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COMPACT TOEPLITZ OPERATORS WITH UNBOUNDED SYMBOLS

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ABSTRACT. We construct bounded Toeplitz operators on the Bergman space L^2_a on the unit disk, whose symbols are unbounded functions. These operators can be compact and in some cases Hilbert-Schmidt. In fact we show that for any (essentially unbounded) function $H \in L^2$ there is a set Γ in the unit disk such that the (essentially unbounded) function given by $h = \chi_\Gamma H$ is the symbol for a compact Toeplitz operator on L^2_a.

KEYWORDS: *Toeplitz operators, compact operators, Hilbert-Schmidt operators.*

MSC (2000): 47B35.

INTRODUCTION

In this paper $\mathbb{D}$ denotes the open unit disk in $\mathbb{C}$ and L^2_a denotes the Bergman space of all holomorphic functions f on $\mathbb{D}$ that are square-integrable with respect to the normalized Lebesgue area measure dA on $\mathbb{D}$. For $f \in L^2(\mathbb{D}, dA)$ we let T_f denote the Toeplitz operator with symbol f on L^2_a defined by $T_f g = P(fg)$, where P denotes the orthogonal projection of $L^2(\mathbb{D}, dA)$ onto L^2_a. This operator is densely defined only, and it is well-known that a bounded symbol f induces a bounded operator T_f on L^2_a (Zhu's book [6] is a good reference for Toeplitz operators).

There are five papers that are related to our work and to this topic. In [1] Axler and Zheng characterize bounded symbols on $\mathbb{D}$ that induce compact Toeplitz operators on the Bergman space (also see [5]). Their result involves the Berezin transform of the symbol. In [2] Grudsky and Vasilevski proved that Toeplitz operators T_f with radial symbols are bounded (compact) on the Bergman space L^2_a if and only if the sequence

$$\gamma_n(f) \equiv \int_0^1 f(r^{1/(2n+2)})\,\mathrm{d}r$$

belongs to $l_\infty(\mathbb{Z}_+)$ (respectively to $C_0(\mathbb{Z}_+)$). In [7], Zorboska defined the so-called radial operators and found an oscillation criterion that guarantees that the compactness of a radial operator is equivalent to the vanishing of the Berezin transform on the unit circle. Finally, there is the result of Luecking [3]. In his paper Luecking considers measures μ on the disk and the induced Toeplitz operators (for a large class of Banach spaces of analytic functions) to determine which trace classes they belong to. In terms of our interests, his result uses a condition for measures relating the convergence of a series of terms of the form $\left[\frac{\mu(R_i)}{|R_i|^2}\right]^2$, where the sum is taken over certain dyadic cubes R_i related to Carleson squares in the disk. Luecking's conditions are also necessary if the measure is positive.

Our main results are related to these earlier results in the following way. First, we produce a family of "essentially" unbounded functions (see Section 2 for a definition) with a restricted growth, which are parametrized by real parameters, which depend on a geometric construction for sets in the disk and these induce Hilbert-Schmidt Toeplitz operators on L^2_a. We then show that for any function F in $L^p(\mathbb{D}, dA)$, $p \geqslant 2$ we can choose a subset of $\mathbb{D}$, say E, for which the function $f = \chi_E F$ is essentially unbounded and induces a compact operator on L^2_a. This is our second main result. Also in the first set of examples we have been unable to adjust Luecking's method to apply to our work.

1. COMPACT AND HILBERT-SCHMIDT EXAMPLES

In this section we work with unbounded functions on the unit disk $\mathbb{D}$ that satisfy certain growth conditions. We want to construct specific types of subdomains in $\mathbb{D}$ such that these unbounded functions "blow up" on a set of positive measure on $\partial\mathbb{D}$, but the corresponding Toeplitz operators remain bounded, compact or even Hilbert-Schmidt. We start with a basic "cusp" domain $\Delta_0 = \{z \in \mathbb{D} : 0 < m < \operatorname{Re} z < 1, 0 < \operatorname{Arg} z < (1-|z|)^b, b \geqslant 1\}$ with m and b to be specified later. For an arbitrary ξ_j on the unit circle, let Δ_j be the rotation of Δ_0 to this point and dilate Δ_0 so that $0 < m_j < |z| < 1$, for some m_j, and $0 < \operatorname{Arg} z - \operatorname{Arg} \xi_j < (1-|z|)^b$. The sequence $\{\xi_j\}$ will correspond to the end points of a certain Cantor set. We construct the Cantor set in $[0,1]$ first, by taking $0 < \delta < 1$ and remove the middle centered interval of length $\frac{\delta}{2}$. From the remaining set consisting of two disjoint intervals, remove the center of these two intervals of length $\frac{\delta}{4}$ and repeat. This process produces a compact set of positive measure in $[0,1]$. Using the function $f(x) = e^{2\pi i x}$, we map this Cantor set to the unit circle and ξ_j's will be the images of the end points. Hence every Δ_j reaches the unit circle at ξ_j and we can choose Δ_j's so that they are disjoint.

