,q], u \neq \infty$.

Proof. Let $u \in [p,q], u \neq \infty$. Then $(e^{tA})_{t\in\mathbb{R}_+}$ is bounded analytic on $L_u(\Omega)$ by interpolation; see e.g. the arguments in [13], [21]. Hence the spectral mapping theorem $\sigma(e^{tA}) \setminus \{0\} = e^{t\sigma(A)}$ holds on $L_u(\Omega)$. But $\sigma(e^{tA})$ on $L_u(\Omega)$ is independent of u by Proposition 1.4. ∎

Detailed examples of elliptic operators A whose semigroup (e^{tA}) satisfies GGEs are given in [22], [13], [19], [20] and summarized e.g. in [6], [4] where operators of the following types are discussed: higher order operators with complex coefficients, Schrödinger operators with singular potentials and (more generally) second order operators with real and singular lower order coefficients.

2. MODIFICATIONS OF THE MAIN RESULT

For arbitrary spaces of homogeneous type (i.e. without condition (1.4)), one obtains the following version of Theorem 1.2.

PROPOSITION 2.1. *Let (Ω, μ, d) be a space of homogeneous type. Let $1 \leqslant p \leqslant q \leqslant \infty$ and $\gamma \geqslant 0$. Let R be a linear operator and $r > 0$.*

(i) *The following are equivalent:*

(1) *We have for $(u, v) = (p, q)$ and some $\alpha, \beta \geqslant 0$ such that $\alpha + \beta = \gamma$:*

$$\|\chi_{B(x,r)} R \chi_{B(y,r)}\|_{u \to v} \leqslant v_r(x)^{-\alpha} v_r(y)^{-\beta} g(d(x,y) r^{-1}) \quad \textit{for all } x, y \in \Omega.$$

(2) *We have for $(u, v) = (p, q)$ and some $\alpha, \beta \geqslant 0$ such that $\alpha + \beta = \gamma$:*

$$\|\chi_{B_1} v_r^{\alpha} R v_r^{\beta} \chi_{B_2}\|_{u \to v} \leqslant g(d(B_1, B_2) r^{-1}) \quad \textit{for all balls } B_1, B_2 \subset \Omega.$$

(3) *We have for $(u, v) = (p, q)$ and some $\alpha, \beta \geqslant 0$ such that $\alpha + \beta = \gamma$:*

$$\|\chi_{B(x,r)} R \chi_{A(x,r,k)}\|_{u \to v} \leqslant v_r(x)^{-(\alpha+\beta)} g(k) \quad \textit{for all } x \in \Omega, k \in \mathbb{N}.$$

In (1), (2), *and* (3), *one may replace "for $(u, v) = (p, q)$ and some $\alpha, \beta \geqslant 0$ such that $\alpha + \beta = \gamma$" equivalently by "for all $p \leqslant u \leqslant v \leqslant q$ and all $\alpha, \beta \geqslant 0$ such that $\alpha + \beta - \gamma = \frac{1}{u} + \frac{1}{q} - \frac{1}{v} - \frac{1}{p}$".*

(ii) *If* (1) *holds then we have for all $p \leqslant u \leqslant v \leqslant q$ and all $\alpha, \beta \geqslant 0$ such that $\alpha + \beta - \gamma = \frac{1}{u} + \frac{1}{q} - \frac{1}{v} - \frac{1}{p}$:*

$$\|v_r^{\alpha} R v_r^{\beta}\|_{u \to v} \leqslant C_0 \sum_{k=0}^{\infty} (k+1)^{\lambda_0} g(k).$$

Here the constants $C_0, \lambda_0 \geqslant 0$ are independent of u, v, α, β and R, r, g.

(iii) *If* (1) *holds for $\gamma = \frac{1}{p} - \frac{1}{q}$ then, for all $u \in [p, q]$, we have $R \in \mathfrak{L}(L_u(\Omega))$ and*

$$\sup_{x \in \Omega} \|\mathrm{e}^{\rho d(x,\cdot)} R \mathrm{e}^{-\rho d(x,\cdot)} - R\|_{u \to u} \to 0 \quad \textit{for } \rho \searrow 0.$$

Here $A(x, r, k)$ denotes the annular set $B(x, (k+1)r) \setminus B(x, kr)$. The statement (i) is written modulo identification of g and $\widetilde{g}$, where $\widetilde{g}(u) := a g(bu)^c$ for some constants $a, b, c > 0$ independent of R and r.

Note that $d(B(x,r), A(x,r,k)) r^{-1} \in [k-1, k+1]$, i.e. the conditions (2) and (3) are of a similar type as condition (iii) in Theorem 1.2.

3. PROOFS

Recall that the function $h : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ is convex, hence $(h^{\#})^{\#} = h$ by the Fenchel-Moreau theorem. We extend h to $\mathbb{R}_{-}$ by setting $h(-t) := h(0), t > 0$. This extension is convex since h is increasing on $\mathbb{R}_{\geqslant 0}$.

