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COMMUTANTS AND HYPOREFLEXIVE CLOSURE OF OPERATORS

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ABSTRACT. We show that the commutants of several classes of operators are boundedly reflexive; including Hilbert space triangular operators and Banach space compact cyclic operators, the latter gives an affirmative answer to a question of Don Hadwin and Deguang Han. Under a mild condition on the spectrum of a Banach space operator, we show that the hyporeflexive closure of the operator is boundedly reflexive. With a simpler proof, we obtain a stronger version of a theorem of David Larson and Warren Wogen on algebraic extensions of bitriangular operators. We also show that the commutant of a bitriangular operator on a Hilbert space is reflexive if and only if the bitriangular operator is quasisimilar to a diagonal operator.

KEYWORDS: *Commutant, reflexivity, bounded reflexivity, separating vector, cyclic vector.*

MSC (2000): 47L05, 46B28.

1. INTRODUCTION

In this paper, we use X to denote a complex separable Banach space and $B(X)$ to denote the set of all bounded linear operators on X. If $\mathcal{S}$ is a subset of $B(X)$, for any $r > 0$, define $\mathcal{S}_r = \{T \in \mathcal{S} : \|T\| \leqslant r\}$. Let $\mathcal{S}$ be a subspace of $B(X)$, and let $\mathrm{ref}(\mathcal{S}) = \{T \in B(X) : Tx \in [\mathcal{S}x]$, for all $x \in X\}$, where $[\,\cdot\,]$ denotes the norm closure. A subspace $\mathcal{S} \subseteq B(X)$ is said to be *reflexive* if $\mathrm{ref}(\mathcal{S}) = \mathcal{S}$. A subspace $\mathcal{S} \subseteq B(X)$ is called *n-reflexive* if $\mathcal{S}^{(n)}$, the n-fold inflation of $\mathcal{S}$, is reflexive in $B(X^{(n)})$, where $X^{(n)}$ is the direct sum of n copies of X. Let $\mathrm{ref_b}(\mathcal{S}) = \{T \in B(X) :$ there exists an $M_T > 0$ such that $Tx \in [\mathcal{S}_{M_T}x]$, for all $x \in X\}$. Then $\mathcal{S}$ is called *boundedly reflexive* if $\mathrm{ref_b}(\mathcal{S}) = \mathcal{S}$. We say $\mathcal{S}$ is *hereditarily boundedly reflexive* if every weakly closed subspace of $\mathcal{S}$ is boundedly reflexive. Note that if $\mathcal{S}$ is a weakly closed subset of $B(X)$, then $[\mathcal{S}_M x] = \mathcal{S}_M x$; in this case, the closure $[\,\cdot\,]$ is not needed in the definition of $\mathrm{ref_b}\mathcal{S}$. Bounded reflexivity plays a very important role in the study of positivity of elementary operators on C^*-algebras; see [14] and its references for more details.

For any $x \in X$, define the map $\phi_x : \mathcal{S} \to X$ by $\phi_x(T) = Tx$, for all $T \in \mathcal{S}$. A vector $x \in X$ is called a *separating vector* of $\mathcal{S}$ if ϕ_x is injective; x is called a *strictly separating vector* of $\mathcal{S}$ if ϕ_x is bounded below on $\mathcal{S}$; and x is called a *cyclic vector* of $\mathcal{S}$ if the range of ϕ_x is dense in X. A subset E of X is called a *separating set* of $\mathcal{S}$ if the only operator $S \in \mathcal{S}$ such that $Sx = 0$, $\forall x \in E$ is $S = 0$. For any operator $T \in B(X)$, we use $w(T)$ to denote the weakly closed algebra generated by T and the identity operator. We say T is reflexive (boundedly reflexive, respectively) if $w(T)$ is reflexive (boundedly reflexive, respectively). For any subset $\mathcal{S} \subseteq B(X)$, we use Lat$\mathcal{S}$ to denote the set of all common invariant subspaces of operators in $\mathcal{S}$ and AlgLat$\mathcal{S}$ to denote the set of operators that leave all subspaces of Lat$\mathcal{S}$ invariant. When $\mathcal{S}$ is an algebra that contains the identity operator, then ref$\mathcal{S}$ = AlgLat$\mathcal{S}$. Let $\mathcal{S}'$ denote the commutant of $\mathcal{S}$. We call $\mathcal{S}' \cap \text{AlgLat}\mathcal{S}$ the *hyporeflexive closure* of $\mathcal{S}$. When $\mathcal{S} = \{T\}$, we use $\{T\}' \cap \text{AlgLat}T$ to denote the hyporeflexive closure of T. To show certain operators are boundedly reflexive, sometimes, it is easier to show the hyporeflexive closure of the operator is hereditarily boundedly reflexive. An operator algebra $\mathcal{A}$ is called *hyporeflexive* if $\mathcal{A}' \cap \text{AlgLat}\mathcal{A} = \mathcal{A}$. Hyporeflexivity was studied in [8], [15], and [16].

In the case of Hilbert spaces, we use H to denote a complex separable Hilbert space and $B(H)$ for the set of bounded linear operators on H. Let $T(H)$ be the set of all trace-class operators and $F_1(H)$ be the set of all rank-1 operators. For any subspace $\mathcal{S}$ of $B(H)$, the *preannihilator* of $\mathcal{S}$, denoted by $\mathcal{S}_\perp$, is defined by $\mathcal{S}_\perp = \{T \in T(H) : \text{trace}(TS) = 0, \ \forall\ S \in \mathcal{S}\}$. A subspace $\mathcal{S}$ of $B(H)$ is called *elementary* if $\mathcal{S}_\perp + F_1(H) = T(H)$ and $\mathcal{S}$ is called *approximately elementary* if $\mathcal{S}_\perp + F_1(H)$ is trace-norm dense in $T(H)$.

