J. OPERATOR THEORY
54:2(2005), 239–250

CONJUGATION, THE BACKWARD SHIFT, AND TOEPLITZ KERNELS

STEPHAN RAMON GARCIA

Communicated by William B. Arveson

ABSTRACT. For each outer function Ω in the Smirnov class and each $p \in (0,\infty)$, we define a subspace $\mathcal{N}_{\Omega}^{p}$ of H^p that carries an operation analogous to complex conjugation. Using these subspaces, we explicitly describe the invariant subspaces and noncyclic functions for the backward shift operator on H^p for $p \in [1,\infty)$ and $p \in (0,\infty)$, respectively. We also discuss pseudocontinuations, the Darlington synthesis problem from electrical network theory, and the kernels of Toeplitz operators.

KEYWORDS: *Invariant subspaces, backward shift operator, Toeplitz operators, pseudocontinuation, Smirnov class, unitary matrices, Darlington synthesis, cyclic functions, noncyclic functions, inner functions.*

MSC (2000): 30D55, 47A15.

1. INTRODUCTION

In this note we introduce a family of subspaces of the H^p spaces for $p \in (0,\infty)$ upon which a certain conjugation operation is defined. Using these subspaces, we obtain explicit representations for the invariant subspaces and noncyclic functions for the backward shift operator on H^p for $p \in [1,\infty)$ and $p \in (0,\infty)$, respectively. We discuss applications of these results to the theory of pseudocontinuations and to the Darlington synthesis problem from electrical network theory. Furthermore, we obtain explicit representations for the kernels of Toeplitz operators on H^p for $p \in (1,\infty)$ and relate this to results of Hayashi [8] and Dyakonov [6].

The first portion of this note concerns the construction and structure theory of our subspaces. The second portion involves applications of these results. The building blocks of our construction are the so-called real Smirnov functions, first described by Helson [9] and later by Sarason and the author [7].

Recall that each H^p space is contained in the Smirnov class N^+ which consists of all functions of the form h_1/h_2 where h_1, h_2 are in H^∞ and h_2 is outer. If

f belongs to N^+, then f possesses nontangential limiting values a.e. on $\partial\mathbb{D}$. Each f in N^+ factors as $f = I_f F$ where I_f is inner and F is outer. This factorization is unique up to unimodular constant factors. We freely identify Smirnov functions with their boundary values on $\partial\mathbb{D}$ and we frequently suppress the "a.e."

2. REAL SMIRNOV FUNCTIONS

A function f belonging to the Smirnov class N^+ is called a *real Smirnov function* if its boundary function is real valued a.e. on $\partial\mathbb{D}$. The set R^+ of all real Smirnov functions is a real subalgebra of N^+ that was explicitly described by Helson [9]. He showed that if ψ_1 and ψ_2 are relatively prime inner functions such that $\psi_1 - \psi_2$ is outer, then the function

$$f(z) = \mathrm{i}\frac{\psi_1 + \psi_2}{\psi_1 - \psi_2} \tag{2.1}$$

is a real Smirnov function and every real Smirnov function arises this way. Although elegant, this representation has its limitations. For example, the inner functions ψ_1 and ψ_2 are often difficult to identify and there are no general criteria describing when the difference of inner functions is outer.

In [7], Sarason and the author proved that any outer function F belonging to R^+ can be represented as a locally uniformly convergent product

$$f(z) = |f(0)| \prod_{n=1}^{\infty} \frac{T(\varphi_n^+)}{T(\varphi_n^-)}$$

where T denotes the linear fractional transformation

$$T(z) := \mathrm{i}\frac{1 - \mathrm{i}z}{1 + \mathrm{i}z}$$

and the inner functions φ_n^+ and φ_n^- are naturally associated with the boundary values of $\arg F$ on $\partial\mathbb{D}$.

If φ is a nonconstant inner function and $k(z) = \frac{z}{(1-z)^2}$ denotes the Koebe function, then $k(\varphi)$ belongs to R^+. The inner factor of $k(\varphi)$ is precisely φ and hence the construction of the general function in R^+ can be reduced to the consideration of outer functions in R^+. In short, real Smirnov functions can be described explicitly in terms of inner functions.

3. THE SPACES $\mathcal{N}_\Omega^p$

Let C^+ denote the complex subalgebra $R^+ + \mathrm{i}R^+$ of N^+. With respect to the translation invariant metric

$$\rho(f, g) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(1 + |f(\mathrm{e}^{\mathrm{i}t}) - g(\mathrm{e}^{\mathrm{i}t})|)\, \mathrm{d}t$$

on N^+ [3], C^+ is a complete metric space and a topological algebra.