We use the symbols $\preceq$ and $\succeq$ to indicate domination up to constants independent of the relevant parameters. The symbol $\sim$ indicates the validity of $\preceq$ and $\succeq$.

Proof of Proposition 1.1. Since the operator R has integral kernel $k(x,y)$, the operator $\mathrm{e}^{-\rho\phi}R\mathrm{e}^{\rho\phi}$ has integral kernel $(x,y)\mapsto \mathrm{e}^{-\rho\phi(x)}k(x,y)\mathrm{e}^{\rho\phi(y)}$.

(ii)⇒(i): We obtain from (ii) that

$$|k(x,y)|\mathrm{e}^{\rho(\phi(y)-\phi(x))}\leqslant \mathrm{e}^{h^{\#}(\rho r)}\qquad \text{for all } x,y\in\Omega, \phi\in\mathcal{A}, \rho\geqslant 0.$$

Optimizing with respect to $\phi\in\mathcal{A}$ yields

$$|k(x,y)|\leqslant \mathrm{e}^{h^{\#}(\rho r)-\rho d(x,y)}\qquad \text{for all } x,y\in\Omega, \rho\geqslant 0.$$

Finally, optimizing with respect to $\rho\geqslant 0$ yields

$$|k(x,y)|\leqslant \mathrm{e}^{-(h^{\#})^{\#}(d(x,y)r^{-1})}=\mathrm{e}^{-h(d(x,y)r^{-1})}\qquad \text{for all } x,y\in\Omega.$$

(i)⇒(ii): We have for all $\phi\in\mathcal{A}, \rho\geqslant 0$:

$$\begin{aligned}\|\mathrm{e}^{-\rho\phi}R\mathrm{e}^{\rho\phi}\|_{1\to\infty} &= \sup_{x,y}\mathrm{e}^{-\rho\phi(x)}|k(x,y)|\mathrm{e}^{\rho\phi(y)}\\ &\leqslant \sup_{x,y}\mathrm{e}^{\rho d(x,y)}\mathrm{e}^{-h(d(x,y)r^{-1})} &&\text{[by (i)]}\\ &\leqslant \sup_{t\geqslant 0}\mathrm{e}^{\rho r t-h(t)}=\mathrm{e}^{h^{\#}(\rho r)}. \quad ∎\end{aligned}$$

For operators R without integral kernel, an adaptation of the above proof (ii)⇒(i) yields the following.

REMARK 3.1. Let $(\Omega,\mathfrak{A},\mu,d)$ be a metric measure space, $E,F\in\mathfrak{A}$ and $\mathcal{A}$ a set of measurable functions $\phi:\Omega\to\mathbb{R}$ such that

$$d(E,F)=\sup_{\phi\in\mathcal{A}}\Big(\inf_F\phi-\sup_E\phi\Big).$$

Let $1\leqslant p,q\leqslant\infty$, let R be a linear operator and $r>0$ such that

$$\|\mathrm{e}^{-\rho\phi}R\mathrm{e}^{\rho\phi}\|_{p\to q}\leqslant \mathrm{e}^{h(\rho r)}\qquad \text{for all } \phi\in\mathcal{A}, \rho\geqslant 0.$$

Then $\|\chi_E R\chi_F\|_{p\to q}\leqslant \mathrm{e}^{-h^{\#}(d(E,F)r^{-1})}$.

Proof. By hypothesis, we have for all $\phi\in\mathcal{A}$ and $\rho\geqslant 0$:

$$\begin{aligned}\|\chi_E R\chi_F\|_{p\to q}&\leqslant \|\mathrm{e}^{\rho\phi}\|_{L_\infty(E)}\|\mathrm{e}^{-\rho\phi}R\mathrm{e}^{\rho\phi}\|_{p\to q}\|\mathrm{e}^{-\rho\phi}\|_{L_\infty(F)}\\ &\leqslant \mathrm{e}^{\rho\sup_E\phi}\mathrm{e}^{h(\rho r)}\mathrm{e}^{-\rho\inf_F\phi}.\end{aligned}$$

Optimizing with respect to $\phi\in\mathcal{A}$ yields for all $\rho\geqslant 0$:

$$\|\chi_E R\chi_F\|_{p\to q}\leqslant \mathrm{e}^{h(\rho r)-\rho d(E,F)}.$$

Optimizing with respect to $\rho\geqslant 0$ yields:

$$\|\chi_E R\chi_F\|_{p\to q}\leqslant \mathrm{e}^{-h^{\#}(d(E,F)r^{-1})}. \quad ∎$$

LEMMA 3.2. *Let (Ω, μ, d) be a space of homogeneous type. Then there exists a constant $C_0 > 0$ such that*

$$C_0^{-1} \leqslant \frac{|B(x,r)|}{|B(y,r)|} \leqslant C_0 \quad \text{for all } r > 0, x \in \Omega, y \in B(x,r).$$

Proof. This is clear since $|B(x,r)| \leqslant |B(y,2r)| \leqslant C_0|B(y,r)|$ if $y \in B(x,r)$. ∎

We will use the following notations:

$$N_{p,r}f(x) := |B(x,r)|^{-1/p}\|f\|_{L_p(B(x,r))}$$
$$N_{p,q,r}f(x) := |B(x,r)|^{-1/q}\|f\|_{L_p(B(x,r))}.$$

LEMMA 3.3. *Let Ω be a space of homogeneous type, $1 \leqslant p \leqslant q \leqslant \infty$ and $r > 0$.*

(i) $N_{p,r}f(x) \leqslant N_{q,r}f(x)$.