In Section 2, we show that if $T \in B(X)$ is a cyclic operator with "thin" approximate point spectrum, then the commutant of T is hereditarily boundedly reflexive. As a consequence, compact cyclic operators are boundedly reflexive; for example, the Volterra operator is boundedly reflexive. This affirmatively answers a question raised by D. Hadwin and D. Han in [7]. For any Banach space operator T with "thin" approximate point spectrum, we show that the hyporeflexive closure of T is boundedly reflexive. Finally, we show that there exists a Hilbert space operator T such that $w(T)$ has strictly separating vectors, yet $\{T\}' \cap \text{AlgLat}T$ is not approximately elementary. The main results of Section 2 are Theorem 2.1, Theorem 2.5, and Theorem 2.7.

In Section 3, we show that if $T \in B(H)$ is triangular or co-triangular, then the commutant of T is boundedly reflexive. In this case, we also prove the hyporeflexive closure of T has a dense set of separating vectors. This strengthens a theorem of Larson and Wogen ([13], Theorem 10) and our proof is much simpler. As for reflexivity, we show that the commutant of a bitriangular operator on a Hilbert space is reflexive if and only if the bitriangular operator is quasisimilar to a diagonal operator; this generalizes the main result of [5]. The main results of Section 3 are Theorem 3.8, Theorem 3.9, and Theorem 3.15.

2. OPERATORS WITH "THIN" APPROXIMATE POINT SPECTRA

For $T \in B(X)$, we use $\sigma_{\mathrm{p}}(T)$ and $\sigma_{\mathrm{ap}}(T)$ to denote the point spectrum and approximate point spectrum of T, respectively. We say T has *thin approximate point spectrum* if $\mathbb{C} \setminus \sigma_{\mathrm{ap}}(T)$ is dense in $\mathbb{C}$.

THEOREM 2.1. *Suppose $T \in B(X)$ is a cyclic operator and T has thin approximate point spectrum. Then $\{T\}'$ is hereditarily boundedly reflexive; in particular, T is boundedly reflexive.*

Proof. Suppose $S \in \mathrm{ref}_{\mathrm{b}}(\{T\}')$ and x_0 is a cyclic vector of T. Then x_0 is necessarily a separating vector of $\{T\}'$.

Adding a scalar multiple of identity if necessary, we can assume T is invertible. Since $\{T\}'$ is weakly closed and x_0 is a separating vector of $\{T\}'$, there exists a unique $A_0 \in \{T\}'$ so that $Sx_0 = A_0x_0$. Subtracting A_0 from S and still call it S, we may assume $Sx_0 = 0$. We need to show $S = 0$.

First, we will show $STx_0 = 0$. Clearly, $S(T-\lambda)x_0 = STx_0$, $\forall\, \lambda \in \mathbb{C}$. Also, note that $(T-\lambda)x_0$ is a separating vector of $\{T\}'$, for any $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{p}}(T)$. Since $S \in \mathrm{ref}_{\mathrm{b}}(\{T\}')$, there exists a scalar $M_S > 0$ so that for any $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{p}}(T)$ there exists a unique $A_\lambda \in \{T\}'$ with $\|A_\lambda\| \leqslant M_S$ satisfying $S(T-\lambda)x_0 = A_\lambda(T-\lambda)x_0$; that is, A_λ defines a bounded map from $\mathbb{C} \setminus \sigma_{\mathrm{p}}(T)$ to $B(X)$. Moreover, fix a $\lambda_0 \in \mathbb{C} \setminus \sigma_{\mathrm{ap}}(T)$, then $A_\lambda(T-\lambda)x_0 = A_{\lambda_0}(T-\lambda_0)x_0$, $\forall\, \lambda \in \mathbb{C} \setminus \sigma_{\mathrm{p}}(T)$. Since x_0 is a separating vector of $\{T\}'$, we have $A_\lambda(T-\lambda) = A_{\lambda_0}(T-\lambda_0)$, $\forall\, \lambda \in \mathbb{C} \setminus \sigma_{\mathrm{p}}(T)$. It follows that

$$(T-\lambda_0)(A_\lambda - A_{\lambda_0}) = A_\lambda(\lambda - \lambda_0). \tag{2.1}$$

Since A_λ is uniformly bounded by M_S, (2.1) implies $\|(T-\lambda_0)(A_\lambda - A_{\lambda_0})\| \to 0$, as $\lambda \to \lambda_0$. Since $\lambda_0 \in \mathbb{C} \setminus \sigma_{\mathrm{ap}}(T)$, we have $\|A_\lambda - A_{\lambda_0}\| \to 0$, as $\lambda \to \lambda_0$. Thus A_λ is continuous at λ_0.

By (2.1) again, we have

$$(T-\lambda_0)\frac{A_\lambda - A_{\lambda_0}}{\lambda - \lambda_0} = A_\lambda. \tag{2.2}$$

The continuity of A_λ at λ_0 implies $\left\{(T-\lambda_0)\frac{A_\lambda - A_{\lambda_0}}{\lambda-\lambda_0}\right\}$ is a Cauchy net in $B(X)$ as $\lambda \to \lambda_0$. Again, since $\lambda_0 \in \mathbb{C} \setminus \sigma_{\mathrm{ap}}(T)$, it follows that $\left\{\frac{A_\lambda - A_{\lambda_0}}{\lambda-\lambda_0}\right\}$ is a Cauchy net in $B(X)$ as $\lambda \to \lambda_0$. Thus $\left\{\frac{A_\lambda - A_{\lambda_0}}{\lambda-\lambda_0}\right\}$ is convergent; this implies A_λ is analytic at λ_0. Since λ_0 is arbitrary from $\mathbb{C} \setminus \sigma_{\mathrm{ap}}(T)$, we have shown that A_λ is analytic in $\mathbb{C} \setminus \sigma_{\mathrm{ap}}(T)$. The uniform boundedness of A_λ and the density of $\mathbb{C} \setminus \sigma_{\mathrm{ap}}(T)$ in $\mathbb{C}$ allow us to extend A_λ to a bounded analytic function on $\mathbb{C}$. By the Liouville's Theorem, A_λ is a constant operator, say $A_\lambda = C$. Now the fact that $C(T-\lambda)x_0$ is a constant vector independent of λ and x_0 is a separating vector of $\{T\}'$ implies $C = 0$. Thus, $STx_0 = 0$.

