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CONTINUOUS VERSIONS OF THE LITTLEWOOD-RICHARDSON RULE, SELFADJOINT OPERATORS, AND INVARIANT SUBSPACES

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Dedicated to Ciprian Foiaş on the occasion of his 70th Birthday.
Many Happy Returns of the Day!

Communicated by Florian-Horia Vasilescu

ABSTRACT. We establish the equivalence of two continuous versions of the Littlewood-Richardson rule. We also show how these rules give alternate characterizations for the eigenvalues of a sum of compact selfadjoint operators. Finally, applications to the invariant subspaces of nilpotent one parameter operator semigroups are given.

KEYWORDS: *Horn inequalities, Littlewood-Richardson rule, invariant subspaces, selfadjoint operator.*

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1. INTRODUCTION

Consider a nilpotent $N \times N$ matrix T, which we view as a linear operator on $\mathbb{C}^N$, and an invariant subspace M for T. Denote the sizes of the Jordan cells of T (respectively, $T|M$ and $T^*|M^\perp$) by $\nu_1 \geqslant \nu_2 \geqslant \cdots$ (respectively, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ and $\mu_1 \geqslant \mu_2 \geqslant \cdots$). The integers λ_i, μ_i, and ν_i which can arise this way were characterized in terms of the Littlewood-Richardson rule by Green and Klein [9], [12]. A similar question arises in the study of invariant subspaces for operators of class C_0. (These are operators modeled by direct sums of restrictions of the backward unilateral shift.) In this case, two substitutes for the Littlewood-Richardson rule have been proposed. The main result of [2] provides a necessary condition for the Jordan models of $T, T|M$, and $T^*|M^\perp$. A necessary and sufficient condition is given in [15]. We will show that these two substitutes of the Littlewood-Richardson rule are in fact equivalent. In particular, we deduce that the converse of the main result of [2] is true.

Returning to the case of nilpotent operators, recent work of Klyachko and others on the question of eigenvalue inequalities yields the following curious fact. The integers λ_i, μ_i, and ν_i arise as Jordan cell sizes of operators of the form $T, T|M$, and $T^*|M^\perp$ if and only if there exist selfadjoint matrices A, B, and C such that $A + B = C$, and the nonzero eigenvalues of A (respectively, B, C) are the numbers λ_i (respectively, μ_i, ν_i). This fact does extend to nonnegative real numbers λ_i, μ_i, ν_i, provided that T is replaced by an appropriate operator of class C_0. For the purposes of this paper, it will suffice to work with a subclass of C_0 operators, whose study is equivalent to the consideration of nilpotent one-parameter operator semigroups.

Klyachko's work gives a different solution to the eigenvalue problem, based on inequalities. These inequalities are easily extended to the nonnegative eigenvalues of compact operators A, B, and C satisfying $A + B = C$. This is Theorem 6 in [7], and one of our results here is the converse of that theorem. Along the way we present a somewhat different proof of a recent result (cf. [8]) on eigenvalues of matrices satisfying the inequality $A + B \geqslant C$. This proof makes use of the inductive structure of the Horn inequalities, with no additional reference to the intersections of Schubert varieties. We also point out a larger class of inequalities which the eigenvalues of A, B, and C must satisfy.

The paper is organized as follows. In Section 2 we describe briefly the classical Littlewood-Richardson rule which occurs in the solution of the nilpotent question (and in related questions about primary decompositions of finitely generated modules). Sections 3 and 4 describe the two extensions of the Littlewood-Richardson rule. In particular, it is shown that the two extensions are equivalent. In Section 5 we show that the new rule answers the eigenvalue problem for sums of nonnegative compact operators on a Hilbert space. Section 6 discusses the Horn inequalities and their consequences, including an alternate proof of Theorem 1 in [8], as well as the converse of Theorem 6 in [7]. We wrap up in Section 7 with a discussion of invariant subspaces for nilpotent one-parameter semigroups of operators on Hilbert space.

2. THE CLASSICAL LITTLEWOOD-RICHARDSON RULE

We recall here a few basic facts about Young tableaux which can be found either in [5] or [16]. An *integral partition* is simply a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ and $\lambda_i = 0$ for sufficiently large i. The last number n such that $\lambda_n \neq 0$ is called the *length* of λ, while $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ is called the *weight* of λ. Associated with a partition λ there is a *Young diagram*; this is a left-justified collection of n rows of square boxes with λ_j boxes in the jth row. Formally, the Young diagram of λ can be viewed as a set in the plane defined as

$$D_\lambda = \{(x, y) : 0 < x \leqslant \lambda_j \text{ and } j - 1 < y \leqslant j \text{ for some } j = 1, 2, \ldots, n\}.$$

Here we may think of the y-axis pointing down, so that the longest row of boxes is at the top of the diagram. The conjugate $\widetilde{\lambda}$ is another partition defined by the fact that its Young diagram is symmetrical with that of λ relative to the main diagonal $x = y$. If ν is another partition such that $\nu_j \geqslant \lambda_j$ for $j = 1, 2, \ldots, n$ then we have $D_\lambda \subset D_\nu$, and the difference $D_\nu \setminus D_\lambda$ is called a *skew diagram*. To label a skew diagram $D_\nu \setminus D_\lambda$ simply means to associate a positive integer with each box of the diagram. Thus a labeling consists of a family of integers $n = \{n_{i,j}\}$, where $i \geqslant 1$ and $\lambda_i < j \leqslant \nu_i$. The *content* $c(n)$ of such a labeling is the sequence $(c_1, c_2, \ldots)$, where c_k is the number of labels $n_{i,j}$ which are equal to k. The labeling n is said to be a *column-strict tableau* if $n_{i,j+1} > n_{i,j}$ and $n_{i+1,j} \geqslant n_{i,j}$ whenever the labels appearing in the inequalities are defined. A *row-strict tableau* is the symmetric image (relative to the main diagonal) of a column-strict tableau. Given a tableau n, we may also consider the restricted tableaux $n_k = \{n_{i,j} : i \geqslant k\}$ for $k = 1, 2, \ldots$. A column-strict tableau n will be called a *Littlewood-Richardson tableau* if the content $c(n_k)$ is a partition for all $k \geqslant 1$. A triple (λ, μ, ν) of partitions is said to satisfy the *Littlewood-Richardson rule* if there exists a Littlewood-Richardson tableau n on $D_\nu \setminus D_\lambda$ such that $c(n) = \mu$. In general, the number of such tableaux is denoted $c^{\nu}_{\lambda\mu}$; thus (λ, μ, ν) satisfies the Littlewood-Richardson rule if and only if $c^{\nu}_{\lambda\mu} \neq 0$. A remarkable property of these numbers is as follows:

THEOREM 2.1. *For all partitions λ, μ and ν we have*

$$c^{\nu}_{\lambda\mu} = c^{\nu}_{\mu\lambda} = c^{\widetilde{\nu}}_{\widetilde{\lambda}\widetilde{\mu}}.$$

See, for instance, Chapter 5 of [5] for a proof.

Another result we will use is the following *saturation theorem* which was conjectured by Klyachko [13] and proved by Knutson and Tao [14] (cf. also [18] for a discussion of this conjecture). We will use the notation $N\lambda$ in the obvious sense of componentwise multiplication.

THEOREM 2.2. *For all partitions λ, μ and ν, and all positive integers N, we have $c^{\nu}_{\lambda\mu} \neq 0$ if and only if $c^{N\nu}_{N\lambda\, N\mu} \neq 0$.*

3. REAL YOUNG TABLEAUX

In this paper we will need to work with a generalization of the integer partitions. We will call a *partition* any sequence $\lambda = (\lambda_j)_{j=1}^{\infty}$ of real numbers such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$ and $\lim\limits_{j\to\infty} \lambda_j = 0$. The (possibly infinite) sum $|\lambda| = \sum\limits_{j=1}^{\infty} \lambda_j$ is called the *weight* of λ. The (possibly infinite) number of nonzero numbers λ_j is called the *length* of λ, and is denoted $\ell(\lambda)$. As in the integral case, we define the Young diagram associated with a partition λ by

$$D_\lambda = \{(x, y) : 0 < x \leqslant \lambda_j \text{ and } j - 1 < y \leqslant j \text{ for some } j = 1, 2, \ldots\}.$$

Thus a Young diagram is no longer a collection of square boxes, but rather a stack of rectangular boxes with unit height, with the biggest box at the top (if we point the y-axis down). Given a second partition ν such that $\nu_j \geqslant \mu_j$ for all j one can again form the *skew diagram* $D_\nu \setminus D_\lambda$. (We will write $\nu \geqslant \mu$ to indicate componentwise inequality.)

DEFINITION 3.1. Given partitions $\nu \geqslant \lambda$, a (column-strict) *Young tableau* on $D_\nu \setminus D_\lambda$ is a function $\tau : D_\nu \setminus D_\lambda \to \mathbb{N}^* = \{1, 2, \ldots\}$ satisfying the following properties:

(i) $\tau(x, y) = \tau(x, \lceil y \rceil)$, where $\lceil y \rceil$ denotes the least integer greater than or equal to y;

(ii) $\tau(x, y) < \tau(x, y + 1)$ if $(x, y), (x, y + 1) \in D_\nu \setminus D_\lambda$;

(iii) $\tau(x', y) \geqslant \tau(x, y)$ if $(x, y), (x', y) \in D_\nu \setminus D_\lambda$ and $x' \geqslant x$.

Given a tableau τ, the *content* of τ is the sequence $c(\tau) = (c_1, c_2, \ldots)$, where c_n is the area of the set $\{(x, y) \in D_\nu \setminus D_\lambda : \tau(x, y) = n\}$.