If Ω is an outer function, then $\Omega C^+ \cap H^p$ is closed in H^p for every $p \in (0,\infty)$. We define $\mathcal{N}^p_\Omega$ to be $\Omega C^+ \cap H^p$, regarded as a subspace of H^p. Although $\mathcal{N}^p_\Omega$ may be trivial for certain choices of Ω and p, it is clearly nontrivial if Ω belongs to H^p. An obvious condition for equality of these spaces is the following.

PROPOSITION 3.1. *If Ω_1, Ω_2 belong to H^p for some $p \in (0,\infty)$, then $\mathcal{N}^p_{\Omega_1} = \mathcal{N}^p_{\Omega_2}$ if and only if $\Omega_1 = h\Omega_2$ for some invertible h in C^+.*

The example $1 + \mathrm{i}\big(\mathrm{i}\frac{z+1}{z-1}\big)$ shows that outer functions in C^+ need not be invertible in C^+.

Each h in C^+ possesses a unique representation $h = a + \mathrm{i}b$ where a and b belong to R^+. Since the algebra of boundary functions corresponding to C^+ is closed under complex conjugation, the expression $\overline{h}$ will refer to the analytic function $a - \mathrm{i}b$ and its boundary values on $\partial\mathbb{D}$, rather than the complex conjugate of the analytic function h on $\mathbb{D}$. This induces a natural involution of $\mathcal{N}^p_\Omega$ which we also refer to as *conjugation*.

Suppose that $f = h\Omega$ belongs to $\mathcal{N}^p_\Omega$ for some outer function Ω and $p \in (0,\infty)$. Since the functions f and $\widehat{f} := \overline{h}\Omega$ have the same modulus on $\partial\mathbb{D}$, they share the same outer factor, say F. Therefore $\widehat{f}$ belongs to $\mathcal{N}^p_\Omega$ and we may write $f = I_f F$ and $\widehat{f} = I_{\widehat{f}} F$ where I_f and $I_{\widehat{f}}$ denote the inner factors of f and $\widehat{f}$, respectively. We refer to f and $\widehat{f}$ as *conjugate functions*. The obvious equation $\widehat{f} = \overline{f}\Omega/\overline{\Omega}$ completely characterizes conjugate pairs in $\mathcal{N}^p_\Omega$.

PROPOSITION 3.2. *Two functions f and g in H^p satisfy $g = \overline{f}\Omega/\overline{\Omega}$ a.e. on $\partial\mathbb{D}$ for some outer function Ω if and only if f and g belong to $\mathcal{N}^p_\Omega$ and are conjugates.*

Proof. If f, g belong to H^p and satisfy $g = \overline{f}\Omega/\overline{\Omega}$, then they have the same modulus on $\partial\mathbb{D}$ and hence the same outer factor, say F. Writing $f = I_f F$ and $g = I_g F$, it follows that $I_f I_g F/\overline{F} = \Omega/\overline{\Omega}$. Since Ω is outer and

$$
\begin{aligned}
I_f I_g &= \tfrac{1}{2}(I_f + I_g) \Big/ \overline{\tfrac{1}{2}(I_f + I_g)} \\
&= \tfrac{1}{2\mathrm{i}}(I_f - I_g) \Big/ \overline{\tfrac{1}{2\mathrm{i}}(I_f - I_g)}
\end{aligned}
$$

on $\partial\mathbb{D}$, both $\frac{1}{2}(I_f + I_g)F/\Omega$ and $\frac{1}{2\mathrm{i}}(I_f - I_g)F/\Omega$ belong to R^+. Hence there exist a and b in R^+ such that $\frac{1}{2}(I_f + I_g)F = a\Omega$ and $\frac{1}{2\mathrm{i}}(I_f - I_g)F = b\Omega$. Thus $f = (a + \mathrm{i}b)\Omega$ and $g = (a - \mathrm{i}b)\Omega$ belong to $\mathcal{N}^p_\Omega$ and are conjugates. The converse is trivial. ∎

If $f = I_f F$ and $\widehat{f} = I_{\widehat{f}} F$ are conjugates in $\mathcal{N}^p_\Omega$, then $I_f I_{\widehat{f}} F = \overline{F}\Omega/\overline{\Omega}$. This shows that F belongs to $\mathcal{N}^p_\Omega$ and satisfies $\widehat{F} = I_f I_{\widehat{f}} F$. Moreover, the inner function $I_f I_{\widehat{f}}$ depends only upon F and Ω. We call this inner function the *associated inner* *function of* F and denote it $\mathcal{I}_F$. The functions $f = I_f F$ in $\mathcal{N}^p_\Omega$ with outer factor F are precisely those functions whose inner factors I_f divide $\mathcal{I}_F$.