Since x_0 is a separating vector of $\{T\}'$ and T is invertible, it follows that Tx_0 is a separating vector of $\{T\}'$. Replacing x_0 by Tx_0 and repeating the above argument inductively, we can obtain $ST^n x_0 = 0$ for every positive integer n. This implies $S = 0$, since x_0 is a cyclic vector of T.

Hereditary bounded reflexivity of $\{T\}'$ follows from the fact that $\{T\}'$ has separating vectors. ∎

COROLLARY 2.2. *If $T \in B(X)$ is a compact cyclic operator then $\{T\}'$ is hereditarily boundedly reflexive.*

REMARKS 2.3. (i) In [7], D. Hadwin and D. Han asked whether the Volterra operator is boundedly reflexive. Corollary 2.2 answers the above question affirmatively. For, the Volterra operator is compact and it has abundant cyclic vectors. In fact, the set of cyclic vectors for any unicellular operator is a second category set, by Theorem 2 of [18]. Therefore, any unicellular operator with a dense resolvent set is boundedly reflexive; in particular, the Donahue operators are boundedly reflexive. Note that, on the contrary, a unicelluar operator can never be reflexive.

(ii) It is known and easy to show that reflexive operators must have plenty of nontrivial invariant subspaces. However, a boundedly reflexive Banach space operator may be transitive; for instance, take the quasinilpotent transitive operator constructed in [17], it follows immediately from Theorem 2.1 that the operator is boundedly reflexive.

THEOREM 2.4. *Suppose T is a cyclic operator and $\mathbb{C} \setminus (\sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{ap}}(T^*))$ is dense in $\mathbb{C}$. Then $\{T\}'$ is hereditarily boundedly reflexive.*

Proof. We can modify the proof of Theorem 2.1 by taking $\lambda_0, \lambda \in \mathbb{C} \setminus (\sigma_{\mathrm{p}}(T) \cup \sigma_{\mathrm{ap}}(T^*))$ and taking the adjoint of equation (2.1). The rest is similar. ∎

Using Theorem 2.1, we can also prove the following:

THEOREM 2.5. *If T has thin approximate point spectrum then $\{T\}' \cap \mathrm{AlgLat}T$ is boundedly reflexive.*

Proof. First, note that if E is any invariant subspace of T then $\sigma_{\mathrm{ap}}(T|_E) \subseteq \sigma_{\mathrm{ap}}(T)$, where $T|_E$ is the restriction of T to E. Thus $T|_E$ has thin approximate point spectrum. Since

$$\mathrm{ref}_{\mathrm{b}}(\{T\}' \cap \mathrm{AlgLat}T) \subseteq \mathrm{ref}(\{T\}' \cap \mathrm{AlgLat}T) \subseteq \mathrm{ref}(\mathrm{AlgLat}T) = \mathrm{AlgLat}T,$$

E is an invariant subspace of $\mathrm{ref}_{\mathrm{b}}(\{T\}' \cap \mathrm{AlgLat}T)$. Clearly, $(\{T\}' \cap \mathrm{AlgLat}T)|_E \subseteq \{T|_E\}'$, where $(\{T\}' \cap \mathrm{AlgLat}T)|_E$ is the restriction of $\{T\}' \cap \mathrm{AlgLat}T$ to E.

Suppose $S \in \mathrm{ref}_{\mathrm{b}}(\{T\}' \cap \mathrm{AlgLat}T)$. Then trivially $S \in \mathrm{AlgLat}T$, so we only need to show $S \in \{T\}'$. For any $x \in X$, let E_x be the cyclic invariant subspace of T generated by x. Since $T|_{E_x}$ is cyclic and it has thin approximate point spectrum, $\{T|_{E_x}\}'$ is boundedly reflexive, by Theorem 2.1; that is, $\mathrm{ref}_{\mathrm{b}}(\{T|_{E_x}\}') = \{T|_{E_x}\}'$.

It follows that

$$S|_{E_x} \in \mathrm{ref_b}((\{T\}' \cap \mathrm{AlgLat}T)|_{E_x}) \subseteq \mathrm{ref_b}(\{T|_{E_x}\}') = \{T|_{E_x}\}'.$$

Thus, $ST|_{E_x} = S|_{E_x}T|_{E_x} = T|_{E_x}S|_{E_x} = TS|_{E_x}$. Since x is arbitrary, we conclude $ST = TS$. ∎

The C_0-contractions on Hilbert spaces have been extensively studied in the literature (see [2] and its references for details). It is well-known that not all C_0-contractions are reflexive. The situation is different for bounded reflexivity.

COROLLARY 2.6. *All C_0-contractions on a Hilbert space are boundedly reflexive.*

Proof. Suppose T is a C_0-contraction on a Hilbert space. Then by Theorem 1.2 on page 74 in [2] it follows that $\{T\}' \cap \mathrm{AlgLat}T = w(T)$. Also, from Theorem 4.11 on page 32 in [2] we get that T has thin approximate point spectrum. The conclusion now follows from Theorem 2.5. ∎

For any $T \in B(H)$, it is not hard to see that $w(T) \subseteq \{T\}' \cap \mathrm{AlgLat}T$. However, up until Wogen's counterexamples in [19], it had been a longstanding open question whether the inclusion could be proper. Not much is known about the structure of the hyporeflexive closure of a Hilbert space operator. In fact, it is still an open question whether $\{T\}' \cap \mathrm{AlgLat}T$ is always an abelian algebra; see [9], [11], and [12] for related questions. Our next theorem further differentiates $w(T)$ and $\{T\}' \cap \mathrm{AlgLat}T$, and it may shed some light on the above question.

