J. OPERATOR THEORY
53:2(2005), 381–397

SEPARATING PARTIAL NORMALITY CLASSES WITH COMPOSITION OPERATORS

CHARLES BURNAP, IL BONG JUNG and ALAN LAMBERT

Communicated by William B. Arveson

ABSTRACT. Complete measure-theoretic characterizations are given for ∞-, p-, and weakly hyponormal composition operators. Examples are then presented showing that composition operators distinguish between these classes as well as those of quasinormality and subnormality.

KEYWORDS: *Composition operators, p-hyponormal operators, weakly hyponormal operators, conditional expectation.*

MSC (2000): 47B20, 47B38.

1. INTRODUCTION

In this article we will be concerned with characterizing certain operators on an L^2 space in terms of membership in the various partial normality classes. Here is a brief review of what constitutes membership for an operator A on a Hilbert space in each class:

(i) A is *normal* if $A^*A = AA^*$.

(ii) A is *quasinormal* if $A(A^*A) = (A^*A)A$.

(iii) A is *subnormal* if A is the restriction of a normal operator to an invariant subspace.

(iv) A is *hyponormal* if $A^*A \geqslant AA^*$.

(v) For $0 < p < \infty$, A is *p-hyponormal* if $(A^*A)^p \geqslant (AA^*)^p$ [10].

(vi) A is ∞*-hyponormal* if it is p-hyponormal for all p [22].

(vii) A is *weakly hyponormal* if $|\widetilde{A}| \geqslant |A| \geqslant |(\widetilde{A})^*|$, where $\widetilde{A}$ is the Aluthge transform of A as defined later in this note.

There has been considerable interest in recent years in p- and weak hyponormality; a small sample of the related articles are found in our list of references (in particular, [9], [11], [13], and [14]). There are several basic relationships between these classes. The ones of concern in this note are as follows:

$$\text{normal} \Rightarrow \text{quasinormal} \Rightarrow \text{subnormal} \Rightarrow \text{hyponormal;}$$
$$\text{quasinormal} \Rightarrow \infty\text{-hyponormal} \Rightarrow p\text{-hyponormal} \Rightarrow \text{weakly hyponormal;}$$
$$p\text{-hyponormal} \Rightarrow q\text{-hyponormal for } q < p.$$

There is no containment between the subnormal and ∞-hyponormal classes [22]. Later in the paper, we provide examples showing that composition operators separate these classes as well.

We will be concerned with operators defined by point transformations in a measure space environment, and their relationships to the various classes of partial normality. We now present the basic structure of the objects to be examined, along with restrictions placed on these objects:

(i) $(X, \mathcal{F}, \mu)$ is a σ-finite measure space.
(ii) $T : X \to X$ is measurable: $T^{-1}\mathcal{F} \subset \mathcal{F}$.
(iii) $\mu \circ T^{-1} \ll \mu$.
(iv) $h = \frac{d\mu \circ T^{-1}}{d\mu} \in L^\infty$. ($h < \infty$ almost everywhere is equivalent to asserting that $T^{-1}\mathcal{F}$ is σ-finite. The essential boundedness condition assures continuity of C.)
(v) $Cf = f \circ T$ on $L^2(X, \mathcal{F}, \mu)$.
(vi) $Ef = E(f|T^{-1}\mathcal{F})$, the *conditional expectation of f with respect to* $T^{-1}\mathcal{F}$.

References [23] and [26] provide good sources for the general properties of (measure based) composition operators. More topic specific references are provided below where appropriate. All stated equations, set relations, etc. are taken up to μ null sets. Most almost everywhere disclaimers are not specifically displayed.

Some basic properties:

(i) E is the self adjoint projection onto $L^2(X, T^{-1}\mathcal{F}, \mu)$.
(ii) Every $T^{-1}\mathcal{F}$ measurable function has the form $F \circ T$.
(iii) $F \circ T = G \circ T$ if and only if $hF = hG$; in fact, $F \circ T \geqslant G \circ T$ if and only if $F\chi_S \geqslant G\chi_S$, where $S = \text{support}\, h$ [5]. In particular, the notation $h(Ef) \circ T^{-1}$ is well defined.
(iv) $C^*f = h(Ef) \circ T^{-1}$, cf. [18].
(v) $C^*Cf = hf$.
(vi) $CC^*f = (h \circ T)Ef$, cf. [15].
(vii) (Change of Variables Formula) $\int\limits_{T^{-1}A} f \circ T d\mu = \int\limits_{A} hf d\mu$.
(viii) $h \circ T > 0$, cf. [15].

In addition to properties inherent to its nature as a projection, we shall make repeated use of the following properties of a conditional expectation $E(\cdot|\mathcal{A})$, where $\mathcal{A}$ is a sub σ-algebra of $\mathcal{F}$. M.M. Rao's text [25] is an excellent source for properties of conditional expectation. One specialized part of the list below is referenced separately:

(i) For $\mathcal{A}$ measurable a and $\mathcal{F}$ measurable f, $E(af|\mathcal{A}) = aE(f|\mathcal{A})$.

(ii) E is strictly monotone:

$$f \geqslant g \Rightarrow Ef \geqslant Eg, \quad f \geqslant 0 \quad \text{and} \quad Ef = 0 \Rightarrow f = 0.$$

(iii) For any $\mathcal{A}$ set A and L^2 function f, $\int\limits_A f \mathrm{d}\mu = \int\limits_A E(f|\mathcal{A})\mathrm{d}\mu$.

(iv) For any nonnegative function f in L^2 and any $r > 0$, support Ef = support $E(f^r)$ is the smallest (up to null sets) $\mathcal{A}$ set containing support f [21].