Let $\Delta = \bigcup_{j=1}^{\infty} \Delta_j$. We consider a measurable function $H(z)$ satisfying $H(z) = O((1-|z|)^{-c})$, where $0 < c < \frac{1}{2}$ and such that for every open ball $B(\xi_j, r)$ with center ξ_j and radius r, we have

$$\operatorname{ess\,sup}\{|H(z)| : z \in B(\xi_j, r) \cap \Delta_j\} = \infty$$

for all r. Of course, it is the small r we are concerned with. We define $h(z) = \chi_\Delta(z)H(z)$, where χ_Δ is the characteristic function of Δ. It is clear that h goes to ∞ at every ξ_j, but also at every other point of the Cantor set in $\partial\mathbb{D}$, since these other points are the limit points of the endpoints ξ_j. We therefore conclude that h "blows up" on this set of positive measure on $\partial\mathbb{D}$. It is easy to see that $H \in L^2(\mathbb{D}, dA)$ and consequently $h \in L^2(\mathbb{D}, dA)$ too for an appropriate choice of b and the m_j's (for example $b = 2c + 1$ and $m_j \geqslant 1 - \frac{1}{j^2}$).

We want to study boundedness of the Toeplitz operator T_h. We have the following theorem.

THEOREM 1.1. *Let $H(z)$ be a function measurable on $\mathbb{D}$ with a growth $O((1 - |z|)^{-c})$ where $0 < c < \frac{1}{2}$. For any Cantor set in $\partial\mathbb{D}$ with "end points" $\xi_j \in \partial\mathbb{D}$ there are numbers $0 < m_j < 1$ and disjoint sets Δ_j, which are rotations of the set*

$$\{z \in \mathbb{D} : m_j < \operatorname{Re} z < 1,\ 0 < \operatorname{Arg} z < (1 - |z|)^b,\ b > 0\}$$

to ξ_j, with $\sum\limits_{j=1}^{\infty} m_j^{(b-2c-3)/2}$ finite so that if $\Delta = \bigcup\limits_{j=1}^{\infty} \Delta_j$, then the symbol $h = \chi_\Delta H$ produces a bounded Toeplitz operator on L^2_a. In particular, we can choose $b = 2c + 5$ and the m_j so that $m_j \geqslant 1 - \frac{1}{j^2}$.

Proof. For $f \in L^2_a$, we denote

$$F(z) = (T_h f)(z) = \sum_{j=1}^{\infty} \int\limits_{\Delta_j} \frac{H(w)f(w)}{(1 - z\overline{w})^2} \mathrm{d}A(w).$$

For each j,

$$\begin{aligned} \Big| \int\limits_{\Delta_j} \frac{H(w)f(w)}{(1 - z\overline{w})^2} \mathrm{d}A(w) \Big| &\leqslant \|f\| \Big[\int\limits_{\Delta_j} \frac{|H(w)|^2}{|1 - z\overline{w}|^4} \mathrm{d}A(w) \Big]^{1/2} \\ &\leqslant C\|f\| \Big[\int\limits_{m_j}^{1} \frac{(1 - |w|)^b}{(1 - |w|)^{2c+4}} \mathrm{d}|w| \Big]^{1/2} \\ &= C\|f\| (1 - m_j)^{(b-2c-3)/2} \end{aligned}$$

if $b - 2c - 3 > 0$ for some constant C. It now follows that

$$|F(z)| \leqslant C \Big[\sum_{j=1}^{\infty} (1 - m_j)^{(b-2c-3)/2} \Big] \|f\|.$$

Hence for $b = 2c + 5$ and $m_j \geqslant 1 - \frac{1}{j^2}$, the series will converge. Notice that the bound above is independent of z, hence $F \in H^\infty$. We conclude that T_h is bounded on L_a^2. ∎

Hence, in spite of the fact that the symbol goes to infinity on a set of positive measure on $\partial\mathbb{D}$, we obtain a bounded Toeplitz operator. In fact, we can even make it compact. This is the content of our next theorem.