(ii) $C_1^{-1}\|f\|_p \leqslant \|N_{p,r}f\|_p \leqslant C_1\|f\|_p$.

(iii) *Let $1 \leqslant u \leqslant v \leqslant \infty$ be such that $\frac{1}{q} + \frac{1}{u} = \frac{1}{p} + \frac{1}{v}$. Then $\|N_{p,q,r}f\|_v \leqslant C_1\|f\|_u$.*

In (ii) *and* (iii) *the constant C_1 is independent of $r > 0$.*

Proof. (i) follows directly from Hölder's inequality.

(ii) A simple calculation using Fubini's Theorem shows $\|f\|_p = \|\widetilde{N}_{p,r}f\|_p$ for the

$$\widetilde{N}_{p,r}f(x) := \Big(\int\limits_{B(x,r)} |f(y)|^p \frac{\mathrm{d}\mu(y)}{|B(y,r)|}\Big)^{1/p}.$$

But on the other hand we have $C_0^{-1/p}N_{p,r}f(x) \leqslant \widetilde{N}_{p,r}f(x) \leqslant C_0^{1/p}N_{p,r}f(x)$, where C_0 is the constant in Lemma 3.2.

(iii) Note that we have for $K_r(x,y) := |B(x,r)|^{-p/q}\chi_{B(x,r)}(y)$:

$$N_{p,q,r}f(x)^p = \int\limits_{\Omega} K_r(x,y)|f(y)|^p\mathrm{d}y = K_r(|f|^p)(x).$$

On the other hand, we have $\|K_r(x,\cdot)\|_{q/p}, \|K_r(\cdot,x)\|_{q/p} \leqslant C_0$ for all $x \in \Omega$ and all $r > 0$:

$$\|K_r(x,\cdot)\|_{q/p} = |B(x,r)|^{-p/q}\Big(\int\limits_{B(x,r)} 1\mathrm{d}y\Big)^{p/q} = 1$$

$$\|K_r(\cdot,x)\|_{q/p} = \Big(\int\limits_{B(x,r)} |B(y,r)|^{-1}\mathrm{d}y\Big)^{p/q} \leqslant C_0 \qquad \text{[by Lemma 3.2]}.$$

Hence the assertion follows from a standard Young-type argument:

$$\begin{aligned}\|N_{p,q,r}f\|_v &= \|K_r(|f|^p)\|_{v/p}^{1/p} \leqslant (\|K_r\|_{u/p\to v/p}\||f|^p\|_{u/p})^{1/p}\\ &\leqslant C_0^{1/p}\|f\|_u.\end{aligned}$$ ∎

LEMMA 3.4. *Let Ω be a space of homogeneous type and $p, p_0, q \in [1,\infty]$. Then we have for all linear operators R, S and all $r, s > 0$:*

$$\|RS\|_{p\to q} \leqslant C_0 \int_\Omega \|R\chi_{B(z,r)}\|_{p_0\to q}\|\chi_{B(z,s)}S\|_{p\to p_0}|B(z, r\wedge s)|^{-1}\mathrm{d}z$$

The constant C_0 is independent of R, S and r, s.

Proof. Denoting $t := r \wedge s$, we can estimate in the following way:

$$\begin{aligned}
\langle r, RSf\rangle = \langle R'g, Sf\rangle &= \int_\Omega (R'g)(x) \int_\Omega \chi_{B(z,t)}(x)\mathrm{d}z v_t(x)^{-1}(Sf)(x)\mathrm{d}x \\
&= \int_\Omega \langle R'g, \chi_{B(z,t)}v_t^{-1}Sf\rangle \mathrm{d}z && \text{[Fubini]} \\
&\leqslant \|g\|_{q'} \int_\Omega \|R\chi_{B(z,t)}\|_{p_0\to q}\|\chi_{B(z,t)}v_t^{-1}Sf\|_{p_0}\mathrm{d}z \\
&\preceq \|g\|_{q'} \int_\Omega \|R\chi_{B(z,r)}\|_{p_0\to q}\|\chi_{B(z,t)}Sf\|_{p_0}v_t(z)^{-1}\mathrm{d}z && [t\leqslant r, \text{Lemma 3.2}] \\
&\leqslant \|g\|_{q'}\|f\|_p \int_\Omega \|R\chi_{B(z,r)}\|_{p_0\to q}\|\chi_{B(z,s)}S\|_{p\to p_0}v_t(z)^{-1}\mathrm{d}z && [t\leqslant s]. \quad \blacksquare
\end{aligned}$$