We will have need of the following special case: If $\mathcal{A}$ is the purely atomic σ-subalgebra of $\mathcal{F}$ generated by the partition of X into sets of positive measure $\{A_k\}_{k\geqslant 0}$, then

$$E(f|\mathcal{A}) = \sum_{k=0}^{\infty} \frac{1}{\mu(A_k)}\Big(\int\limits_{A_k} f\mathrm{d}\mu\Big)\chi_{A_k}.$$

The plan for the remainder of this article is to present characterizations of composition operators in the various classes mentioned above. We then give specific examples illustrating the separation of the various classes by composition operators. We start with the previously known classes:

(i) C is normal if and only if $T^{-1}\mathcal{F} = \mathcal{F}$ and $h = h \circ T$ [15].

(ii) C is quasinormal if and only if $h = h \circ T$ [15].

(iii) C is hyponormal if and only if $h > 0$ and $E\Big(\frac{1}{h}\Big) \leqslant \frac{1}{h \circ T}$ [18].

(iv) C is subnormal if and only if the sequence $\{\mu \circ T^{-k}A\}$ is a moment sequence for each $\mathcal{F}$ set A of positive finite measure, or equivalently, if and only if the sequence $\{h_n\}$ is a moment sequence almost everywhere, where $h_n = \frac{\mathrm{d}\mu \circ T^{-n}}{\mathrm{d}\mu}$ [19].

2. CLASSIFICATIONS

To establish a characterization of p-hyponormality for $p \in (0, \infty)$, we first examine the operators $(C^*C)^p$ and $(CC^*)^p$.

LEMMA 2.1. *$(C^*C)^p f = h^p f$ and $(CC^*)^p f = (h^p \circ T)Ef$.*

Proof. C^*C is the multiplication operator M_h, and the stated formula is a direct consequence of the functional calculus for normal operators. As for the second part,

$$CC^* = M_{h\circ T}E,$$

and since $h \circ T$ is $T^{-1}\mathcal{F}$ measurable, the positive operator $M_{h\circ T}$ commutes with the self adjoint projection E. ∎

It was shown in [15] that if C is hyponormal, then $h > 0$ almost everywhere. Below, we show that this remains valid for p-hyponormal composition operators. We will show in Section 3 of this note that this conclusion is not justified if C is assumed to be only weakly hyponormal (or any of its generalizations described later).

It will be convenient to establish two general (not composition operator specific) results. The results in the following proposition were proved by J. Herron as part of his doctoral dissertation [16]. As noted earlier, for any nonnegative function f, support $f \subset$ support Ef^r for any $r > 0$. For this reason we adopt the notational convention of writing expressions such as $\frac{f}{Ef^r}$ for $\left[\frac{f}{Ef^r}\right]\chi_{\text{support}\, f}$. In some of the more involved calculations we will display the appropriate characteristic functions where there would otherwise be zero division problems.

PROPOSITION 2.2. ([16]) *Let $E = E(\cdot|\mathcal{A})$ and let ϕ be a nonnegative $\mathcal{F}$ measurable function.*

(i) *Define the positive operator P_ϕ by $P_\phi f = \phi E(\phi f)$. Let $\widehat{\phi} = \frac{\phi}{(E(\phi^2))^{1/4}}$. Then $P_\phi^{1/2} = P_{\widehat{\phi}}$.*

(ii) *Define the operator R_ϕ by $R_\phi f = E(\phi f)$. Then $\|R_\phi\| = \|\sqrt{E(\phi^2)}\|_\infty$.*

LEMMA 2.3. *Let α and β be nonnegative functions, with $S =$ support α. Then the following are equivalent:*

(i) *for every $f \in L^2(X, \mathcal{F}, \mu)$,*

$$\int_X \alpha |f|^2 \mathrm{d}\mu \geqslant \int_X |E(\beta f|\mathcal{A})|^2 \mathrm{d}\mu;$$

(ii) support $\beta \subset S$ *and* $E\Big(\frac{\beta^2}{\alpha}\chi_S|\mathcal{A}\Big) \leqslant 1$ *almost everywhere.*

Proof. Let $E(\cdot) = E(\cdot|\mathcal{A})$ and let R be the operator given by

$$Rf = E\Big(\frac{\beta}{\alpha^{1/2}}\chi_S f\Big).$$

We first show (i) implies (ii). Let $M \subset X \sim S$ with $\mu M < \infty$. Then setting $f = \chi_M$ in (i) we have

$$0 = \int \alpha\, \chi_M \, \mathrm{d}\mu \geqslant \int |E(\beta \chi_M)|^2 \mathrm{d}\mu.$$

Consequently, $E(\beta\chi_M) = 0$. Because E is strictly monotone, we have $\beta\chi_M = 0$. Since $(X, \mathcal{F}, \mu)$ is σ-finite, this proves support $\beta \subset S$. Now define

$$\mathcal{G} = \{g \in L^2(\mathcal{F}) : \alpha^{-1/2}\chi_S g \in L^2(\mathcal{F})\}.$$

By σ-finiteness, $\mathcal{G}$ is dense in L^2. For $g \in \mathcal{G}$ and $f = \alpha^{-1/2}\chi_S g$, the inequality given in (i) implies

$$\int |g|^2 \mathrm{d}\mu \geqslant \int \Big|E\Big(\frac{\beta}{\alpha^{1/2}}\chi_S g\Big)\Big|^2 \mathrm{d}\mu.$$

Then (since $\mathcal{G}$ is dense), the operator R must be a contraction. By Herron's proposition, the inequality in (ii) holds.