THEOREM 1.2. *Let h and Δ be as in Theorem 1. Then we can choose constants b and m_j so that T_h is a compact operator on L_a^2.*

Proof. We assume that a sequence $f_n \in L_a^2$ satisfies $\|f_n\| \leqslant 1$ and f_n converges uniformly to zero on compact subsets of $\mathbb{D}$. We want to show $\|T_h f_n\| \to 0$ as $n \to \infty$. Let $\{m_j\}$ be chosen in a way that makes $\sum_{j=1}^{\infty}(1 - m_j) < \infty$. Let $\varepsilon > 0$ and choose J such that $\sum_{j=J}^{\infty}(1 - m_j) < \varepsilon$. Then we can find a positive integer N so that $|f_n(z)| < \varepsilon$ for all $n \geqslant N$ and $|z| < m_J$. We now consider

$$
\begin{aligned}
\|T_h f_n\|^2 &= \int_{\mathbb{D}} \Big| \int_{\mathbb{D}} \frac{f_n(w)h(w)}{(1 - z\overline{w})^2} \mathrm{d}A(w) \Big|^2 \mathrm{d}A(z) \\
&\leqslant \int_{\mathbb{D}} \Big[\Big| \int_{|w|<m_J} \frac{f_n(w)h(w)}{(1 - z\overline{w})^2} \mathrm{d}A(w) \Big| \\
&\qquad\qquad + \Big| \int_{m_J<|w|<1} \frac{f_n(w)h(w)}{(1 - z\overline{w})^2} \mathrm{d}A(w) \Big| \Big]^2 \mathrm{d}A(z) \\
&\leqslant 2 \int_{\mathbb{D}} \Big[\int_{|w|<m_J} \frac{|f_n(w)||h(w)|}{|1 - z\overline{w}|^2} \mathrm{d}A(w) \Big]^2 \mathrm{d}A(z) \\
&\qquad\qquad + 2 \int_{\mathbb{D}} \Big[\int_{m_J<|w|<1} \frac{|f_n(w)||h(w)|}{|1 - z\overline{w}|^2} \mathrm{d}A(w) \Big]^2 \mathrm{d}A(z) \\
&\equiv 2I_1 + 2I_2.
\end{aligned}
$$

Now let us evaluate the integrals I_1 and I_2. For any $n > N$,

$$
\begin{aligned}
I_1 &\leqslant \varepsilon^2 \int_{\mathbb{D}} \Big[\int_{\{|w|\leqslant m_J\}\cap\Delta} \frac{|H(w)|}{|1 - z\overline{w}|^2} \mathrm{d}A(w) \Big]^2 \mathrm{d}A(z) \\
&\leqslant C\pi\varepsilon^2 \Big[\sum_{j=1}^{J-1} \int_{\Delta_j} \frac{1}{|1 - |w||^{c+2}} \mathrm{d}A(w) \Big]^2
\end{aligned}
$$

$$\leqslant C\pi\varepsilon^2\Big[\sum_{j=1}^{J-1}\int_{m_j}^{1}(1-r)^{b-c-2}\mathrm{d}r\Big]^2 = C\pi\varepsilon^2\Big[\sum_{j=1}^{J-1}(1-m_j)^{b-c-1}\Big]^2$$

$$\leqslant C\pi\varepsilon^2\Big[\sum_{j=1}^{\infty}(1-m_j)^{b-c-1}\Big]^2. \tag{1.1}$$

For the second integral we use the Cauchy-Schwarz inequality to obtain

$$\int_{\mathbb{D}}\Big[\int_{|w|>m_J}\frac{|f_n(w)||h(w)|}{|1-z\overline{w}|^2}\mathrm{d}A(w)\Big]^2\mathrm{d}A(z)$$

$$\leqslant \int_{\mathbb{D}}\int_{|w|>m_J}\frac{|h(w)|^2}{|1-z\overline{w}|^4}\mathrm{d}A(w)\mathrm{d}A(z)$$

$$\leqslant \pi\sum_{j=J}^{\infty}\int_{m_j}^{1}\frac{(1-|w|)^b}{(1-|w|)^{2c+4}}\mathrm{d}|w| = \frac{\pi}{b-2c-3}\sum_{j=J}^{\infty}(1-m_j)^{b-2c-3}. \tag{1.2}$$

