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SUMS OF SMALL NUMBER OF COMMUTATORS

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ABSTRACT. For many C^*-algebras $\mathcal{A}$, techniques have been developed to show that all elements which have trace zero with respect to all tracial states can be written as a sum of finitely many commutators, and that the number of commutators required depends only upon the algebra, and not upon the individual elements. In this paper, we show that if the same holds for $q\mathcal{A}q$ whenever q is a "sufficiently small" projection in $\mathcal{A}$, then every element that is a sum of finitely many commutators in $\mathcal{A}$ is in fact a sum of two. We then apply this commutator reduction argument to certain C^*-algebras of real rank zero with a unique trace, as well as to a class of approximately homogeneous C^*-algebras whose K_0 group has large denominators. Finally, we use these results to show that many C^*-algebras are linearly spanned by their projections.

KEYWORDS: *Commutators, C^*-algebra, real rank zero, approximately homogeneous, projections.*

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1. INTRODUCTION

1.1. Let R be a unital ring. A *commutator* in R is an element of the form $[a,b] := ab - ba$. We shall denote by $\mathfrak{c}(R)$ the set of all commutators in R. Of particular interest historically has been the study of commutators in matrix rings $\mathbb{M}_n(R)$ over R. Given $n \in \mathbb{N}$ and a ring R as above, we may define a trace on $\mathbb{M}_n(R)$ via: $\mathrm{tr}_n : \mathbb{M}_n(R) \to R$, $\mathrm{tr}_n([r_{ij}]) = \sum\limits_{i=1}^{n} r_{ii}$. One of the earliest results is due to Shoda [41] who showed in 1936 that if $\mathbb{K}$ is a field of characteristic 0 and $n \geqslant 1$ is an integer, then $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{K})$ is a commutator if and only if $\mathrm{tr}_n(A) = 0$. This was extended to fields of arbitrary characteristic by Albert and Muckenhoupt [1] in 1957.

It is tempting to believe that such a result should easily extend to $\mathbb{M}_n(R)$. Alas, like so many of the best temptations, this one too should be avoided. As recently as six years ago, M. Rosset and S. Rosset [40] exhibited a class of examples of unital, abelian rings R and $A = [a_{ij}] \in \mathbb{M}_2(R)$ such that $\mathrm{tr}_2(A) = a_{11} + a_{22} = 0$, but $A \notin \mathfrak{c}(\mathbb{M}_2(R))$. (As they prove, one can choose $R = \mathbb{C}[x, y, z]$, where x, y and z are indeterminates.) In a positive direction, they also showed that if R is an abelian, unital ring, $n \geqslant 1$, $B \in \mathbb{M}_n(R)$ and $\mathrm{tr}_n(B) = 0$, then B can be expressed as a sum of two commutators from $\mathbb{M}_n(R)$.

Commutators first became of interest in operator theory because of their connection with physics. According to the postulates of quantum mechanics, the one-dimensional physical states of a quantum system at time t are represented by wave functions which correspond to continuously differentiable, normalized vectors in $L^2(\mathbb{R}, \mathrm{d}x)$, while observables such as the quantum analogs Q of position and P of momentum are described by hermitian linear maps acting on these wave functions. The states which can be observed are the eigenvectors of the observable, and for the states of position and momentum to be simultaneously observable, Q and P would need to be simultaneously diagonalizable, which would in turn imply that they commute. However, the action of Q is given by $Qf(x) = xf(x)$, while $Pf(x) = -\mathrm{i}\hbar f'(x)$ (where $\hbar$ is Planck's constant). A simple calculation shows that $[P, Q] = -\mathrm{i}\hbar I \neq 0$, which is the basis of one formulation of the Heisenberg Uncertainty Principle.

Let $\mathcal{H}$ be a complex, infinite-dimensional, separable Hilbert space, $\mathcal{B}(\mathcal{H})$ denote the set of bounded operators acting on $\mathcal{H}$, and by $\mathcal{K}(\mathcal{H})$ let us denote the closed, two-sided ideal of compact operators in $\mathcal{B}(\mathcal{H})$. In 1947, Wintner [45] proved that if $P = P^*, Q = Q^* \in \mathcal{B}(\mathcal{H})$, then $[P, Q] \notin \{\lambda I : 0 \neq \lambda \in \mathbb{C}\}$. Four years later, C.R. Putnam [36] observed that Wintner's proof works even if P and Q are not self-adjoint. In the meantime, Wielandt [44] had developed a new method to show that if $\mathcal{A}$ is any unital normed algebra, then $\mathfrak{c}(\mathcal{A}) \cap \{\lambda 1 : 0 \neq \lambda \in \mathbb{C}\} = \varnothing$.

It was P.R. Halmos [18] who proved that, given $A \in \mathcal{B}(\mathcal{H})$, the operator $A \oplus 0 \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is a commutator, and he used this to conclude that every operator in $\mathcal{B}(\mathcal{H})$ is a sum of two commutators. He also observed [19] that by applying Wielandt's result to the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, one could deduce that no operator of the form $\lambda I + K$, where $0 \neq \lambda \in \mathbb{C}$ and $K \in \mathcal{K}(\mathcal{H})$, is a commutator. The study of commutators in $\mathcal{B}(\mathcal{H})$ culminated in 1965 with the *tour de force* of A. Brown and C. Pearcy [3] who demonstrated that Wielandt's Theorem provides the only obstruction to membership in $\mathfrak{c}(\mathcal{B}(\mathcal{H}))$. That is, they proved that $T \in \mathcal{B}(\mathcal{H})$ is a commutator if and only if $T \notin \{\lambda I + K : 0 \neq \lambda \in \mathbb{C}, K \in \mathcal{K}(\mathcal{H})\}$.

The question of describing commutators and their spans in ideals of compact operators has generated a large amount of study, with too many results to describe here. We refer the reader to the excellent paper of K. Dykema, T. Figiel, G. Weiss and M. Wodzicki [10] for a survey of this vast area.

1.2. With these results in hand, it was natural to consider the problem of describing commutators and their spans in subalgebras of operators. Of course, the results obtained vary depending upon the context, and we mention but a few examples. Consider a von Neumann algebra $\mathcal{M}$. If $\mathcal{M}$ is a factor, then as we have already seen, $\mathfrak{c}(\mathcal{M})$ was characterized by Shoda in the Type I_n case and by Brown and Pearcy in the Type I_∞ case. The Type II_∞ case was handled by H. Halpern [21], while Brown and Pearcy [4] showed that if $\mathcal{M}$ is a Type III factor acting on a separable Hilbert space, then the commutators in $\mathcal{M}$ coincide with the set of non-scalar elements of $\mathcal{M}$, together with 0 — that is, Wielandt's criteria is once again the only factor determining adherence to $\mathfrak{c}(\mathcal{M})$. When $\mathcal{M}$ is not a factor, Pearcy and Topping [32] showed that every selfadjoint element with (canonical) central trace zero in a finite Type I von Neumann algebra is a commutator, so that every element with central trace zero is a sum of two commutators. Subsequently, T. Fack and P. de la Harpe [15] showed that in any finite von Neumann algebra with central trace τ, an element $T \in \mathcal{M}$ satisfies $\tau(T) = 0$ if and only if T can be expressed as $T = \sum_{k=1}^{10} [X_k, Y_k]$, where $X_k, Y_k \in \mathcal{M}$ and $\|X_k\| \leqslant 12\,\|T\|$, $\|Y_k\| \leqslant 12$ for all $1 \leqslant k \leqslant 10$.