Conversely, suppose (ii) holds. Put

$$\mathcal{H} = \{f \in L^2 : \sqrt{\alpha} f \in L^2\}.$$

Unless $f \in \mathcal{H}$, (i) holds trivially. So, suppose $f \in \mathcal{H}$. Then, since (ii) implies that the operator R is a contraction (Proposition 2.2), we have $\|\sqrt{\alpha}f\| \geqslant \|R\sqrt{\alpha}f\|$. Equivalently,

$$\int \alpha |f|^2 \mathrm{d}\mu \geqslant \int |E(\beta\chi_S f)|^2 \mathrm{d}\mu = \int |E(\beta f)|^2 \mathrm{d}\mu.$$

Hence the proof is complete. ∎

Here is our classification of p-hyponormal composition operators:

THEOREM 2.4. *C is p-hyponormal if and only if $h > 0$ and $E(\frac{1}{h^p}) \leqslant \frac{1}{h^p \circ T}$.*

Proof. First notice that

$$\langle (C^*C)^p f, f \rangle = \int_X h^p |f|^2 \mathrm{d}\mu$$

and

$$\langle (CC^*)^p f, f \rangle = \int_X h^p \circ T (Ef)(\overline{f}) \mathrm{d}\mu = \int_X |E(h^{p/2} \circ Tf)|^2 \mathrm{d}\mu.$$

Let $S = \operatorname{support} h$. By Lemma 2.3, C is p-hyponormal if and only if $\operatorname{support} h^{p/2} \circ T \subset \operatorname{support} h^p$ and $E\Big(\frac{\chi_S h^p \circ T}{h^p}\Big) \leqslant 1$. Since $h \circ T > 0$, the condition involving supports is true if and only if $h > 0$ (so that $\chi_S = 1$). The inequality is then equivalent to $E\Big(\frac{1}{h^p}\Big) \leqslant \frac{1}{h^p \circ T}$ because $h^p \circ T$ is $T^{-1}\mathcal{F}$-measurable. ∎

As mentioned earlier, D. Harrington and R. Whitley showed in [15] that C is quasinormal if and only if $h \circ T = h$. But, quasinormality $\Rightarrow$ ∞-hyponormality $\Rightarrow$ p-hyponormality for all $p \in (0, \infty)$. It is not too surprising then, that the measure-theoretic characterization of this class clearly exhibits this lineage.

THEOREM 2.5. *C is ∞-hyponormal if and only if $h \circ T \leqslant h$.*

Proof. Suppose first that $h \circ T \leqslant h$. Then for any $p > 0$, $\frac{1}{h^p} \leqslant \frac{1}{h^p \circ T}$. Since E is order preserving, we have

$$\forall p > 0,\ E\frac{1}{h^p} \leqslant E\frac{1}{h^p \circ T} = \frac{1}{h^p \circ T};$$

so that C is ∞-hyponormal.

Now suppose that C is ∞-hyponormal. It was shown in [20] that for any sub σ-algebra $\mathcal{A}$, and for each nonnegative ($\mathcal{F}$ measurable) function f, the sequence $(E(f^n|\mathcal{A}))^{1/n}$ converges almost everywhere to a function $\widehat{f}$ with the following properties:

$\widehat{f}$ is $\mathcal{A}$ measurable (because it is a pointwise limit of such functions) and $f \leqslant \widehat{f}$ almost everywhere. If a is any $\mathcal{A}$ measurable function with $f \leqslant a$ almost everywhere, then $\widehat{f} \leqslant a$ almost everywhere.

For these reasons, $\widehat{f}$ is called *the minimal $\mathcal{A}$ majorant of f*. Let w be the minimal $\mathcal{A}$ majorant of $\frac{1}{h}$. For every positive integer n,

$$E\frac{1}{h^n} \leqslant \frac{1}{h^n \circ T} \quad (n\text{-hyponormality}).$$

Thus $\left(E\frac{1}{h^n}\right)^{1/n} \leqslant \frac{1}{h\circ T}$, so $w \leqslant \frac{1}{h\circ T}$. But $\frac{1}{h} \leqslant w$, so $h \circ T \leqslant h$. ∎

Our final general classification concerns weak hyponormality and some of its generalizations. For a function w, define the linear transformation W by $Wf = w(f \circ T)$. The transformation W is called a *weighted composition operator*. We will make use of several properties of such operators. Detailed analysis of these operators is found in [5], [6], and [7].

PROPOSITION 2.6. ([5]) *For $w \geqslant 0$:*

(i) $W^*Wf = h \cdot [E(w^2)] \circ T^{-1} f$;

(ii) $WW^*f = w \cdot h \circ TE(wf)$.

It follows from the preceding proposition that

$$|W|f = \sqrt{h \cdot [E(w^2)] \circ T^{-1}} f.$$

As for $|W^*|$, note that

$$WW^*f = w \cdot h \circ TE(wf) = w \cdot \sqrt{h \circ T} E(w\sqrt{h \circ T} f);$$

i.e., with the notation from Herron's proposition, $WW^* = P_{w\sqrt{h\circ T}}$. We then have

$$|W^*| = P_v, \quad \text{where } v = \frac{w\sqrt{h \circ T}}{[E(w\sqrt{h \circ T})^2]^{1/4}}.$$

THEOREM 2.7. *Let W be a weighted composition operator with weight $w \geqslant 0$, and let S be the support of h:*

(i) $|W| \geqslant |C|$ *if and only if* $E(w^2) \geqslant 1$;

(ii) $|C| \geqslant |W^*|$ *if and only if* support $w \subset S$ *and* $E\left(\frac{v^2}{\sqrt{h}}\chi_S\right) \leqslant 1$.

Proof. We adopt the notation given directly before the statement of this theorem.

(i) Since $|W|$ and $|C|$ are multiplication operators, we need only compare their symbols. After squaring and composing with T, we obtain $|W| \geqslant |C|$ if and only if $(h \circ T)E(w^2) \geqslant h \circ T$. Because $h \circ T > 0$ almost everywhere, we obtain (i).