First, note that for a weak*-closed linear subspace of $B(H)$, the existence of separating vectors implies the subspace is approximately elementary by Proposition 1.2 of [9]. The converse of the above is not true. In fact, Wogen constructed an operator T so that $w(T)$ is elementary and $w(T)$ has no finite separating set ([19], Example 10).

THEOREM 2.7. *There exists an operator $T \in B(H)$ such that $w(T)$ has strictly separating vectors and $\{T\}' \cap \mathrm{AlgLat}T$ is not approximately elementary.*

Proof. First, we construct a weakly closed subspace $\mathcal{S} \subset B(H)$ so that $\mathcal{S}$ has strictly cyclic vectors and $\mathcal{S}$ is not boundedly reflexive.

For any $u, v \in H$, we use $u \otimes v$ to denote the rank-1 operator with $(u \otimes v)x = \langle x, v\rangle u, \forall\, x \in H$, where $\langle \cdot, \cdot \rangle$ denotes the inner product. Let $\{e_n\}_1^\infty$ be an orthonormal basis of H and S be the backward unilateral shift with respect to $\{e_n\}_1^\infty$. Define

$$\mathcal{S} = \{x \otimes e_1 + Sx \otimes e_2 : x \in H\}.$$

It is straight forward to check that $e_1 \otimes e_2 \notin \mathcal{S}, \mathcal{S}$ is weakly closed, and for any $x \otimes e_1 + Sx \otimes e_2 \in \mathcal{S}, \|x \otimes e_1 + Sx \otimes e_2\| \leqslant 2\|x\|$.

Take any fixed $y = \sum\limits_{n=1}^{\infty} y_n e_n$ with $|y_1| > |y_2|$, then $y_1 I + y_2 S$ is invertible. Thus, $\|(x \otimes e_1 + Sx \otimes e_2)y\| = \|(y_1 I + y_2 S)x\| \geqslant \|(y_1 I + y_2 S)^{-1}\|^{-1}\|x\| \geqslant$

$\frac{1}{2}\|(y_1I + y_2S)^{-1}\|^{-1}\|x \otimes e_1 + Sx \otimes e_2\|$. It follows that y is a strictly separating vector of $\mathcal{S}$.

To see $\mathcal{S}$ is not boundedly reflexive, we show $e_1 \otimes e_2 \in \mathrm{ref_b}\mathcal{S}$. This yields that $\mathcal{S} \neq \mathrm{ref_b}\mathcal{S}$, thus $\mathcal{S}$ is not boundedly reflexive.

To this end, we show that for any $y = \sum\limits_{n=1}^{\infty} y_n e_n \in H$, there exists an $x \in H$ with $\|x\| \leqslant 2$ so that $(x \otimes e_1 + Sx \otimes e_2)y = (e_1 \otimes e_2)y$. Without loss of generality, we can assume y_1 and y_2 are not both zero; otherwise, we could choose $x = 0$. If $|y_1| \geqslant \frac{1}{2}|y_2|$, choose $x_1 = \frac{y_2}{y_1}$ and $x_n = 0, n = 2, 3, \ldots$; if $|y_1| < \frac{1}{2}|y_2|$, choose $x_1 = 0, x_2 = 1$ and $x_n = \left(-\frac{y_1}{y_2}\right)^{n-2}, n = 3, 4, \ldots$.

Using $\mathcal{S}$ constructed above and follow the same procedure as that of Construction 3.2 of [1], we can construct an operator T (see Proposition 3.3 of [1]) with the desired properties. A similar argument to that of Proposition 3.6 of [1] shows that T is not boundedly reflexive. From the way T is constructed (see especially page 577, Form (3.2) of [1]), one could easily see that all strictly separating vectors of $\mathcal{S}$ are embedded into strictly separating vectors of $w(T)$ also.

Next, we show that $\{T\}' \cap \mathrm{AlgLat}T$ is not approximately elementary. Note that, by Theorem 3.10 of [14], a boundedly reflexive operator space is hereditarily boundedly reflexive if and only if it is approximately elementary. By Proposition 3.5 of [1], $\{T\}' \cap \mathrm{AlgLat}T = \mathrm{AlgLat}T$, so it is boundedly reflexive (in fact, reflexive). Since $w(T)$ is not boundedly reflexive, $\{T\}' \cap \mathrm{AlgLat}T$ is not hereditarily boundedly reflexive, thus it is not approximately elementary. ∎

From the proof of Theorem 2.7, one can see that the existence of strictly separating vectors for $w(T)$ does not guarantee bounded reflexivity of $w(T)$. Even though, the existence of "abundant" strictly separating vectors for a weakly closed subspace of operators implies bounded reflexivity for the subspace of operators, by Theorem 3.5 of [14].

3. TRIANGULAR OPERATORS

Suppose $T \in B(H)$. A vector $x \in H$ is called an *algebraic vector* of T if there exists a nonzero polynomial $p(z)$ such that $p(T)x = 0$. An operator is called a *triangular operator* if the set of its algebraic vectors is dense in H. Let $\ker(\mu - T)^n = \{x \in H : (\mu - T)^n x = 0\}$, where μ is any complex scalar and n is any positive integer. It can be easily verified that T is a triangular operator if and only if H is the closed linear span of $\ker(\mu - T)^n$, over all μ and n. The adjoint of a triangular operator is called a *co-triangular operator*.