Proof of Theorem 1.2. We will use condition (3) of Proposition 2.1 in the proof. We will show the following implications:

$$\text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(i)} \Rightarrow (3) \Rightarrow \text{(ii)}.$$

(ii)⇒(iii): This implication follows directly from Remark 3.1, applied for $h^\#$ and $v_r^\alpha R v_r^\beta$ instead of h and R. From now on we suppose that Ω is of homogeneous type. In particular, Ω is of some dimension $D > 0$, i.e.

$$v_{\lambda r}(x) \leqslant C\lambda^D v_r(x) \quad \text{for all } x\in\Omega, \lambda\geqslant 1, r>0. \tag{3.1}$$

(iii)⇒(i): This implication can be seen as follows, using in the second step that $d(B(x,r), B(y,r)) \geqslant d(x,y) - 2r$ and h is increasing:

$$\begin{aligned}
\|\chi_{B(x,r)}R\chi_{B(y,r)}\|_{p\to q} &\sim \left\|\chi_{B(x,r)}\Big(\frac{v_r}{v_r(x)}\Big)^\alpha R\Big(\frac{v_r}{v_r(y)}\Big)^\beta \chi_{B(y,r)}\right\|_{p\to q} && \text{[by Lemma 3.2]} \\
&\leqslant v_r(x)^{-\alpha}v_r(y)^{-\beta}\mathrm{e}^{-h(d(x,y)r^{-1}-2)} && \text{[by (iii)]}.
\end{aligned}$$

(i)⇒(3): This implication can be seen as follows:

$$
\begin{aligned}
&\|\chi_{B(x,r)} R \chi_{A(x,y,k)}\|_{p\to q} \\
&\quad\preceq \int_{\Omega} \|\chi_{B(x,r)} R \chi_{B(y,r)}\|_{p\to q} \|\chi_{B(y,r)} \chi_{A(x,r,k)}\|_{p\to p} v_r(y)^{-1}\mathrm{d}y \quad \text{[by Lemma 3.4]} \\
&\quad\leqslant \int_{B(x,(k+2)r)\setminus B(x,(k-1)r)} v_r(x)^{-\alpha} v_r(y)^{-\beta-1} \mathrm{e}^{-h(d(x,y)r^{-1})}\mathrm{d}y \quad \text{[by (i)]} \\
&\quad\leqslant v_r(x)^{-\alpha}\mathrm{e}^{-h(k)} \int_{B(x,(k+2)r)} v_r(y)^{-\beta-1}\mathrm{d}y \quad \text{[h is incr.]} \\
&\quad\preceq v_r(x)^{-(\alpha+\beta)}\mathrm{e}^{-h(k)}(k+1)^{D(\beta+1)} \quad \text{[by Lemma 3.2 and (3.1)].}
\end{aligned}
$$

(3)⇒(ii): Let $\alpha, \beta \geqslant 0$ such that $\alpha+\beta = \gamma$. Fix $\phi \in \mathcal{A}, \rho \geqslant 0, \varepsilon > 0$ and define $w(x) := \mathrm{e}^{-\rho\phi(x)}$. Denoting $\kappa := \left(\frac{1}{q}+\gamma-\alpha\right), q_0 := \left(\frac{1}{q}+\gamma-\alpha-\beta\right)^{-1}, b_k := \mathrm{e}^{-\varepsilon h(k-2)} - \mathrm{e}^{\varepsilon h(k-1)} \geqslant 0, h_1(s) := (1-\varepsilon)h(s)$ we will show at first the following claim:

$$
N_{q,r}(w v_r^{\alpha} R v_r^{\beta} w^{-1} f) \preceq \mathrm{e}^{2\rho r + h_1^{\#}(\rho r)} \Big(\sum_{k=1}^{\infty} b_k k^{\kappa} (N_{p,q_0,kr} f)^p \Big)^{1/p}.
$$

Indeed, this is obtained as follows by making double use of $\frac{w(x)}{w(y)} \leqslant \mathrm{e}^{\rho d(x,y)}$:

