J. OPERATOR THEORY
56:1(2006), 3–15


# NORM ESTIMATIONS FOR FINITE SUMS OF POSITIVE OPERATORS

DAN POPOVICI and ZOLTÁN SEBESTYÉN

*Communicated by Nikolai K. Nikolski*

ABSTRACT. We propose some norm estimations for sums of positive operators on Hilbert spaces, extending the ones given by Davidson-Power and Kittaneh for two operators. Such inequalities are useful in the theory of best approximations in $C^*$-algebras, complex interpolation, the theory of generalized inverses and operator approximation. We prove that the equality case in generalized triangle inequalities is obtained when equality holds in the corresponding Cauchy-Schwarz type inequalities, extending a recent result of Kittaneh. Certain applications concerning orthogonal projections or operators having orthogonal ranges are given.

KEYWORDS: *Norm estimation, operator matrix, triangle inequality, positive operator.*

MSC (2000): 47A30, 47B65.

## 1. INTRODUCTION

If $\mathscr{J}$ is a (closed, two-sided) ideal of a $C^*$-algebra $\mathscr{A}$ and $a \in \mathscr{A} \setminus \mathscr{J}$ is the $\mathscr{J}$-strict limit of a bounded net $\{j_\alpha\} \subset \mathscr{J}$, Davidson and Power [1] proved that the norm of $a + \mathscr{J}$ can be attained by an element in the closed convex hull of $\{j_\alpha\}$. One of the main ingredients was the following norm inequality for positive operators $A$ and $B$ on a Hilbert space $\mathscr{H}$:

$$\|A+B\| \leqslant \max\{\|A\|, \|B\|\} + \|AB\|^{1/2}, \tag{1.1}$$

where $\|\cdot\|$ denotes the usual operator norm. Kittaneh [5] proved that $\|AB\|^{1/2}$ can be replaced in (1.1) by $\|A^{1/2}B^{1/2}\|$:

$$\|A+B\| \leqslant \max\{\|A\|, \|B\|\} + \|A^{1/2}B^{1/2}\|. \tag{1.2}$$

However, another inequality, which is sharper than both (1.2) and the triangle inequality, have been recently discovered by the same author [6]:

$$\|A+B\| \leqslant \frac{1}{2}\Big(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\|A^{1/2}B^{1/2}\|^2}\Big). \tag{1.3}$$

Kittaneh observed in [6] that a similar inequality holds true if our operators $A$ and $B$ have orthogonal ranges (instead of being positive):

$$\|A+B\|^2 \leqslant \frac{1}{2}\Big(\|A\|^2+\|B\|^2+\sqrt{(\|A\|^2-\|B\|^2)^2+4\|AB^*\|^2}\Big). \tag{1.4}$$

It is our aim in Section 2 to extend the Kittaneh's result (1.3) for arbitrarily finite sums of positive operators (Theorem 2.1, Corollaries 2.6 and 2.9). More precisely, if $A_1, A_2, \ldots, A_n$ are positive operators on $\mathscr{H}$ we prove that

$$\Big\|\sum_{k=1}^{n} A_k\Big\| \leqslant \|(\|A_i^{1/2}A_j^{1/2}\|)_{1\leqslant i,j\leqslant n}\|, \tag{1.5}$$

inequality which is sharper than the corresponding triangle inequality. In particular, we deduce that

$$\Big\|\sum_{k=1}^{n} A_k\Big\| \leqslant \max_{j=1}^{n}\sum_{i=1}^{n}\|A_i^{1/2}A_j^{1/2}\|, \tag{1.6}$$

which extends (1.1) and (1.2).

As applications of these norm estimations we provide some sufficient conditions on operator invertibility and show how our inequalities for sums of positive operators give rise to inequalities for sums of operators having orthogonal ranges. More exactly, if $A_1, A_2, \ldots, A_n$ have pairwise orthogonal ranges we prove that

$$\Big\|\sum_{k=1}^{n} A_k\Big\|^2 \leqslant \|(\|A_iA_j^*\|)_{1\leqslant i,j\leqslant n}\|, \tag{1.7}$$

inequality which is an extension of (1.4). The given proofs are simpler than the original ones (for two operators) obtained in [1], [5] and [6].

If $A$ and $B$ are positive operators on $\mathscr{H}$ then equality holds in the associated triangle inequality (i.e. $\|A+B\| = \|A\|+\|B\|$) if and only if the equality case holds in a Cauchy-Schwarz type inequality (i.e. $\|AB\| = \|A\|\|B\|$; cf. Propositions 3.3 and 3.4 of [6]). We prove that, more generally, for positive operators $A_1, A_2, \ldots, A_n$ on $\mathscr{H}$ the equality

$$\Big\|\sum_{k=1}^{n} A_k\Big\| = \sum_{k=1}^{n}\|A_k\| \tag{1.8}$$

can be still expressed in terms of the product $A_1A_2\cdots A_n$ norm:

$$\Big\|\prod_{k=1}^{n} A_k\Big\| = \prod_{k=1}^{n}\|A_k\|.$$

In addition, we show that (1.8) is equivalent to

$$\Big\|\sum_{k=1}^{n} a_kA_k^{r_k}\Big\| = \sum_{k=1}^{n} a_k\|A_k\|^{r_k},$$

for certain (and also for any) $a_k, r_k > 0$ $(k = 1, 2, \ldots, n)$. Further applications and extensions conclude the paper.