The first main result of this section states that the commutant of a triangular operator is boundedly reflexive. We break the proof of this result into Lemmas 3.1 through 3.7, leading to Theorem 3.8. First, it is more convenient for us to extend the definitions of reflexivity and bounded reflexivity to subspaces of $B(K, L)$, the set of all bounded linear operators from a Hilbert space K to a Hilbert space L. Since the generalizations are obvious, we omit the formal definitions.

Suppose $\mathcal{S} \subseteq B(H)$ and H_0 is another Hilbert space. We define the left augmentation of $\mathcal{S}$ to be

$$\widetilde{\mathcal{S}} = \{(\mathrm{O} \quad S) \in B(H_0 \oplus H, H) : S \in \mathcal{S}\},$$

where O is the zero transformation from H_0 to H. Similarly, we define the bottom augmentation of $\mathcal{S}$ to be

$$\overline{\mathcal{S}} = \left\{ \begin{pmatrix} S \\ \mathrm{O} \end{pmatrix} \in B(H, H \oplus H_0) : S \in \mathcal{S} \right\},$$

where O is the zero transformation from H to H_0.

The following lemma is immediate from the definitions.

LEMMA 3.1. *A subspace of $B(H)$ is boundedly reflexive if and only if any left augmentation of the subspace is boundedly reflexive if and only if any bottom augmentation of the subspace is boundedly reflexive.*

The next lemma may not be new. We provide a proof here for completeness, also due to the fact that we could not find a reference for it.

LEMMA 3.2. *Let $T = \begin{pmatrix} J_1 + \lambda & 0 \\ 0 & J_2 \end{pmatrix}$, where J_1 and J_2 are upper triangular nilpotent Jordan blocks of sizes m and n, respectively, and λ is a complex scalar. Suppose $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, where B_{11} and B_{22} are $m \times m$ and $n \times n$ matrices, respectively.*

(i) *If $\lambda \neq 0$ then $B \in \{T\}'$ if and only if B_{12} and B_{21} are both zero matrices, and B_{11} and B_{22} are both upper triangular square matrices with constant diagonals.*

(ii) *If $\lambda = 0$ then $B \in \{T\}'$ if and only if B_{11} and B_{22} are both upper triangular square matrices with constant diagonals, and B_{12} and B_{21} are both augmentations of upper triangular square matrices, with constant diagonals, of size $k \times k$, where $k = \min(m, n)$.*

Proof. Since $TB = BT$, a quick computation yields $J_1 B_{11} = B_{11} J_1$, $J_2 B_{22} = B_{22} J_2$, $(J_1 + \lambda) B_{12} = B_{12} J_2$, and $J_2 B_{21} = B_{21}(J_1 + \lambda)$. It is well-known and can be easily checked that commutant of a Jordan block consists of all upper triangular matrices with constant diagonals, so B_{11} and B_{22} are both upper triangular square matrices with constant diagonals.

If $\lambda \neq 0$, using $(J_1 + \lambda) B_{12} = B_{12} J_2$ repeatedly will give us $(J_1 + \lambda)^N B_{12} = B_{12} J_2^N$. When $N \geqslant n$, the right hand side of the last equation is 0. The invertibility of $(J_1 + \lambda)^N$ implies $B_{12} = 0$. A similar argument shows $B_{21} = 0$.

If $\lambda = 0$ and $m = n$, then $J_1 = J_2$. Thus B_{12} and B_{21} are in the commutant of $J_1 = J_2$, so they are both upper triangular square matrices with constant diagonals (with no augmentation, or trivial augmentation).

Suppose $\lambda = 0$ and $m > n$. Repeated use of $J_1 B_{12} = B_{12} J_2$ gives us $J_1^{m-1} B_{12} = B_{12} J_2^{m-1} = 0$. It follows that the entries in the last row of B_{12} are all 0. If $m-1>n$, then $J_1^{m-2}B_{12}=B_{12}J_2^{m-2}=0$. This implies that the entries in the row that is second from the bottom of B_{12} are all 0. Repeating the above until $m-k=n$. This shows that B_{12} can be written in the form $B_{12}=\begin{pmatrix} B'_{12} \\ \mathrm{O} \end{pmatrix}$, where O is the zero $k\times n$ matrix and B'_{12} is a $n\times n$ square matrix. Write J_1 in the following form: $J_1=\begin{pmatrix} J_2 & C \\ 0 & D \end{pmatrix}$. Using the above forms for J_1 and B_{12} in equation $J_1B_{12}=B_{12}J_2$, we can obtain $J_2B'_{12}=B'_{12}J_2$, i.e. B'_{12} is in the commutant of J_2. Thus B'_{12} is an upper triangular square matrix with constant diagonals. The argument for B_{21}, as well as for the case $m<n$, is similar. ∎

In the following, for convenience, we will allow N to be either a positive integer or ∞.

LEMMA 3.3. *Let J_i be an upper triangular nilpotent Jordan block acting on a finite dimensional Hilbert space H_i, $H=\bigoplus_{i=1}^{N} H_i$, and P_i be the orthogonal projection from H onto H_i. Suppose $T=\bigoplus_{i=1}^{N}(J_i+\lambda_i)\in B(H)$. Then for any $B\in B(H)$, $B\in\{T\}'$ if and only if, for any m and n,*

(i) *if $\lambda_m\neq\lambda_n$ then $P_mBP_n=0$;*

(ii) *if $\lambda_m=\lambda_n$ then P_mBP_n is an augmentation of a $k\times k$ upper triangular matrix with constant diagonals, where $k=\min(\dim H_m,\dim H_n)$.*

Proof. Note that P_mH and P_nH are reducing subspaces of T. Thus P_m+P_n is an orthogonal projection and $(P_m+P_n)H$ is a reducing subspace of T. It follows that $TB=BT$ if and only if $(P_m+P_n)T(P_m+P_n)B(P_m+P_n)=(P_m+P_n)B(P_m+P_n)T(P_m+P_n)$. The conclusion follows by an appeal to Lemma 3.2. ∎

LEMMA 3.4. *Let J_i be an upper triangular nilpotent Jordan block acting on a finite dimensional Hilbert space H_i, $H=\bigoplus_{i=1}^{N} H_i$, and P_i be the orthogonal projection from H onto H_i. Suppose $T=\bigoplus_{i=1}^{N}(J_i+\lambda_i)\in B(H)$. Then $\{T\}'$ is boundedly reflexive.*

Proof. First notice that the subspace of all upper triangular square matrices with constant diagonals is a unital algebra generated by a Jordan block. It is well-known and can be easily verified that a Jordan block is a cyclic operator. Therefore, by Theorem 2.1, a unital algebra generated by a Jordan block is boundedly reflexive. Thus any augmentation of the subspace is boundedly reflexive by Lemma 3.1.