(ii) As for this assertion,

$$|C| \geqslant |W^*| \iff \forall f, \quad \int h^{1/2}|f|^2 \mathrm{d}\mu \geqslant \langle |W^*|f, f\rangle = \int vE(vf)\overline{f} \mathrm{d}\mu.$$

However,

$$\int vE(vf)\overline{f}\,\mathrm{d}\mu = \langle E(vf), vf\rangle = \|E(vf)\|^2 = \int |E(vf)|^2\mathrm{d}\mu.$$

Since support v = support w, the desired conclusion follows from Lemma 2.3. ∎

A tool which has been of considerable use in operator theory in recent years is the *Aluthge transform* [1], [17]: For any operator A, let $A = U|A|$ be the canonical polar decomposition for A. The Aluthge transform of A is the operator $\widetilde{A}$ given by

$$\widetilde{A} = |A|^{1/2}U|A|^{1/2}.$$

More generally, we may form the family of operators $\{A_r : 0 < r \leqslant 1\}$, where $A_r = |A|^r U|A|^{1-r}$ [2]. Our first task in this context is to calculate these entities for a composition operator C. One may easily verify that the parts of the polar decomposition $U, |C|$ for C are given by

$$|C|f = \sqrt{h}f, \quad Uf = \frac{1}{\sqrt{h\circ T}}f\circ T.$$

This is valid for all composition operators, even if h vanishes on a set of positive measure. We then have

$$C_r f = h^{r/2}U(h^{(1-r)/2}f) = \frac{h^{r/2}(h^{(1-r)/2}\circ T)}{\sqrt{h\circ T}}f\circ T = \Big(\frac{h}{h\circ T}\Big)^{r/2} f\circ T.$$

We see then that C_r is a weighted composition operator with weight $w_r = \Big(\frac{h}{h\circ T}\Big)^{r/2}$. We now catalog the pieces needed for our analysis:

$$w_r = \Big(\frac{h}{h\circ T}\Big)^{r/2}, \qquad |C_r|f = \Big(\sqrt{h\cdot[E(w_r^2)]\circ T^{-1}}\Big)\cdot f;$$

$$v_r = \frac{w_r\sqrt{h\circ T}}{[E(w_r\sqrt{h\circ T})^2]^{1/4}}, \qquad |(C_r)^*|f = P_{v_r}f = v_r E(v_r f).$$

Note that support v_r = support w_r = support h, which we denote by S.

Our immediate goal is to characterize *weakly hyponormal* composition operators. An operator A is defined to be weakly hyponormal if $|\widetilde{A}| \geqslant |A| \geqslant |(\widetilde{A})^*|$ [3], [4]. We will actually obtain characterizations for the more general situation $|C_r| \geqslant |C| \geqslant |C_r^*|$. If these inequalities hold, we say that C is *r-weakly hyponormal*. (Note that $\widetilde{C} = C_{1/2}$, so that weak hyponormality coincides with $\frac{1}{2}$-weak hyponormality.)

THEOREM 2.8. (i) $|C_r| \geqslant |C| \iff [Eh^r] \geqslant h^r\circ T$;
(ii) $|C| \geqslant |C_r^*| \iff [E(h^{r-1/2}\chi_S)]^2 h^{1-r}\circ T \leqslant Eh^r$.

Proof. Applying Theorem 2.7, $|C_r| \geqslant |C| \geqslant |C_r^*|$ if and only if $E(w_r^2) \geqslant 1 \geqslant E\Big(\frac{v_r^2}{\sqrt{h}}\chi_S\Big)$. Inserting the h based identities of w_r and v_r followed by routine gathering of exponents, etc. leads to the stated inequalities. ∎

REMARK 2.9. In the expression $[E(h^{r-1/2}\chi_S)]^2$ from part (ii) of the preceding theorem, the appearance of χ_S is only needed if $r \leqslant \frac{1}{2}$. This apparent split in the theory at weak ($r = \frac{1}{2}$) hyponormality might be a point for further development.

Of special interest is the case $r = \frac{1}{2}$, that is to say, the weakly hyponormal case:

COROLLARY 2.10. *C is weakly hyponormal if and only if $\sqrt{h \circ T} \leqslant E\sqrt{h}$.*

Proof. In the case $r = \frac{1}{2}$, our inequalities are

$$\sqrt{h \circ T} \leqslant E\sqrt{h}$$

and

$$[E\chi_S]^2\sqrt{h \circ T} \leqslant E\sqrt{h}.$$

But $E\chi_S \leqslant E1 = 1$, so the latter inequality is implied by the former. ∎

To our knowledge, the invariant subspace problem remains open for composition operators. Of course, if we are dealing with a finite measure space then the constant function 1 is an eigenvector for the composition operator. If $T^{-1}\mathcal{F} \neq \mathcal{F}$, then the closure of the range of the composition operator C is a nontrivial invariant subspace for C. Also, if S =support $h \neq X$, then for any set $A \subset X \sim S$ with $0 < \mu(T^{-1}A) < \infty$, $\chi_{T^{-1}A}$ is a nonzero member of the kernel of C^*. These considerations allow us to make a small contribution to the sought general solution of the invariant subspace problem for composition operators:

COROLLARY 2.11. *Suppose that the composition operator C is r-weakly hyponormal for some $r \in (0, 1)$. Then C has a nontrivial invariant subspace.*

Proof. In light of the preceding comments, we may and do assume that $h > 0$ almost everywhere and $E = I$, the identity operator. Then r-weak hyponormality implies $h^r \geqslant h^r \circ T$; and so $h \geqslant h \circ T$. But this means that C is ∞-hyponormal, and consequently C has a nontrivial invariant subspace (cf. [22]). ∎