Each function f belonging to $\mathcal{N}^p_\Omega$ possesses the two representations

(1) $f = I_f F$ where F is outer, I_f is the inner factor of f, and $I_f | \mathcal{I}_F$.
(2) $f = g_1 + \mathrm{i}g_2$ where $\widehat{g_k} = g_k$ for $k = 1, 2$.

To pass from (1) to (2) note that the inner factor $I_{\widehat{f}}$ of $\widehat{f}$ is given by $I_{\widehat{f}} = \mathcal{I}_F / I_f$. Going from (2) to (1) is slightly more complicated. Let $f = g_1 + \mathrm{i}g_2 = I_f F$ where the functions I_f and F are to be determined. Since F is outer, it is determined by the equation $|F|^2 = |g_1|^2 + |g_2|^2$ on $\partial\mathbb{D}$. The inner functions I_f and $I_{\widehat{f}}$ satisfy

$$I_f = \frac{g_1}{F} + \mathrm{i}\frac{g_2}{F},$$
$$I_{\widehat{f}} = \frac{g_1}{F} - \mathrm{i}\frac{g_2}{F}$$

and hence $\mathcal{I}_F$ is given by

$$\mathcal{I}_F = I_f I_{\widehat{f}} = \left(\frac{g_1}{F}\right)^2 + \left(\frac{g_2}{F}\right)^2.$$

To study the structure of the $\mathcal{N}^p_\Omega$ spaces, we require a special class of outer functions. Let us call an outer function *simple* if it is of the form $\Psi = \frac{1}{2}(\psi_1 + \psi_2)$ where ψ_1, ψ_2 are relatively prime inner functions, not both constant. We say that an outer function F in H^p is weakly outer (in H^p) if it is divisible in H^p by a simple outer function. Otherwise we say that F is strongly outer (in H^p).

PROPOSITION 3.3. *An outer function F in $\mathcal{N}^p_\Omega$ satisfies $F = \Psi G$ for some simple outer function $\Psi = \frac{1}{2}(\psi_1 + \psi_2)$ and an outer function G in H^p if and only if G belongs to $\mathcal{N}^p_\Omega$ and $\mathcal{I}_G = \psi_1\psi_2\mathcal{I}_F$.*

Proof. This follows immediately from Proposition 3.2 and the identity $\Psi / \overline{\Psi} = \psi_1\psi_2$ which holds a.e. on $\partial\mathbb{D}$. ∎

PROPOSITION 3.4. *If Ω is an outer function in H^p, then* $\dim \mathcal{N}^p_\Omega > 1$ *if and only if Ω is weakly outer.*

Proof. If $\dim \mathcal{N}^p_\Omega > 1$, then we can find some g belonging to $\mathcal{N}^p_\Omega$ such that $\widehat{g} = g$ and g is not a scalar multiple of Ω. Since the functions $\Omega + \mathrm{i}g$ and $\Omega - \mathrm{i}g$ share the same modulus on $\partial\mathbb{D}$, they share the same outer factor. We may therefore write $\Omega + \mathrm{i}g = \psi_1 F$ and $\Omega - \mathrm{i}g = \psi_2 F$ where ψ_1 and ψ_2 are inner functions, not both constant, and F is an outer function belonging to H^p. Solving for Ω yields $\Omega = \frac{1}{2}(\psi_1 + \psi_2)F$ and hence Ω is weakly outer. Conversely, Propositions 3.2 and 3.3 show that if Ω is weakly outer in H^p, then $\dim \mathcal{N}^p_\Omega > 1$. ∎

Let us call an inner function I a *maximal inner function* for $\mathcal{N}^p_\Omega$ if it is the inner factor of some f belonging to $\mathcal{N}^p_\Omega$ and I does not properly divide the inner factor of any g in $\mathcal{N}^p_\Omega$. Note that a maximal inner function for $\mathcal{N}^p_\Omega$ is necessarily the associated inner function $\mathcal{I}_F$ for some outer function F in $\mathcal{N}^p_\Omega$.

PROPOSITION 3.5. *An inner function I is a maximal inner function for $\mathcal{N}^p_\Omega$ if and only if there exists a strongly outer function F in $\mathcal{N}^p_\Omega$ such that $I = \mathcal{I}_F$.*

Proof. If $I = \mathcal{I}_F$ and F is weakly outer, then Proposition 3.3 implies that I is not maximal. If $I = \mathcal{I}_F$ is not a maximal inner function for $\mathcal{N}^p_\Omega$, then $I\psi = \mathcal{I}_G$ for some outer function G in $\mathcal{N}^p_\Omega$ and a nonconstant inner function ψ. Hence

$$IF/\overline{F} = \Omega/\overline{\Omega} = I\psi G/\overline{G}$$

on $\partial\mathbb{D}$. If Ψ denotes the simple outer function $\Psi = \frac{1}{2}(1+\psi)$, then $\Psi G/F$ belongs to R^+ and the method of Proposition 3.4 applied to the functions $F \pm \mathrm{i}\Psi G$ shows that F is weakly outer. ∎

If F is strongly outer in H^p, then the associated inner functions of F are maximal in *any* subspace $\mathcal{N}^p_\Omega$ that contains F. Another relationship between outer functions and their associated inner functions is given below.