If we choose $b = 2c+4$, then the series on the right-hand side of equation (1.1) converges and the sum in (1.2) is less than a constant times ε by our assumption. Hence

$$\|T_h f_n\|^2 = 2I_1 + 2I_2 \leqslant c_1\varepsilon^2 + c_2\varepsilon \quad \text{if } n > N.$$

Therefore T_h is compact. ∎

REMARK 1.3. Recently, Axler and Zheng [1] proved that for a bounded f, T_f is a compact operator on L_a^2 if and only if the Berezin transform of f

$$Bf(z) = (1-|z|^2)^2\int_{\mathbb{D}}\frac{f(w)}{|1-z\overline{w}|^4}\mathrm{d}A(w) \to 0$$

as $|z| \to 1$. It is not known if this condition is sufficient for unbounded symbols. There are some interesting recent results on this conjecture. Zorboska [8] has recently proved that the condition is sufficient if f belongs to the hyperbolic BMO space, which contains L^∞ . Further Miao and Zheng [4] have introduced a class of functions, called BT, with $L^\infty \subset BT$ for which T_f is compact on L_a^2 if and only if the Berezin transform of f vanishes on the unit circle.

Instead of using the Cantor set, we could position our disjoint cusps at the countable set $\{\xi_j\}$ that form a dense set in $\partial\mathbb{D}$. In this situation for every Δ_j, we define $g_j(z) = (1-|z|)^{-c}\chi_{\Delta_j}(z)$ and let $T_j = T_{g_j}$. As in the proof of Theorem 1.1, it is easy to show that $\|T_j\| \leqslant 1$ for all j, provided b is large enough. As in Theorem 1.2, each T_j is compact. Now form the operator $T = \sum\limits_{j=0}^{\infty}\frac{1}{2^j}T_j$.

We will show that T is a compact Toeplitz operator. An obvious candidate for the symbol of this operator is $\sum\limits_{j=0}^{\infty} \frac{1}{2^j} g_j$.

For $f \in L_a^2$, $\|f\| = 1$, we estimate

$$\Big\| \sum_{j=N}^{M} \frac{1}{2^j} T_j f \Big\| \leqslant \sum_{j=N}^{M} \frac{1}{2^j} \|T_j f\| \leqslant \|f\| \sum_{j=N}^{M} \frac{1}{2^j}$$

which goes to 0 as $N, M \to \infty$. Thus we have a Cauchy sequence of operators that converges in norm to T so T must be compact. For $\varepsilon > 0$, we estimate

$$\Big\| T - T_{\sum_{j=0}^{\infty} \frac{1}{2^j} g_j} \Big\| \leqslant \Big\| T - T_{\sum_{j=0}^{N} \frac{1}{2^j} g_j} \Big\| + \Big\| T_{\sum_{j=N+1}^{\infty} \frac{1}{2^j} g_j} \Big\|.$$

The first norm is less than ε since $T_{\sum_{j=0}^{N} \frac{1}{2^j} g_j} = \sum\limits_{j=0}^{N} \frac{1}{2^j} T_j$ converges in norm to T and we choose N large enough. For an arbitrary polynomial p, using the estimates we have done before, one can show that

$$\Big\| T_{\sum_{j=N+1}^{\infty} \frac{1}{2^j} g_j} p \Big\|^2 \leqslant \|p\|^2 \sum_{j=N+1}^{\infty} \frac{1}{2^j} (1 - m_j)^{b-2c-3}$$

which is smaller than $\varepsilon^2 \|p\|^2$ for a proper choice of b, m_j and N. By the density argument we see that $\Big\| T_{\sum_{j=N+1}^{\infty} \frac{1}{2^j} g_j} \Big\| < \varepsilon$ and the claim is proved.

So now we have an example of a compact Toeplitz operator whose symbol is unbounded on a set of positive measure in a neighborhood of each point of the unit circle. In fact, T is not just compact.