As mentioned before such norm inequalities are important tools in the theory of best approximation in $C^*$-algebras [1], in complex interpolation ([11], Chapter 2) or in the theory of generalized inverses and operator approximation [7], [8], [9]. It is expected that our generalizations and extensions would provide a larger set of applications.

## 2. A GENERALIZED DAVIDSON-POWER-KITTANEH INEQUALITY

Let $\mathscr{H}_k$, $k = 1, 2, \ldots, n$ be complex Hilbert spaces. Any linear and bounded operator $T$ on $\bigoplus_{k=1}^{n} \mathscr{H}_k$ can be represented as a $n \times n$ operator matrix, namely $T = (T_{ij})_{1 \leqslant i,j \leqslant n}$ with $T_{ij} \in \mathcal{B}(\mathscr{H}_j, \mathscr{H}_i)$ (the Banach space of all bounded linear operators from $\mathscr{H}_j$ into $\mathscr{H}_i$ with the norm topology). In their study on norm-inequalities for operator matrices, Hou and Du showed in [3] that, for such an operator matrix,

$$\|T\| \leqslant \|(\|T_{ij}\|)_{1 \leqslant i,j \leqslant n}\|. \tag{2.1}$$

This is the main ingredient in proving a general norm inequality for sums of positive operators which is sharper than the triangle inequality and extends (1.1), (1.2) and (1.3). We want to remark that our proofs are even simpler than the ones given, for two operators, in [1], [5] and [6].

THEOREM 2.1. *Let $T_k \in \mathcal{B}(\mathscr{H}_k, \mathscr{H})$, $k = 1, 2, \ldots, n$ be bounded linear Hilbert space operators. Then*

$$\Big\| \sum_{k=1}^{n} T_k T_k^* \Big\| \leqslant \|(\|T_i^* T_j\|)_{1 \leqslant i,j \leqslant n}\|. \tag{2.2}$$

*In particular, for $n = 2$, we have*

$$\|T_1 T_1^* + T_2 T_2^*\| \leqslant \frac{1}{2}\Big( \|T_1\|^2 + \|T_2\|^2 + \sqrt{(\|T_1\|^2 - \|T_2\|^2)^2 + 4\|T_1^* T_2\|^2} \Big). \tag{2.3}$$

*Proof.* Let $T = (T_1, T_2, \ldots, T_n) \in \mathcal{B}\Big( \bigoplus_{k=1}^{n} \mathscr{H}_k, \mathscr{H} \Big)$. Then

$$\Big\| \sum_{k=1}^{n} T_k T_k^* \Big\| = \|TT^*\| = \|T^*T\| = \|(T_i^* T_j)_{1 \leqslant i,j \leqslant n}\|.$$

In addition, by (2.1),

$$\|(T_i^* T_j)_{1 \leqslant i,j \leqslant n}\| \leqslant \|(\|T_i^* T_j\|)_{1 \leqslant i,j \leqslant n}\|,$$

which completes the proof of (2.2).

The matrix in the right-hand side of (2.2) is hermitian. Its norm equals its spectral radius which, for $n = 2$, is

$$\frac{1}{2}\Big( \|T_1\|^2 + \|T_2\|^2 + \sqrt{(\|T_1\|^2 - \|T_2\|^2)^2 + 4\|T_1^* T_2\|^2} \Big). \qquad \blacksquare$$

REMARK 2.2. (i) As mentioned before, (2.2) is sharper than the triangle inequality. To see this note that, for any unit vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$,

$$\begin{aligned}\|(\|T_i^* T_j\|)_{1\leqslant i,j\leqslant n}\alpha^T\|^2 &= \sum_{i=1}^n \Big|\sum_{j=1}^n \|T_i^*T_j\|\alpha_j\Big|^2 \leqslant \sum_{i=1}^n \Big(\sum_{j=1}^n \|T_i\|\|T_j\||\alpha_j|\Big)^2 \\ &= \|(\|T_i\|\|T_j\|)_{1\leqslant i,j\leqslant n}(|\alpha_1|,\dots,|\alpha_n|)^T\|^2 \\ &\leqslant \|(\|T_i\|\|T_j\|)_{1\leqslant i,j\leqslant n}\|^2.\end{aligned}$$

Consequently,

$$\begin{aligned}\|(\|T_i^*T_j\|)_{1\leqslant i,j\leqslant n}\| \leqslant \|(\|T_i\|\|T_j\|)_{1\leqslant i,j\leqslant n}\| &= \|(\|T_1\|,\dots,\|T_n\|)^*(\|T_1\|,\dots,\|T_n\|)\| \\ &= (\|T_1\|,\dots,\|T_n\|)(\|T_1\|,\dots,\|T_n\|)^* = \sum_{k=1}^n \|T_k\|^2.\end{aligned}$$

(ii) If $\|(\|T_i^*T_j\|)_{1\leqslant i,j\leqslant n}\| < \sum_{k=1}^n \|T_k\|^2$ then

$$\begin{aligned}\Big\langle \sum_{k=1}^n (\|T_k\|^2 - T_kT_k^*)h, h\Big\rangle &= \sum_{k=1}^n \|T_k\|^2 - \Big\langle \Big(\sum_{k=1}^n T_kT_k^*\Big)h, h\Big\rangle \\ &\geqslant \sum_{k=1}^n \|T_k\|^2 - \Big\|\sum_{k=1}^n T_kT_k^*\Big\| \\ &\geqslant \sum_{k=1}^n \|T_k\|^2 - \|(\|T_i^*T_j\|)_{1\leqslant i,j\leqslant n}\| > 0,\end{aligned}$$

for every unit vector $h \in \mathscr{H}$. We deduce that $\sum_{k=1}^n (\|T_k\|^2 - T_kT_k^*)$ is invertible.