Let us note that the numbers c_n in the preceding definition are always finite; in fact $c_n \leqslant \nu_1$ because τ is assumed to be column-strict. Observe also that, given a number $t \geqslant 0$, the restriction

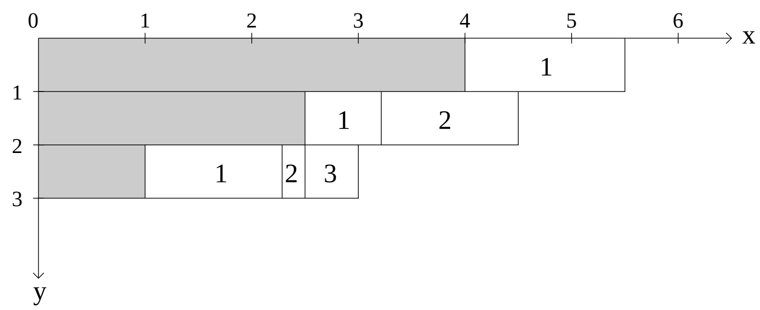
$$\tau_t = \tau|\{(x, y) \in D_\nu \setminus D_\lambda : x > t\}$$

is also a Young tableau, and hence has a content $c(\tau_t)$.

DEFINITION 3.2. A Young tableau τ will be called a *Littlewood-Richardson tableau* if $c(\tau_t)$ is a partition for every $t \geqslant 0$. Three partitions λ, μ, and ν will be said to satisfy the *Littlewood-Richardson rule* if $D_\lambda \subset D_\nu$, and there exists a Littlewood-Richardson tableau τ defined on $D_\nu \setminus D_\lambda$ such that $c(\tau) = \mu$.

It is obvious that a triple (λ, μ, ν) of integral partitions such that $c^\nu_{\lambda\mu} \neq 0$ satisfies the newly defined Littlewood-Richardson rule. (The converse statement is also true; cf. the remark following Lemma 3.6 below.)

For some intuitive support, we provide below a Littlewood-Richardson tableau confirming that $\lambda = (4, 2.5, 1)$, $\mu = (3.5, 1.5, .5))$, and $\nu = (5.5, 4.5, 3)$ satisfy the Littlewood-Richardson rule. The reader will easily verify that there is a one-parameter family of tableaux associated with this triple (λ, μ, ν).



The analogue of the saturation theorem is trivial for this Littlewood-Richardson rule.

LEMMA 3.3. *If λ, μ and ν are partitions, and s is a positive real number, then (λ, μ, ν) satisfies the Littlewood-Richardson rule if and only if $(s\lambda, s\mu, s\nu)$ does.*

Proof. Indeed, if τ is a Littlewood-Richardson tableau for (λ, μ, ν), then $\tau(x, y) = \tau(x/s, y)$ is a Littlewood-Richardson tableau for $(s\lambda, s\mu, s\nu)$. ∎

The reader will verify the following result with no difficulty.

LEMMA 3.4. *Assume that (λ, μ, ν) satisfies the Littlewood-Richardson rule.*

(i) *We have $\mu_j \leqslant \nu_j$ for all j.*

(ii) $|\lambda| = \sum_{n=1}^{\infty} (\nu_j - \mu_j)$*; in particular* $|\nu| = |\lambda| + |\mu|$.

Note that the two conditions in (ii) are not equivalent if $\sum_{j=1}^{\infty} \nu_j$ is infinite.

Consider a Littlewood-Richardson tableau τ on $D_\nu \setminus D_\lambda$. It is easy to see that $\tau(x, y) \leqslant j$ when $j - 1 < y \leqslant j$, and thus there exist points

$$\lambda_j = a_{j0} \leqslant a_{j1} \leqslant \cdots \leqslant a_{jj} = \nu_j$$

such that $\tau(x, y) = k$ for $j - 1 < y \leqslant j$ and $a_{j,k-1} < x \leqslant a_{jk}$, $k = 1, 2, \ldots, j$. We will call the numbers $\{a_{jk} : j \geqslant 1, 0 \leqslant k \leqslant j\}$ the *breakpoints* of τ. The Littlewood-Richardson property of τ can be reformulated in terms of these breakpoints.

LEMMA 3.5. *Let $\lambda \leqslant \nu$ be two partitions, and let $\{a_{jk} : j \geqslant 1, 0 \leqslant k \leqslant j\}$ be real numbers such that*

$$\lambda_j = a_{j0} \leqslant a_{j1} \leqslant \cdots \leqslant a_{jj} = \nu_j$$

for $j \geqslant 1$. Define $\tau : D_\nu \setminus D_\lambda \to \mathbb{N}^*$ by setting $\tau(x, y) = k$ for $j - 1 < y \leqslant j$ and $a_{j,k-1} < x \leqslant a_{jk}$, $k = 1, 2, \ldots, j$. Then τ is a Littlewood-Richardson tableau if and only if the following conditions are satisfied for all $j \geqslant 1$ and $1 \leqslant k \leqslant j$:

(i) $a_{j,k-1} \geqslant a_{j+1,k}$*;*

(ii) $\sum_{\ell=k+1}^{j+1} (a_{\ell,k+1} - a_{\ell k}) \leqslant \sum_{\ell=k}^{j} (a_{\ell k} - a_{\ell,k-1})$.

Proof. Condition (i) simply means that τ is column-strict (i.e., a Young tableau). The left (respectively, right) hand side of condition (ii) represents the area of $\{(x, y) : x \geqslant a_{j+1,k}, \tau(x, y) = k + 1\}$ (respectively, $\{(x, y) : x \geqslant a_{j+1,k}, \tau(x, y) = k\}$). To conclude we only need to observe that (if we let m_2 denote area)

$$m_2(\{(x, y) : x \geqslant t, \tau(x, y) = k\}) - m_2(\{(x, y) : x \geqslant t, \tau(x, y) = k + 1\})$$

is a continuous, piecewise linear function of t, equal to zero for $t = \nu_1$, and which attains each of its local minima at a point of the form $t = a_{j+1,k}$ for some $j \geqslant k$. ∎

Convergence of a sequence of partitions to another partition will be understood in the componentwise sense; this is equivalent to uniform convergence.

LEMMA 3.6. *Assume that for $n \geqslant 1$ we are given triples $(\lambda^n, \mu^n, \nu^n)$ of partitions satisfying the Littlewood-Richardson rule. If $\lambda^n \to \lambda$, $\mu^n \to \mu$, and $\nu^n \to \nu$ as $n \to \infty$, then (λ, μ, ν) also satisfies the Littlewood-Richardson rule.*

Proof. Consider Littlewood-Richardson tableaux τ^n for $(\lambda^n, \mu^n, \nu^n)$, and denote by a^n_{jk} the breakpoints of τ^n. Passing, if necessary, to a subsequence, we may assume that the limits $a_{jk} = \lim_{n\to\infty} a^n_{jk}$ exist for all $j \geqslant 1$ and $0 \leqslant k \leqslant j$. Clearly

$$\lambda_j = a_{j0} \leqslant a_{j1} \leqslant \cdots \leqslant a_{jj} = \nu_j$$

for $j \geqslant 1$, and therefore we can define a function τ as in Lemma 3.5. The fact that each τ^n is a Littlewood-Richardson tableau implies that conditions (i) and (ii) of Lemma 3.5 are satisfied by the numbers a^n_{jk}. We conclude that these conditions are satisfied by a_{jk} as well, and hence τ is a Littlewood-Richardson tableau. To conclude the proof we need to show that the content of τ is μ. This content is given by $(c(\tau))_k = \sum_{j=k}^{\infty}(a_{jk} - a_{j,k-1})$. Since $a^n_{jk} - a^n_{j,k-1} \leqslant \nu^n_j$, it follows from uniform convergence that

$$(c(\tau))_k = \lim_{n\to\infty} \sum_{j=k}^{\infty}(a^n_{jk} - a^n_{j,k-1}) = \lim_{n\to\infty} \mu^n_k = \mu_k,$$

as desired. ∎

We show next that the Littlewood-Richardson rule is determined in some sense by finite rational partitions.

LEMMA 3.7. *Assume that the triple (λ, μ, ν) satisfies the Littlewood-Richardson rule. There exist then triples $(\lambda^n, \mu^n, \nu^n)$, $n = 1, 2, \ldots$, such that:*

(i) *each $(\lambda^n, \mu^n, \nu^n)$ satisfies the Littlewood-Richardson rule;*

(ii) *there is a Littlewood-Richardson tableau for $(\lambda^n, \mu^n, \nu^n)$ with only rational breakpoints;*

(iii) *$\lambda^n \to \lambda$, $\mu^n \to \mu$, and $\nu^n \to \nu$ as $n \to \infty$;*

(iv) *for each n, λ^n, μ^n, and ν^n have finite length.*

Proof. First we observe that (λ, μ, ν) can be approximated arbitrarily well by finite partitions. To do this, let ε be a positive number, and choose N so that $\nu_N < \varepsilon$. We can then consider the partitions λ' and ν' given by $\lambda'_j = \max\{\lambda_j - \varepsilon, 0\}$ and $\nu'_j = \max\{\nu_j - \varepsilon, 0\}$. The function $\tau'(x, y) = \tau(x + \varepsilon, y)$ is then a Littlewood-Richardson tableau with content μ' equal to the content of τ'. It is clear that λ'_j, μ'_j and ν'_j are within ε from λ_j, μ_j and ν_j, respectively, while $\nu'_N = 0$. We can therefore restrict ourselves to the case of finite partitions. Assume therefore that $\nu_N = 0$, and τ is a Littlewood-Richardson tableau for (λ, μ, ν) with breakpoints a_{jk}. Fix also a positive number $\varepsilon < 1/4N$. Since only finitely many of the numbers a_{jk} are different from zero, it follows from a theorem of Dirichlet (see Theorem 201 in [10]) that there exist integers M and A_{jk} such that $|A_{jk} - Ma_{jk}| < \varepsilon$. Define now partitions λ' and ν' by setting $\lambda'_j = A_{0j}/M$ and $\nu'_j = A_{jj}/M$, and define a function $\tau' : D_{\nu'} \setminus D_{\lambda'}$ by setting $\tau'(x,y) = k$ if $j-1 < y \leqslant j$ and $A_{j,k-1}/M < x \leqslant A_{jk}/M$. First observe that τ' is a Young tableau. Indeed, we have $a_{j+1,k} \leqslant a_{j,k-1}$ which implies $A_{j+1,k} \leqslant A_{j,k-1}$ because $\varepsilon < 1/2$. We next verify that τ' is in fact a Littlewood-Richardson tableau. To do this we must consider the content of τ_t, and it will suffice to do this for $t = T/M$ with T an integer (other values can be done simply by linear interpolation). In this case we have

$$M(c(\tau'_t))_k = \sum_{j=k}^{N}[\max\{A_{jk}, T\} - \max\{A_{j,k-1}, T\}],$$

and this sum differs from $M(c(\tau_t))_k$ by less than $N\varepsilon < 1/2$. It follows that $M(c(\tau'_t))_k$ is the closest integer to $M(c(\tau_t))_k$, and thus $c(\tau')$ is indeed a partition. Finally, if we denote by μ' the content of τ', we have $|\mu'_j - \mu_j| < 2N\varepsilon$. The lemma follows. ∎