In non-selfadjoint operator algebras, A.R. Sourour and the author [27] have observed that Halmos' proof (see for eg., Problem 234 of [20]) that every element of $\mathcal{B}(\mathcal{H})$ is a sum of two commutators extends *mutatis mutandis* to any unital, weakly closed subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ such that $\mathcal{A}$ has infinite multiplicity in the sense that $\mathcal{A} \simeq \mathcal{A} \otimes \mathcal{B}(\mathcal{H})$ (i.e. the weak operator closure of the span of the elementary tensors $A \otimes B$ acting on $\mathcal{H} \otimes \mathcal{H}$). In particular, this holds for nest algebras $\mathcal{A} = \mathrm{Alg}(\mathcal{N})$, where $\mathcal{N}$ is a nest with no finite-dimensional atoms. (We refer the reader to [8] for more information about nest algebras.)

1.3. The study of commutators (and their linear spans) in the context of C^*-algebras is in part related to an attempt to extend the Murray-von Neumann equivalence theory of projections which has proven so useful in the study of von Neumann algebras. In 1979, J. Cuntz and G.K. Pedersen [7] defined a relation (which we shall denote by $\approx$) on the positive cone $\mathcal{A}_+$ of a C^*-algebra $\mathcal{A}$ by setting $h \approx k$ if there exists a sequence $(u_n)_{n=1}^\infty$ in $\mathcal{A}$ so that $h = \sum_{n=1}^\infty u_n^* u_n$ and $k = \sum_{n=1}^\infty u_n u_n^*$, the sums converging in norm. Recall that a *tracial state* on $\mathcal{A}$ is a positive linear functional τ on $\mathcal{A}$ such that $\|\tau\| = 1$ and $\tau(xy) = \tau(yx)$ for all $x, y \in \mathcal{A}$. We denote by $\mathcal{T}(\mathcal{A})$ the set of tracial states on $\mathcal{A}$, which is weak*-compact in the case where $\mathcal{A}$ is unital. With $\mathcal{A}_{\mathrm{sa}} := \{a \in \mathcal{A} : a = a^*\}$ and $\mathcal{A}_0 = \{h - k : h, k \in \mathcal{A}_{\mathrm{sa}}, h \approx k\}$, they showed that $\mathcal{A}_0$ is a closed subspace of $\mathcal{A}_{\mathrm{sa}}$ and that $\mathcal{A}_0 = \{a \in \mathcal{A}_{\mathrm{sa}} : \tau(a) = 0 \text{ for all } \tau \in \mathcal{T}(\mathcal{A})\}$. If we denote by $\mathfrak{sl}(\mathcal{A})$ the set $\bigcap\{\ker \tau : \tau \in \mathcal{T}(\mathcal{A})\}$, then $\mathfrak{sl}(\mathcal{A}) = \mathcal{A}_0 + \mathrm{i}\mathcal{A}_0 = \overline{[A, A]}$, where $[A, A] = \operatorname{span} \mathfrak{c}(\mathcal{A})$.

For von Neumann algebras, $\mathcal{A}_0$ is always spanned by finite sums of self-commutators [15]. (A *self-commutator* in $\mathcal{A}$ is an element of the form $[x, x^*]$ -

we denote the set of self-commutators in $\mathcal{A}$ by $\mathfrak{sc}(\mathcal{A})$.) The corresponding statement for C^*-algebras was shown by G.K. Pedersen and N.H. Petersen to fail [34]. T. Fack then considered the question of finding C^*-algebras for which this property does hold in [14], and devised an ingenious method to prove that if $\mathcal{A}$ is a unital, simple AF C^*-algebra, then every element of $\mathcal{A}_0$ can be expressed as a sum of 7 self-commutators. We shall return to this in Sections 3 and 4.

Recall that a C^*-algebra $\mathcal{A}$ is said to be *stable* if $\mathcal{A} \simeq \mathcal{A} \otimes \mathcal{K}(\mathcal{H})$. In the paper cited above, Fack also showed that if $\mathcal{A}$ is stable, then any selfadjoint element in $\mathcal{A}$ can be written as a sum of five self-commutators. Finally, suppose that $\mathcal{B}$ is a unital C^*-algebra and there exist mutually orthogonal projections p and q in $\mathcal{A}$ with $p \sim q \sim 1$ (we say that the identity 1 of $\mathcal{A}$ is *properly infinite*, and hence that $\mathcal{A}$ is *properly infinite*). Fack showed that the same conclusion holds in this case as well, namely, that every selfadjoint element of $\mathcal{A}$ is a sum of five self-commutators. An immediate consequence is that every element of a properly infinite unital C^*-algebra, or of a stable algebra, is a sum of at most 10 commutators.

Fack's method for simple, unital AF C^*-algebras served as the basis for K. Thomsen's extension of Fack's results to inductive limits of finite direct sums of homogeneous C^*-algebras [42]. More precisely, when $\mathcal{A}$ is a C^*-algebra of the type considered in Subsection 4.2 of the present paper, Thomsen proved — using an adaptation of Fack's Theorem — that every element of $\mathcal{A}_0$ is a sum of a fixed, finite number of self-commutators. (See Theorem 4.2 below.)

More recently, C. Pop [35] has shown that if $\mathcal{A}$ is a unital C^*-algebra, then $\mathcal{A}$ has no tracial states if and only if there exists some $n \geqslant 2$ so that $h \in \mathcal{A}_{\mathrm{sa}}$ implies that h is a sum of n self-commutators. Moreover, if $\mathcal{A}$ is properly infinite, then every element of $\mathcal{A}$ can be expressed as a sum of two commutators.

1.4. In Section 2 of this paper, we show that in certain unital C^*-algebras $\mathcal{B}$, knowing that there exists a fixed $m \in \mathbb{N}$ such that every element in a corner subalgebra $\mathfrak{sl}(q\mathcal{B}q)$ (where q is a "sufficiently small projection" in $\mathcal{B}$) can be expressed as a sum of m commutators is sufficient to conclude that every such element can be expressed as sums of two commutators. The main result of this section is the commutator reduction argument, Theorem 2.3.