PROPOSITION 3.6. *Fix $p \in [1, \infty)$ and let F be an outer function in $\mathcal{N}^p_\Omega$ that is bounded away from 0. If G is an outer function in $\mathcal{N}^p_\Omega$ such that $\mathcal{I}_G = \mathcal{I}_F$, then F and G are real scalar multiples of each other.*

Proof. If $\mathcal{I}_G = \mathcal{I}_F$, then G/F belongs to R^+ since

$$\mathcal{I}_G G/\overline{G} = \Omega/\overline{\Omega} = \mathcal{I}_F F/\overline{F}.$$

The quotient G/F belongs to $R^+ \cap H^1 = \mathbb{R}$ since F is bounded away from 0. ∎

4. THE BACKWARD SHIFT OPERATOR

Our first application of the spaces $\mathcal{N}^p_\Omega$ involves the backward shift on the Hardy spaces. The backward shift of a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic on the unit disk is the function

$$Bf(z) := \frac{f(z) - f(0)}{z}.$$

In terms of Taylor coefficients at the origin, the backward shift sends the coefficient sequence $(a_0, a_1, a_2, \ldots)$ to $(a_1, a_2, a_3, \ldots)$. The mapping $f \mapsto Bf$ is a continuous linear operator (also denoted by B) on each H^p space.

A subspace $\mathcal{M} \subseteq H^p$ is called *B-invariant* (or simply *invariant*) if $B\mathcal{M} \subseteq \mathcal{M}$. A straightforward application of Beurling's theorem and the F. and M. Riesz theorem yields the following well-known result.

THEOREM. *The proper B-invariant subspaces of H^p for $p \in (1, \infty)$ are precisely the subspaces $H^p \cap \varphi \overline{zH^p}$ where φ is a nonconstant inner function.*

Aleksandrov extended the theorem above to include the case $p = 1$. For $p \in (0,1)$, the situation is more difficult. Although the subspaces $H^p \cap \varphi\overline{zH^p}$ are still B-invariant, many more invariant subspaces exist. Nevertheless, Aleksandrov proved that every proper B-invariant subspace of H^p for $p \in (0,1)$ is contained in a subspace of the form $H^p \cap \varphi\overline{zH^p}$. For a detailed account of these results see [3].

Suppose that φ is a nonconstant inner function. For each ζ in $\partial\mathbb{D}$ such that φ has a nontangential limiting value at ζ of unit modulus, we define

$$K_\zeta(z) := \frac{1 - \overline{\varphi(\zeta)}\varphi(z)}{1 - \overline{\zeta}z}.$$

Fix $p \in (0,\infty)$ and let c be a unimodular constant such that $c^2 = \overline{\zeta}\varphi(\zeta)$. Consider the space $\mathcal{N}^p_\Omega$ where $\Omega = cK_\zeta$. By Proposition 3.2, a function f belongs to $\mathcal{N}^p_\Omega$ if and only if

$$f = \frac{\overline{g}cK_\zeta}{\overline{cK_\zeta}} = \overline{gz}\varphi$$

for some g in H^p. If such a function g exists, then f and g are conjugates. Since a function f belongs to $H^p \cap \varphi\overline{zH^p}$ if and only if there exists a function g in H^p such that $f = \overline{gz}\varphi$, we have proved the following proposition.

PROPOSITION 4.1. *If φ is a nonconstant inner function and $p \in (0,\infty)$, then* $H^p \cap \varphi\overline{zH^p} = \mathcal{N}^p_{cK_\zeta}$.

Therefore every function in $H^p \cap \varphi\overline{zH^p}$ is of the form $(a + ib)K_\zeta$ where a and b belong to R^+ and the invariant subspaces $H^p \cap \varphi\overline{zH^p}$ enjoy the description

$$H^p \cap \varphi\overline{zH^p} = \{(a + ib)K_\zeta : a, b \in R^+\} \cap H^p.$$