$$
\begin{aligned}
&N_{q,r}(w v_r^{\alpha} R v_r^{\beta} w^{-1} f)(x) \\
&\quad= v_r(x)^{-1/q} \Big\| \chi_{B(x,r)} \frac{w}{w(x)} v_r^{\alpha} R v_r^{\beta} \frac{w(x)}{w} f \Big\|_q \\
&\quad\leqslant v_r(x)^{\alpha-1/q}\mathrm{e}^{\rho r} \sum_{k=0}^{\infty} \Big\| \chi_{B(x,r)} v_r^{\alpha} R v_r^{\beta} \chi_{A(x,r,k)} \frac{w(x)}{w} f \Big\|_q \\
&\quad\leqslant v_r(x)^{\alpha-1/q-\gamma}\mathrm{e}^{\rho r} \sum_{k=0}^{\infty} \mathrm{e}^{-h(k)} \Big\| \chi_{A(x,r,k)} v_r^{\beta} \frac{w(x)}{w} f \Big\|_p \quad \text{[by (3)]} \\
&\quad\leqslant v_r(x)^{\alpha-1/q-\gamma}\mathrm{e}^{\rho r} \sum_{k=0}^{\infty} \mathrm{e}^{-h(k)+(k+1)\rho r} \|\chi_{A(x,r,k)} v_r^{\beta} f\|_p \\
&\quad\leqslant v_r(x)^{\alpha-1/q-\gamma}\mathrm{e}^{2\rho r} \Big(\sup_{k\geqslant 0} \mathrm{e}^{k\rho r - h_1(k)} \Big) \sum_{k=0}^{\infty} \mathrm{e}^{-\varepsilon h(k)} \|\chi_{A(x,r,k)} v_r^{\beta} f\|_p \\
&\quad\preceq v_r(x)^{\alpha-1/q-\gamma}\mathrm{e}^{2\rho r + h_1^{\#}(\rho r)} \Big(\sum_{k=1}^{\infty} b_k \|\chi_{B(x,kr)} v_r^{\beta} f\|_p^p \Big)^{1/p} \quad \text{[summ. by parts]} \\
&\quad\preceq \mathrm{e}^{2\rho r + h_1^{\#}(\rho r)} \Big(\sum_{k=1}^{\infty} b_k k^{\kappa} |B(x,kr)|^{-p/q_0} \|\chi_{B(x,kr)} f\|_p^p \Big)^{1/p} \quad \text{[by (3.1)].}
\end{aligned}
$$

Now the proof of (ii) can be finished as follows:

$$\begin{aligned}
&\mathrm{e}^{-2\rho r-h_1^\#(\rho r)}\|wv_r^\alpha R v_r^\beta w^{-1}f\|_q\\
&\qquad\sim \mathrm{e}^{-2\rho r-h_1^\#(\rho r)}\|N_{q,r}(wv_r^\alpha R v_r^\beta w^{-1}f)\|_q && \text{[by Lemma 3.3(ii)]}\\
&\qquad\preceq \Big\|\Big(\sum_{k=1}^\infty b_k k^\kappa (N_{p,q_0,kr}f)^p\Big)^{1/p}\Big\|_q && \text{[by claim]}\\
&\qquad\leqslant \Big(\sum_{k=1}^\infty b_k k^\kappa \|N_{p,q_0,kr}f\|_q^p\Big)^{1/p}\\
&\qquad\preceq \Big(\sum_{k=1}^\infty b_k k^\kappa \|f\|_p^p\Big)^{1/p} && \text{[by Lemma 3.3(iii)]}\\
&\qquad\preceq \|f\|_p. \quad ∎
\end{aligned}$$

Proof of Proposition 2.1. (ii) Let $p \leqslant u \leqslant v \leqslant q$ and $\alpha, \beta \geqslant 0$ such that $\alpha + \beta - \gamma = \frac{1}{u} + \frac{1}{q} - \frac{1}{v} - \frac{1}{p}$. Denoting $\kappa := Dp\Big(\frac{1}{q} + \gamma - \alpha\Big), \lambda_0 := \max(\kappa - 1, 0), K := \sum\limits_{k=0}^\infty (k+1)^{\lambda_0} g(k), q_0 := \Big(\frac{1}{q} + \gamma - \alpha - \beta\Big)^{-1}, b_k := g(k-1) - g(k) \geqslant 0$, one obtains as in the first part of the implication (3)⇒(ii) in the proof of Theorem 1.2 (for $\rho = 0$, i.e. $w \equiv 1$):

$$N_{q,r}(v_r^\alpha R v_r^\beta f) \preceq K^{1/p'}\Big(\sum_{k=1}^\infty b_k k^\kappa (N_{p,q_0,kr}f)^p\Big)^{1/p}.$$

Now arguing similarly to the second part of the implication (3)⇒(ii) in the proof of Theorem 1.2 yields

$$\|v_r^\alpha R v_r^\beta f\|_v \;\preceq\; \|N_{q,r}(v_r^\alpha R v_r^\beta f)\|_v \;\preceq\; K^{1/p'}\Big(\sum_{k=1}^\infty b_k k^\kappa\Big)^{1/p}\|f\|_u \;\preceq\; K\|f\|_u.$$

(i) We denote by (1′), (2′), (3′) the "for-all-versions" of the conditions (1), (2), (3). Note that the implications (1′)⇒(1), (2′)⇒(2) and (3′)⇒(3) are clear. The implications (2′)⇒(1′) and (2)⇒(1) are obvious since $d(B(x,r), B(y,r)) \geqslant d(x,y) - 2r$ and g is decreasing.