3. EXAMPLES

In this section we show that composition operators provide examples precisely marking the distinctions between the different partial normality classes. This is especially noteworthy because composition operators are often viewed as somewhat generalized weighted shifts, and weighted shifts have long been used to concretely illustrate various operator traits, from compact and quasinilpotent to hyponormal and subnormal. Shifts, however, prove to be essentially useless in the exploration of p-hyponormality; indeed, all levels of hyponormality (but not subnormality) for a weighted shift hold together or not at all. In fact, even the square of a weighted shift is not a good candidate for this type of analysis because the square of a shift is (unitarily equivalent to) the orthogonal direct sum of two weighted shifts, and p-hyponormality is easily seen to be inherited by such direct summands. Thus it may be somewhat surprising that the class of composition operators we shall use to distinguish the respective p-hyponormal classes are unitarily equivalent to rank one perturbations of the direct sum of two weighted shifts.

EXAMPLE 3.1. *Separating p-hyponormal classes.*
Let X be the set of nonnegative integers, let $\mathcal{F}$ be the σ-algebra of all subsets of X, and take μ to be the measure determined by the strictly positive sequence $\{m_k\}_{k\geqslant 0}$ defined below. Our point transformation T is defined as follows

$$T(k) = \begin{cases} 0 & k = 0,1,2; \\ k-2 & k \geqslant 3. \end{cases}$$

The action of T may be viewed as two paths leading back to 0, with 0 tied to itself. We specify our point mass measure m as follows (initializing the sequence at m_0):

$$m = 1,\ 1,\ 1,\ c,\ d,\ c^2,\ d^2,\ c^3,\ d^3,\ \ldots;$$

where c and d are fixed positive numbers. The powers of c occur for odd integers and those of d for even integers. The precise formula for power vs position will not be of consequence in our calculations. It follows that the σ-algebra $T^{-1}\mathcal{F}$ is generated by the atoms

$$\{0,1,2\},\ \{3\},\ \{4\},\ \ldots.$$

We now calculate the Radon-Nikodym derivative h:

$$\mu\circ T^{-1}(0) = \mu(\{0,1,2\}) = 3, \quad h(0) = \frac{\mu\circ T^{-1}(0)}{m_0} = 3.$$

For $k \geqslant 1$, $T^{-1}\{k\} = \{k+2\}$ so that

$$h(k) = \frac{m_{k+2}}{m_k} = \begin{cases} c & \text{for odd } k \geqslant 1; \\ d & \text{for even } k \geqslant 2. \end{cases}$$

In sequence form

$$h = 3,\ c,\ d,\ c,\ d,\ \ldots$$

and consequently

$$h\circ T = 3,\ 3,\ 3,\ c,\ d,\ c,\ \ldots.$$

In order to compute the necessary conditional expectations, recall the model for conditioning with respect to a partition $\{A_k\}_{k\geqslant 0}$ listed earlier:

$$E(f|\mathcal{A}) = \sum_{k=0}^{\infty} \frac{1}{\mu(A_k)}\Big(\int_{A_k} f\,\mathrm{d}\mu\Big)\chi_{A_k}.$$

So with respect to our current situation we have

$$Ef = \Big(\frac{f_0 m_0 + f_1 m_1 + f_2 m_2}{m_0 + m_1 + m_2}\Big)\chi_{\{0,1,2\}} + \sum_{k\geqslant 3} f_k \chi_{\{k\}}$$
$$= \frac{f_0 + f_1 + f_2}{3}\chi_{\{0,1,2\}} + \sum_{k\geqslant 3} f_k \chi_{\{k\}}.$$

In sequence form:

$$Ef = \frac{f_0+f_1+f_2}{3},\ \frac{f_0+f_1+f_2}{3},\ \frac{f_0+f_1+f_2}{3},\ f_3,\ f_4,\ \ldots.$$

Now fix a number $p > 0$, and let us consider $E\Big(\frac{1}{h^p}\Big)$ and $\frac{1}{h^p \circ T}$:

$$E\Big(\frac{1}{h^p}\Big) = \frac{\frac{1}{3^p}+\frac{1}{c^p}+\frac{1}{d^p}}{3},\ \frac{\frac{1}{3^p}+\frac{1}{c^p}+\frac{1}{d^p}}{3},\ \frac{\frac{1}{3^p}+\frac{1}{c^p}+\frac{1}{d^p}}{3},\ \frac{1}{c^p},\ \frac{1}{d^p},\ \frac{1}{c^p},\ \ldots. \tag{3.1}$$

In particular, $\frac{1}{h^p\circ T}$ and $E\Big(\frac{1}{h^p}\Big)$ agree for $k \geqslant 3$, so we need only compare their values for $k = 0$, or, to the same ends, consider $(h^p \circ T(0))E\Big(\frac{1}{h^p}\Big)(0)$. This product is

$$3^p \cdot \frac{\frac{1}{3^p}+\frac{1}{c^p}+\frac{1}{d^p}}{3} = \frac{1+\Big(\frac{3}{c}\Big)^p + \Big(\frac{3}{d}\Big)^p}{3}.$$