EXAMPLE. Let φ denote a nonconstant inner function and let $p \in (0,\infty)$. For each w in the unit disk $\mathbb{D}$, the function

$$K_w(z) := \frac{1 - \overline{\varphi(w)}\varphi(z)}{1 - \overline{w}z}$$

is a bounded outer function belonging to $H^p \cap \varphi\overline{zH^p}$. Its conjugate is the function

$$\widehat{K}_w(z) = \frac{\varphi(z) - \varphi(w)}{z - w}$$

and its associated inner function $\mathcal{I}_{K_w}$ is given by

$$\mathcal{I}_{K_w}(z) = \frac{B_{\varphi(w)}(\varphi(z))}{B_w(z)} \tag{4.1}$$

where $B_a(z)$ denotes the linear fractional transformation

$$B_a(z) = \frac{z - a}{1 - \overline{a}z}.$$

If φ is not a Möbius transformation, then $\mathcal{I}_{K_w}$ is nonconstant for every w in $\mathbb{D}$. Indeed, if $\mathcal{I}_{K_w}$ is constant, then $B_{\varphi(w)}(\varphi(z))$ and $B_w(z)$ differ by a unimodular constant and Pick's Lemma shows that φ is a Möbius transformation. Since each K_w is bounded away from 0, Proposition 3.6 implies that the only outer functions F in $H^p \cap \varphi\overline{zH^p}$ for $p \in (1, \infty)$ with $\mathcal{I}_F = \mathcal{I}_{K_w}$ are the real scalar multiples of K_w.

We say that a function f in H^p is *noncyclic* if the closed linear span $[f]_p$ of the set $\{f, Bf, B^2f, \ldots\}$ is not all of H^p. This is equivalent to asserting that f belongs to $H^p \cap \varphi\overline{zH^p}$ for some nonconstant inner function φ. We now explicitly describe the noncyclic functions in H^p.

PROPOSITION 4.2. *If $p \in (0, \infty)$, then the linear manifold of noncyclic functions in H^p is exactly*

$$H^p \cap \left\{ \mathrm{i}\frac{\psi_1 - \psi_2}{\psi_1 + \psi_2}\frac{1-\varphi_1}{1-\varphi_2} : \psi_1, \psi_2, \varphi_1, \varphi_2 \text{ inner} \right\}.$$

Proof. That the noncyclic functions in H^p form a linear manifold is well-known [3]. Proposition 3.2 implies that a function $f = I_f F$ belongs to $H^p \cap \varphi\overline{zH^p}$ if and only if $f/\overline{f} = \varphi_1\overline{\varphi}_2$ where $\varphi_1 = \varphi I_f$ and $\varphi_2 = zI_{\widehat{f}}$. Since $f\frac{1-\varphi_2}{1-\varphi_1}$ belongs to R^+, the result follows from (2.1). ∎

In light of a well-known result of Douglas, Shapiro, and Shields, cf. Theorem 2.2.1 in [4], Proposition 4.2 explicitly characterizes all H^2 (and more generally H^p for $p \in (1, \infty)$) functions which are *pseudocontinuable of bounded type*. We refer the reader to [10] and the original paper [4] for the relevant definitions and background.

A function f in $H^p \cap \varphi\overline{zH^p}$ is called a *generator* (of $H^p \cap \varphi\overline{zH^p}$) if $[f]_p = H^p \cap \varphi\overline{zH^p}$. Let (φ, ψ) denote the greatest common divisor of the inner functions φ and ψ. Interpreting Theorem 3.1.5 of [4] in terms of conjugation, we obtain the following proposition.

PROPOSITION 4.3. *If f belongs to $H^p \cap \varphi\overline{zH^p}$ for some $p \in [1, \infty)$ and $(I_{\widehat{f}}, \varphi) = 1$, then $[f]_p = H^p \cap \varphi\overline{zH^p}$.*

COROLLARY. *Let $p \in [1, \infty)$. If F is an outer function belonging to $H^p \cap \varphi\overline{zH^p}$, then $\widehat{F}$ generates $H^p \cap \varphi\overline{zH^p}$. In particular, $\widehat{K}_w$ is a generator for each w in $\mathbb{D}$.*

If φ is a singular inner function, then Equation (4.1) and Frostman's Theorem imply that $(\mathcal{I}_{K_w}, \varphi) = 1$ and hence K_w is a generator for almost every w. More generally, if $\varphi(w) \neq 0$ and $\varphi(w)$ does not lie in the exceptional set for φ (the set of all w in the unit disk such that $B_w(\varphi)$ is not a Blaschke product), then $(\mathcal{I}_{K_w}, \varphi) = 1$ and K_w generates $H^p \cap \varphi\overline{zH^p}$.

COROLLARY. *Let $p \in [1, \infty)$. If F is an outer function belonging to $H^p \cap \varphi\overline{zH^p}$ and $(\mathcal{I}_F, \varphi) = 1$, then every f in $H^p \cap \varphi\overline{zH^p}$ with outer factor F generates $H^p \cap \varphi\overline{zH^p}$. In particular, every self-conjugate outer function is a generator.*

If $p \in [1,\infty)$, then every outer function F in $H^p \cap \varphi\overline{zH^p}$ is the sum of two generators. Indeed,

$$F = \frac{1}{2}(1+\mathcal{I}_F)F + \mathrm{i}\frac{1}{2\mathrm{i}}(1-\mathcal{I}_F)F$$

is such a representation.