In Section 3, we use the outline of Fack's proof for simple, unital AF C^*-algebras to show that if $\mathcal{A}$ is a simple, unital C^*-algebra of real rank zero such that $\mathcal{A}$ satisfies B. Blackadar's FCQ2 (see Subsection 3.2 for the definition of this property), and if $\mathcal{A}$ has a unique tracial state, then $\mathcal{A}$ satisfies the conditions of the commutator reduction argument, and thus every element of $\mathfrak{sl}(\mathcal{A})$ can be expressed as a sum of two commutators. In Section 4, we "fine tune" Thomsen's proof for the C^*-algebras of Subsection 4.2 to reveal that the commutator reduction argument applies to these as well, whence the same conclusion may once again be drawn.

Finally, in Section 5, we apply these results on commutators along with a construction from [25] to show that in the algebras considered above, every element can be expressed as a linear combination of a (relatively small) *fixed* finite number of projections. In some cases, we improve the bounds obtained in [25].

2. THE REDUCTION ARGUMENT

2.1. Let us gather some of the previous notations and definitions together in one place for convenience later on. For a C^*-algebra $\mathcal{A}$, we set $[a,b] = ab - ba$, $a,b \in \mathcal{A}$, $\mathfrak{c}(\mathcal{A}) = \{[a,b] : a,b \in \mathcal{A}\}$, and $\mathfrak{sc}(\mathcal{A}) = \{[a^*,a] : a \in \mathcal{A}\}$. A *tracial state* on $\mathcal{A}$ is a positive, norm one linear functional on $\mathcal{A}$ satisfying $\tau(xy) = \tau(yx)$ for all $x,y \in \mathcal{A}$. We let $\mathfrak{sl}(\mathcal{A}) = \{a \in \mathcal{A} : \tau(a) = 0 \text{ for all } \tau \in \mathcal{T}(A)\}$. If $\mathcal{T}(\mathcal{A})$ is empty, by convention we set $\mathfrak{sl}(\mathcal{A}) = \mathcal{A}$. Let $\mathcal{A}_{\mathrm{sa}} := \{a \in \mathcal{A} : a = a^*\}$. As mentioned in Subsection 1.3 above, we define an equivalence relation on the positive elements $\mathcal{A}_+$ of $\mathcal{A}$ via $x \approx y$ if there exist a sequence (u_n) such that $x = \sum\limits_{n=1}^{\infty} u_n^* u_n$ and $y = \sum\limits_{n=1}^{\infty} u_n u_n^*$. Letting $\mathcal{A}_0 = \{h - k : h,k \in \mathcal{A}_{\mathrm{sa}}, h \approx k\}$, $\mathcal{A}_0$ is a closed subspace of $\mathcal{A}_{\mathrm{sa}}$ and $\mathcal{A}_0 = \mathcal{A}_{\mathrm{sa}} \cap \mathfrak{sl}(\mathcal{A})$. Thus $\mathfrak{sl}(\mathcal{A}) = \mathcal{A}_0 + \mathrm{i}\mathcal{A}_0 = \overline{[A,A]}$, where $[A,A] = \operatorname{span} \mathfrak{c}(\mathcal{A})$. For a subset $\mathcal{S}$ of $\mathcal{A}$ and $m \geqslant 1$, let us write $\Sigma_m \mathcal{S}$ to mean $\{s_1 + \cdots + s_m : s_k \in \mathcal{S}, 1 \leqslant k \leqslant m\}$, and by $\mathbb{C}\mathcal{S}$ we denote the set $\{\lambda s : \lambda \in \mathbb{C}, s \in \mathcal{S}\}$. Thus if $\mathfrak{P}(\mathcal{A})$ denotes the set of projections in $\mathcal{A}$, then the algebraic span of $\mathfrak{P}(\mathcal{A})$ coincides with $\bigcup\limits_{k=1}^{\infty} \Sigma_k \mathbb{C}\mathfrak{P}(\mathcal{A})$. Given projections $p,q \in \mathfrak{P}(\mathcal{A})$, we write $p \preceq q$ if there exists $u \in \mathcal{A}$ such that $u^*u = p, uu^* = p' \leqslant q$. If in fact $p' < q$, we write $p \prec q$.

2.2. The original setting for the following lemma in [10] is simpler to visualize, and the crux of the idea is there (Lemma 6.2). The conclusion there is that if $\mathcal{A}$ is an algebra and $\mathcal{B} = \mathbb{M}_n(\mathcal{A})$, $\beta = \operatorname{diag}(b_1, b_2, \ldots, b_n) \in \mathcal{B}$ and $\sum\limits_{k=1}^{n} b_k = 0$, then $\beta \in \mathfrak{c}(\mathcal{B})$. In passing to C^*-algebras — even those of real rank zero — we do not know that a given algebra $\mathcal{B}$ is "divisible" in the sense of Rieffel [38]; that is, that $\mathcal{B} \simeq \mathbb{M}_n(\mathcal{A})$ for some algebra $\mathcal{A}$. On the other hand, when $\mathcal{A}$ is a simple unital C^*-algebra of real rank zero, then S. Zhang [46], [47] has shown the existence of mutually orthogonal projections $q_1 \preceq q_2 \preceq \cdots \preceq q_n$ so that $\sum\limits_{k=1}^{n} q_k = 1$.

The device being used here is to replace the canonical matrix units $\{E_{ij}\}$ of $\mathbb{M}_n(\mathbb{C})$ with appropriate partial isometries w_{ij} (playing the role of E_{ij}, $1 \leqslant i,j \leqslant n$), constructed from partial isometries v_k (playing the role of $E_{k,k+1}$, $1 \leqslant k \leqslant n-1$).

LEMMA 2.1. *Let $\mathcal{B}$ be a unital C^*-algebra and suppose that $q_1, q_2, \ldots, q_n$ are mutually orthogonal projections in $\mathcal{B}$ satisfying:*

(i) $1 = q_1 + q_2 + \cdots + q_n$;
(ii) $q_1 \preceq q_2 \preceq \cdots \preceq q_n$.

Fix partial isometries v_k, $1 \leqslant k \leqslant n-1$ so that $v_k v_k^* = q_k$, $v_k^* v_k \leqslant q_{k+1}$. For $1 \leqslant i < j \leqslant n$, let $w_{ij} = v_i v_{i+1} \cdots v_{j-1} \in q_i \mathcal{B} q_j$ and let $w_{jj} = q_j$, $1 \leqslant j \leqslant n$. Suppose $b_k \in q_k \mathcal{B} q_k$, $1 \leqslant k \leqslant n$ and that $\sum\limits_{k=1}^{n} w_{kn}^* b_k w_{kn} = 0$. Then $\beta = \sum\limits_{k=1}^{n} b_k \in \mathfrak{c}(\mathcal{B})$.