(3)⇒(2′): By (ii), applied for $g\Big(\frac{d(B_1,B_2)}{r} - 2\Big)^{-1/2}\chi_{B_1}R\chi_{B_2}$ and $g^{1/2}$ instead of R and g, it suffices to show

$$\|\chi_{B(x,r)}\chi_{B_1}R\chi_{B_2}\chi_{A(x,r,k)}\|_{p\to q} \leqslant v_r(x)^{-\gamma} g\Big(\frac{d(B_1,B_2)}{r} - 2\Big)^{1/2} g(k)^{1/2}.$$

Since g is decreasing, the latter is direct from (i) once we show

$$B(x,r)\cap B_1, B_2 \cap A(x,r,k) \neq \emptyset \Longrightarrow k \geqslant \frac{d(B_1,B_2)}{r} - 2.$$

Hence suppose $a \in B(x,r)\cap B_1$ and $b \in B_2 \cap A(x,r,k)$. Then

$$d(B_1,B_2) \leqslant d(a,b) \leqslant d(a,x) + d(x,b) \leqslant (k+2)r.$$

(1′)⇒(3′) and (1)⇒(3): Since $v_r(x) \leqslant v_{(l+2)r}(z) \leqslant C(l+2)^D v_r(z)$ whenever $z \in A(x,r,l)$, we can estimate as follows:

$$
\begin{aligned}
&\|\chi_{B(x,r)} R \chi_{A(x,r,k)}\| \\
&\quad\preceq \int_{\Omega} \|\chi_{B(x,r)} R \chi_{B(z,r)}\|_{u\to v} \|\chi_{B(z,r)} \chi_{A(x,r,k)}\|_{u\to u} v_r(z)^{-1} \mathrm{d}z \quad \text{[by Lemma 3.4]} \\
&\quad\leqslant \int_{B(x,(k-1)_+ r)^c} v_r(x)^{-\alpha} v_r(z)^{-\beta-1} g(d(x,z)r^{-1}) \mathrm{d}z \qquad \text{[(1′) resp. (1)]} \\
&\quad\preceq v_r(x)^{-\alpha} \sum_{l=[k-1]_+}^{\infty} \int_{A(x,r,l)} [(l+2)^{-D} v_r(x)]^{-\beta-1} g(l-2) \mathrm{d}z \\
&\quad\preceq v_r(x)^{-(\alpha+\beta)} \sum_{l=[k-1]_+}^{\infty} (l+2)^{D(\beta+2)} g(l-2) = v_r(x)^{-(\alpha+\beta)} \widetilde{g}(k).
\end{aligned}
$$

(iii) First observe that $R \in \mathfrak{L}(L_u(\Omega))$ for all $u \in [p,q]$ by (ii). Let $\rho \geqslant 0, x, y \in \Omega$ and denote $e_{\rho,x} := \mathrm{e}^{\rho d(x,\cdot)}$. Then

$$
\begin{aligned}
&N_{q,r}(e_{\rho,y} R e_{-\rho,y} - R) f(x) \\
&\quad\leqslant N_{q,r}(e_{\rho,y} e_{-\rho,y}(x) R(e_{\rho,y}(x) e_{-\rho,y} - 1) f)(x) + N_{q,r}((e_{\rho,y} e_{-\rho,y}(x) - 1) R f)(x).
\end{aligned}
$$

We estimate the first term on the right hand side where we use

$$
e_{\rho,y} e_{-\rho,y}(x) \leqslant e_{\rho,x} \quad \text{and} \quad |e_{\rho,y}(x) e_{-\rho,y} - 1| \leqslant e_{\rho,x} - 1
$$

which follow from the triangle inequality. Our assumption yields

$$
\begin{aligned}
N_{q,r}&(e_{\rho,y} e_{-\rho,y}(x) R(e_{\rho,y}(x) e_{-\rho,y} - 1) f)(x) \\
&\leqslant \mathrm{e}^{\rho r} v_r(x)^{-1/p} \sum_{k=0}^{\infty} g(k) \|\chi_{A(x,r,k)} (e_\rho(x) e_{-\rho,y} - 1) f\|_p \\
&\leqslant \mathrm{e}^{\rho r} v_r(x)^{-1/p} \sum_{k=0}^{\infty} g(k) (\mathrm{e}^{\rho(k+1)r} - 1) \|\chi_{B(x,(k+1)r)} f\|_p \\
&\preceq \sum_{k=0}^{\infty} a_k(\rho) N_{p,(k+1)r} f(x) \qquad \text{[dim. } D\text{]},
\end{aligned}
$$

where $a_k(\rho) := \mathrm{e}^{\rho r} g(k)(k+1)^{D/p}(\mathrm{e}^{\rho(k+1)} - 1)$. The second term on the right hand side is easier to estimate. We use $|e_{\rho,y} e_{-\rho,y}(x) - 1| \leqslant e_{\rho,x} - 1$ again and obtain

$$
\begin{aligned}
N_{q,r}((e_{\rho,y} e_{-\rho,y}(x) - 1) R f)(x) &\leqslant (\mathrm{e}^{\rho r} - 1) N_q R f(x) \\
&\leqslant (\mathrm{e}^{\rho r} - 1) v_r(x)^{-1/p} \sum_{k=0}^{\infty} g(k) \|\chi_{A(x,r,k)} f\|_p \\
&\preceq \sum_{k=0}^{\infty} b_k(\rho) N_{p,(k+1)r} f(x) \qquad \text{[dim. } D\text{]},
\end{aligned}
$$