5. DARLINGTON SYNTHESIS

Our motivation for this discussion stems from results of Arov [1] and Douglas and Helton [5] connecting the Darlington synthesis problem from electrical network theory to the theory of pseudocontinuations [10] and hence to the backward shift operator.

The simplest case of the Darlington synthesis problem is the following. Given a function a in H^∞, do there exist b, c, d belonging to H^∞ such that

$$U = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \tag{5.1}$$

is unitary a.e. on $\partial\mathbb{D}$? Arov, and Douglas and Helton considered the general (operator valued) problem and showed that the scalar valued problem is solvable if and only if (via Theorem 2.2.1 in [4]) a is noncyclic for the backward shift on H^2.

If U is of the form (5.1) and unitary a.e. on $\partial\mathbb{D}$, then $\det U$ is an inner function. A necessary and sufficient condition is the following.

PROPOSITION 5.1. *If φ is a nonconstant inner function, then U is unitary a.e. on $\partial\mathbb{D}$ and $\det U = \varphi$ if and only if:*

(i) *a, b, c, d belong to $(z\varphi H^2)^\perp = H^2 \cap \varphi\overline{H^2}$;*
(ii) *$\widehat{a} = d$ and $\widehat{b} = c$;*
(iii) *$|a|^2 + |b|^2 = 1$ a.e. on $\partial\mathbb{D}$.*

Proof. ($\Rightarrow$) If U is unitary on $\partial\mathbb{D}$, then $\varphi = ad + bc$ is inner. Comparing entries in the identity $U = (U^*)^{-1}$ yields the equations $a = \overline{d}\varphi$ and $b = \overline{c}\varphi$. Proposition 4.1 implies that a, b, c, d are in $(z\varphi H^2)^\perp$, $\widehat{a} = d$, and $\widehat{b} = c$. Finally, the identity $UU^* = I$ shows that $|a|^2 + |b|^2 = 1$ a.e. on $\partial\mathbb{D}$.

($\Leftarrow$) Let $a = I_a F$, $b = I_b G$, $c = I_c G$, and $I_d F$ where I_a, I_b, I_c, I_d are inner and F, G are outer. Consider the entries in the product

$$UU^* = \begin{pmatrix} I_a F & -I_b G \\ I_c G & I_d F \end{pmatrix} \begin{pmatrix} \overline{I_a F} & \overline{I_c G} \\ \overline{-I_b G} & \overline{I_d F} \end{pmatrix}.$$

Since $|F|^2 + |G|^2 = 1$ a.e. on $\partial\mathbb{D}$, the entries on the main diagonal of the product are both 1. The upper right corner of the product is the function $X = I_a F\overline{I_c G} - I_b G\overline{I_d F}$ which we wish to show is identically 0.

A few manipulations lead to the equation

$$I_a I_d \frac{F}{\overline{F}} - I_b I_c \frac{G}{\overline{G}} = X \frac{I_c I_d}{\overline{FG}}.$$

Since a, d and b, c are conjugates, we see that $I_a I_d = \mathcal{I}_F$ and $I_b I_c = \mathcal{I}_G$. Therefore

$$X \frac{I_c I_d}{\overline{FG}} = \varphi - \varphi = 0$$

and hence X vanishes identically. A similar argument applies to the bottom left corner of the product. Thus U is unitary a.e. on $\partial\mathbb{D}$ and

$$\begin{aligned} \det U &= ad + bc \\ &= I_a I_d F^2 + I_b I_c G^2 \\ &= \mathcal{I}_F F^2 + \mathcal{I}_G G^2 \\ &= |F|^2 \varphi + |G|^2 \varphi \\ &= \varphi \end{aligned}$$

a.e. on $\partial\mathbb{D}$ which completes the proof. ∎

If the 2×2 scalar valued Darlington synthesis problem is solvable for some a in H^∞, then there exists a function b in H^∞ such that $|a|^2 + |b|^2 = 1$ on $\partial\mathbb{D}$. Moreover, b must also lie in $(z\varphi H^2)^\perp = H^2 \cap \varphi\overline{H^2}$, the B-invariant subspace generated by φ.