In particular, if $P_1, \dots, P_n$ are orthogonal projections on a Hilbert space and $\|(\|P_iP_j\|)_{1\leqslant i,j\leqslant n}\| < n$ then $\sum_{k=1}^n (1 - P_k)$ is invertible. ∎

The right-hand side of (2.2) is usually difficult to compute (for large $n$'s). The following estimation could be, in this sense, more useful by the applications point of view:

COROLLARY 2.3. *Let $T_k \in \mathcal{B}(\mathscr{H}_k, \mathscr{H})$, $k = 1, 2, \dots, n$ be bounded linear Hilbert space operators. Then*

$$\Big\|\sum_{k=1}^n T_kT_k^*\Big\| \leqslant \max_{j=1}^n \sum_{i=1}^n \|T_i^*T_j\|. \tag{2.4}$$

*Proof.* The operator norm of the hermitian matrix $(T_i^*T_j)_{1\leqslant i,j\leqslant n}$ is majorized by any of its complete algebra norms on $\mathcal{B}(\mathbb{C}^n)$ (being spectral norms), in particular by its $\mathcal{B}(\ell^\infty)$ norm, which is

$$\max_{j=1}^n \sum_{i=1}^n \|T_i^*T_j\|. \quad ∎$$

REMARK 2.4. Let $A_k$, $k = 1, 2, \dots, n$ be positive operators on a Hilbert space. If $T_k = A_k^{1/2}$, $k = 1, 2, \dots, n$, then (2.2) becomes

$$\Big\| \sum_{k=1}^{n} A_k \Big\| \leqslant \| (\| A_i^{1/2} A_j^{1/2} \|)_{1 \leqslant i,j \leqslant n} \|,$$

which is a generalization of (1.3) for arbitrarily finite sums of positive operators. In addition, by (2.4), we obtain the inequality

$$\Big\| \sum_{k=1}^{n} A_k \Big\| \leqslant \max_{j=1}^{n} \sum_{i=1}^{n} \| A_i^{1/2} A_j^{1/2} \|,$$

which extends (1.1) and (1.2). ∎

The following two results, obtained as consequences of Theorem 2.1 and/or Corollary 2.3 are, in a certain sense, extensions of (2.2) and/or (2.4):

COROLLARY 2.5. *Let $T_k \in \mathcal{B}(\mathscr{H}_k, \mathscr{H})$, $k = 1, 2, \dots, n$ be bounded linear operators. Then*

$$\Big\| \sum_{k=1}^{n} T_k T_k^* \Big\| \leqslant \max_{k=1}^{m} \sum_{l=1}^{m} \Big\| \Big( \sum_{i \in P_k} T_i T_i^* \Big)^{1/2} \Big( \sum_{j \in P_l} T_j T_j^* \Big)^{1/2} \Big\|,$$

*for any partition $\mathcal{P} = \{P_1, P_2, \dots, P_m\}$ of $\{1, 2, \dots, n\}$.*

*Proof.* We apply Corollary 2.3 for operators $S_1, S_2, \dots, S_m$ defined by

$$S_k = (T_i)_{i \in P_k} : \bigoplus_{i \in P_k} \mathscr{H}_i \to \mathscr{H}, \quad k = 1, 2, \dots, m.$$

We just have to observe that $S_k S_k^* = \sum_{i \in P_k} T_i T_i^*$ and

$$\begin{aligned} \| S_l^* S_k \|^2 = \| S_l^* S_k S_k^* S_l \| &= \Big\| S_l^* \Big( \sum_{i \in P_k} T_i T_i^* \Big) S_l \Big\| \\ &= \Big\| \Big( \sum_{i \in P_k} T_i T_i^* \Big)^{1/2} S_l S_l^* \Big( \sum_{i \in P_k} T_i T_i^* \Big)^{1/2} \Big\| \\ &= \Big\| \Big( \sum_{i \in P_k} T_i T_i^* \Big)^{1/2} \Big( \sum_{j \in P_l} T_j T_j^* \Big)^{1/2} \Big\|^2, \end{aligned}$$

for $k, l = 1, 2, \dots, m$. ∎

COROLLARY 2.6. *Let $T_k \in \mathcal{B}(\mathscr{H}_k, \mathscr{H})$ and $S_k \in \mathcal{B}(\mathscr{H}, \mathscr{H}_k)$, $k = 1, 2, \dots, n$ be bounded linear Hilbert space operators. Then*

$$\Big\| \sum_{k=1}^{n} T_k S_k \Big\|^2 \leqslant \| (\| T_i^* T_j \|)_{1 \leqslant i,j \leqslant n} \|^{1/2} \| (\| S_i S_j^* \|)_{1 \leqslant i,j \leqslant n} \|^{1/2}.$$