Proof. Set $s_j = \sum\limits_{k=1}^{j} (w_{kj}^* b_k w_{kj}) v_j$, $1 \leqslant j \leqslant n-1$. Then $\Big[\sum\limits_{j=1}^{n-1} s_j, \sum\limits_{k=1}^{n-1} v_k^*\Big] = \beta$, which is best seen by calculating

$$\left[\begin{bmatrix} 0 & s_1 & & \\ & \ddots & \ddots & \\ & & \ddots & s_{n-1} \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & & \\ v_1^* & \ddots & & \\ & \ddots & \ddots & \\ & & v_{n-1}^* & 0 \end{bmatrix}\right]$$

$$= \begin{bmatrix} s_1 v_1^* & & & & \\ & s_2 v_2^* - v_1^* s_1 & & & \\ & & \ddots & & \\ & & & s_{n-1} v_{n-1}^* - v_{n-2}^* s_{n-2} & \\ & & & & -v_{n-1}^* s_{n-1} \end{bmatrix}.$$

Then for $2 \leqslant j \leqslant n-1$, we have

$$\begin{aligned} s_j v_j^* - v_{j-1}^* s_{j-1} &= \sum_{k=1}^{j} w_{kj}^* b_k w_{kj} - \sum_{k=1}^{j-1} v_{j-1}^* (w_{k,j-1}^* b_k w_{k,j-1}) v_{j-1} \\ &= \sum_{k=1}^{j} w_{kj}^* b_k w_{kj} - \sum_{k=1}^{j-1} w_{kj}^* b_k w_{kj} = w_{jj}^* b_j w_{jj} = b_j. \end{aligned}$$

Moreover, $s_1 v_1^* = b_1$, while

$$-v_{n-1}^* s_{n-1} = -\sum_{k=1}^{n-1} w_{kn}^* b_k w_{kn} = w_{nn}^* b_n w_{nn} = b_n,$$

completing the proof. ∎

LEMMA 2.2. *Let $\mathcal{B}$ be a unital C^*-algebra and suppose that $q_1, q_2, \ldots, q_n$ are mutually orthogonal projections in $\mathcal{B}$ satisfying $1 = q_1 + q_2 + \cdots + q_n$. Let $b \in \mathcal{B}$ and suppose that $q_k b q_k \in \mathfrak{c}(q_k \mathcal{B} q_k)$, $1 \leqslant k \leqslant n$. Then $b \in \mathfrak{c}(\mathcal{B})$.*

Proof. Suppose that $q_k b q_k = [x_k, y_k]$ where $x_k, y_k \in q_k \mathcal{B} q_k$ for each $1 \leqslant k \leqslant n$. If $x_k \neq 0$, then we can replace x_k by $x_k / \|x_k\|$ and y_k by $y_k \|x_k\|$, which allows us to assume *a priori* that $\|x_k\| = 1$. (If $x_k = 0$, we first replace x_k by q_k.) In particular, we have that $\sigma_{q_k \mathcal{B} q_k}(x_k) \subseteq \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leqslant 1\}$.

Let $d_k = x_k + 3kq_k$, so that $\sigma_{q_k \mathcal{B} q_k}(d_k) \subseteq 3k + \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z - 3k| \leqslant 1\}$. If $j \neq k$, then $\sigma_{q_j \mathcal{B} q_j}(d_j) \cap \sigma_{q_k \mathcal{B} q_k}(d_k)$ is empty. At the same time, we still have that $[d_k, y_k] = [x_k, y_k] = q_k b q_k$ for all k. From this it follows that for $1 \leqslant j \neq k \leqslant n$, the Rosenblum operator

$$\begin{array}{rccc} \tau_{d_j,d_k} : & q_j \mathcal{B} q_k & \to & q_j \mathcal{B} q_k \\ & z & \mapsto & d_j z - z d_k \end{array}$$

is invertible, as follows easily from, for e.g. Corollary 3.2 of [22]. As such, for each $1 \leqslant j \neq k \leqslant n$, we can find $z_{jk} \in q_j \mathcal{B} q_k$ so that $\tau_{d_j,d_k}(z_{jk}) = q_j b q_k$.

For $1 \leqslant j \leqslant n$, let $z_{jj} = y_j$. Setting

$$d = \sum_{k=1}^{n} d_k, z = \sum_{1 \leqslant j,k \leqslant n} z_{jk} = \sum_{1 \leqslant j,k \leqslant n} q_j z_{jk} q_k,$$

a routine calculation shows that

$$[d, z] = \sum_{1 \leqslant j,k \leqslant n} d_j z_{jk} - z_{jk} d_k = \sum_{1 \leqslant j,k \leqslant n} q_j b q_k = b. \quad \blacksquare$$

As pointed out in [39], the formula for z_{jk} is given by

$$z_{jk} = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} (d_j - w1)^{-1} (q_j b q_k)(d_k - w1)^{-1} \mathrm{d}w,$$

where Γ is any closed contour in $\mathbb{C}$ such that $\sigma_{q_j \mathcal{B} q_j}(d_j)$ lies in the bounded component of $\mathbb{C} \backslash \Gamma$ and $\sigma_{q_k \mathcal{B} q_k}(d_k)$ lies in the unbounded component. If we choose Γ to be a circle of radius $3/2$ centered at $3j$, then an elementary calculation shows that $\|z_{jk}\| \leqslant 6 \|q_j b q_k\|$. This estimate will be included in the calculation of Remark 5.3.

A simple case of the following result is handled in Proposition 2.7 below, and the reader may prefer to look at that result first.

THEOREM 2.3 (The commutator reduction argument). *Let $\mathcal{B}$ be a unital C^*-algebra and suppose that $q_1, q_2, \ldots, q_n$ are mutually orthogonal projections in $\mathcal{B}$ satisfying:*

(i) $1 = q_1 + q_2 + \cdots + q_n$;
(ii) $q_1 \preceq q_2 \preceq \cdots \preceq q_{n-m} \sim q_{n-m+1} \sim \cdots \sim q_n$ *for some* $1 \leqslant m \leqslant n-1$.

As before, let v_k be partial isometries for which $v_k v_k^* = q_k$, $v_k^* v_k \leqslant q_{k+1}$, $1 \leqslant k \leqslant n-m-1$, and $v_k v_k^* = q_k$, $v_k^* v_k = q_{k+1}$ if $n-m \leqslant k \leqslant n-1$. For $1 \leqslant i < j \leqslant n$, let $w_{ij} = v_i v_{i+1} \cdots v_{j-1} \in q_i \mathcal{B} q_j$ and let $w_{jj} = q_j$, $1 \leqslant j \leqslant n$. For $b \in \mathcal{B}$, write $b_{jk} = q_j b q_k$, $1 \leqslant j,k \leqslant n$ and suppose that $\sum\limits_{k=1}^{n} w_{kn}^* b_{kk} w_{kn} \in \Sigma_m \mathfrak{c}(q_n \mathcal{B} q_n)$. Then $b \in \Sigma_2 \mathfrak{c}(\mathcal{B})$.