Using Theorem 2.4, this shows that we have p-hyponormality if and only if

$$\Big(\frac{3}{c}\Big)^p + \Big(\frac{3}{d}\Big)^p \leqslant 2. \tag{3.2}$$

First consider the extreme case,

$$\Big(\frac{3}{c}\Big)^p + \Big(\frac{3}{d}\Big)^p = 2.$$

We must have $2 - \Big(\frac{3}{d}\Big)^p > 0$; equivalently, $d > 3\cdot 2^{-1/p}$. Choose any such $d < 3$, and let $c = 3\cdot\Big[2-\Big(\frac{3}{d}\Big)^p\Big]^{-1/p}$. Then the corresponding composition operator is p-hyponormal. In fact, it satisfies the equality $h^p \circ TE\Big(\frac{1}{h^p}\Big) = 1$. With c and d chosen in this way, and $p > 0$ fixed, we show that for any $q > p$ our composition operator is not q-hyponormal. To this end we must show that for $q > p$,

$$\Big(\frac{3}{c}\Big)^q + \Big(\frac{3}{d}\Big)^q > 2.$$

For positive numbers A and B, consider the following functions of the nonnegative variable $x : u(x) = A^x$ and $v(x) = 2 - B^x$. Their graphs cross when $x = 0$, and they may cross at no more than one other point (unless $A = B = 1$). In our case ($A = \frac{3}{c}$ and $B = \frac{3}{d}$) we have found that point of intersection; namely $x = p$. For all larger x their difference (in the order presented) is positive.

According to Theorem 2.5, Corollary 2.10, (3.1), and (3.2), we have:

(i) C is quasinormal if and only if $c = d = 3$;

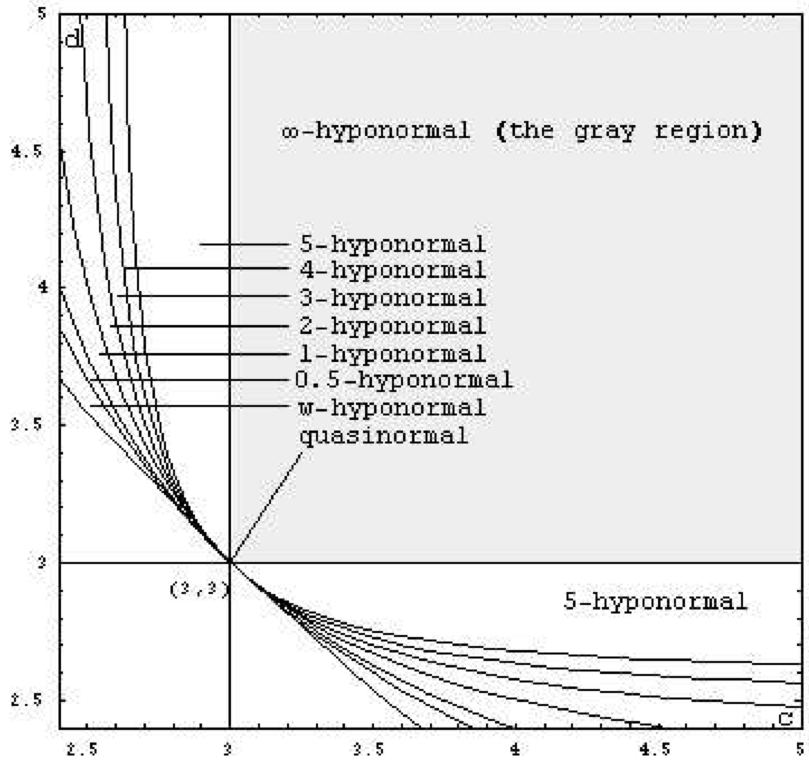

FIGURE 1. Figure 3.1

(ii) C is ∞-hyponormal if and only if $c \geqslant 3$ and $d \geqslant 3$;

(iii) C is p-hyponormal if and only if $\left(\frac{3}{c}\right)^p + \left(\frac{3}{d}\right)^p \leqslant 2$;

(iv) C is w-hyponormal if and only if $\sqrt{\frac{c}{3}} + \sqrt{\frac{d}{3}} \geqslant 2$.

Hence we have Figure 3.1, which clearly shows the separation of these partial normality classes.

We asserted earlier that the specific type of composition operator used above to separate the p-hyponormality classes is a rank one perturbation of the direct sum of two weighted shifts. To see this, let χ_k be the characteristic function of the singleton $\{k\}$. Then $\left\{e_k = \frac{1}{\sqrt{m_k}}\chi_k : k \geqslant 0\right\}$ is an orthonormal basis for our weighted l^2 space. Now our construction for T may be rephrased as

$$\chi_0 \circ T = \chi_0 + \chi_1 + \chi_2, \quad \chi_k \circ T = \chi_{k+2} \quad \text{for } k \geqslant 1.$$

In terms of the given orthonormal basis, these take the form

$$Ce_k = \begin{cases} e_0 + \sqrt{\frac{m_1}{m_0}}e_1 + \sqrt{\frac{m_2}{m_0}}e_2 & \text{for } k = 0; \\ \sqrt{\frac{m_{k+2}}{m_k}}e_{k+2} & \text{for } k \geqslant 1. \end{cases}$$

Suppose that $\{\alpha_k\}_{k\geqslant 0}$ is a bounded sequence of nonzero complex numbers. Let

$$\mathcal{H} = \bigvee\{e_{2k} : k \geqslant 0\} \quad \text{and} \quad \mathcal{K} = \bigvee\{e_{2k+1} : k \geqslant 0\}.$$

We then define shifts A and B on $\mathcal{H}$ and $\mathcal{K}$ respectively by

$$Ae_{2k} = \alpha_{2k}e_{2k+2}, \quad Be_{2k+1} = \alpha_{2k+1}e_{2k+3}.$$

Then the operator W given by $We_k = \alpha_k e_{k+2}; k \geqslant 0$ is unitarily equivalent to $A \oplus B$. Returning to our composition operator setting, let $\alpha_k = \sqrt{\frac{m_{k+2}}{m_k}}; k \geqslant 0$, and let F be the rank one operator $e_0 \otimes \left(e_0 + \sqrt{\frac{m_1}{m_0}}e_1\right)$. Then C is $F + W$.