The preceding proof incidentally shows that the Littlewood-Richardson rule for integer partitions is the same as the rule introduced in this section. Indeed, let (λ, μ, ν) be finite integer partitions satisfying the rule of Definition 3.2. The procedure of the preceding proof in this case does not change the partitions (since they are already integer partitions), but rather it modifies the Littlewood-Richardson tableau so that its break points are rational. Thus, there exists a Littlewood-Richardson tableau τ for (λ, μ, ν) with rational break points. For a certain natural number M, the tableau $\tau(x/M, y)$ will only have integral break points, and this amounts to saying that $(M\lambda, M\mu, M\nu)$ satisfies the classical Littlewood-Richardson rule. Therefore $c^{\nu}_{\lambda\mu} \neq 0$ by the saturation Theorem 2.2. The symmetry of the classical rule yields now the following result.

PROPOSITION 3.8. *A triple (λ, μ, ν) satisfies the Littlewood-Richardson rule if and only if (μ, λ, ν) does.*

Proof. By Lemmas 3.6 and 3.7 we can restrict ourselves to the case of finite rational partitions. If λ, μ and ν are such partitions then $M\lambda, M\mu$ and $M\nu$ are integer partitions for some natural number M. Since (λ, μ, ν) satisfies the Littlewood-Richardson rule if and only if $(M\lambda, M\mu, M\nu)$ does, we can restrict ourselves to the finite integral case. The proposition follows from Theorem 2.1 and the remark preceding the statement. ∎

4. LITTLEWOOD-RICHARDSON FUNCTIONS

The reflection of a Young diagram D_λ in the main diagonal is not generally a Young diagram, and this precludes the formation of a dual partition $\widetilde{\lambda}$. However there is a form of the Littlewood-Richardson rule which extends the classical one for dual partitions.

DEFINITION 4.1. Assume we are given for each $t \geqslant 0$ a partition $\sigma(t)$. We will say that σ is a *Littlewood-Richardson function* if the following conditions are satisfied:

(i) $0 \leqslant \sigma_j(t+h) - \sigma_j(t) \leqslant h$ for all $t, h \geqslant 0$ and $j = 1, 2, \ldots$; in particular the functions σ_j are absolutely continuous;

(ii) $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_j(t) \in \{0, 1\}$ for all j and almost every $t \geqslant 0$;

(iii) $\sum\limits_{j=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_j(t)$ is finite for almost all $t \geqslant 0$, and equal to zero for large values of t;

(iv) $\sum\limits_{j=p}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_j(t)$ is nonincreasing for $p = 1, 2, \ldots$.

The above conditions imply the existence of a partition μ such that

$$\sum_{j=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_j(t) = \sum_{j=1}^{\infty} \chi_{(0,\mu_j)}(t)$$

for almost all $t \geqslant 0$. The partition μ will be called the *content* of σ and will be denoted $c(\sigma)$. We will say that a triple (λ, μ, ν) satisfies the Littlewood-Richardson Rule if there exists a Littlewood-Richardson function σ such that $\sigma(0) = \lambda$, $\sigma(t) = \nu$ for large t, and $c(\sigma) = \mu$.

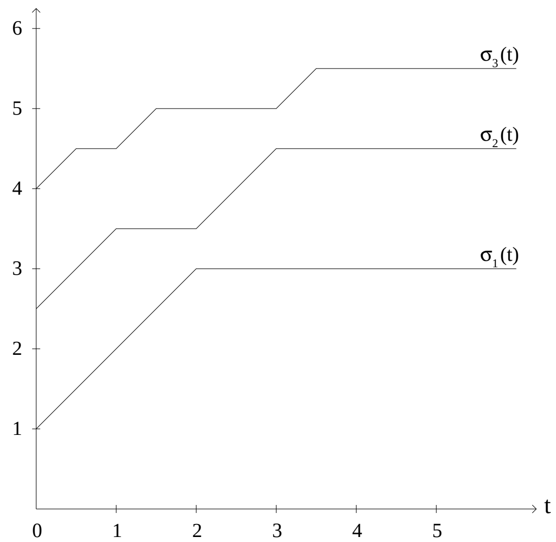
The Littlewood-Richardson Rule will eventually be shown to be equivalent to the Littlewood-Richardson rule, which is why we did not distinguish it beyond the capital R. Littlewood-Richardson functions appeared, in a somewhat more restrictive form, in the study of invariant subspaces for operators on Hilbert space [2].

The figure from the next page illustrates the graphs of a Littlewood-Richardson function confirming that $\lambda = (4, 2.5, 1)$, $\mu = (3.5, 1.5, .5)$, and $\nu = (5.5, 4.5, 3)$ satisfy the Littlewood-Richardson Rule. Again, one can verify the existence of a one-parameter family of such functions associated with this triple.

LEMMA 4.2. *If λ, μ, and ν are finite integer partitions, and $c^{\nu}_{\lambda\mu} \neq 0$, then (λ, μ, ν) satisfies the Littlewood-Richardson Rule.*

Proof. The reflection in the main diagonal of a Littlewood-Richardson tableau for $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu})$ is a row-strict Young tableau n on $D_\nu \setminus D_\lambda$ with content $c(n) = \widetilde{\mu}$. We define a Littlewood-Richardson function σ by specifying its values at integer points, and extending it by linear interpolation. We set $\sigma_j(0) = \lambda_j$, $\sigma_j(k) = \sigma_j(k-1) + 1$ if $n(j, \sigma_j(k-1) + 1) = k$, and $\sigma_j(k) = \sigma_j(k-1)$ otherwise. The reader should have no difficulty to show that σ is a Littlewood-Richardson function with content μ. ∎

The saturation property for the Littlewood-Richardson Rule is easily verified.



LEMMA 4.3. *If λ, μ, and ν are partitions, and s is a positive real number, then (λ, μ, ν) satisfies the Littlewood-Richardson Rule if and only if $(s\lambda, s\mu, s\nu)$ does.*

Proof. It suffices to observe that $t \mapsto s\sigma(t/s)$is a Littlewood-Richardson function whenever σ is one. ∎

The Littlewood-Richardson function constructed in the proof of Lemma 4.2 is piecewise linear. We will see that this is true for all such functions, except possibly in an arbitrarily small neighborhood of $t = 0$. It will be convenient to write $\sigma(\infty)$ for $\sigma(t)$ if t is so large that σ is constant on (t, ∞).

PROPOSITION 4.4. *Assume that (λ, μ, ν) satisfies the Littlewood-Richardson Rule, and let σ be a Littlewood-Richardson function with $\sigma(0) = \lambda$, $\sigma(\infty) = \nu$, and $c(\sigma) = \mu$.*

(i) *We have $\sigma(\nu_1) = \sigma(\infty)$.*

(ii) *For all $j \geqslant 1$, $\mu_j \leqslant \nu_j$.*

(iii) *If ε is a positive number, then the function σ fails to be differentiable only at a finite number of points in the interval $[\varepsilon, \infty)$.*

Proof. For each j the function σ_j is eventually constant, so there exists a lowest point θ_j with the property that $\sigma_j(\theta_j) = \sigma_j(\infty)$. All the points θ_j are $\leqslant \mu_1$, but they generally do not form a partition. Fix a point $\varepsilon > 0$ of differentiability for σ, and chose N minimal so that $\sum\limits_{j=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_j(\varepsilon) = \sum\limits_{j=1}^{N} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_j(\varepsilon)$. Since $\sum\limits_{j=N+1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_j(t)$ is decreasing, this sum must be zero for $t \geqslant \varepsilon$. In particular we have $\theta_j \leqslant \varepsilon$ for

$j > N$, but $\theta_N > \varepsilon$. Now, for $t > \varepsilon$, the function

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_N(t) = \sum_{j=N}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_j(t)$$

must be concave, hence $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_N(t) = 1$ for $t \in (\varepsilon, \theta_N)$. We deduce that $\sigma_N(\theta_N) = \sigma_N(\varepsilon) + \theta_N - \varepsilon \geqslant \theta_N - \varepsilon$. In an analogous fashion, if N_1 is the greatest integer such that $\theta_{N_1} > \theta_N$, the function σ_{N_1} must have derivative one on (θ_N, θ_{N_1}), whence

$$\sigma_{N_1}(\theta_{N_1}) = \sigma_{N_1}(\theta_N) + \theta_{N_1} - \theta_N \geqslant \sigma_N(\theta_N) + \theta_{N_1} - \theta_N \geqslant \theta_{N_1} - \varepsilon.$$