Suppose $S\in\mathrm{ref_b}(\{T\}')$. Then $P_mSP_n\in P_m\mathrm{ref_b}(\{T\}')P_n\subseteq\mathrm{ref_b}(P_m\{T\}'P_n)$. It follows from Lemma 3.1, Lemma 3.3, and the previous paragraph that $P_m\{T\}'P_n$ is boundedly reflexive. Thus $P_mSP_n\in P_m\{T\}'P_n$. Another appeal to Lemma 3.3 yields $S\in\{T\}'$. ∎

An operator $L \in B(H)$ is called a *quasiaffinity* if L is injective and has a dense range. An operator $A \in B(H)$ is called *quasisimilar to* an operator $T \in B(H)$ if there exist quasiaffinities L and R so that $LT = AL$ and $TR = RA$.

LEMMA 3.5. *Suppose $A \in B(H)$ is quasisimilar to $T \in B(H)$. Then $\{A\}'$ is boundedly reflexive (reflexive, respectively) if and only if $\{T\}'$ is boundedly reflexive (reflexive, respectively).*

Proof. By symmetry, we only have to prove one direction. Suppose $\{T\}'$ is boundedly reflexive and L and R are quasiaffinities such that $LT = AL$ and $TR = RA$. For any $B \in \{T\}'$, we have $LBRA = LBTR = LTBR = ALBR$. This implies $L\{T\}'R \subseteq \{A\}'$. Similarly, $R\{A\}'L \subseteq \{T\}'$. Since $LRA = LTR = ALR, LR \in \{A\}'$. Suppose $S \in \mathrm{ref_b}(\{A\}')$. It follows that $RSL \in \mathrm{ref_b}(\{T\}')$. Since $\{T\}'$ is boundedly reflexive, there exists a $C \in \{T\}'$ such that $RSL = C$. Hence, $LRSLR = LCR \in \{A\}'$. Since $LR \in \{A\}'$, $LRASLR = ALRSLR = ALCR = LCRA = LRSLRA = LRSALR$. The density of the ranges of L and R implies $AS = SA$. Therefore, $S \in \{A\}'$, i.e. $\{A\}'$ is boundedly reflexive.

The argument for reflexivity is similar. ∎

A Hilbert space operator is called *bitriangular* if both the operator and its adjoint are triangular. A Hilbert space operator C is called an *algebraic operator* if there exists a nonzero polynomial $p(z)$ so that $p(C) = 0$. Clearly, algebraic operators are bitriangular.

LEMMA 3.6. *The commutant of a bitriangular operator is boundedly reflexive. In particular, the commutant of an algebraic operator is boundedly reflexive.*

Proof. Let A be any bitriangular operator. By Theorem 4.6 of [3], A is quasisimilar an operator T, which is the direct sum of Jordan blocks. By Lemma 3.4, $\{T\}'$ is boundedly reflexive. Thus $\{A\}'$ is boundedly reflexive by Lemma 3.5. ∎

A subspace $E \subseteq H$ is called a *semi-invariant* subspace of an operator T if there exist invariant subspaces $M_1 \subseteq M_2$ of T so that $E = M_2 \ominus M_1$. A subspace $E \subseteq H$ is called a *hyperinvariant* subspace of T if E is an invariant subspace of $\{T\}'$.

LEMMA 3.7. *Suppose $T \in B(H)$. Let $\{E_\lambda, \lambda \in \Lambda\}$ be a family of hyperinvariant subspaces of T whose linear span is dense in H and $T|_{E_\lambda}$ be the restriction of T to E_λ. If $\{T|_{E_\lambda}\}'$ is boundedly reflexive for all λ then $\{T\}'$ is boundedly reflexive.*

Proof. Since E_λ is a hyperinvariant subspace of T, it follows that $\{T\}'|_{E_\lambda} \subseteq \{T|_{E_\lambda}\}'$. Take an arbitrary $A \in \mathrm{ref_b}(\{T\}')$. Note that $\mathrm{ref_b}(\{T\}') \subseteq \mathrm{ref}(\{T\}') = \mathrm{AlgLat}(\{T\}')$, thus $E_\lambda \in \mathrm{Lat}A$. It is not hard to check that $A|_{E_\lambda} \in \mathrm{ref_b}(\{T\}'|_{E_\lambda}) \subseteq \mathrm{ref_b}(\{T|_{E_\lambda}\}') = \{T|_{E_\lambda}\}'$, the last equality is due to the hypothesis that $\{T|_{E_\lambda}\}'$ is boundedly reflexive. Thus, $AT|_{E_\lambda} = A|_{E_\lambda}T|_{E_\lambda} = T|_{E_\lambda}A|_{E_\lambda} = TA|_{E_\lambda}$. Now, the density of the linear span of E_λ implies $AT = TA$. ∎

THEOREM 3.8. *If T is triangular or co-triangular then the commutant of T is boundedly reflexive.*

Proof. First, suppose T is triangular. Let $E_{n,\mu} = \ker(\mu - T)^n$. Then $E_{n,\mu}$ is a hyperinvariant subspace of T. Since T is triangular, H is the closed linear span of these subspaces. Clearly, $T|_{E_{n,\mu}}$ is algebraic. By Lemma 3.6, $\{T|_{E_{n,\mu}}\}'$ is boundedly reflexive. The conclusion now follows from Lemma 3.7.