Proof. We shall set

$$\beta_1 = \sum_{1\leqslant j\neq k\leqslant n} q_j b q_k = \begin{bmatrix} 0 & & & \\ & \ddots & b_{jk} & \\ & b_{jk} & \ddots & \\ & & & 0 \end{bmatrix};$$

$$\beta_2 = \operatorname{diag}(b_{11}, b_{22}, \ldots, b_{n-1\,n-1}, -\sum_{k=1}^{n-1} w_{kn}^* b_{kk} w_{kn});$$

$$\beta_3 = \operatorname{diag}(0, 0, \ldots, 0, \sum_{k=1}^{n} w_{kn}^* b_{kk} w_{kn}).$$

Then $b = \beta_1 + \beta_2 + \beta_3$. By assumption, $\sum\limits_{k=1}^{n} w_{kn}^* b_{kk} w_{kn} \in \Sigma_m \mathfrak{c}(q_n \mathcal{B} q_n)$, and so we can find $x_i, y_i \in q_n \mathcal{B} q_n$, $1 \leqslant i \leqslant m$ so that $\sum\limits_{k=1}^{n} w_{kn}^* b_{kk} w_{kn} = \sum\limits_{i=1}^{m} [x_i, y_i]$.

Let $\beta_4 = \sum\limits_{k=1}^{m} -[x_k', y_k'] + \sum\limits_{k=1}^{m} [x_i, y_i]$, where $x_k' = w_{(n-m-1)+k,n} x_k w_{(n-m-1)+k,m}^*$, $y_k' = w_{(n-m-1)+k,n} y_k w_{(n-m-1)+k,n}^*$, $1 \leqslant k \leqslant m$, and $\beta_5 = \sum\limits_{k=1}^{n} [x_k', y_k']$. That is, we have:

$$\beta_4 = \operatorname{diag}(0, 0, \ldots, 0, -[x_1', y_1'], \ldots, -[x_m', y_m'], \sum_{k=1}^{m} [x_i, y_i])$$

$$\beta_5 = \operatorname{diag}(0, 0, \ldots, 0, [x_1', y_1'], \ldots, [x_m', y_m'], 0).$$

Clearly $\beta_3 = \beta_4 + \beta_5$, whence $b = (\beta_1 + \beta_5) + (\beta_2 + \beta_4)$.

By Lemma 2.2, $\beta_1 + \beta_5 \in \mathfrak{c}(\mathcal{B})$. By Lemma 2.1, $\beta_2 + \beta_4 \in \mathfrak{c}(\mathcal{B})$. Hence $b \in \Sigma_2(\mathfrak{c}(\mathcal{B}))$. ∎

THEOREM 2.4. *Let $\mathcal{B}$ be a C^*-algebra. Suppose that $z = z^* \in \mathfrak{c}(\mathcal{A})$. Then $z \in \Sigma_2 \mathfrak{sc}(\mathcal{A})$. Thus $\Sigma_m \mathfrak{c}(\mathcal{A}) \subseteq \Sigma_{2m} \mathfrak{sc}(\mathcal{A})$ for all $m \geqslant 1$.*

Proof. Choose $x, y \in \mathcal{A}$ so that $z = [x, y]$. Let $x = a + \mathrm{i}b$, $y = c + \mathrm{i}d$ be the Cartesian decomposition of x and y into their real and imaginary parts. Then

$$z = [a + \mathrm{i}b, c + \mathrm{i}d] = [a, c] - [b, d] + \mathrm{i}([a, d] + [b, c]).$$

Noting that $h = h^*, k = k^*$ implies that $[h, k]^* = -[h, k]$ and that $(\mathrm{i}[h, k])^* = \mathrm{i}[h, k] = (1/2)[(h + \mathrm{i}k)^*, (h + \mathrm{i}k)]$, we get

$$\begin{aligned} z &= \frac{z + z^*}{2} = \mathrm{i}([a, d] + [b, c]) \\ &= \Big[\Big(\frac{1}{\sqrt{2}}(a + \mathrm{i}d)\Big)^*, \Big(\frac{1}{\sqrt{2}}(a + \mathrm{i}d)\Big)\Big] + \Big[\Big(\frac{1}{\sqrt{2}}(b + \mathrm{i}c)\Big)^*, \Big(\frac{1}{\sqrt{2}}(b + \mathrm{i}c)\Big)\Big] \\ &\in \Sigma_2 \mathfrak{sc}(\mathcal{A}). \end{aligned}$$

The second statement follows trivially from the first. ∎

Although the following result is subsumed by the results of Section 4, nevertheless we produce it here because those results involve a number of technicalities which are only required in the more general setting, and which obscure the simplicity of the underlying technique in this more restricted yet still interesting case.

THEOREM 2.5 (Un apéritif). *Let $\mathcal{B}$ be a unital, simple, infinite dimensional* AF *C^*-algebra. Then every element in $\mathfrak{sl}(\mathcal{B})$ is a sum of two commutators, and every selfadjoint element of $\mathfrak{sl}(\mathcal{B})$ is a sum of* 4 *self-commutators. In particular, $\mathfrak{sl}(\mathcal{B}) = \Sigma_2(\mathcal{B})$ and $\mathcal{B}_0 = \Sigma_4\mathfrak{sc}(\mathcal{B})$.*

Proof. Note that under these conditions on $\mathcal{B}$, it was shown by Fack [14] that every element of $\mathcal{B}_0$ can be expressed as a sum of 7 or fewer self-commutators. From this it clearly follows that every trace zero element is a sum of at most 14 commutators.

Since $\mathcal{B}$ is an infinite dimensional, simple unital AF C^*-algebra, it has real rank zero and so by [47], for all $n \geqslant 1$ we can find mutually orthogonal projections $q_1 \preceq q_2 \sim q_3 \sim \cdots \sim q_{2^n+1}$ so that $1 = \sum_{k=1}^{2^n+1} q_k$. For our purposes, choosing $n = 4$ is sufficient. Suppose $b \in \mathfrak{sl}(\mathcal{B})$ and let $b_{jk} = q_j b q_k$, $1 \leqslant j, k \leqslant 17$. If we let v_k, w_{jk} denote the operators from Theorem 2.3, then for any trace τ on $\mathcal{B}$, we get

$$0 = \tau(b) = \sum_{k=1}^{17} \tau(b_{kk}) = \sum_{k=1}^{17} \tau(w_{k,17}^* b_{kk} w_{k,17}).$$

Since any trace on $q_{17}\mathcal{B}q_{17}$ extends to a unique trace on $\mathcal{B}$ (as pointed out by Fack, q_{17} is an order unit in $K_0(\mathcal{B})_+$, or alternatively, see Lemma 3.8 below), it follows that $\sum_{k=1}^{17} w_{k17}^* b_{kk} w_{k17} \in \mathfrak{sl}(q_{17}\mathcal{B}q_{17})$.