Continuing in this fashion we find $N_1 > N_2 > \cdots > N_k \geqslant 0$ such that $\theta_{N_1} < \theta_{N_2} < \cdots < \theta_{N_k}$ but $\theta_n \leqslant \theta_{N_j}$ for $N_{j+1} < n < N_j$ if $j < k$, and $\theta_n \leqslant \theta_{N_k}$ for $n < N_k$. The argument above will show inductively that $\sigma_{N_j}(\theta_{N_j}) \geqslant \theta_{N_j} - \varepsilon$. In particular,

$$\theta_1 \leqslant \theta_{N_k} \leqslant \sigma_{N_k}(\theta_{N_k}) + \varepsilon \leqslant \sigma_1(\infty) + \varepsilon.$$

Letting ε tend to zero we obtain the inequality $\theta_1 \leqslant \sigma_1(\infty) = \nu_1$. This proves assertion (i) of the proposition. The same argument applies to the Littlewood-Richardson function $(\sigma_p, \sigma_{p+1}, \ldots)$ and it yields $\theta_p \leqslant \sigma_p(\infty) \leqslant \sigma_1(\infty)$. Since μ_1 is the supremum of the θ_p, we deduce that $\mu_1 \leqslant \nu_1$. To prove assertion (ii) for all values of j observe again that $\theta_j \leqslant \nu_j$ for all j, and therefore

$$\sum_{j=1}^{\infty} \chi_{(0,\mu_j)} = \sum_{j=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_j(t) \leqslant \sum_{j=1}^{\infty} \chi_{(0,\theta_j)} \leqslant \sum_{j=1}^{\infty} \chi_{(0,\nu_j)}.$$

Finally, to prove assertion (iii), let ε and N be chosen as above. For $p \leqslant N$ and $t > \varepsilon$, the function $\sum\limits_{j=p}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_j(t) = \sum\limits_{j=p}^{N} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_j(t)$ is decreasing and takes at most $N - p + 2$ values; it must therefore have at most $N - p + 1$ discontinuities. It follows that $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_p(t) = \sum\limits_{j=p}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_j(t) - \sum\limits_{j=p-1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_j(t)$ also has only a finite number of discontinuities for $t > \varepsilon$. The proposition follows. ∎

The preceding proposition shows that, given a Littlewood-Richardson function σ, there exist sequences $\theta_{j1} > \theta_{j2} > \cdots \geqslant 0$ with the property that $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_j$ is constant on the intervals $(\theta_{j,k+1}, \theta_{jk})$ and (θ_{j1}, ∞). In case σ_j only has finitely many points of nondifferentiability we will set $\theta_{jk} = 0$ for large k. In any case, we have $\theta_{jk} \to 0$ as $k \to \infty$. The points θ_{jk} will be called the *breakpoints* of the function σ. Let us also remark that there are no more than N^2 break points greater than ν_N; this can easily be seen from the argument concluding the proof of Proposition 4.4.

LEMMA 4.5. *Assume that for $n \geqslant 1$ we are given triples $(\lambda^n, \mu^n, \nu^n)$ of partitions satisfying the Littlewood-Richardson Rule. If $\lambda^n \to \lambda$, $\mu^n \to \mu$, and $\nu^n \to \nu$ as $n \to \infty$, then (λ, μ, ν) also satisfies the Littlewood-Richardson Rule.*

Proof. Let σ^n be Littlewood-Richardson functions for $(\lambda^n, \mu^n, \nu^n)$, and denote by θ^n_{jk} the breakpoints of σ^n. Passing to a subsequence, we may assume that the limits $\theta_{jk} = \lim_{n\to\infty} \theta^n_{jk}$ exists. Clearly then $\nu_j \geqslant \theta_{jk} \geqslant \theta_{j,k+1} \geqslant 0$, and at most N^2 of these points are above ν_N. It follows that $\theta_{jk} \to 0$ as $k \to \infty$. Passing possibly to a smaller subsequence, we may also assume that for each j the sequence σ^n_j has a uniform limit σ_j; this follows from equicontinuity. It is now easily seen that the function σ_j has constant derivative equal to 0 or 1 on intervals of the form $(\theta_{j,k+1}, \theta_{jk})$ and (θ_{j1}, ∞). In fact, $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_j$ is the limit of $\frac{\mathrm{d}}{\mathrm{d}t}\sigma^n_j$ at all points other than the θ_{jk}. From these observations, and from the fact that each σ^n is a Littlewood-Richardson function we see that σ is a Littlewood-Richardson function as well. We leave to the reader to verify that $\sigma(0) = \lambda, \sigma(\infty) = \nu$, and $c(\sigma) = \mu$. ∎

LEMMA 4.6. *Assume that the triple (λ, μ, ν) satisfies the Littlewood-Richardson Rule. There exist then triples $(\lambda^n, \mu^n, \nu^n)$, $n = 1, 2, \ldots$, such that:*

(i) *each $(\lambda^n, \mu^n, \nu^n)$ satisfies the Littlewood-Richardson Rule;*

(ii) *there is a Littlewood-Richardson function for $(\lambda^n, \mu^n, \nu^n)$ with only rational breakpoints;*

(iii) *$\lambda^n \to \lambda$, $\mu^n \to \mu$, and $\nu^n \to \nu$ as $n \to \infty$;*

(iv) *for each n, λ^n, μ^n, and ν^n have finite length.*

Proof. Let σ be a Littlewood-Richardson function for (λ, μ, ν). If $\varepsilon > 0$ then the function $\sigma'(t) = \sigma(t + \varepsilon)$ is also a Littlewood-Richardson function such that $\sigma'(0), \sigma'(\infty)$, and $c(\sigma')$ are finite partitions within ε of λ, μ, and ν, respectively. Thus we may restrict ourselves to the case of finite partitions. Assume, for instance, that $\sigma_N = 0$ so that σ has at most N^2 nonzero breakpoints θ_{jk}. Fix now a positive number $\varepsilon < 1/2$ and apply Dirichlet's theorem to find integers M and Θ_{jk} such that $|\Theta_{jk} - M\theta_{jk}| < \varepsilon$ for all j, k. Define a new function σ' with breakpoints Θ_{jk}/M, such that $\frac{\mathrm{d}}{\mathrm{d}t}\sigma'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(t)$ whenever t is farther away than ε/M from one of these breakpoints. The function $\sum_{j=p}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\sigma'_j(t)$ coincides with $\sum_{j=p}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_j(t)$ if Mt differs by more than ε from an integer and hence it is decreasing for such t. It is immediate then that this property is preserved for all t with Mt not an integer. Similar considerations show that σ' is a Littlewood-Richardson function. To conclude the proof it suffice to observe that $\sigma'(0)$, $\sigma'(\infty)$ and $c(\sigma')$ are within $N^2\varepsilon$ from λ, μ, and ν, respectively. ∎

As in Section 3, the preceding proof provides the converse of Lemma 4.2. Indeed, it suffices to remark that a Littlewood-Richardson function with only integral breakpoints is obtained from the transpose of a Littlewood-Richardson tableau via the construction of Lemma 4.2.

PROPOSITION 4.7. *A triple (λ, μ, ν) satisfies the Littlewood-Richardson Rule if and only if it satisfies the Littlewood-Richardson rule.*

Proof. Lemmas 3.3, 3.6, 3.7, 4.3, 4.5, and 4.6 allow us to restrict ourselves to finite integer partitions. For these partitions the result is a consequence of the equality $c^{\nu}_{\lambda\mu} = c^{\widetilde{\nu}}_{\widetilde{\lambda}\widetilde{\mu}}$. ∎

5. EIGENVALUES FOR SUMS OF NONNEGATIVE MATRICES

In this section we will consider compact selfadjoint operators acting on a complex, separable Hilbert space $\mathfrak{H}$. If A is such an operator, we will denote by $\Lambda_+(A)$ the sequence of its nonnegative eigenvalues, arranged in decreasing order and repeated according to their multiplicities. If A only has finitely many nonnegative eigenvalues, we will append a string of zeroes so as to make $\Lambda_+(A)$ an infinite sequence. One of the questions addressed in [7] concerns the relations among the sets $\Lambda_+(A), \Lambda_+(B)$, and $\Lambda_+(C)$ if $C = A + B$ (cf. Theorem 6 in [7]). Observe that $\Lambda_+(A)$ is always a partition in our terminology, and it is a finite partition if A has finite rank. We will have more to say about the results described in [7]. For the moment we concentrate on a result which we reformulate below.

PROPOSITION 5.1. *Let λ, μ and ν be finite integer partitions. Then we have $c^{\nu}_{\lambda\mu} \neq 0$ if and only if there exist finite rank nonnegative operators A, B, and C such that $\Lambda_+(A) = \lambda, \Lambda_+(B) = \mu, \Lambda_+(C) = \nu$, and $A + B = C$. Moreover, A, B, and C can be chosen to be zero on the orthogonal complement of a given space of dimension $\ell(\nu)$.*

The purpose of this section is to extend this result to arbitrary partitions. Note that the next result can be extended to more than three sequences of partitions. We will denote by $\|A\|$ the usual operator norm of A.