where $b_k(\rho) := (\mathrm{e}^{\rho r}-1)g(k)(k+1)^{D/p}$. Putting everything together, we deduce for all $u \in [p,q]$:

$$\begin{aligned}
\|(e_{\rho,y}Re_{-\rho,y}-R)f\|_u &\sim \|N_{u,r}(e_{\rho,y}Re_{-\rho,y}-R)f\|_u && \text{[Lemma 3.3(ii)]}\\
&\leqslant \|N_{q,r}(e_{\rho,y}Re_{-\rho,y}-R)f\|_u && \text{[Lemma 3.3(i)]}\\
&\preceq \sum_{k=0}^{\infty}(a_k(\rho)+b_k(\rho))\|N_{p,(k+1)r}f\|_u\\
&\leqslant \sum_{k=0}^{\infty}(a_k(\rho)+b_k(\rho))\|N_{u,(k+1)r}f\|_u && \text{[Lemma 3.3(i)]}\\
&\sim \sum_{k=0}^{\infty}(a_k(\rho)+b_k(\rho))\|f\|_u. && \text{[Lemma 3.3(ii)].}
\end{aligned}$$

Finally, note that $\sum_k(a_k(\rho)+b_k(\rho)) \to 0$ for $\rho \searrow 0$ by dominated convergence and the growth condition on g. ∎

We quote a lemma ([19], Lemma 9) which, for linear operators, reduces the proof of bounded invertibility to a problem of boundedness. Let E, F, G be Hausdorff spaces with $E, F \hookrightarrow G$ such that $E \cap F$ is dense in both E and F, and let $D \subset E \cap F$ be dense for the initial topology induced by the embeddings $E \cap F \hookrightarrow E$ and $E \cap F \hookrightarrow F$.

LEMMA 3.5. *Let $S_E : E \to E$ and $S_F : F \to F$ be continuous mappings that coincide on D. Assume that S_E is continuously invertible and that the restriction of $(S_E)^{-1}$ to D extends to a continuous mapping $T : F \to F$. Then S_F is continuously invertible, and $(S_F)^{-1} = T$.*

Proof. Since D is dense in $E \cap F$ and $E, F \hookrightarrow G$, we have $S_E = S_F$ on $E \cap F$ and $(S_E)^{-1}$ and T coincide on $E \cap F$. Hence $TS_F = S_F R = \mathrm{Id}$ on $E \cap F$. The density of $E \cap F$ in F yields the claim. ∎

Proof of Proposition 1.4. By hypothesis (1.5) and Theorem 2.1(ii), we have $R \in \mathfrak{L}(L_u(\Omega))$ for all $u \in [p,q]$. We fix $s, u \in [p,q]$, assume that $\lambda \in \rho_{L_s}(R)$ and have to show that $\lambda \in \rho_{L_u}(R)$. By Lemma 3.5 this amounts to showing that $(\lambda - R)^{-1}|_{L_{\infty,c}(\Omega)} \in \mathfrak{L}(L_u(\Omega))$. We consider only the case $s \leqslant u$, the other case $s \geqslant u$ can be seen by simple modifications of our arguments. If $\lambda \neq 0$ then

$$(\lambda - R)^{-1} = \lambda^{-1}I + \lambda^{-1}R(\lambda - R)^{-1}$$

by the resolvent equation, hence it remains to prove $S := R(\lambda - R)^{-1} \in \mathfrak{L}(L_u(\Omega))$. Since $u \in [s,q]$, it suffices to show for some $\varepsilon > 0$:

$$\|\mathrm{e}^{-\rho d(x,\cdot)} v_r^{1/s-1/q} S \mathrm{e}^{\rho d(x,\cdot)}\|_{s\to q} \leqslant C \quad \text{for all } x \in \Omega, \rho \in [0,\varepsilon]. \tag{3.2}$$

Indeed, the latter means $\|\mathrm{e}^{-\rho\phi} v_r^{1/s-1/q} S \mathrm{e}^{\rho\phi}\|_{s\to q} \leqslant \mathrm{e}^{h_1^{\#}(\rho r)}$ for all $\phi \in \mathcal{A}, \rho \geqslant 0$ and for $\mathcal{A} := \{d(x,\cdot) : x \in \Omega\}, h_1(t) := \varepsilon r t - C$ (i.e. $h_1^{\#}(\rho r) = \infty \cdot \chi_{(\varepsilon,\infty)}(\rho) + C$), which implies $S \in \mathfrak{L}(L_u(\Omega))$ by Theorem 1.2(ii)⇒(i) (for $\mathcal{S}$ = set of balls in Ω) and Proposition 2.1(ii). We factor the LHS in (3.2):