PROPOSITION 5.2. *If a belongs to $(z\varphi H^2)^\perp$ for some nonconstant inner function φ, $\| a \|_\infty \leqslant 1$, and a is not an inner function, then there exists an outer function $G \in (z\varphi H^2)^\perp$ such that $|a|^2 + |G|^2 = 1$ a.e. on $\partial\mathbb{D}$.*

Proof. As before, let $a = I_a F$ where I_a is inner and F is outer. By Proposition 4.1, it follows that $\mathcal{I}_F F = \overline{F}\varphi$. This implies that $\mathcal{I}_F F^2 = |F|^2 \varphi$ which yields $\varphi - \mathcal{I}_F F^2 = \varphi(1 - |F|^2)$. Since F is not an inner function and $\varphi - \mathcal{I}_F F^2$ belongs to H^∞, it follows that

$$\int_{\partial\mathbb{D}} \log(1 - |F|^2) = \int_{\partial\mathbb{D}} \log|\varphi - \mathcal{I}_F F^2| > -\infty$$

and there exists an outer function G in H^∞ such that $|G|^2 = 1 - |F|^2$. Thus

$$\psi G^2 = \varphi - \mathcal{I}_F F^2 = \varphi |G|^2$$

for some inner function ψ. Since $\psi G = \overline{G}\varphi$, Proposition 3.2 implies that G belongs to $(z\varphi H^2)^\perp$ and $\psi = \mathcal{I}_G$. ∎

We now parametrize all nonconstant matrices U of the form (5.1) which are unitary on $\partial\mathbb{D}$:

(i) Pick a nonconstant inner function φ.

(ii) Select a function a in $(z\varphi H^2)^\perp$ satisfying $\| a \|_\infty \leqslant 1$.

(a) If a is inner, then $b = c = 0$, a divides φ, and U is diagonal.

(b) If a is not inner, then select b in $(z\varphi H^2)^\perp$ such that $|a|^2 + |b|^2 = 1$ a.e. on $\partial\mathbb{D}$.

(iii) $\widehat{b}$ and $\widehat{a}$ complete the bottom row.

6. KERNELS OF TOEPLITZ OPERATORS

If u belongs to L^∞, then the *Toeplitz operator* with symbol u is the operator $T_u : H^2 \to H^2$ defined by $T_u f = P(uf)$ where P denotes the orthogonal projection from L^2 onto H^2. The projection P is a special case of the Riesz projection $P : L^p \to H^p$ which is continuous whenever $p \in (1, \infty)$. Since the multiplication operator with symbol u is bounded for such p, the map $f \mapsto P(uf)$ defines a bounded linear operator (also denoted T_u) on H^p for each $p \in (1, \infty)$. These operators will be referred to as Toeplitz operators on H^p and the kernel of T_u as an operator on H^p will be denoted $\ker_p T_u$. We show that the nontrivial kernels of Toeplitz operators $\ker_p T_u$ for $p \in (1, \infty)$ coincide with the spaces $\mathcal{N}^p_\Omega$ where Ω is an outer function in H^p. We require the following result of Hayashi [8].

THEOREM (Hayashi). *Let T_u be a Toeplitz operator on H^p for some $p \in (1, \infty)$. If $\ker_p T_u$ is nontrivial, then there exists an outer function Ω in H^p such that $\ker_p T_u = \ker_p T_{\overline{z\Omega}/\Omega}$.*

PROPOSITION 6.1. *If $p \in (1, \infty)$ and Ω in H^p is outer, then $\ker_p T_{\overline{z\Omega}/\Omega} = \mathcal{N}^p_\Omega$.*

Proof. If $f = I_f F$ belongs to $\ker_p T_{\overline{z\Omega}/\Omega}$, then $I_f F \overline{z\Omega}/\Omega = \overline{zIF}$ for some inner function I. Hence $IF = \overline{I_f F}\, \Omega/\overline{\Omega}$ which implies that f is in $\mathcal{N}^p_\Omega$. If f belongs to $\mathcal{N}^p_\Omega$, then $f = h\Omega$ for some h in C^+. Since $P((\overline{z\Omega}/\Omega)\, f) = P(\overline{z(\overline{h}\Omega)})$ and $\overline{h}\Omega$ is the boundary function for the H^p function $\widehat{f}$, we see that $T_{\overline{z\Omega}/\Omega} f = 0$. ∎

If Ω is an outer function in H^p, then Bourgain's Factorization Theorem [2] provides Blaschke products φ, b and an invertible outer function G in H^∞ such that

$$\Omega/\overline{\Omega} = \frac{\varphi G}{b\overline{G}}.$$

Our final proposition relates $\mathcal{N}^p_\Omega$ spaces, B-invariant subspaces, and the kernels of Toeplitz operators for $p \in (1, \infty)$. The equality of (4.1) and (5.1) is from [6].