LEMMA 5.2. *Consider partitions λ^n, μ^n, ν^n, $n = 1, 2, \ldots$, and λ, μ, ν such that $\lambda^n \to \lambda$, $\mu^n \to \mu$, and $\nu^n \to \nu$ as $n \to \infty$. Assume that nonnegative compact operators A_n, B_n, C_n are given such that $\Lambda_+(A_n) = \lambda^n$, $\Lambda_+(B_n) = \mu^n$, and $\Lambda_+(C_n) = \nu^n$. Then there exist a subsequence $n_1 < n_2 < \cdots$, unitary operators $U_1, U_2, \ldots$, and nonnegative compact operators A, B, and C, such that $\|A - U_k A_{n_k} U_k^*\| \to 0$, $\|B - U_k B_{n_k} U_k^*\| \to 0$, and $\|C - U_k C_{n_k} U_k^*\| \to 0$ as $k \to \infty$. (The continuous dependence of the eigenvalues yields $\Lambda_+(A) = \lambda$, $\Lambda_+(B) = \mu$, and $\Lambda_+(C) = \nu$.)*

Proof. Fix an orthonormal basis $e_1, e_2, \ldots$, and consider orthonormal sequences e^n_j, f^n_j, g^n_j such that $A_n e^n_j = \lambda^n_j e^n_j$, $B_n f^n_j = \mu^n_j f^n_j$, $C_n g^n_j = \nu^n_j g^n_j$ for all n and j. We can then choose unitary operators $U_1, U_2, \ldots$ such that the vectors $U_n e^n_j$, $U_n f^n_j$ and $U_n g^n_j$ belong to the linear space $\mathfrak{H}_n$ generated by $e_1, e_2, \ldots, e_{3n}$ for $j \leqslant n$. Replacing the operators A_n, B_n, C_n by $U_n A_n U_n^*$, $U_n B_n U_n^*$, $U_n C_n U_n^*$, we may assume from the beginning that the vectors e^n_j, f^n_j and g^n_j belong to $\mathfrak{H}_n$ for $j \leqslant n$. Denote by P_n the orthogonal projection onto $\mathfrak{H}_n$. The condition on the eigenvalues of A_n implies that $\|A_n - P_n A_n P_n\| \leqslant \mu^n_{n+1}$, and this implies that the set $\{A_n\}_{n=1}^{\infty}$ is relatively compact. The same argument applies to B_n and C_n, and therefore we can find integers $n_1 < n_2 < \cdots$ such that the sequences $(A_{n_k})_{k=1}^{\infty}$, $(B_{n_k})_{k=1}^{\infty}$, and $(C_{n_k})_{k=1}^{\infty}$ converge in norm. The limits A, B, C of these sequences will then satisfy the conclusion of the lemma. ∎

THEOREM 5.3. *Given partitions λ, μ, and ν, the following conditions are equivalent:*

(i) *the triple (λ, μ, ν) satisfies the Littlewood-Richardson rule;*

(ii) *there exist compact nonnegative operators A, B, and C such that $\Lambda_+(A) = \lambda, \Lambda_+(B) = \mu, \Lambda_+(C) = \nu$, and $A + B = C$.*

Proof. The theorem is true for finite integer partitions. Lemma 3.3, along with the fact that $\Lambda_+(sA) = s\Lambda_+(A)$ for $s > 0$, implies that the theorem is true for finite rational partitions as well. The implication (i)⇒(ii) follows then by approximation from Lemmas 3.7 and 5.2. Conversely, assume that (ii) holds. Let $e_1, e_2, \ldots$ be an orthonormal basis for $\mathfrak{H}$, and denote by P_n the orthogonal projection onto the linear span of $e_1, e_2, \ldots, e_n$. Then clearly $\Lambda_+(P_nAP_n) \to \lambda$ as $n \to \infty$, and corresponding assertions hold for B and C. Lemma 3.6 shows now that it suffices to prove (i) under the additional condition that A, B, and C have finite rank. The argument of [6] shows then that we may also assume that the eigenvalues of A, B, and C are rational, and (i) follows from the remark at the beginning of the proof. ∎

6. THE HORN INEQUALITIES

The solution of the general eigenvalue problem for sums of selfadjoint matrices is expressed in [7] in terms of eigenvalue inequalities rather than the Littlewood-Richardson rule (except, in the case of integer eigenvalues). These inequalities are expressed in terms of subsets of $\{1, 2, \ldots, n\}$. Such a subset I of cardinality r can also be viewed as an increasing function $\{1, 2, \ldots, r\} \to \{1, 2, \ldots, n\}$, and as such it has a corresponding partition $\Lambda(I) = (\lambda_1, \lambda_2, \ldots)$ given by $\lambda_1 = I(r) - r$, $\lambda_2 = I(r-1) - (r-1), \ldots, \lambda_r = I(1) - 1$, and $\lambda_k = 0$ for $k > r$. It will be convenient to use the notation Σ_I for the sum of the elements of I:

$$\Sigma_I = \sum_{k=1}^{r} I(k).$$

Following the notation of [7], we consider the following collection of triples of subsets of $\{1, 2, \ldots, n\}$ of cardinality $r \leqslant n$:

$$U_r^n = \left\{(I, J, K) : \Sigma_I + \Sigma_J = \Sigma_K + \frac{r(r+1)}{2}\right\}.$$

Next define inductively on r a subset T_r^n of U_r^n as follows. Set $T_1^n = U_1^n$ and, for $1 < r \leqslant n$,

$$T_r^n = \Big\{(I,J,K) \in U_r^n : \Sigma_{I\circ I'} + \Sigma_{J\circ J'} \leqslant \Sigma_{K\circ K'} + \frac{p(p+1)}{2} \quad \text{for all } p < r \text{ and } (I',J',K') \in T_p^r\Big\}.$$

Observe that the set T_n^n consists only of the triplet (I,I,I), where $I = \{1,2,\ldots,n\}$. We will also need a larger set of triples defined as

$$\overline{T}_r^n = \Big\{(I,J,K) : \Sigma_{I\circ I'} + \Sigma_{J\circ J'} \leqslant \Sigma_{K\circ K'} + \frac{p(p+1)}{2} \quad \text{for all } p \leqslant r \text{ and } (I',J',K') \in T_p^r\Big\}.$$

Note that the requirement that (I,J,K) belong to U_r^n has been replaced by the weaker inequality $\Sigma_I + \Sigma_J \leqslant \Sigma_K + r(r+1)/2$ in the definition of $\overline{T}_r^n$.

The set T_r^n is characterized as follows (cf. Theorem 12 in [7]).

THEOREM 6.1. *Let I, J, and K be subsets of cardinality r of $\{1,2,\ldots,n\}$. Then the triple (I,J,K) belongs to T_r^n if and only if the triple $(\Lambda(I),\Lambda(J),\Lambda(K))$ satisfies the Littlewood-Richardson rule.*

One of the main results of [7] can be formulated as follows (cf. Theorem 1.) (Of course, the results in [7] are formulated for arbitrary eigenvalues, not just positive ones; the general case can be reduced to positive eigenvalues by adding constant multiples of the identity matrix to A, B, and C.)

THEOREM 6.2. *Let λ, μ, and ν be partitions of length at most n. The following are equivalent:*

(i) *there exist nonnegative selfadjoint $n \times n$ matrices A, B, and C such that $\Lambda_+(A) = \lambda, \Lambda_+(B) = \mu, \Lambda_+(C) = \nu$, and $A + B = C$;*

(ii) *we have:*

(a) *trace identity: $|\lambda| + |\mu| = |\nu|$;*

(b) *for all $r < n$, and all $(I,J,K) \in T_r^n$, the Horn inequality*

$$(*_{IJK}) \qquad \sum_{k\in K} \nu_k \leqslant \sum_{i\in I} \lambda_i + \sum_{j\in J} \mu_j$$

is satisfied.

The following result was, as mentioned in the introduction, also proved by Fulton [8] (cf. also Friedland [4] for a closely related result). Our derivation uses a somewhat different path, and it includes some amusing properties of the sets T_r^n.

THEOREM 6.3. *For arbitrary partitions λ, μ, and ν, the following conditions are equivalent:*

(i) *there exist nonnegative compact operators A, B, and C such that $\Lambda_+(A) = \lambda, \Lambda_+(B) = \mu, \Lambda_+(C) = \nu$, and $A + B \geqslant C$;*

(ii) *for all natural numbers* $r \leqslant n$ *and all* $(I, J, K) \in T_r^n$*, the Horn inequality* $(*_{IJK})$ *holds.*

If (ii) *holds and* $\ell(\nu) \leqslant n$*, then* A, B, C *can be chosen to be* $n \times n$ *matrices.*

The proof is done first in the finite length case, and it involves an inductive procedure. This makes it more convenient to state that case separately.

THEOREM 6.4.n. *Let* λ, μ*, and* ν *be partitions of length at most* n*. The following are equivalent:*

(i) *there exist selfadjoint* $n \times n$ *matrices* A, B*, and* C *such that* $\Lambda_+(A) = \lambda, \Lambda_+(B) = \mu, \Lambda_+(C) = \nu$*, and* $A + B \geqslant C$*;*

(ii) *for all* $r \leqslant n$*, and all* $(I, J, K) \in T_r^n$*, the Horn inequality* $(*_{IJK})$ *is satisfied.*

Note that condition (2)(a) in Theorem 6.2 has been replaced by the inequality $|\lambda| + |\mu| \geqslant |\nu|$ in Theorem 6.4.

Let us note some equivalent forms of Theorem 6.4. First, given $(I, J, K) \in T_r^n$ let us consider the inequaltity

$$(\maltese_{nIJK}) \qquad \sum_{i \in I} \lambda_{n+1-i} + \sum_{j \in J} \mu_{n+1-j} \leqslant \sum_{k \in K} \nu_{n+1-k}.$$

THEOREM 6.5.n. *Let* λ, μ*, and* ν *be partitions of length at most* n*. The following are equivalent:*

(i) *there exist selfadjoint* $n \times n$ *matrices* A, B*, and* C *such that* $\Lambda_+(A) = \lambda, \Lambda_+(B) = \mu, \Lambda_+(C) = \nu$*, and* $A + B \leqslant C$*;*

(ii) *for all* $r \leqslant n$*, and all* $(I, J, K) \in T_r^n$*, the inequality* $(\maltese_{nIJK})$ *is satisfied.*

Proof. (that Theorems 6.5 and 6.4 are equivalent) Assume that (λ, μ, ν) is a triple of partitions of length at most n. Fix numbers L, M, and N such that $N = L + M$ and $L \geqslant \lambda_n$, $M \geqslant \mu_n$, and $N \geqslant \nu_n$. We can then consider the partitions λ', μ', and ν' defined by $\lambda' = (L - \lambda_n, L - \lambda_{n-1}, \ldots, L - \lambda_1, 0, 0, \ldots)$, with analogous definitions for μ' and ν'. Clearly, (λ, μ, ν) satisfies condition (ii) of Theorem 6.5 if and only if (λ', μ', ν') satisfies condition (ii) of Theorem 6.4. The desired equivalence follows from the fact that the inequality $A + B \leqslant C$ is also equivalent to $(L1_n - A) + (M1_n - B) \geqslant (N1_n - C)$ (where 1_n denotes the identity matrix of order n). ∎

The following result is also contained in Theorem 5 of [8].