$$\begin{aligned}
&\|\mathrm{e}^{-\rho d(x,\cdot)} v_r^{1/s-1/q} S \mathrm{e}^{\rho d(x,\cdot)}\|_{s\to q} \\
&\quad\le \|\mathrm{e}^{-\rho d(x,\cdot)} v_r^{1/s-1/q} R \mathrm{e}^{\rho d(x,\cdot)}\|_{s\to q} \|\mathrm{e}^{-\rho d(x,\cdot)} (\lambda - R)^{-1} \mathrm{e}^{\rho d(x,\cdot)}\|_{s\to s} \\
&\quad= \|\mathrm{e}^{-\rho d(x,\cdot)} v_r^{1/s-1/q} R \mathrm{e}^{\rho d(x,\cdot)}\|_{s\to q} \|(\lambda - \mathrm{e}^{\rho d(x,\cdot)} R \mathrm{e}^{-\rho d(x,\cdot)})^{-1}\|_{s\to s}\,. \qquad (3.3)
\end{aligned}$$

The first term in (3.3) can be estimated uniformly in $x \in \Omega$ as follows:

$$\begin{aligned}
&\|\mathrm{e}^{-\rho d(x,\cdot)} v_r^{1/s-1/q} R \mathrm{e}^{\rho d(x,\cdot)}\|_{s\to q} \\
&\quad\leqslant \mathrm{e}^{h^\#(\rho r)} \quad \text{for all } \rho \geqslant 0 \qquad \text{[by hyp. (1.5), Prop. 2.1, Thm. 1.2]} \\
&\quad\leqslant C \quad \text{for all } \rho \in [0,\varepsilon] \qquad \text{[by growth cond. on } h\text{].}
\end{aligned}$$

The second term in (3.3) is uniformly bounded in $x \in \Omega$ and $\rho \geqslant 0$ small enough since $(\lambda - R)^{-1} \in \mathfrak{L}(L_s(\Omega))$, inversion is continuous on the open set of bounded invertible operators on $L_s(\Omega)$ and

$$\sup_{x\in\Omega} \|\mathrm{e}^{\rho d(x,\cdot)} R \mathrm{e}^{-\rho d(x,\cdot)} - R\|_{s\to s} \to 0 \quad \text{for} \rho \searrow 0$$

by hypothesis (1.5) and Proposition 2.1(iii). If $\lambda = 0$ we simply write $R^{-1} = R((-R)^{-1})^2$ and argue as before. ∎

Finally, we prove the equivalence of GGEs for the special case $(p_0, q_0) = (1, \infty)$ and GEs. It has already been shown in Proposition 2.9 of [5]; we give the proof for the sake of completeness.

PROPOSITION 3.6. *Let (Ω, d, μ) be a space of homogeneous type, $R \in \mathfrak{L}(L_1(\Omega)$, and $L_\infty(\Omega))$ have the integral kernel $k \in L_\infty(\Omega^2)$. Furthermore, let $g : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a decreasing function and $r > 0$. Then the following are equivalent:*

(i) *for all $x, y \in \Omega$ we have*

$$\|\chi_{B(x,r)} R \chi_{B(y,r)}\|_{1\to\infty} \leqslant v_r(x)^{-1} g\Big(\frac{d(x,y)}{r}\Big);$$

(ii) *for all $x, y \in \Omega$ we have*

$$|k(x,y)| \leqslant v_r(x)^{-1} g\Big(\frac{d(x,y)}{r}\Big).$$

Here the statement is written modulo identification of g and $\widetilde{g}$, where $\widetilde{g}(s) = Cg((s-2)_+)$ and C is the doubling constant from (1.2).

Proof. (ii)⇒(i) We fix $x, y \in \Omega$ and observe that the operator $\chi_{B(x,r)} R \chi_{B(y,r)}$ has the integral kernel

$$(u,v) \mapsto \chi_{B(x,r)}(u) k(u,v) \chi_{B(y,r)}(v).$$

Hence we can estimate as follows:

$$\begin{aligned}\|\chi_{B(x,r)} R \chi_{B(y,r)}\|_{1\to\infty} &= \sup_{u,v\in\Omega} \chi_{B(x,r)}(u)|k(u,v)|\chi_{B(y,r)}(v) \\ &\leqslant \sup_{u\in B(x,r)} \sup_{v\in B(y,r)} v_r(u)^{-1} g\Big(\frac{d(u,v)}{r}\Big) && \text{[by (ii)]} \\ &\leqslant v_r(x)^{-1} \widetilde{g}\Big(\frac{d(x,y)}{r}\Big) \quad [d(x,y) \leqslant d(u,v)+2].\end{aligned}$$

(i)$\Rightarrow$(ii) A reformulation of (i) is the following:

$$\sup_{u,v\in\Omega} \chi_{B(x,r)}(u)|k(u,v)|\chi_{B(y,r)}(v) \leqslant v_r(x)^{-1} g\Big(\frac{d(x,y)}{r}\Big)$$

for all $x, y \in \Omega$. Applying this for $u = x$ and $v = y$ yields

$$|k(x,y)| \leqslant v_r(x)^{-1} g\Big(\frac{d(x,y)}{r}\Big). \quad \blacksquare$$

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