But $q_{17}\mathcal{B}q_{17}$ is both simple (see, for eg. Theorem 3.2.8 of [29]) and an AF-C^*-algebra ([11] or [30]). Furthermore, q_{17} serves as an identity for this algebra. By Fack's result ([14], Theorem 3.1), every trace zero element of $q_{17}\mathcal{B}q_{17}$ lies in $\Sigma_{14}\mathfrak{c}(q_{17}\mathcal{B}q_{17})$. By the commutator reduction argument, Theorem 2.3, $b \in \Sigma_2\mathfrak{c}(\mathcal{B})$. Since $b \in \mathfrak{sl}(\mathcal{B})$ was arbitrary, $\mathfrak{sl}(\mathcal{B}) \subseteq \Sigma_2\mathfrak{c}(\mathcal{B})$. By Theorem 2.4, $\mathcal{B}_0 = \mathfrak{sl}(\mathcal{B}) \cap \mathcal{B}_{\text{sa}} \subseteq \Sigma_4\mathfrak{sc}(\mathcal{B})$.

Finally, we note that $\Sigma_2\mathfrak{c}(\mathcal{B}) \subseteq \mathfrak{sl}(\mathcal{B})$ and $\Sigma_4\mathfrak{sc}(\mathcal{B}) \subseteq \mathcal{B}_0$ are both clear, whence the second statement easily follows. ∎

DEFINITION 2.6. Let $\mathcal{B}$ be a C^*-algebra. If $\mathfrak{sl}(\mathcal{B}) = \Sigma_m\mathfrak{c}(\mathcal{B})$ for some $m \in \mathbb{N}$, then we set $\gamma(\mathcal{B}) = \min\{m \in \mathbb{N} : \mathfrak{sl}(\mathcal{B}) = \Sigma_m\mathfrak{c}(\mathcal{B})\}$. Otherwise we set $\gamma(\mathcal{B}) = \infty$.

We also define $\gamma_{\text{sa}}(\mathcal{B}) = \min\{m \in \mathbb{N} : \mathcal{B}_0 = \Sigma_m\mathfrak{sc}(\mathcal{B})\}$ if such an m exists, and $\gamma_{\text{sa}}(\mathcal{B}) = \infty$ otherwise.

We shall refer to $\gamma(\mathcal{B})$ (respectively $\gamma_{\text{sa}}(\mathcal{B})$) as the *commutator index* (respectively *self-commutator index*) of $\mathcal{B}$.

PROPOSITION 2.7. *Let $\mathcal{B}$ be a unital C^*-algebra and suppose that $\gamma(\mathcal{B}) < \infty$. Then the commutator index of $\mathbb{M}_{\gamma(\mathcal{B})+1}(\mathcal{B})$ is less than or equal to 2, and the self-commutator index of $\mathbb{M}_{\gamma(\mathcal{B})+1}(\mathcal{B})$ is less than or equal to 4.*

Proof. We begin with a couple of well-known and straightforward observations. First note that for all $n \geqslant 1$, $\mathbb{M}_n(\mathcal{B}) \simeq \mathcal{B} \otimes \mathbb{M}_n(\mathbb{C})$ and there is a bijective correspondence between the tracial states of $\mathcal{B}$ and those of $\mathbb{M}_n(\mathcal{B})$ given by

$$\begin{array}{cccc} \Phi: & \mathcal{T}(\mathcal{B}) & \rightarrow & \mathcal{T}(\mathcal{B} \otimes \mathbb{M}_n) \\ & \tau & \mapsto & \tau_n := \tau \otimes \overline{\mathrm{tr}}_n \end{array}$$

where $\overline{\mathrm{tr}}_n$ denotes the unique normalized tracial state on $\mathbb{M}_n(\mathbb{C})$. As a consequence, for $b = [b_{ij}] \in \mathbb{M}_n(\mathcal{B})$ and $\tau \in \mathcal{T}(\mathcal{B})$, $\tau_n(b) = \sum\limits_{j=1}^{n} \tau(b_{jj})$.

Let $\{E_{ij}\}$ denote the canonical matrix units for $\mathbb{M}_{\gamma(\mathcal{B})+1}(\mathbb{C})$, $1 \leqslant i, j \leqslant \gamma(\mathcal{B}) + 1$. We set $q_i = 1 \otimes E_{ii}$, $1 \leqslant i \leqslant \gamma(\mathcal{B}) + 1$ and $v_k = 1 \otimes E_{k,k+1}$, $1 \leqslant k \leqslant \gamma(\mathcal{B})$ as in Theorem 2.3. We then define w_{ij} as in that theorem and observe that $w_{k,\gamma(\mathcal{B})+1} w^*_{k,\gamma(\mathcal{B})+1} = q_{kk}$ for all $1 \leqslant k \leqslant \gamma(\mathcal{B}) + 1$. Hence for $b \in \mathfrak{sl}(\mathbb{M}_{\gamma(\mathcal{B})+1}(\mathcal{B}))$, we have

$$\begin{aligned} \tau\Big(\sum_{k=1}^{\gamma(\mathcal{B})+1} w^*_{k,\gamma(\mathcal{B})+1} b_{kk} w_{k,\gamma(\mathcal{B})+1} \Big) &= \sum_{k=1}^{\gamma(\mathcal{B})+1} \tau(w_{k,\gamma(\mathcal{B})+1} w^*_{k,\gamma(\mathcal{B})+1} b_{kk}) \\ &= \sum_{k=1}^{\gamma(\mathcal{B})+1} \tau(b_{kk}) = \tau_{\gamma(\mathcal{B})+1}(b) = 0. \end{aligned}$$

Thus $\sum\limits_{k=1}^{\gamma(\mathcal{B})+1} w^*_{k,\gamma(\mathcal{B})+1} b_{kk} w_{k,\gamma(\mathcal{B})+1} \in \mathfrak{sl}(\mathcal{B}) = \Sigma_{\gamma(\mathcal{B})}\mathfrak{c}(\mathcal{B})$. By the Reduction Argument, Theorem 2.3, $b \in \Sigma_2\mathfrak{c}(\mathbb{M}_{\gamma(\mathcal{B})+1}(\mathcal{B}))$, which completes the proof of the first statement. The second statement follows immediately from Theorem 2.4. ∎

2.3. Suppose that $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{T}(\mathcal{A})$ is empty. As mentioned in paragraph 2.1, $\mathcal{A} = \mathfrak{sl}(\mathcal{A})$. In [35], C. Pop proves that when $\mathcal{A}$ is such an algebra, the commutator index $\gamma(\mathcal{A})$ is finite, and he remarks that, assuming that the value of $\gamma(\mathcal{A})$ is known, it would be interesting to determine the smallest value of m such that the commutator index of $\mathbb{M}_m(\mathcal{A})$ is 2; i.e. $\mathbb{M}_m(\mathcal{A}) = \Sigma_2\mathfrak{c}(\mathbb{M}_m(\mathcal{A}))$. The previous proposition implies that the smallest such integer m can be no larger than $\gamma(\mathcal{A}) + 1$.