THEOREM 6.6.n. *Let* λ, μ*, and* ν *be partitions of length at most* n*. The following are equivalent:*

(i) *there exists a partition* $\nu' \geqslant \nu$ *such that* (λ, μ, ν') *satisfies the Littlewood-Richardson rule;*

(ii) *for all* $r \leqslant n$*, and all* $(I, J, K) \in T_r^n$*, the Horn inequality* $(*_{IJK})$ *is satisfied.*

Proof. (that Theorem 6.6 is equivalent to Theorem 6.4) Assume first that ν' is chosen as in (i). Then Theorem 6.2 implies that (λ, μ, ν') satisfies all the Horn inequalities $(*_{IJK})$ for $(I, J, K) \in T_r^n$ and $r \leqslant n$ Since $\nu \leqslant \nu'$, it follows that (ii) is true as well. Conversely, assume that (ii) is verified, and choose (by virtue of Theorem 6.4) matrices A, B, C satisfying the conditions of that theorem. Then the partition $\nu' = \Lambda(A+B)$ will satisfy condition (i). ∎

An analogous argument proves that the following result is equivalent to Theorem 6.4.n.

THEOREM 6.7.n. *Let λ, μ, and ν be partitions of length at most n. The following are equivalent:*

(i) *there exists a partition $\nu' \leqslant \nu$ such that (λ, μ, ν') satisfies the Littlewood-Richardson rule;*

(ii) *for all $r \leqslant n$, and all $(I, J, K) \in T_r^n$, the inequality $(\maltese_{nIJK})$ is satisfied.*

It will be of interest to us that the partitions ν' in the preceding two results can be chosen to be integral provided that the original partitions λ, μ, and ν are integral. We will only provide the proof for the result corresponding with Theorem 6.6 since the other result can be obtained using the reflection in the argument of Theorem 6.5 above. Note that the following proof does not depend on Theorem 6.4.

If λ is a partition, and I is a set of integers, we will use $\lambda \circ I$ to denote the partition $(\lambda_{i_1}, \lambda_{i_2}, \ldots)$, where $i_1, i_2, \ldots$ is the increasing enumeration of I; this partition is continued with zeros if I is a finite set. We also denote by I^c the complement of I in the set of natural numbers.

PROPOSITION 6.8.n. *Let λ, μ, and ν be integral partitions of length at most n. If there exists a partition $\nu' \geqslant \nu$, of length at most n, such that (λ, μ, ν') satisfies the Littlewood-Richardson rule, then there also exists an integral partition $\nu'' \geqslant \nu$ such that (λ, μ, ν'') satisfies the Littlewood-Richardson rule.*

Proof. Assume that Proposition 6.8.k is true for all $k < n$, and (λ, μ, ν') satisfies the Littlewood-Richardson rule for some $\nu' \geqslant \nu$. As seen in the proof of Lemma 3.6, there exists a natural number N such that, upon replacing the elements of $N\nu'$ with their closest integers one obtains a partition $\widetilde{\nu}$ such that $(\lambda, \mu, \widetilde{\nu}/N)$ satisfies the Littlewood-Richardson rule; clearly we still have $\widetilde{\nu}/N \geqslant \nu$. In other words, we can restrict ourselves from the beginning to the case in which ν' only has rational entries. Assume therefore that N is a natural number such that $N\nu'$ is an integral partition. If the entries of ν' are not all integers, then there must be at least two entries which are not integers. Indeed, the weight $|\nu'| = |\lambda| + |\mu|$ is an integer. Denote by p (respectively, q) the first (respectively, last) index for which ν'_p (respectively, ν'_q) is not an integer. We may assume that ν' has been chosen so that q is as small as possible. Now, all the Horn inequalities associated with T_r^n, $r < n$, are satisfied by (λ, μ, ν') We consider first the case in which the inequality

$$\sum_{k \in K} \nu'_k \leqslant \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j$$

is strict for every $r < n$ and every $(I,J,K) \in T_r^n$. In this case the difference between the two sides in this inequality is at least $1/N$. It follows that we may replace ν'_p and ν'_q by $\nu'_p + 1/N$ and $\nu'_q - 1/N$ and still preserve the validity of all of these inequalities. Proceeding in this way (note that the value of p may need to be changed) we cannot arrive at a situation where ν'_q is an integer; this follows from the minimality of q. Therefore we arrive at a situation in which, for some $r < n$ and some $(I,J,K) \in T_r^n$ we have

$$\sum_{k\in K} \nu'_k = \sum_{i\in I} \lambda_i + \sum_{j\in J} \mu_j.$$

Consider now $n \times n$ selfadjoint matrices A, B, and C' such that $\Lambda_+(A) = \lambda$, $\Lambda_+(B) = \mu$, $\Lambda_+(C') = \nu'$, and $A + B = C'$. It follows from Theorem 5 in [7] (and its proof) that the matrices A, B, C have a common reducing subspace M such that $A|M$ has eigenvalues $\{\lambda_i : i \in I\}$, $A|M^\perp$ has eigenvalues $\{\lambda_i : i \in \{1,2,\ldots,n\} \setminus I\}$, and with analogous properties for $B|M$, $B|M^\perp$, $C'|M$, and $C'|M^\perp$. Since $\nu_k \leqslant \nu'_k$ for every k, it follows that the triples $(\lambda \circ I, \mu \circ J, \nu \circ K)$ and $(\lambda \circ I^c, \mu \circ J^c, \nu \circ K^c)$ satisfy the hypothesis of Proposition 6.8.r and 6.8.$(n-r)$, respectively. The inductive hypothesis implies the existence of integral partitions $\nu^1 \geqslant \nu \circ I$ and $\nu^2 \geqslant \nu \circ I^c$ such that $(\lambda \circ I, \mu \circ J, \nu^1)$ and $(\lambda \circ I^c, \mu \circ J^c, \nu^2)$ satisfy the Littlewood-Richardson rule. Define ν'' to be the partition obtained by listing in decreasing order the elements of ν^1 and ν^2 (informally, $\nu'' = \nu^1 \cup \nu^2$). It is obvious that $\nu'' \geqslant \nu$, and the proof is concluded if we show that (λ, μ, ν'') satisfies the Littlewood-Richardson rule. This however is immediate from the matrix formulation of the rule. Indeed, choose matrices A_1, A_2, B_1, B_2, C_1, and C_2 with eigenvalues $\lambda \circ I$, $\lambda \circ I^c$, $\mu \circ J$, $\mu \circ J^c$, ν^1, and ν^2, respectively, such that $A_1 + B_1 = C_1$ and $A_2 + B_2 = C_2$. Then the matrices $A_1 \oplus A_2$, $B_1 \oplus B_2$, and $C_1 \oplus C_2$ have eigenvalues λ, μ, and ν''. ∎

Theorem 6.4.n is trivial for $n = 1$. From this point on we will assume that $n > 1$, and Theorem 6.4.k has been proved for all $k < n$. The proof of Theorem 6.4.n depends on a number of consequences of this assumption. First we prove an important property of the elements in $\overline{T}_r^n$. The inequality $K' \leqslant K$ in the following statement is understood as an inequality between functions, i.e., $k'_i \leqslant k_i$ for $i = 1,2,\ldots,r$, where the k_i and k'_i are the elements of K and K', respectively, listed in increasing order.

PROPOSITION 6.9. *Fix natural numbers $r < n$, and a triple (I,J,K) in $\overline{T}_r^n$. There exists a set $K' \leqslant K$ such that $(I,J,K') \in T_r^n$.*

Proof. One can verify without difficulty that (I,J,K) belongs to $\overline{T}_r^n$ if and only if $(\Lambda(I), \Lambda(J), \Lambda(K))$ satisfies the hypothesis of Theorem 6.5.r. Therefore there exists a partition $\nu' \leqslant \Lambda(K)$ such that $(\Lambda(I), \Lambda(J), \nu')$ satisfies the Littlewood-Richardson rule. The partition ν' may be assumed integral by (the analogue of)

Proposition 6.8. Next define K' by the equality $\Lambda(K') = \nu'$. Clearly $K' \leqslant K$, and the proposition follows from Theorem 5.1. ∎

COROLLARY 6.10. *Fix positive integers $r < n < N$.*

(i) *If a triple (λ, μ, ν) satisfies all the Horn inequalities $(*_{IJK})$ for $(I, J, K) \in T_r^n$, then it must also satisfy $(*_{IJK})$ for $(I, J, K) \in \overline{T}_r^n$.*

(ii) *If $(I, J, K) \in \overline{T}_n^N$ and $(I', J', K') \in \overline{T}_r^n$ then $\Sigma_{I \circ I'} + \Sigma_{J \circ J'} \leqslant \Sigma_{K \circ K'} + r(r+1)/2$.*

Proof. Part (i) follows immediately from the previous result because partitions are decreasing, while (ii) follows because $I, J, \ldots$ are increasing (when viewed as functions). ∎

COROLLARY 6.11. *r. Fix positive integers $r < n < N$. If $(I, J, K) \in \overline{T}_n^N$ and $(I', J', K') \in \overline{T}_r^n$ then $(I \circ I', J \circ J', K \circ K')$ belongs to $\overline{T}_r^N$.*

Proof. By part (ii) of the preceding corollary, it suffices to show that for every $s < r$, and every $(I'', J'', K'') \in T_s^r$, we have $\Sigma_{(I \circ I') \circ I''} + \Sigma_{(J \circ J') \circ J''} \leqslant \Sigma_{(K \circ K') \circ K''} + s(s+1)/2$. Assuming that Corollary 6.11.*s* has been proved for all $s < r$, we see that $(I' \circ I'', J' \circ J'', K' \circ K'') \in \overline{T}_s^n$, and the required inequality follows from part (ii) of Theorem 6.6.*s*. ∎

Given $I \subset \{1, 2, \ldots, n\}$, we set I_n^c the relative complement $I^c \cap \{1, 2, \ldots, n\}$. Also consider the reflection $\rho_n : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ defined by $\rho_n(i) = n - i + 1$. As noted in [7] (cf. Lemma 4), a triple (I, J, K) belongs to T_r^n if and only if $(\rho_n(I_n^c), \rho_n(J_n^c), \rho_n(K_n^c))$ belongs to T_{n-r}^n. Observe that $K \leqslant K'$ if and only if $\rho_n(K_n^c) \leqslant \rho_n(K_n'^c)$. Therefore Proposition 6.9 (also applied with r replaced by $n - r$) yields the following result.