*Proof.* Let $T = (T_1, T_2, \dots, T_n)$ and $S = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{pmatrix}$. Then $TS = \sum_{k=1}^{n} T_k S_k$ and

$$\begin{aligned} \|TS\|^2 \leqslant \|T\|^2 \|S\|^2 &= \Big\| \sum_{k=1}^{n} T_k T_k^* \Big\| \Big\| \sum_{k=1}^{n} S_k^* S_k \Big\| \\ &\leqslant \|(\|T_i^* T_j\|)_{1 \leqslant i,j \leqslant n}\|^{1/2} \|(\|S_i S_j^*\|)_{1 \leqslant i,j \leqslant n}\|^{1/2}. \quad \blacksquare \end{aligned}$$

Let $P_k$, $k = 1, 2, \dots, n$ be orthogonal projections on a Hilbert space. Then, by Theorem 2.1 and Corollary 2.3,

$$\Big\| \sum_{k=1}^{n} P_k \Big\| \leqslant \|(\|P_i P_j\|)_{1 \leqslant i,j \leqslant n}\| \leqslant \max_{j=1}^{n} \sum_{i=1}^{n} \|P_i P_j\|. \tag{2.5}$$

If $P_1$ and $P_2$ are orthogonal projections (not both null) on a Hilbert space then $\|P_1 + P_2\| = 1 + \|P_1 P_2\|$ ([2], [12]). We can observe that, in this case, equality holds in the inequalities (2.5). This need not be the case for $n \geqslant 3$ as stated by the following example:

EXAMPLE 2.7. Let $P \neq 0, 1$ be an orthogonal projection on a Hilbert space, $P_1 = 1$, $P_2 = P$ and $P_3 = 1 - P$. Then $\|P_1 + P_2 + P_3\| = 2$, while

$$\|(\|P_i P_j\|)_{1 \leqslant i,j \leqslant n}\| = \left\| \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right\| = 3. \quad \blacksquare$$

We shall give some applications of the main theorem concerning operators having orthogonal ranges. Such operators and their corresponding norm estimations are usually encountered when minimizing norms of linear polynomials in terms of generalized inverses [7], [8], [9].

If operators $T_k$, $k = 1, 2, \dots, n$ have orthogonal ranges ($\operatorname{ran} T_i \perp \operatorname{ran} T_j$ for $i \neq j$) then it is obvious that

$$\max_{j=1}^{n} \sum_{i=1}^{n} \|T_i^* T_j\| = \max_{j=1}^{n} \|T_j\|^2 = \Big\| \sum_{k=1}^{n} T_k T_k^* \Big\|.$$

Therefore the equality case holds in the inequalities (2.2) and (2.4). On the other hand, since

$$\Big( \sum_{k=1}^{n} T_k \Big)^* \Big( \sum_{k=1}^{n} T_k \Big) = \sum_{k=1}^{n} T_k^* T_k,$$

one can apply Theorem 2.1 to operators $T_k^*$, $k = 1, 2, \dots, n$ to obtain the following extension of the Kittaneh's result (1.4):

COROLLARY 2.8. *Let* $T_k$, $k = 1, 2, \ldots, n$ *be bounded linear operators on a Hilbert space* $\mathscr{H}$ *with* $\operatorname{ran} T_i \perp \operatorname{ran} T_j$, *for* $i \neq j$. *Then*

$$\Big\| \sum_{k=1}^{n} T_k \Big\|^2 \leqslant \| (\| T_i T_j^* \|)_{1 \leqslant i,j \leqslant n} \|.$$

Employing operator matrix techniques one can further deduce:

COROLLARY 2.9. *Let* $T = \{T_{ij}\}_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} \subset \mathcal{B}(\mathscr{H})$ *be a given* $m \times n$ *operator matrix on a Hilbert space such that* $\operatorname{ran} T_{pj} \perp \operatorname{ran} T_{qj}$, *for* $p, q = 1, 2, \ldots, m$, $p \neq q$ *and* $j = 1, 2, \ldots, n$. *Then*

$$\max_{i=1}^{m} \Big\| \sum_{j=1}^{n} T_{ij}^* T_{ij} \Big\| \leqslant \Big\| \Big( \Big\| \sum_{k=1}^{m} T_{ki} T_{kj}^* \Big\| \Big) \Big\|. \tag{2.6}$$

*Proof.* One can apply Theorem 2.1 for operators

$$S_1 = \begin{pmatrix} T_{11}^* \\ T_{21}^* \\ \vdots \\ T_{m1}^* \end{pmatrix}, \; S_2 = \begin{pmatrix} T_{12}^* \\ T_{22}^* \\ \vdots \\ T_{m2}^* \end{pmatrix}, \ldots, S_n = \begin{pmatrix} T_{1n}^* \\ T_{2n}^* \\ \vdots \\ T_{mn}^* \end{pmatrix}.$$

Observe that, by hypothesis,

$$S_j S_j^* = \begin{pmatrix} T_{1j}^* T_{1j} & 0 & \ldots & 0 \\ 0 & T_{2j}^* T_{2j} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & T_{mj}^* T_{mj} \end{pmatrix}, \quad j = 1, 2, \ldots, n.$$

Then

$$\Big\| \sum_{j=1}^{n} S_j S_j^* \Big\| = \max_{i=1}^{m} \Big\| \sum_{j=1}^{n} T_{ij}^* T_{ij} \Big\|.$$

Finally,

$$S_i^* S_j = \sum_{k=1}^{m} T_{ki} T_{kj}^*, \quad i, j = 1, 2, \ldots, n. \qquad \blacksquare$$

We remark that (2.2) can be reobtained by taking $m = 1$ in (2.6).