THEOREM 1.4. *Let Δ_j be a rotation of the set*

$$\{z \in \mathbb{D} : m_j < \operatorname{Re} z < 1, 0 < \operatorname{Arg} z < (1 - |z|)^b, b > 0\}$$

to the point ξ_j and assume $\{\xi_j\}$ is a dense subset of the unit circle. If $g_j(z) = (1 - |z|)^{-c} \chi_{\Delta_j}(z)$ for $c < 1$ and all $j = 0, 1, \ldots$, then we can find b and $\{m_j\}$ so that $T = T_{\sum_{j=0}^{\infty} \frac{1}{2^j} g_j}$ is a Hilbert-Schmidt operator on L_a^2.

Proof. Let $e_k(z) = \sqrt{k+1} z^k$ be the standard orthonormal basis for L_a^2. We have to show that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |\langle T e_k, e_n \rangle|^2 < \infty.$$

At first, we compute the following:

$$\big| \langle T_j e_k, e_n \rangle \big| = \Big| \int\limits_{\mathbb{D}} \Big[\int\limits_{\mathbb{D}} \frac{g_j(w) \sqrt{k+1} w^k}{(1 - z\overline{w})^2} \mathrm{d}A(w) \Big] \sqrt{n+1}\, \overline{z}^n \mathrm{d}A(z) \Big|$$

$$
\begin{aligned}
&= \sqrt{k+1}\sqrt{n+1}\Big|\int_{\mathbb{D}} g_j(w)w^k\overline{w}^n \mathrm{d}A(w)\Big| \\
&\leqslant c\sqrt{nk}\int_{m_j}^{1}(1-r)^{5-c}r^{k+n+1}\mathrm{d}r
\end{aligned}
$$

if we choose $b = 5$. Choosing the sequence m_j as in the proof of Theorem 1.2 and using a repeated integration by parts yields that

$$
\int_{m_j}^{1}(1-r)^4 r^{k+n+1}\mathrm{d}r \leqslant d\frac{(1-m_j)^{k+n+6}}{(k+n)^5} < \frac{d}{(k+n)^5}\,,
$$

for some constant d. Hence

$$
\begin{aligned}
\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\big|\langle Te_k, e_n\rangle\big|^2 &\leqslant \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\Big[\sum_{j=0}^{\infty}\frac{1}{2^j}|\langle T_j e_k, e_n\rangle|\Big]^2 \\
&\leqslant d^2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{nk}{(k+n)^{10}} = d^2\sum_{k=1}^{\infty}k\sum_{n=1}^{\infty}\frac{n}{(n+k)^{10}} \\
&= d^2\sum_{k=1}^{\infty}\frac{k}{k^{10}}\sum_{n=1}^{\infty}\frac{n}{(1+\frac{n}{k})^{10}}\,.
\end{aligned}
$$

Using the integral test, we show $\int_1^\infty \frac{x\mathrm{d}x}{(1+\frac{x}{k})^{10}}$ converges to Ik^2, where $I = \int_1^\infty \frac{u\mathrm{d}u}{(1+u)^{10}}$. Hence

$$
\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}|\langle Te_k, e_n\rangle|^2 \leqslant d^2 I\sum_{k=1}^{\infty}\frac{1}{k^7} < \infty
$$

and T is Hilbert-Schmidt. ∎

2. COMPACTNESS VIA LEBESGUE POINTS OF THE SYMBOL

Let H be a measurable function on $\mathbb{D}$ and for positive integers n and k set $A_n = \{z \in \mathbb{D} : n+1 > \ |H(z)|\ \geqslant n\}$ and the annulus $B_k = \Big\{z : 1 - \frac{1}{k} < |z| < 1\Big\}$.

DEFINITION 2.1. The function H is essentially unbounded near $\partial\mathbb{D}$ if and only if for every positive integer k we have

$$
\text{ess sup}\{|H(z)| : z \in B_k\} = \infty,
$$

where the ess sup norm is calculated with respect to $\mathrm{d}A$.

Assume the function H is essentially unbounded near $\partial\mathbb{D}$. The sets A_n are disjoint. We set $C_{n,k} = A_n \cap B_k$. Since

$$\Big(\bigcup_{n=1}^{\infty} A_n\Big) \cap B_k = \bigcup_{n,k=1}^{\infty} C_{n,k}$$

and the measure dA of this set is positive we may choose a subsequence $n(k)$ so that $dA(C_{n(k),k}) > 0$. We may choose it so that it is increasing. Also we write $C_{n(k),k}$ as $C_{n(k)}$.