If T is co-triangular, then the above paragraph shows that $\{T^*\}'$ is boundedly reflexive. Now the conclusion follows from Corollary 2.7 of [14] which states that a subspace $\mathcal{S} \subseteq B(H)$ is boundedly reflexive if and only if $\mathcal{S}^*$ is boundedly reflexive. ∎

THEOREM 3.9. *If $T \in B(H)$ is triangular or co-triangular then $\{T\}' \cap \mathrm{AlgLat}T$ has a dense set of separating vectors; in particular, $w(T)$ has a dense set of separating vectors.*

Proof. If T is triangular, let $\{e_n\}_1^\infty$ be an orthonormal basis of H so that $E_n = \mathrm{span}\{e_1, \ldots, e_n\}$ are invariant subspaces of T. Let $\mathcal{A} = \{T\}' \cap \mathrm{AlgLat}T$. Clearly, E_n are invariant subspaces of $\mathcal{A}$. Take any $A \in \mathcal{A}$, $A|_{E_n}T|_{E_n} = AT|_{E_n} = TA|_{E_n} = T|_{E_n}A|_{E_n}$, so $A|_{E_n} \in \{T|_{E_n}\}'$. Note that for any $E \in \mathrm{Lat}(T|_{E_n})$, then $E \in \mathrm{Lat}T$. Thus, $E \in \mathrm{Lat}A$, which implies $E \in \mathrm{Lat}(A|_{E_n})$. Therefore, $A|_{E_n} \in \{T|_{E_n}\}' \cap \mathrm{AlgLat}(T|_{E_n}) = w(T|_{E_n})$ (The last equality holds for all algebraic operators, in fact for all C_0-contractions, by Theorem 1.2 on page 74 of [2].), yielding $\mathcal{A}|_{E_n} \subseteq w(T|_{E_n})$. It is well known that a singly generated unital algebra on a finite dimensional Hilbert space has separating vectors, so $w(T|_{E_n})$ has separating vectors. Hence, $\mathcal{A}|_{E_n}$ has separating vectors. By Theorem 11 of [6], $\mathcal{A}$ has a dense set of separating vectors.

If T is co-triangular, let $\{e_n\}_1^\infty$ be an orthonormal basis of H such that $E_n^\perp$ are invariant subspaces of T, where $E_n = \mathrm{span}\{e_1, \ldots, e_n\}$. Thus each E_n is semi-invariant subspace of T. Again, assume $\mathcal{A} = \{T\}' \cap \mathrm{AlgLat}T$. Let P_n be the orthogonal projection of H onto E_n. Take any $A \in \mathcal{A}$, then each E_n is a semi-invariant subspace of A also. It follows that $P_nA|_{E_n}P_nT|_{E_n} = P_nAT|_{E_n} = P_nTA|_{E_n} = P_nT|_{E_n}P_nA|_{E_n}$, so $P_nA|_{E_n} \in \{P_nT|_{E_n}\}'$. It can be easily verified that for any $E \in \mathrm{Lat}(P_nT|_{E_n})$, then $E \oplus E_n^\perp \in \mathrm{Lat}T$. Thus $E \oplus E_n^\perp \in \mathrm{Lat}A$, which implies $E \in \mathrm{Lat}(P_nA|_{E_n})$. Therefore $P_n\mathcal{A}|_{E_n} \subseteq \{P_nT|_{E_n}\}' \cap \mathrm{AlgLat}(P_nT|_{E_n}) = w(P_nT|_{E_n})$. Since $w(P_nT|_{E_n})$ has separating vectors, $P_n\mathcal{A}|_{E_n}$ has separating vectors. By Theorem 11 of [6], $\mathcal{A}$ has a dense set of separating vectors. ∎

COROLLARY 3.10. *If T is triangular or co-triangular then $\{T\}' \cap \mathrm{AlgLat}T$ is hereditarily boundedly reflexive; in particular, T is boundedly reflexive.*

Proof. Clearly, $\mathrm{AlgLat}T$ is reflexive, so it is boundedly reflexive. By Theorem 3.8, $\{T\}'$ is boundedly reflexive. Therefore, $\{T\}' \cap \mathrm{AlgLat}T$ is boundedly reflexive. The hereditary bounded reflexivity follows from the fact that $\{T\}' \cap \mathrm{AlgLat}T$ has separating vectors, by Theorem 3.9. ∎

REMARK 3.11. For perspective, it should be noted that every operator on a finite dimensional space is boundedly reflexive, an immediate consequence of Corollary 2.6 or Corollary 3.10.

Let E and F be closed subspaces of H so that $H = E \oplus F$. An operator $A \in B(H)$ is called an *algebraic extension* of an operator $T \in B(E)$ if A has the following form:

$$A = \begin{pmatrix} T & B \\ 0 & C \end{pmatrix}$$

with respect to the decomposition $H = E \oplus F$, where $C \in B(F)$ is an algebraic operator.

The main result of [13] states that if A is an algebraic extension of a bitriangular operator then $w(A)$ has a separating vector ([13], Theorem 10). Our next corollary improves this result.

COROLLARY 3.12. *If A is an algebraic extension of a bitriangular operator then $w(A)$ has a dense set of separating vectors.*

Proof. Suppose E and F are closed subspaces of H such that $H = E \oplus F$ and A has the following form with respect to the orthogonal decomposition:

$$A = \begin{pmatrix} T & B \\ 0 & C \end{pmatrix},$$

where T is bitriangular and C is algebraic.