## 3. THE TRIANGLE "EQUALITY"

Our next aim is to provide necessary and sufficient conditions on a finite sequence $\{A_1, A_2, \ldots, A_n\}$ of positive operators in order to obtain the equality case (1.8) in a generalized triangle inequality.

The first step was already made:

PROPOSITION 3.1. ([6]) *Let $A$ and $B$ be positive operators on $\mathscr{H}$. The following conditions are equivalent:*

(i) $\|A+B\| = \|A\| + \|B\|$;

(ii) $\|A^{1/2}B^{1/2}\| = \|A\|^{1/2}\|B\|^{1/2}$;

(iii) $\|AB\| = \|A\|\|B\|$.

The following lemma has been an important tool in proving arithmetic-geometric mean type inequalities [4], [10], but also for norm inequalities for sums of positive operators [5], [6]. Though it holds for every unitarily invariant norm, we state it (as only needed here) for the usual operator norm:

LEMMA 3.2. *If $A$ and $B$ are operators on $\mathscr{H}$ such that $AB$ is selfadjoint then*

$$\|AB\| \leqslant \|\mathrm{Re}\,(BA)\|.$$

The main ingredients for our main theorem are gathered in two lemmas involving triples of positive operators.

The proof of the first one is similar to that in Proposition 3.3 of [6]:

LEMMA 3.3. *Let $A, B, C$ be positive operators on $\mathscr{H}$. Then if $\|A^{1/2}CB^{1/2}\| = \|C^{1/2}A^{1/2}\|\|C^{1/2}B^{1/2}\|$ we have*

$$\|C^{1/2}(A+B)^{1/2}\|^2 = \|C^{1/2}A^{1/2}\|^2 + \|C^{1/2}B^{1/2}\|^2. \tag{3.1}$$

*Proof.* Take $x_n, y_n$ $(n \geqslant 0)$ unit vectors in $\mathscr{H}$ such that

$$\langle A^{1/2}CB^{1/2}y_n, x_n\rangle \to \|A^{1/2}CB^{1/2}\|.$$

Then, by passing to limit in the following set of inequalities,

$$\begin{aligned}|\langle C^{1/2}B^{1/2}y_n, C^{1/2}A^{1/2}x_n\rangle| &\leqslant \|C^{1/2}A^{1/2}x_n\|\|C^{1/2}B^{1/2}\| \\ &\leqslant \|C^{1/2}A^{1/2}\|\|C^{1/2}B^{1/2}\|,\end{aligned}$$

we obtain that $\|C^{1/2}A^{1/2}x_n\| \to \|C^{1/2}A^{1/2}\|$ and, similarly, $\|C^{1/2}B^{1/2}y_n\| \to \|C^{1/2}B^{1/2}\|$.

Moreover,

$$\begin{aligned}\|C^{1/2}(A+B)^{1/2}\|^2 &= \|C^{1/2}(A+B)C^{1/2}\| \\ &= \left\|\begin{pmatrix} C^{1/2}A^{1/2} & C^{1/2}B^{1/2} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A^{1/2}C^{1/2} & 0 \\ B^{1/2}C^{1/2} & 0 \end{pmatrix}\right\| \\ &= \left\|\begin{pmatrix} A^{1/2}C^{1/2} & 0 \\ B^{1/2}C^{1/2} & 0 \end{pmatrix}\begin{pmatrix} C^{1/2}A^{1/2} & C^{1/2}B^{1/2} \\ 0 & 0 \end{pmatrix}\right\| \\ &= \left\|\begin{pmatrix} A^{1/2}CA^{1/2} & A^{1/2}CB^{1/2} \\ B^{1/2}CA^{1/2} & B^{1/2}CB^{1/2} \end{pmatrix}\right\|.\end{aligned}$$

We deduce that

$$\left|\left\langle \begin{pmatrix} A^{1/2}CA^{1/2} & A^{1/2}CB^{1/2} \\ B^{1/2}CA^{1/2} & B^{1/2}CB^{1/2} \end{pmatrix}\begin{pmatrix} ax_n \\ by_n \end{pmatrix}, \begin{pmatrix} ax_n \\ by_n \end{pmatrix}\right\rangle\right| \leqslant \|C^{1/2}(A+B)^{1/2}\|^2, \tag{3.2}$$

for $a = \frac{\|C^{1/2}A^{1/2}\|}{(\|C^{1/2}A^{1/2}\|^2+\|C^{1/2}B^{1/2}\|^2)^{1/2}}$ and $b = \frac{\|C^{1/2}B^{1/2}\|}{(\|C^{1/2}A^{1/2}\|^2+\|C^{1/2}B^{1/2}\|^2)^{1/2}}$. Expanding and letting $n \to \infty$ in (3.2) we obtain that

$$\|C^{1/2}(A+B)^{1/2}\|^2 \geqslant \|C^{1/2}A^{1/2}\|^2 + \|C^{1/2}B^{1/2}\|^2.$$

The converse inequality is obvious. Therefore, (3.1) holds, as required. ∎

LEMMA 3.4. *Let $A, B, C$ be positive operators on $\mathscr{H}$. If $\|(A+B)C\| = (\|A\| + \|B\|)\|C\|$ then $\|ACB\| = \|A\|\|C\|\|B\|$.*

*Proof.* Observe that, in our hypothesis,

$$(\|A\| + \|B\|)\|C\| = \|(A+B)C\| \leqslant \|(A+B)C^{1/2}\|\|C\|^{1/2} \leqslant (\|A\| + \|B\|)\|C\|.$$