For each $n(k)$ choose a point of density $P_{n(k)}$ in the set $C_{n(k)}$. We also let $r_{n(k)}$ be a positive radius to be chosen later. At this point we choose the $r_{n(k)}$ so small that the disks $D_k = D(P_{n(k)}, r_{n(k)})$ with center $P_{n(k)}$ and radius $r_{n(k)}$ are all pairwise disjoint, and so that $r_{n(k)} < \frac{1-|P_{n(k)}|}{2}$. Here $|P_{n(k)}|$ denotes the absolute value of the complex number (point) $P_{n(k)}$. Denote by $D_k^* = D_k \cap C_{n(k)}$, and recall that each D_k^* has positive measure. Set $\Gamma = \bigcup D_k^*$ and let $h(z) = \chi_\Gamma(z)H(z)$, where χ_Γ denotes the characteristic function of the set in question. One can check that the function h is essentially unbounded near $\partial\mathbb{D}$. We are now ready to state the second main result of the paper.

THEOREM 2.2. *Let $H \in L^2(\mathbb{D}, dA)$ be an essentially unbounded function near $\partial\mathbb{D}$. Then there is a set Γ in $\mathbb{D}$ such that the function $h = \chi_\Gamma H$ is essentially unbounded near $\partial\mathbb{D}$ and the Toeplitz operator T_h is compact on L_a^2.*

Proof. We show that the Toeplitz operator

$$(T_h f)(z) \equiv \int_{\mathbb{D}} \frac{h(w)f(w)}{(1-z\overline{w})^2} \mathrm{d}A(w) = g(z)$$

maps L_a^2 into itself. To this end let $z \in \mathbb{D}$, let f be a polynomial with $\|f\|_2 \leqslant 1$ and compute

$$\int_{\mathbb{D}} |(T_h f)(z)|^2 \mathrm{d}A(z) = \int_{\mathbb{D}} |g(z)|^2 \mathrm{d}A(z).$$

First we do an estimate on $|g(z)|$, showing that g is a bounded (holomorphic) function on $\mathbb{D}$. Since f is in L_a^2, the pointwise estimate

$$|f(z)| \leqslant \frac{\|f\|}{1-|z|}$$

holds for all z in the unit disk. Hence,

$$\begin{aligned}|g(z)| &\leqslant \|f\| \int_{\mathbb{D}} \frac{|h(w)|}{|1-z\overline{w}|^2(1-|w|)} \mathrm{d}A(w)\\ &\leqslant \sum_{k=1}^{\infty} \int_{D_k^*} \frac{n(k)+1}{(1-|z||w|)^2(1-|w|)} \mathrm{d}A(w).\end{aligned}$$

For $w \in D_k^*$ we have that

$$|1 - z\overline{w}| \geqslant 1 - |z||w| \geqslant 1 - (|P_{n(k)}| + r_{n(k)})$$

and hence,

$$|g(z)| \leqslant \sum_{k=1}^{\infty} \int_{D_k^*} \frac{n(k)+1}{(1 - |P_{n(k)}| - r_{n(k)})^3} \mathrm{d}A(w).$$

The last sum is smaller than

$$C \sum_{k=1}^{\infty} \frac{(n(k)+1) r_{n(k)}^2}{(1 - |P_{n(k)}|)^3}.$$

We make the final choice of $r_{n(k)}^2 \leqslant \frac{(1-|P_{n(k)}|)^3}{n(k)^2(n(k)+1)}$. (This is consistent with our earlier requirements on $r_{n(k)}$). Since $n(k) \geqslant k$, we have

$$|g(z)| \leqslant C \sum_{k=1}^{\infty} \frac{1}{k^2} \equiv M$$

and this is independent of z in $\mathbb{D}$.

We have

$$\int_{\mathbb{D}} |g(z)|^2 \mathrm{d}A(z) \leqslant M^2$$

so T_h is densely defined with uniform estimates on L_a^2 and so is continuous there. Using the argument similar to the one in the proof of Theorem 1.2, we can show that T_h defined above is in fact compact. ∎

We hope these examples will shed more light on this problem of characterizing compact or Schatten class Toeplitz operators with unbounded symbols. There are other related questions involving our operators. For example, it would be interesting to study the spectrum of these operators. We may return to this problem in the future.

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