PROPOSITION 6.2. *Let $p \in (1, \infty)$ and let Ω be an outer function in H^p. If $\Omega/\overline{\Omega} = \frac{\varphi G}{b\overline{G}}$ where φ, b are Blaschke products and G is an invertible H^∞ function, then the following subspaces of H^p are identical:*

(i) $\mathcal{N}^p_\Omega$;

(ii) $\ker T_{\overline{z\Omega}/\Omega}$;

(iii) $\frac{G}{b}(bH^p \cap \varphi\overline{H^p})$;
(iv) $\{f \in G(H^p \cap \varphi\overline{H^p}) : bI_f | \mathcal{I}_{F/G}\}$.

The notation $\mathcal{I}_{F/G}$ *in* (iv) *refers to the associated inner function for the outer function* F/G *in* $H^p \cap \varphi\overline{H^p}$.

Proof. The equality of (i) and (ii) follows from Proposition 6.1.

If $f = I_f F$ belongs to $\mathcal{N}^p_\Omega$, then

$$\widehat{f} = I_{\widehat{f}} F = \overline{I_f F}\,\Omega/\overline{\Omega} = \overline{I_f F}\,\frac{\varphi G}{b\overline{G}}.$$

Hence $bI_f I_{\widehat{f}}(F/G) = \overline{(F/G)}\varphi$ and the outer function F/G belongs to $H^p \cap \varphi\overline{H^p}$ and has the associated inner function $\mathcal{I}_{F/G} = bI_f I_{\widehat{f}}$. Hence (i) yields (iv).

If f belongs to $G(H^p \cap \varphi\overline{H^p})$ and $bI_f|\mathcal{I}_{F/G}$, then $bI_f J(F/G) = \overline{(F/G)}\varphi$ for some inner function J. This is equivalent to

$$I_f J F = \overline{F}\Big(\frac{\varphi G}{b\overline{G}}\Big) = \overline{F}\,\Omega/\overline{\Omega}.$$

Hence f belongs to $\mathcal{N}^p_\Omega$ (with conjugate JF) and (iv) yields (i).

If $f = I_f F$ belongs to $\frac{G}{b}(bH^p \cap \varphi\overline{H^p})$, then $f = \frac{G}{b}(bI\frac{F}{G})$ for some inner function I. It follows that $I_f = I$ and $bI_f\frac{F}{G}$ belongs to $H^p \cap \varphi\overline{H^p}$. Therefore $bI_f|\mathcal{I}_{F/G}$ and hence (iii) gives (iv).

If f belongs to $G(H^p \cap \varphi\overline{H^p})$ and $bI_f|\mathcal{I}_{F/G}$, then $bI_f\frac{F}{G}$ is in $bH^p \cap \varphi\overline{H^p}$. Hence $f = \frac{G}{b}(bI_f\frac{F}{G})$ belongs to $\frac{G}{b}(bH^p \cap \varphi\overline{H^p})$ which proves that (iv) yields (iii). ∎

REFERENCES

[1] D.Z. AROV, Darlington's method in the study of dissipative systems, *Dokl. Akad. Nauk SSSR* **201**(1971), 559–562.

[2] J. BOURGAIN, A problem of Douglas and Rudin on factorization, *Pacific J. Math.* **121**(1986), 47–50.

[3] J.A. CIMA, W.T. ROSS, *The Backward Shift on the Hardy Space*, Amer. Math. Soc., Providence RI 2000.

[4] R.G. DOUGLAS, H.S. SHAPIRO, A.L. SHIELDS, Cyclic vectors and invariant subspaces for the backward shift operator, *Ann. Inst. Fourier (Grenoble)* **20**(1970), 37–76.

[5] R.G. DOUGLAS, J.W. HELTON, Inner dilations of analytic matrix functions and Darlington synthesis, *Acta Sci. Math (Szeged)* **34**(1973), 61–67.

[6] K.M. DYAKONOV, Kernels of Toeplitz operators via Bourgain's factorization theorem, *J. Funct. Anal.* **170**(2000), 93–106.

[7] S.R. GARCIA, D. SARASON, Real outer functions, *Indiana Univ. Math. J.* **52**(2003), 1397–1412.

[8] E. HAYASHI, The solution sets of extremal problems in H^1, *Proc. Amer. Math. Soc.* **93**(1985), 690–696.

[9] H. HELSON, Large analytic functions. II, in *Analysis and Partial Differential Equations*, Lecture Notes in Pure and Appl. Math., vol. 122, Marcel Dekker, New York 1990, pp. 217–220.

[10] W.T. ROSS, H.S. SHAPIRO, *Generalized Analytic Continuation*, Univ. Lecture Ser., vol. 25, Amer. Math. Soc., Providence RI 2002.

STEPHAN RAMON GARCIA, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT SANTA BARBARA, SANTA BARBARA, CALIFORNIA, 93106–3080, USA
E-mail address: garcias@math.ucsb.edu

Received November 5, 2003; revised February 23, 2004.