Hence $\|(A+B)C^{1/2}\| = (\|A\| + \|B\|)\|C\|^{1/2}$. A similar argument shows that $\|(A+B)^{1/2}C^{1/2}\| = (\|A\| + \|B\|)^{1/2}\|C\|^{1/2}$ also holds. Consequently,

$$\begin{aligned}(\|A\| + \|B\|)\|C\| &= \|C^{1/2}(A+B)C^{1/2}\| \leqslant \|C^{1/2}AC^{1/2}\| + \|C^{1/2}BC^{1/2}\| \\ &\leqslant \|C^{1/2}AC^{1/2}\| + \|B\|\|C\| \leqslant (\|A\| + \|B\|)\|C\|.\end{aligned}$$

This forces (3.1), $\|C^{1/2}AC^{1/2}\| = \|C\|\|A\|$ and, by symmetry, $\|C^{1/2}BC^{1/2}\| = \|C\|\|B\|$.

Since $C^{1/2}AC^{1/2}$ and $C^{1/2}BC^{1/2}$ are positive, we deduce that

$$\|C^{1/2}ACBC^{1/2}\| = \|C^{1/2}AC^{1/2}\|\|C^{1/2}BC^{1/2}\| = \|A\|\|C\|^2\|B\|$$

(by Proposition 3.1 (i)⇒(iii)). Moreover,

$$\|A\|\|C\|^2\|B\| = \|C^{1/2}ACBC^{1/2}\| \leqslant \|C\|^{1/2}\|ACB\|\|C\|^{1/2} \leqslant \|A\|\|C\|^2\|B\|$$

imply that $\|ACB\| = \|A\|\|C\|\|B\|$. The proof is complete. ∎

REMARK 3.5. If $A, B, C$ are as above then

$$\|A+B+C\| \leqslant \|A+B\| + \|C\| \leqslant \|A\| + \|B\| + \|C\|.$$

We obtain that

$$\|A+B+C\| = \|A\| + \|B\| + \|C\| \tag{3.3}$$

if and only if $\|A+B\| = \|A\| + \|B\|$ (or $\|AB\| = \|A\|\|B\|$) and $\|A+B+C\| = \|A+B\| + \|C\|$ (or $\|(A+B)C\| = \|A+B\|\|C\|$). Consequently, (3.3) can be rewritten in equivalent form as

$$\|(A+B)C\| = (\|A\| + \|B\|)\|C\|. \tag{3.4}$$ ∎

COROLLARY 3.6. *Let $A, B, C$ be positive operators on $\mathscr{H}$. The following conditions are equivalent:*

(i) $\|A+B+C\| = \|A\| + \|B\| + \|C\|$;
(ii) $\|ABC\| = \|A\|\|B\|\|C\|$;
(iii) $\|BCA\| = \|B\|\|C\|\|A\|$;
(iv) $\|CAB\| = \|C\|\|A\|\|B\|$.

*Proof.* If (i) holds then, by (3.4) and Lemma 3.4, we obtain (iii).

Conversely, by (iii), the following inequalities

$$\|B\|\|C\|\|A\| = \|BCA\| \leqslant \|B\|^{1/2}\|B^{1/2}CA^{1/2}\|\|A\|^{1/2} \leqslant \|B\|^{1/2}\|B^{1/2}C^{1/2}\|\|C^{1/2}A^{1/2}\|\|A\|^{1/2} \leqslant \|B\|\|C\|\|A\|$$

allow that $\|B^{1/2}CA^{1/2}\| = \|B^{1/2}A^{1/2}\|\|A^{1/2}C^{1/2}\|$, $\|B^{1/2}C^{1/2}\| = \|B\|^{1/2}\|C\|^{1/2}$ and $\|A^{1/2}C^{1/2}\| = \|A\|^{1/2}\|C\|^{1/2}$. We use Lemma 3.3 to prove that

$$\|C^{1/2}(A+B)C^{1/2}\| = (\|A\| + \|B\|)\|C\|.$$

Moreover, by Lemma 3.2,

$$\|(A+B)C\| \geqslant \|C^{1/2}(A+B)C^{1/2}\| = (\|A\| + \|B\|)\|C\|.$$

The converse inequality being obvious we obtain (3.4). Hence, by Remark 3.5, we get the conclusion. ∎

We are now in position to prove our main result in this section:

THEOREM 3.7. *Let $A_k$, $k = 1, 2, \dots, n$ be positive operators on $\mathscr{H}$. The following conditions are equivalent:*

(i) $\left\| \sum_{k=1}^{n} A_k \right\| = \sum_{k=1}^{n} \|A_k\|$;

(ii) $\left\| \prod_{k=1}^{n} A_{\sigma(k)} \right\| = \prod_{k=1}^{n} \|A_{\sigma(k)}\|$, *for a certain $\sigma \in S_n$;*

(iii) $\left\| \prod_{k=1}^{n} A_{\sigma(k)} \right\| = \prod_{k=1}^{n} \|A_{\sigma(k)}\|$, *for every $\sigma \in S_n$.*