EXAMPLE 3.2. *A composition operator which is subnormal but not ∞-hyponormal.*

As stated earlier, a method for constructing subnormal composition operators was presented in [19]. We now recall several important details of that construction and generate a subnormal, non ∞-hyponormal composition operator. We will have need of the following notational conventions for a composition operator on $L^2(X, \mathcal{F}, \mu)$. For each positive integer n, let T^n denote the n-fold composition of T with itself. Then C^n is the operator of composition by T^n. Let $h_n = \frac{\mathrm{d}\mu\circ T^{-n}}{\mathrm{d}\mu}$. Then

$$h_0 = 1, \quad h_1 = h, \quad h_{n+1} = h \cdot (Eh_n) \circ T^{-1}.$$

REMARK. The operation of composition by T acts continuously on all L^p spaces so long as $h \in L^\infty$. Let D be this operator on L^1. Then $D^*g = hE(g) \circ T^{-1}$ for $g \in L^\infty$. Our recursion formula takes the form $h_{n+1} = D^*h_n$; equivalently, $h_n = D^{*n}1$.

We now make use of the characterization of subnormality presented at the beginning of this note: C is subnormal if and only if for almost every x, the sequence $\{h_n(x)\}_{n\geqslant 0}$ is a moment sequence. Let $X = \mathbb{Z} \cup \{1^*, 2^*, \ldots\}$ where $\{1^*, 2^*, \ldots\}$ is a countable set disjoint from $\mathbb{Z}$. The transformation T is given by

$$T(k) = k - 1 \quad \text{for all } k, \quad T(k^*) = (k-1)^* \quad \text{for } k \geqslant 2, \quad T(1^*) = 0.$$

The point masses are

$$m_k = \int_1^3 t^k \mathrm{d}t \quad \text{for } k \leqslant 0, \quad m_k = \int_1^2 t^k \mathrm{d}t \quad \text{for } k > 0, \quad m_{k^*} = \int_2^3 t^k \mathrm{d}t \quad \text{for } k > 0.$$

Then

$$h_k(j) = \int_1^3 t^k \Big(\frac{t^j \mathrm{d}t}{m_j}\Big) \quad \text{for } j \leqslant 0,$$

$$h_k(j) = \int_1^2 t^k \Big(\frac{t^j \mathrm{d}t}{m_j}\Big) \quad \text{for } j > 0,$$

$$h_k(j^*) = \int_2^3 t^k \Big(\frac{t^j \mathrm{d}t}{m_j}\Big) \quad \text{for } j > 0.$$

This guarantees that C is subnormal. To see that C is not ∞-hyponormal, we show that $h(T1) > h(1)$. Indeed, $h(T1) = h(0) = \frac{m_1+m_{1^*}}{m_0} = 2$ while $h(1) = \frac{m_2}{m_1} = \frac{14}{9}$.

EXAMPLE 3.3. *A composition operator which is ∞-hyponormal, but not subnormal.*

Since we will be looking for a composition operator which is ∞-hyponormal but not subnormal, we need a method for ruling out certain sequences as moment sequences. Suppose that $\{\lambda_k\}$ is the moment sequence $\{\int t^k \mathrm{d}\beta\}$, and that β is a probability measure. Suppose that $\lambda_1^2 = \lambda_2$. Then by the equality clause in the Cauchy-Schwarz Theorem, the function $\phi(t) = t$ is constant almost everywhere $\mathrm{d}\beta$. But this can occur if and only if β is a point mass δ_r. It follows that $r = \lambda_1$. Thus for all $k, \lambda_k = \lambda_1^k$.

We are now in position to present an ∞-hyponormal composition operator that is not subnormal. Let X and T be as in Example 3.2. Fix positive numbers a and c. The point masses are

$$m_k = c^{k+2} \quad -\infty < k \leqslant 0, \quad m_k = m_{k^*} = a^k \quad k \geqslant 1.$$

The same type of calculations used previously now lead to the following values for h and $h \circ T$:

$$h(k) = c \quad \text{for } k \leqslant -1, \quad h(0) = 2\frac{a}{c^2}, \quad h(k) = h(k^*) = a \quad \text{for } k \geqslant 1.$$

The function $h \circ T$ is then a one place shift of h :

$$h \circ T(k) = c \quad \text{for } k \leqslant 0,$$
$$h \circ T(1) = h \circ T(1^*) = 2\frac{a}{c^2},$$
$$h \circ T(k) = h \circ T(k^*) = a \quad \text{for } k \geqslant 2.$$

To ensure ∞-hyponormality, we must have $h \circ T \leqslant h$. These functions are in fact equal at all points except $0, 1$, and 1^*. So

$$h \circ T \leqslant h \text{ if and only if } c \leqslant \frac{2a}{c^2} \leqslant a.$$

These last inequalities are valid if and only if

$$\sqrt{2} \leqslant c \quad \text{and} \quad \frac{c^3}{2} \leqslant a.$$

Note that we have actual equality, hence quasinormality if and only if $a = c = \sqrt{2}$. Although there is ample room to choose a and c satisfying the inequalities, we wish to make our choice so that the composition operator is not subnormal. To this end, we examine the sequence $\{h_k(-2)\}_{k\geqslant 0}$:

$$h_0(-2) = m_{-2} = 1, \qquad h_1(-2) = \frac{m_{-1}}{m_{-2}} = c,$$

$$h_2(-2) = \frac{m_0}{m_{-2}} = c^2, \qquad h_3(-2) = \frac{m(\{1,1^*\})}{m_{-2}} = 2a.$$