If $1 \in \mathcal{A}$ is a C^*-algebra and 1 is properly infinite, then Fack [14] showed that $\mathcal{A}_{\mathrm{sa}} = \mathcal{A}_0 = \Sigma_5\mathfrak{sc}(\mathcal{A})$, whence $\mathcal{A} = \mathfrak{sl}(\mathcal{A}) = \Sigma_{10}\mathfrak{c}(\mathcal{A})$. The estimate was reduced from 10 to 2 by Pop in [35], using entirely different means. The result below reproduces Pop's estimate, but first starting from Fack's.

COROLLARY 2.8 ([35]). *Suppose that $1 \in \mathcal{A}$ is properly infinite. Then $\mathcal{A} = \Sigma_2\mathfrak{c}(\mathcal{A})$.*

Proof. Since $1 \in \mathcal{A}$ is properly infinite, we can find a sequence $p_1, p_2, \ldots, p_{16}$ of mutually orthogonal projections, each equivalent to 1 and hence to each other, such that $\sum_{k=1}^{16} p_k < 1$. Let $p_0 = 1 - \sum_{i=1}^{16} p_k$. Then $p_0 < 1 \sim p_{16}$, and so $p_0 \prec p_{16}$.

Moreover, $p_{16}\mathcal{A}p_{16}$ is $*$-isomorphic to $\mathcal{A}$, and hence is properly infinite. By Fack's result as quoted in the paragraph preceding the corollary, $p_{16}\mathcal{A}p_{16} = \Sigma_{10}\mathfrak{c}(p_{16}\mathcal{A}p_{16})$. It then follows from Theorem 2.3 that $\mathcal{A} = \Sigma_2\mathfrak{c}(\mathcal{A})$. ∎

3. REAL RANK ZERO

3.1. As mentioned in Subsection 1.2 above, T. Fack's results to the effect that every element of $\mathcal{A}_0$ lies in $\Sigma_7\mathfrak{c}(\mathcal{A})$ for simple, unital, AF C^*-algebras was extended to certain limits of finite direct sums of homogeneous C^*-algebras by K. Thomsen (see Theorem 4.2 below). In this section, we show that the outline of Fack's proof can also be used to obtain a similar result for a class of simple, unital C^*-algebras of real rank zero, possessing a unique tracial state. While the conclusions of the lemmas and theorems are often direct analogs of those of Fack (indeed, in some cases they are identical), the proofs are quite different. We suspect that the basic methodology can be adapted to other classes of C^*-algebras as well.

DEFINITION 3.1. A C^*-algebra $\mathcal{A}$ is said to be of *real rank zero* if every selfadjoint element of $\mathcal{A}$ can be approximated by selfadjoint elements with finite spectrum.

We remark that although this was not the original definition, it is equivalent to it [5].

3.2. Since every element of a C^*-algebra $\mathcal{A}$ is a linear combination of its real and imaginary parts, it immediately follows that if $\mathcal{A}$ is of real rank zero, then the span of the projections in $\mathcal{A}$ is dense. The converse is known to be false [34]. As an application of the results obtained in this subsection, we shall see that in some cases, the algebra is *linearly* spanned by its projections.

There is a well-established notion of comparability of projections in factor von Neumann algebras [23]. In trying to generalize such a theory to C^*-algebras, B. Blackadar [2] described a number of different notions of comparability which he referred to as *Fundamental Comparability Questions* (FCQ). The second of these, FCQ2, is particularly relevant to the present setting.

DEFINITION 3.2. A simple, unital C^*-algebra $\mathcal{A}$ is said to satisfy FCQ2 if, for all projections $p, q \in \mathcal{A}$, $\tau(p) < \tau(q)$ for all $\tau \in \mathcal{T}(\mathcal{A})$ implies that $p \prec q$.

One result which we shall use repeatedly in the sequel is the following, due to S. Zhang. Recall that a C^*-algebra $\mathcal{A}$ is said to be *non-elementary* if it is neither a matrix algebra over $\mathbb{C}$, nor the algebra of compact operators $\mathcal{K}(\mathcal{H})$ on some infinite dimensional, separable Hilbert space $\mathcal{H}$.

THEOREM 3.3 ([47], Theorem I). *Let $\mathcal{A}$ be a non-elementary, simple C^*-algebra with real rank zero. If p is a projection in $\mathcal{A}$, then for any integer $n \geqslant 1$ and any non-zero projection r of $\mathcal{A}$, there exist subprojections $p_1, p_2, \dots, p_{2^n}, p_{2^n+1}$ of p so that*

$$p = (p_1 \oplus p_2 \oplus \cdots \oplus p_{2^n}) \oplus p_{2^n+1},$$

where $p_1, \dots, p_{2^n}$ are mutually orthogonal, equivalent subprojections of p and p_{2^n+1} is a projection equivalent to both a subprojection of r and a subprojection of p_1.

REMARK 3.4. Note that this implies that if τ is any tracial state on $\mathcal{A}$ and $p \in \mathfrak{P}(\mathcal{A})$ is as above, then $1/(2^n+1)\tau(p) \leqslant \tau(p_k) \leqslant 1/(2^n)\tau(p)$ for all $1 \leqslant k \leqslant 2^n + 1$. From this and the fact that we can choose $n \geqslant 1$ arbitrarily big, it easily follows that for any $0 \leqslant \alpha < \beta \leqslant 1$, there exists a projection $q < p$ so that

$$\alpha\, \tau(p) < \tau(q) < \beta\, \tau(p) \quad \text{for all } \tau \in \mathcal{T}(\mathcal{A}).$$

The following result is surely known, and is included for completeness.

LEMMA 3.5. *Suppose $\mathcal{A}$ is a unital C^*-algebra satisfying* FCQ2. *Let r be a projection in $\mathcal{A}$. Then $r\mathcal{A}r$ satisfies* FCQ2.

Proof. Suppose $p, q \in r\mathcal{A}r$ are projections. Suppose $\varphi(p) < \varphi(q)$ for all $\varphi \in \mathcal{T}(r\mathcal{A}r)$. Given $\tau \in \mathcal{T}(\mathcal{A})$, $\tau_r = \tau|_{r\mathcal{A}r}$ is a trace on $r\mathcal{A}r$, so that $\tau(p) = \tau_r(p) < \tau_r(q) = \tau(q)$. Since $\mathcal{A}$ has FCQ2, there exist $u \in \mathcal{A}$ so that $uu^* = p$, $u^*u = p' < q$.

Since $p, q \in r\mathcal{A}r$, $u = pu = ru$, $u = up' = ur$ and so $p < q$ in $r\mathcal{A}r$. That is, $r\mathcal{A}r$ satisfies FCQ2. ∎

PROPOSITION 3.6. *Suppose that $\mathcal{A}$ is a unital, simple C^*-algebra with real rank zero and a unique tracial state τ. Suppose furthermore that $\mathcal{A}$ satisfies* FCQ2.