COROLLARY 6.12. *Fix positive integers $r < n$ and a triple (I, J, K) of subsets of $\{1, 2, \ldots, n\}$. Then (I, J, K) belongs to $\overline{T}_r^n$ if and only if $(\rho_n(I_n^c), \rho_n(J_n^c), \rho_n(K_n^c))$ belongs to $\overline{T}_{n-r}^n$.*

Assume that $I \subset \{1, 2, \ldots, n\}$ has r elements, and $I' \subset \{1, 2, \ldots, r\}$ has s elements. Then we have the identity

$$\rho_n(I \circ I') = \rho_n(I) \circ \rho_r(I').$$

Indeed, observe that the jth element of $\rho_n(I)$ is $n + 1 - I(r + 1 - j)$. In other words, viewed as an increasing function $\rho_n(I)$ equals $\rho_n \circ I \circ \rho_r$. Thus

$$\rho_n(I \circ I') = \rho_n \circ I \circ I' \circ \rho_s = \rho_n \circ I \circ \rho_r \circ \rho_r \circ I' \circ \rho_s = \rho_n(I) \circ \rho_r(I'),$$

as claimed.

COROLLARY 6.13. *Fix positive integers $r < n$, $s \leqslant n - r$, and triples $(I, J, K) \in T_r^n$, $(I', J', K') \in T_s^{n-r}$. Then the triple*

$$(I \cup (I_n^c \circ I'), J \cup (J_n^c \circ J'), K \cup (K_n^c \circ K'))$$

belongs to $\overline{T}_{r+s}^n$.

Proof. The case $s = n - r$ is trivial, so we assume that $s < n - r$. By Corollary 6.12 (with r replaced by $r + s < n$) it suffices to show that

$$(\rho_n((I \cup (I_n^{\mathrm{c}} \circ I'))_n^{\mathrm{c}}), \rho_n((J \cup (J_n^{\mathrm{c}} \circ J'))_n^{\mathrm{c}}), \rho_n((K \cup (K_n^{\mathrm{c}} \circ K'))_n^{\mathrm{c}})) \in \overline{T}_{n-r-s}^{n}.$$

Note now that $(I \cup (I_n^{\mathrm{c}} \circ I'))_n^{\mathrm{c}} = I_n^{\mathrm{c}} \circ I'^{\mathrm{c}}_{n-r}$ so that

$$\rho_n((I \cup (I_n^{\mathrm{c}} \circ I'))_n^{\mathrm{c}}) = \rho_n(I_n^{\mathrm{c}} \circ I'^{\mathrm{c}}_{n-r}) = \rho_n(I_n^{\mathrm{c}}) \circ \rho_{n-r}(I'^{\mathrm{c}}_{n-r})$$

by the remark preceding this corollary. Now, two applications of Corollary 6.12 show that $(\rho_n(I_n^{\mathrm{c}}), \rho_n(J_n^{\mathrm{c}}), \rho_n(K_n^{\mathrm{c}})) \in \overline{T}_{n-r}^{n}$ and $(\rho_{n-r}(I'^{\mathrm{c}}_{n-r}), \rho_{n-r}(J'^{\mathrm{c}}_{n-r}), \rho_{n-r}(K'^{\mathrm{c}}_{n-r})$ $\in \overline{T}_{n-r-s}^{n-r}$. Therefore an application of Corollary 6.11.$(n - r - s)$ yields the result. ∎

We can now prove Theorem 6.4.n.

Proof. That (i) implies (ii) is immediate by Theorem 6.2. Conversely, assume that (ii) holds. Let ε be the largest nonnegative number with the property that for every $r \leqslant n$, and every $(I, J, K) \in T_r^n$, we have

$$r\varepsilon + \sum_{k \in K} \nu_k \leqslant \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j.$$

Replacing ν_j by $\nu_j + \varepsilon$ for $j = 1, 2, \ldots, n$, we may assume that $\varepsilon = 0$. In this case there must exist $r \leqslant n$ and $(I, J, K) \in T_r^n$ such that

$$\sum_{k \in K} \nu_k = \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j.$$

If $r = n$ this is just the trace identity, and the result follows from Theorem 6.2. If $r < n$ we will show that $(\lambda \circ I, \mu \circ J, \nu \circ K)$ satisfies condition (ii) of Theorem 6.2, while $(\lambda \circ I^{\mathrm{c}}, \mu \circ J^{\mathrm{c}}, \nu \circ K^{\mathrm{c}})$ satisfies condition (ii) of Theorem 6.4.$(n - r)$. Indeed, if $s < r$, and $(I', J', K') \in T_s^r$, then $(I \circ I', J \circ J', K \circ K') \in \overline{T}_s^n$ by Corollary 6.11. Therefore (λ, μ, ν) satisfies $(*_{I \circ I', J \circ J', K \circ K'})$, and this is equivalent to saying that $(\lambda \circ I^{\mathrm{c}}, \mu \circ J^{\mathrm{c}}, \nu \circ K^{\mathrm{c}})$ satisfies $(*_{I'J'K'})$. On the other hand, if $s \leqslant n - r$, and $(I', J', K') \in T_s^{n-r}$, then we can apply Corollaries 6.13 and 6.11 to deduce the inequality

$$\sum_{k \in K} \nu_k + \sum_{k \in K_n^{\mathrm{c}} \circ K'} \nu_k \leqslant \sum_{i \in I} \lambda_i + \sum_{i \in I_n^{\mathrm{c}} \circ I'} \lambda_i + \sum_{j \in J} \mu_j + \sum_{j \in J_n^{\mathrm{c}} \circ J'} \mu_j.$$

Subtracting from this the above equality, we deduce that $(\lambda \circ I^{\mathrm{c}}, \mu \circ J^{\mathrm{c}}, \nu \circ K^{\mathrm{c}})$ satisfies $(*_{I'J'K'})$. We deduce the existence of $r \times r$ matrices A', B', C', and $(n - r) \times (n - r)$ matrices A'', B'', C'' such that $A' + B' = C'$, $A'' + B'' \geqslant C''$, $\Lambda_+(A') = \lambda \circ I$, $\Lambda_+(A'') = \lambda \circ I^{\mathrm{c}}$, and similar conditions on the eigenvalues of the other matrices. Clearly $A = A' \oplus A''$, $B = B' \oplus B''$ and $C = C' \oplus C''$ satisfy condition (i) of Theorem 6.4.n. ∎

Next we prove Theorem 6.3.

Proof. That (i) implies (ii) is known (cf. Theorem 6 in [7]). Assume therefore that condition (ii) holds, and define partitions λ^n, μ^n and ν^n by setting $\lambda_i^n = \lambda_i$ for $i \leqslant n$ and $\lambda_i^n = 0$ otherwise (analogous definitions for μ and ν). Clearly then $(\lambda^n, \mu^n, \nu^n)$ satisfies condition (ii) of Theorem 6.4.n. We deduce the existence of selfadjoint operators A_n, B_n, C_n on $\mathfrak{H}$, with rank at most n, such that $\Lambda_+(A_n) = \lambda^n$, $\Lambda_+(B_n) = \mu^n$, $\Lambda_+(C_n) = \nu^n$, and $A_n + B_n \geqslant C_n$. Lemma 5.2 then yields operators A, B and C satisfying condition (i). ∎

It may be worthwhile to state one form of Theorem 6.2 for compact operators.

THEOREM 6.14. *Let λ, μ, and ν be partitions such that $|\lambda| < \infty$. The following assertions are equivalent:*

(i) *there exist nonnegative compact operators A, B, and C such that $\Lambda_+(A) = \lambda, \Lambda_+(B) = \mu, \Lambda_+(C) = \nu$, and $A + B = C$;*

(ii) *we have:*

(a) *trace identity: $|\lambda| = |\nu - \mu|$;*

(b) *for all $r < n$, and all $(I, J, K) \in T_r^n$, the Horn inequality*

$$(*_{IJK}) \qquad \sum_{k \in K} \nu_k \leqslant \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j.$$

is satisfied.

Proof. It suffices to show that (ii) implies (i), so assume that (ii) is satisfied, and let A, B and C be provided by Theorem 6.3. Set $C' = A + B$ and $\nu' = \Lambda_+(C')$, so that $\nu' \geqslant \nu$, and (λ, μ, ν') satisfies the Littlewood-Richardson rule. By Lemma 3.4 we have

$$|\nu' - \nu| = |\nu' - \mu| - |\nu - \mu| = |\lambda| - |\lambda| = 0,$$

and therefore $C' = C$. ∎

We next consider selfadjoint operators which also have negative eigenvalues. Let us start with the finite dimensional case.