First note that if $\{h_k(-2)\}$ were a moment sequence, then since $h_0(-2) = 1$, the corresponding Borel measure would be a probability measure. Since the $k = 2$ term is the square of the $k = 1$ term, the remark that began this example shows that we must have $h_3(-2) = (h_1(-2))^3$; i.e., $2a = c^3$. Thus C is ∞-hyponormal but not subnormal if and only if $\sqrt{2} \leqslant c$, $\frac{c^3}{2} < a$. For a specific example, take $c = \sqrt{2}, a = 2$. Then

$$\sqrt{2} = c, \quad \frac{c^3}{2} = \sqrt{2} < a.$$

EXAMPLE 3.4. *Separating weak hyponormality from any p-hyponormality.*

We now present a weakly hyponormal composition operator for which h is *zero* on a set of positive measure. This seems particularly interesting to us because the strict positivity of h holds for all the other levels of partial normality so far investigated. As all our examples thus far involved purely atomic measure spaces, we hereby attempt to correct any impression that only such measures supply useful examples in this setting. We present this Lebesgue measure based example:

Let $X = \mathbb{R}$ and let μ be Lebesgue measure. The transformation T is piecewise linear:

$$T(x) = \begin{cases} x - 1 & x \leqslant 2; \\ 3 - x & 2 < x \leqslant 3; \\ \frac{x}{8} + \frac{13}{8} & x > 3. \end{cases}$$

A glance at the graph of T shows that $T^{-1}\mathcal{F}$ consists of all Lebesgue measurable subsets of $(-\infty, 1) \cup (3, \infty)$, together with all Lebesgue measurable subsets of $(1, 3)$ that are symmetric about $x = 2$. The range of T is $(-\infty, 1] \cup (2, \infty)$, so that $\mu \circ T^{-1}(1, 2) = 0$. This forces h to be 0 on $(1, 2)$. In fact

$$h = \chi_{(-\infty,0)} + 2\chi_{(0,1)} + 8\chi_{(2,\infty)},$$
$$\sqrt{h \circ T} = \chi_{(-\infty,1)} + \sqrt{2}\chi_{(1,3)} + \sqrt{8}\chi_{(3,\infty)},$$
$$E\sqrt{h} = \chi_{(-\infty,0)} + \sqrt{2}\chi_{(0,3)} + \sqrt{8}\chi_{(3,\infty)}.$$

In particular, $E\sqrt{h} \geqslant \sqrt{h \circ T}$, guaranteeing weak hyponormality, while $h = 0$ on a set of positive measure rules out all p-hyponormality.

Our final example shows that a composition operator can be both subnormal and ∞-hyponormal without being quasinormal. The composition operator employed is an injective bilateral shift, and one does not need composition operator theory to arrive at the proper conclusion. We make the point of this shift's alter ego as a composition operator to underscore the point that shifts in general are not nearly as good tools for separating the partial normality classes as are composition operators.

EXAMPLE 3.5. *A composition operator which is subnormal and ∞-hyponormal, but not quasinormal.*

Let X be the set of all integers. The point masses are given by

$$m_k = \int_1^2 t^k \mathrm{d}t, \quad -\infty < k < \infty.$$

The transformation T is given by $Tk = k - 1$. It follows that

$$h(k) = \frac{m_{k+1}}{m_k} \quad \text{and} \quad h(Tk) = \frac{m_k}{m_{k-1}}, \quad -\infty < k < \infty.$$

Applications of the Cauchy Schwarz inequality show that for every k, $h(k) > h(Tk)$, so that C is ∞-hyponormal but not quasinormal. As for subnormality, for any k,

$$h_n(k) = \frac{m_{n+k}}{m_k} = \int_1^2 t^n \Big(\frac{t^k}{m_k}\mathrm{d}t\Big), \quad n \geqslant 0,$$

so that C is subnormal.

REMARK 3.6. It might seem that in the examples involving subnormal composition operators, one may choose the specific moment sequences rather arbitrarily. However, only certain measures can occur for specific examples. As an illustration, suppose we construct a subnormal operator with point masses. Specifically, suppose that,

$$h_n(x) = \int t^n \mathrm{d}\delta_{a(x)} \quad \text{for almost every } x.$$

It then follows that $a = h$. Recall that $h > 0$ in any case involving subnormality. Now

$$h \cdot [Eh] \circ T^{-1} = h_2 = h^2 \Rightarrow Eh = h \circ T,$$

and so

$$h \cdot [Eh_2] \circ T^{-1} = h_3 = h^3 \Rightarrow E(h^2) = Eh_2 = h^2 \circ T.$$

We then have a nonnegative function h with conditional variance $E(h^2) = (Eh)^2$; and this happens if and only if $Eh = h$. But then $h = Eh = h \circ T$, so C must be quasinormal. In fact, the converse is true: Suppose C is quasinormal. Then $h = h \circ T$. Because $h_{n+1} = hE(h_n) \circ T^{-1}$, we see that

$$h_n(x) = h^n(x) = \int t^n \mathrm{d}\delta_{h(x)}.$$

So C is quasinormal if and only if $\{h_n(x)\}_{n\geqslant 0}$ is almost everywhere a moment sequence corresponding to a point mass measure. These measures could not be used to separate subnormal and ∞-hyponormal operators.

Acknowledgements. This paper was written while the second author visited the University of North Carolina at Charlotte during the winter of 2003. He wishes to thank the faculty and administration of that institution for their warm hospitality. The second author was supported by KOSEF, R01-2000-00003.

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CHARLES BURNAP, DEPARTMENT OF MATHEMATICS, UNC CHARLOTTE, CHARLOTTE NC 28223, USA
E-mail address: caburnap@email.uncc.edu

IL BONG JUNG, DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA
E-mail address: ibjung@bh.knu.ac.kr

ALAN LAMBERT, DEPARTMENT OF MATHEMATICS, UNC CHARLOTTE, CHARLOTTE NC 28223, USA
E-mail address: allamber@email.uncc.edu

Received April 28, 2003.