First, we show A is co-triangular, i.e. A^* is triangular. Let x be any algebraic vector of T^* and $y \in F$ be arbitrary. Then there is a non-zero polynomial $p(t)$ such that $p(T^*)x = 0$ and $p(C^*) = 0$. Write

$$p(A^*) = \begin{pmatrix} p(T^*) & 0 \\ D & 0 \end{pmatrix}$$

for some $D \in B(E, F)$. Then $p(A^*)(x \oplus y) = 0 \oplus Dx$ and hence $p(A^*)^2(x \oplus y) = 0$. So $x \oplus y$ is an algebraic vector of A^*. Now, the density of algebraic vectors of T^* in E implies the set of algebraic vectors for A^* is dense in H, thus A^* is triangular.

Now by Theorem 3.9, $w(A)$ has a dense set of separating vectors. ∎

COROLLARY 3.13. *If A is an algebraic extension of a bitriangular operator then A is boundedly reflexive.*

Proof. By the proof of Corollary 3.12, A is co-triangular. Now the conclusion follows from Corollary 3.10. ∎

COROLLARY 3.14. *Suppose H is a finite dimensional Hilbert space and $\mathcal{S}$ is any subset of $B(H)$. Then the commutant of $\mathcal{S}$ is boundedly reflexive. In particular, any maximal abelian subalgebra of $B(H)$ is boundedly reflexive.*

Proof. When H is finite dimensional, every operator in $B(H)$ is triangular. Thus, the commutant of each operator in $B(H)$ is boundedly reflexive by Theorem

3.8. Since $\mathcal{S}'$ is the intersection of boundedly reflexive subspaces ($\mathcal{S}' = \cap\{\{S\}' : S \in \mathcal{S}\}$), $\mathcal{S}'$ is boundedly reflexive.

For any maximal abelian subalgebra $\mathcal{A}$ of $B(H)$, $\mathcal{A} = \mathcal{A}'$. Thus, $\mathcal{A}$ is boundedly reflexive. ∎

Using Lemmas 3.3 and 3.5, together with a theorem of Davidson and Herrero ([3], Theorem 4.6), we can give the following necessary and sufficient condition for the commutant of a bitriangular operator to be reflexive.

THEOREM 3.15. *Suppose $A \in B(H)$ is a bitriangular operator. Then $\{A\}'$ is reflexive if and only if A is quasisimilar to a diagonal operator.*

Proof. By Theorem 4.6 of [3], A is quasisimilar to an operator of the form: $T = \bigoplus_{i=1}^{N}(J_i + \lambda_i)$, where each J_i is an upper triangular nilpotent Jordan block acting on a finite dimensional Hilbert space H_i, and $H = \bigoplus_{i=1}^{N} H_i$ (Here $N = \infty$ when $\dim H = \infty$) . By Lemma 3.5, $\{A\}'$ is reflexive if and only if $\{T\}'$ is reflexive. Let P_i be the orthogonal projection from H onto H_i. It follows from Lemma 3.3 that $\{T\}'$ is reflexive if and only if $P_m\{T\}'P_n$ is reflexive for all positive integers m and n; this happens if and only if $\dim H_i = 1$ for each i, since a unital algebra generated by a Jordan block is reflexive if and only if the Jordan block is of size 1×1, by Theorem 2 of [4]. ∎

Theorem 3.15 is a generalization of the following main result of [5].

COROLLARY 3.16. ([5]) *The commutant of an operator on a finite dimensional Hilbert space is reflexive if and only if the operator is diagonalizable.*

Proof. Note that on a finite dimensional Hilbert space every operator is a bitriangular operator and every quasiaffinity is invertible. ∎

N-reflexivity of an operator space measures how close the operator space is to being reflexive. Although the commutant of a Banach space operator might not be reflexive, the following proposition shows that if it is not reflexive, it is never too far from being reflexive.

PROPOSITION 3.17. *The commutant of a Banach space operator is 2-reflexive.*

Proof. Let $T \in B(X)$. For simplicity of notations, we suppose $\mathcal{S} = \{T\}'$. Take any $A \in \text{ref}(\mathcal{S}^{(2)}) \subseteq B(X \oplus X)$, then necessarily A has the form $A = A_1 \oplus A_1$. For any $x \in X$, there exist $S_n \in \mathcal{S}$ such that $S_n x \oplus S_n T x \to A_1 x \oplus A_1 T x$. Since $S_n x \oplus S_n T x = S_n x \oplus T S_n x \to A_1 x \oplus T A_1 x$, it follows that $A_1 T x = T A_1 x$. Since x is arbitrary, we obtain $A_1 T = T A_1$, that is, $A_1 \in \{T\}'$. Therefore, $\{T\}'$ is 2-reflexive. ∎

Proposition 3.17 generalizes Corollary 11 of [10] to Banach spaces.

COROLLARY 3.18. *There exists a* 2*-reflexive transitive abelian unital algebra consisting of entirely boundedly reflexive operators.*

Proof. First note that the intersection of 2-reflexive subspaces is 2-reflexive. Thus, if $\mathcal{S}$ is any subset of $B(X)$ then $\mathcal{S}' = \bigcap\{\{S\}' : S \in \mathcal{S}\}$ is 2-reflexive, by Proposition 3.17.

Let T be the quasinilpotent transitive operator constructed in [17]. Then $\{T\}''$, the double commutant of T, has the desired properties. First of all, $\{T\}''$ is 2-reflexive by the previous paragraph. By Theorem 2.1, $\{T\}'$ is hereditarily boundedly reflexive; in particular, it consists of entirely boundedly reflexive operators. It is not hard to check that $w(T) \subseteq \{T\}'' \subseteq \{T\}'$ and $\{T\}''$ is an abelian unital algebra. Now the transitivity of T implies $\{T\}''$ is transitive. ∎

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