THEOREM 6.15.n. *Let λ, μ, and ν be partitions of length at most n. The following are equivalent:*

(i) *there exist selfadjoint $2n \times 2n$ matrices A, B, and C such that $\Lambda_+(A) = \lambda, \Lambda_+(B) = \mu, \Lambda_+(C) = \nu$, and $A + B = C$;*

(ii) *for all $r \leqslant n$, and all $(I, J, K) \in T_r^n$, the Horn inequality $(*_{IJK})$ is satisfied.*

If these conditions are satisfied, then the matrices in (i) *can be chosen to satisfy the additional conditions $\Lambda_+(-A) = \mu$, $\Lambda_+(-B) = \lambda$, and $\Lambda_+(-C) = \nu$.*

Proof. Theorem 6.15.n is easily verified for $n = 1$. Assume therefore that $n > 1$ and Theorem 6.15.k has been proved for all $k < n$. That (i) implies (ii) follows from Theorem 6.2. Conversely, let us assume that (ii) is satisfied, and consider the set $\mathfrak{L}$ of all partitions ρ of length at most n with the property that there exist $2n \times 2n$ matrices A, B, C such that $\Lambda_+(A) = \lambda$, $\Lambda_+(B) = \mu$, $\Lambda_+(C) = \rho$, $\Lambda_+(-A) = \mu$, $\Lambda_+(-B) = \lambda$, $\Lambda_+(-C) = \rho$, and $A + B = C$. The proof will be concluded by showing that $\nu \in \mathfrak{L}$. Theorem 6.2 implies that $\mathfrak{L}$ is a convex set. Note next that the zero partition (with entries equal to zero) belongs to $\mathfrak{L}$. Indeed, choose any matrix A such that $\Lambda_+(A) = \lambda$ and $\Lambda_+(-A) = \mu$, then set $B = -A$ and $C = 0$. Let t be the greatest positive number such that $t\nu$ satisfies condition (ii) in the statement; such a number exists if $\nu \neq 0$. Since $t \geqslant 1$, the convexity of $\mathfrak{L}$ shows that it suffices to prove that $t\nu \in \mathfrak{L}$. Replacing ν by $t\nu$ we may as well assume that $t = 1$. In this case there must exist $r \leqslant n$, and $(I, J, K) \in T_r^n$ such that

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j = \sum_{k \in K} \nu_k.$$

In case $r = n$, Theorem 6.2 provides nonnegative $n \times n$ matrices A_0, B_0, and C_0 such that $\Lambda_+(A_0) = \lambda$, $\Lambda_+(B_0) = \mu$, $\Lambda_+(C_0) = \nu$, and $A_0 + B_0 = C_0$. The matrices $A = A_0 \oplus (-B_0)$, $B = B_0 \oplus (-A_0)$, and $C = C_0 \oplus (-C_0)$ will then satisfy the conditions of this theorem. In case $r < n$, the argument in the proof of Theorem 6.4 shows that the triples $(\lambda \circ I, \mu \circ J, \nu \circ K)$ and $\lambda \circ I^c, \mu \circ J^c, \nu \circ K^c)$ satisfy condition (ii) of Theorem 6.15.r and Theorem 6.15.$(n - r)$, respectively. The inductive hypothesis then provides $2r \times 2r$ matrices A_1, B_1, C_1 and $2(n-r) \times 2(n-r)$ matrices A_2, B_2, C_2 such that $\Lambda_+(A_1) = \lambda_+(-B_1) = \lambda \circ I$, $\Lambda_+(B_1) = \Lambda_+(-A_1) = \mu \circ J$, $\Lambda_+(C_1) = \Lambda_+(-C_1) = \nu \circ K$, $\Lambda_+(A_2) = \lambda_+(-B_2) = \lambda \circ I^c$, $\Lambda_+(B_2) = \Lambda_+(-A_2) = \mu \circ J^c$, $\Lambda_+(C_2) = \Lambda_+(-C_2) = \nu \circ K^c$, $A_1 + B_1 = C_1$, and $A_2 + B_2 = C_2$. The proof is concluded by setting $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, and $C = C_1 \oplus C_2$. ∎

An appropriate modification of Lemma 5.2 for selfadjoint (but not necessarily nonnegative definite) operators yields the following result. The details of the proof are left to the reader.

THEOREM 6.16. *Given partitions λ, μ, and ν the following are equivalent:*

(i) *there exist selfadjoint compact operators A, B, and C such that $\Lambda_+(A) = \lambda$, $\Lambda_+(B) = \mu, \Lambda_+(C) = \nu$, and $A + B = C$;*

(ii) *for all $r \leqslant n$, and all $(I, J, K) \in T_r^n$, the Horn inequality $(*_{IJK})$ is satisfied.*

If these conditions are satisfied, then the operators in (i) *can be chosen to satisfy the additional conditions $\Lambda_+(-A) = \mu$, $\Lambda_+(-B) = \lambda$, and $\Lambda_+(-C) = \nu$.*

7. INVARIANT SUBSPACES

Consider a strongly continuous operator semigroup $T = \{T(t) : t \geqslant 0\}$ acting on a Hilbert space $\mathfrak{H}$. Such a semigroup will be said to be *nilpotent* if $T(t) = 0$ for some $t > 0$. For each $\lambda \geqslant 0$ there is such a semigroup S_λ defined on $L^2(0, \lambda)$ by right translation: $S_\lambda(t)f(x) = f(x - t)$ if $x \geqslant t$ and $S_\lambda(t)f(x) = 0$ if $x < t$. These semigroups play the role of Jordan cells. To be precise, a semigroup of the form $S_{\lambda_1} \oplus S_{\lambda_2} \oplus \cdots$, with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$, is called a Jordan semigroup. For every nilpotent semigroup T, there exist a unique Jordan semigroup S, and injective operators X, Y with dense ranges such that $XT(t) = S(t)X$ and $YS(t) = T(t)Y$ for all $t \geqslant 0$ (cf. [3] and [1]). One says that T and S are quasisimilar, and S is called the *Jordan model* of T. If $S = S_{\lambda_1} \oplus S_{\lambda_2} \oplus \cdots$ is the Jordan model of T, we will write $\Lambda(T) = (\lambda_1, \lambda_2, \ldots)$.

In this section we will limit ourselves to semigroups for which $\Lambda(T)$ is a partition, i.e., $\lim\limits_{n\to\infty} \lambda_n = 0$. This situation corresponds with finite dimensionality in the following sense: if $\Lambda(T)$ is a partition, and X is an operator on $\mathfrak{H}$ such that $XT(t) = T(t)X$ for all $t \geqslant 0$, then X is one-to-one if and only if it has dense range. In the terminology of [1], T has property (P).

A subspace M is invariant for a semigroup T if $T(t)M \subset M$ for every $t > 0$. Given an invariant subspace, one can form the restricted semigroup $T|M$ and the compressed semigroup $T^*|M^\perp$.

The following result was proved in [2] (the result there is stated in greater generality). Recall that the Littlewood-Richardson Rule requires the existence of a Littlewood-Richardson function.

THEOREM 7.1. *Assume that λ, μ, and ν are partitions such that $|\nu| < \infty$, T is a nilpotent semigroup, and M is an invariant subspace for T such that $\Lambda(T) = \nu$, $\Lambda(T|M) = \lambda$, and $\Lambda(P_{M^\perp}T|M^\perp) = \mu$. Then (λ, μ, ν) satisfies the Littlewood-Richardson Rule.*

The following result is from [15]. The limit in the statement below is taken componentwise, and the empty sum is understood to be zero in condition (c) for $j = 1$.

THEOREM 7.2. *Given partitions λ, μ, and ν, the following conditions are equivalent:*

(i) *there exist a nilpotent semigroup T, and an invariant subspace M for T such that $\Lambda(T) = \nu$, $\Lambda(T|M) = \lambda$, and $\Lambda(P_{M^\perp}T|M^\perp) = \mu$;*

(ii) *there exist partitions ρ^k, $k \geqslant 0$, with the following properties:*

(a) $\rho^{k+1}_{j+1} \leqslant \rho^k_j \leqslant \rho^{k+1}_j$ *for* $k \geqslant 0$ *and* $j \geqslant 1$;

(b) $\rho^0 = \lambda$ *and* $\lim\limits_{k\to\infty} \rho^k = \nu$;

(c) $\sum\limits_{n=1}^{j} (\rho^{k+2}_n - \rho^{k+1}_n) \leqslant \sum\limits_{n=1}^{j-1} (\rho^{k+1}_n - \rho^k_n)$ *for all* $j \geqslant 1$ *and* $k \geqslant 0$;

(d) $\sum\limits_{n=1}^{\infty} (\rho^k_n - \rho^{k-1}_n) = \mu_k$ *for* $k \geqslant 1$.

A brief analysis of these conditions reveals that in fact $\rho^k_j = \rho^{k+1}_j$ if $k \geqslant j$, so the limit (b) is reached rather soon. In fact, setting $a_{jk} = \rho^k_j$ for $0 \leqslant k \leqslant j$, the above conditions amount exactly to saying that the numbers a_{jk} are the breakpoints of a

Littlewood-Richardson tableau with content μ (cf. Lemma 3.5 above). In other words, we have the following result.

THEOREM 7.3. *Given partitions λ, μ, and ν, the following conditions are equivalent:*

(i) *there exist a nilpotent semigroup T, and an invariant subspace M for T such that $\Lambda(T) = \nu$, $\Lambda(T|M) = \lambda$, and $\Lambda(P_{M^\perp}T|M^\perp) = \mu$;*

(ii) *the triple (λ, μ, ν) satisfies the Littlewood-Richardson rule.*

The attentive reader of [2] will notice that in order to prove the converse of the main theorem there one still needs to show that, given a measurable family of triples satisfying the Littlewood-Richardson rule, one can find a measurable family of Littlewood-Richardson functions (or tableaux) associated with these triples. This however does not pose any difficulty.

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