*Proof.* We proceed by induction. Suppose that, for any given positive integer $m$ and any positive operators $A_k$, $k = 1, 2, \dots, m$, on $\mathscr{H}$, $\left\| \sum_{k=1}^{m} A_k \right\| = \sum_{k=1}^{m} \|A_k\|$ if and only if $\left\| \prod_{k=1}^{m} A_k \right\| = \prod_{k=1}^{m} \|A_k\|$. By a similar argument with the one used in Remark 3.5 one can easily deduce that, for any given positive operator $B$ on $\mathscr{H}$,

$$\left\| \sum_{k=1}^{m} A_k + B \right\| = \sum_{k=1}^{m} \|A_k\| + \|B\|$$

if and only if

$$\left\| \Big( \sum_{k=1}^{m} A_k \Big) B \right\| = \Big( \sum_{k=1}^{m} \|A_k\| \Big) \|B\|. \tag{3.5}$$

Let $C = \sum_{k=1}^{m-1} A_k$. Then (3.5) can be rewritten in an equivalent form:

$$\|(C + A_m)B\| = (\|C\| + \|A_m\|)\|B\| \text{ and } \|C\| = \sum_{k=1}^{m-1} \|A_k\|. \tag{3.6}$$

By Remark 3.5 and Corollary 3.6, (3.6) becomes

$$\|CA_mB\| = \|C\|\|A_m\|\|B\| = \Big(\sum_{k=1}^{m-1} \|A_k\|\Big)\|A_m\|\|B\|.$$

Since $\|CA_mB\| = \|C|BA_m|\|$ and $\|A_mB\| = \||BA_m|\|$ we obtain

$$\|C|BA_m|\| = \Big(\sum_{k=1}^{m-1} \|A_k\|\Big)\||BA_m|\| = \Big(\sum_{k=1}^{m-1} \|A_k\|\Big)\|A_m\|\|B\|.$$

Equivalently, by the induction hypothesis,

$$\begin{aligned}\|A_1A_2\cdots A_mB\| &= \|A_1A_2\cdots A_{m-1}|BA_m|\| = \|A_1\|\|A_2\|\cdots\|A_{m-1}\|\||BA_m|\| \\ &= \|A_1\|\|A_2\|\cdots\|A_m\|\|B\|.\end{aligned}$$

The proof is now complete. ∎

To obtain a generalized version of Proposition 3.1 (ii)⇔(iii) (cf. Proposition 3.4 of [6]), we need the following result:

PROPOSITION 3.8. *Let $A_k$, $k = 1,2,\dots,n$ be positive operators on $\mathscr{H}$. If $\Big\|\prod_{k=1}^{n} A_k\Big\| = \prod_{k=1}^{n} \|A_k\|$ then $\Big\|\prod_{k=1}^{n} A_{\sigma(k)}^{r_k}\Big\| = \prod_{k=1}^{n} \|A_{\sigma(k)}\|^{r_k}$, for any $r_1, r_2, \dots, r_n \geqslant 0$ and $\sigma \in S_n$.*

*Proof.* Since, in our hypothesis, $\Big\|\prod_{k=1}^{n} A_k\Big\| = \Big\|\prod_{k=1}^{n} A_{\sigma(k)}\Big\|$, for any $\sigma \in S_n$, it suffices to consider the case $r_1 = r_2 = \dots = r_{n-1} = 1$ and $r_n > 0$.

*Step 1.* If $r_n \leqslant 1$ then the following inequalities

$$\begin{aligned}\|A_1\|\cdots\|A_n\| &= \|A_1\cdots A_n^{r_n}A_n^{1-r_n}\| \\ &\leqslant \|A_1\cdots A_n^{r_n}\|\|A_n\|^{1-r_n} \leqslant \|A_1\|\cdots\|A_n\|^{r_n}\|A_n\|^{1-r_n}\end{aligned}$$

show that

$$\|A_1\cdots A_{n-1}A_n^{r_n}\| = \|A_1\|\cdots\|A_{n-1}\|\|A_n\|^{r_n}. \tag{3.7}$$

*Step 2.* Next suppose that, for a given $r > 0$, (3.7) holds when $r_n$ is replaced by $r$. Then

$$\begin{aligned}(\|A_1\|\cdots\|A_{n-1}\|\|A_n\|^r)^2 = \|A_1\cdots A_{n-1}A_n^r\|^2 &= \|A_1\cdots A_{n-1}A_n^{2r}A_{n-1}\cdots A_1\| \\ &\leqslant \|A_1\cdots A_{n-1}A_n^{2r}\|\|A_{n-1}\|\cdots\|A_1\| \\ &\leqslant (\|A_1\|\cdots\|A_{n-1}\|\|A_n\|^r)^2,\end{aligned}$$

that is $\|A_1\cdots A_{n-1}A_n^{2r}\| = \|A_1\|\cdots\|A_{n-1}\|\|A_n\|^{2r}$.

*Step 3.* If $r_n > 1$ then we can choose a positive integer $m$ with $\frac{r_n}{2^m} < 1$. By Step 1, (3.7) holds when $r_n$ is replaced by $\frac{r_n}{2^m}$. We just have to apply $m$ times Step 2 to finally obtain (3.7). ∎

Our promised generalized version of Proposition 3.4 in [6] follows:

COROLLARY 3.9. *Let $A_k$, $k = 1,2,\dots,n$ be positive operators on $\mathscr{H}$. The following conditions are equivalent:*

(i) $\left\| \prod_{k=1}^{n} A_k \right\| = \prod_{k=1}^{n} \|A_k\|$;

(ii) $\left\| \prod_{k=1}^{n} A_{\sigma(k)}^{r_k} \right\| = \prod_{k=1}^{n} \|A_{\sigma(k)}\|^{r_k}$, *for certain* $r_1, r_2, \dots, r_n > 0$ *and* $\sigma \in S_n$;

(iii) $\left\| \prod_{k=1}^{n} A_{\sigma(k)}^{r_k} \right\| = \prod_{k=1}^{n} \|A_{\sigma(k)}\|^{r_k}$, *for any* $r_1, r_2, \dots, r_n > 0$ *and* $\sigma \in S_n$.

COROLLARY 3.10. *Let $A_k$, $k = 1,2,\dots,n$ be as above. If* $\left\| \sum_{k=1}^{n} A_k \right\| = \sum_{k=1}^{n} \|A_k\|$ *then* $\left\| \sum_{k=1}^{n} a_k A_k^{r_k} \right\| = \sum_{k=1}^{n} a_k \|A_k\|^{r_k}$, *for any* $a_1, a_2, \dots, a_n, r_1, r_2, \dots, r_n \geqslant 0$.

We observe that, for given positive operators $A_j$, $j = 1,2,\dots,n$,

$$\left\| \sum_{j=1}^{n} A_j \right\| \leqslant \left( \max_{k=1}^{n} \|A_k\| \right)^{1/2} \sum_{l=1}^{n} \|A_l\|^{1/2}. \tag{3.8}$$

It is easy to observe that equality holds in (3.8) if and only if (1.8) holds true and

$$\|A_l\| = \max_{k=1}^{n} \|A_k\|, \tag{3.9}$$

for any $l = 1,2,\dots,n$. Hence

$$\left\| \sum_{j=1}^{n} A_j \right\| = n \Big( \max_{k=1}^{n} \|A_k\| \Big) = \sum_{j=1}^{n} \|A_j\| \tag{3.10}$$

or, equivalently, by Theorem 3.7,

$$\left\| \prod_{j=1}^{n} A_j \right\| = \prod_{j=1}^{n} \|A_j\| = \Big( \max_{k=1}^{n} \|A_k\| \Big)^{n}. \tag{3.11}$$

We can now formulate:

COROLLARY 3.11. *Let $A_j$, $j = 1,2,\dots,n$ be positive operators on $\mathscr{H}$. The following conditions are equivalent:*

(i) $\left\| \sum_{j=1}^{n} A_j \right\| = \left( \max_{k=1}^{n} \|A_k\| \right)^{1/2} \sum_{l=1}^{n} \|A_l\|^{1/2}$;

(ii) $\left\| \sum_{j=1}^{n} A_j \right\| = n \Big( \max_{k=1}^{n} \|A_k\| \Big)$;

(iii) $\left\| \prod_{j=1}^{n} A_j \right\| = \Big( \max_{k=1}^{n} \|A_k\| \Big)^{n}$.

*Acknowledgements.* The work of the first author has been supported by the Hungarian Scholarship Board.

The authors are grateful to the referee for his suggestions.

## REFERENCES

[1] K. DAVIDSON, S.C. POWER, Best approximation in $C^*$-algebras, *J. Reine Angew. Math.* **368**(1986), 43–62.

[2] J. DUNCAN, P. J. TAYLOR, Norm inequalities for $C^*$-algebras, *Proc. Roy. Soc. Edinburgh Sect. A* **75**(1975/76), 119–129.

[3] J.C. HOU, H.K. DU, Norm inequalities of positive operator matrices, *Integral Equations Operator Theory* **22**(1995), 281–294.

[4] F. KITTANEH, A note on the arithmetic-geometric mean inequality for matrices, *Linear Algebra Appl.* **171**(1992), 1–8.

[5] F. KITTANEH, Norm inequalities for certain operator sums, *J. Funct. Anal.* **143**(1997), 337–348.

[6] F. KITTANEH, Norm inequalities for sums of positive operators, *J. Operator Theory* **48**(2002), 95–103.

[7] P.J. MAHER, Some operator inequalities concerning generalized inverses, *Illinois J. Math.* **34**(1990), 503–514.

[8] P.J. MAHER, Some norm inequalities concerning generalized inverses, *Linear Algebra Appl.* **174**(1992), 99–110.

[9] A.W. MARSHALL, I. OLKIN, *Inequalities: Theory of Majorization and its Applications*, Academic Press, New York 1979.

[10] A. MCINTOSH, Heinz inequalities and perturbation of spectral families, *Macquarie Math. Rep.* 79-0006, Macquarie Univ., 1979.

[11] G. PISIER, *Similarity Problems and Completely Bounded Maps*, 2nd Edition, Lecture Notes in Math., vol. 1618, Springer-Verlag, Berlin 2001.

[12] I. VIDAV, The norm of the sum of two projections, *Proc. Amer. Math. Soc.* **65**(1977), 297–298.

DAN POPOVICI, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF THE WEST TIMIŞOARA, BD. VASILE PÂRVAN 4, RO-300223 TIMIŞOARA, ROMANIA

*E-mail address*: popovici@math.uvt.ro

ZOLTÁN SEBESTYÉN, DEPARTMENT OF APPLIED ANALYSIS, LORÁND EÖTVÖS UNIVERSITY, PÁZMÁNY PÉTER SÉTÁNY 1/C, H-1117 BUDAPEST, HUNGARY

*E-mail address*: sebesty@cs.elte.hu

Received August 14, 2004.