If $a \in \mathcal{A}_0$, then for all $\varepsilon > 0$ there exist $u, w \in \mathcal{A}$ with $\|u\|, \|w\| \leqslant 2\|a\|^{1/2}$ so that

$$\|a - [u, u^*] - [w, w^*]\| < \varepsilon.$$

In particular, $a \in \overline{\Sigma_2 \mathfrak{sc}(\mathcal{A})}$.

Proof. Let $0 < \varepsilon$. Since $\mathcal{A}$ has real rank zero, we can find $b = b^* \in \mathcal{A}$ with $\|b\| = \|a\|$ so that:

(i) b has finite spectrum $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ with $\lambda_i \neq \lambda_j$ if $i \neq j$;
(ii) $\|a - b\| < \varepsilon$; and
(iii) $\tau(b) = 0$.

Let $\{p_i\}_{i=1}^n$ denote the spectral projections corresponding to λ_i, $1 \leqslant i \leqslant n$. Then $p_i\, p_j = 0 = p_j p_i$ if $i \neq j$, and $b = \sum\limits_{i=1}^{n} \lambda_i p_i$.

Let $\gamma > 0$, and choose $m_0 \in \mathbb{N}$ large enough so that $[2^{m_0}/n] - 1 \geqslant 2^{m_0}/2n > 1/\gamma$, where $[x]$ denotes the maximum integer less than or equal to x.

By Zhang's Theorem 3.3 above, we can find mutually orthogonal projections

$$q_{2^{m_0}+1} \preceq q_{2^{m_0}} \sim q_{2^{m_0}-1} \sim \cdots \sim q_1$$

with $1 = \sum_{i=1}^{2^{m_0}} q_i$. In particular, $\tau(q_{2^{m_0}+1}) \leqslant \tau(q_1) \leqslant 1/2^{m_0}$ and $1/(2^{m_0}+1) \leqslant \tau(q_1)$.

For each $1 \leqslant i \leqslant n$, let κ_i be the maximum positive integer so that $\kappa_i \tau(q_1) < \tau(p_i)$. By FCQ2, we can find mutually orthogonal projections $r_{i,1} \sim r_{i,2} \sim \cdots \sim r_{i,\kappa_i} \sim q_1$ so that

$$r_i := r_{i,1} + r_{i,2} + \cdots + r_{i,\kappa_i} < p_i.$$

Our assumption that $(\kappa_i + 1)\tau(q_1) \geqslant \tau(p_i)$ implies that if $s_i := p_i - r_i$, then $\tau(s_i) \leqslant \tau(q_i) \leqslant 1/2^{m_0}$.

Let $r = r_1 + r_2 + \cdots + r_n$. Then $r\mathcal{A}r \simeq \mathbb{M}_\kappa(q_1\mathcal{A}q_1)$, where $\kappa = \sum_{i=1}^{n} \kappa_i$. Moreover, for all $1 \leqslant i, j \leqslant n$, $p_i r_j = r_j p_i = \delta_{ij} r_j$, where δ_{ij} denotes the Kronecker delta function. Therefore $br = rb$. With respect to the matrix decomposition of $r\mathcal{A}r$ corresponding to $r = r_{1,1} + r_{1,2} + \cdots + r_{1,\kappa_1} + r_{2,1} + \cdots + r_{n,\kappa_n}$, we have

$$rbr = \operatorname{diag}(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_n, \ldots, \lambda_n),$$

where each λ_i appears κ_i times.

Now

$$\begin{aligned} |\tau(rbr)| = |\tau(rbr) - \tau(b)| &= \Big| \sum_{i=1}^{n} \lambda_i \tau(r_i) - \sum_{i=1}^{n} \lambda_i \tau(p_i) \Big| \\ &\leqslant \sum_{i=1}^{n} |\lambda_i| \tau(s_i) \leqslant \sum_{i=1}^{n} \frac{\|b\|}{2^{m_0}} = \frac{n\|b\|}{2^{m_0}} < \gamma\,\|b\|. \end{aligned}$$

Let $b_0 = rbr - (\tau(rbr)/\tau(r))\, r$, so that $\tau(b_0) = 0$. Now $1 - \tau(r) = \sum_{i=1}^{n} \tau(s_i) \leqslant n/2^{m_0} \leqslant \gamma$, so that $\tau(r) \geqslant 1 - \gamma$. Thus

$$\|b_0 - rbr\| \leqslant \Big| \frac{\tau(rbr)}{\tau(r)} \Big| \leqslant \frac{\gamma\,\|b\|}{1-\gamma} = \frac{\gamma\,\|a\|}{1-\gamma}.$$

If we now think of b_0 as an element of $\mathbb{M}_\kappa(\mathbb{C})$ (since $b_0 = q_1 \otimes b_{00}$ for some $b_{00} \in \mathbb{M}_\kappa(\mathbb{C})$), then by Lemma 3.5 of [14], we can find $u \in \mathbb{M}_\kappa(\mathbb{C}) \subseteq \mathbb{M}_\kappa(q_1\mathcal{A}q_1) \subseteq \mathcal{A}$ so that $\|u\| \leqslant \sqrt{2}\|b_0\|^{1/2} \leqslant \sqrt{2}(\|rbr\| + \gamma\,\|a\|/(1-\gamma))^{1/2} \leqslant \sqrt{2}(\|a\| + \gamma\,\|a\|/(1-\gamma))^{1/2} \leqslant \sqrt{2}(((1+\gamma)/(1-\gamma))\|a\|)^{1/2}$, and $b_0 = [u, u^*]$.

Let $s = \sum_{i=1}^{n} s_i$. Then $\tau(s) = \sum_{i=1}^{n} \tau(s_i) \leqslant n/2^{m_0}$. As before, $sb = bs$ and thus $b = rbr + sbs$. By FCQ2, we can find at least $\nu = [2^{m_0}/n] - 1$ mutually orthogonal copies of s, each orthogonal to s, say $s \sim t_1 \sim t_2 \sim \cdots \sim t_\nu$ so that $1 > s + \sum_{i=1}^{\nu} t_i$ and each $t_i = t_{i,1} + t_{i,2} + \cdots + t_{i,n}$ where $t_{i,j} \sim s_j$, $1 \leqslant j \leqslant n$, $1 \leqslant i \leqslant \nu$.

With respect to the decomposition $y = (s_1 + s_2 + \cdots + s_n) + (t_{1,1} + t_{1,2} + \cdots + t_{1,n}) + \cdots + (t_{\nu,1} + t_{\nu,2} + \cdots + t_{\nu,n})$, we can write $sbs \in y\mathcal{A}y$ as $sbs = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n, 0, 0, \ldots, 0)$. Let $\alpha_i = -\lambda_i/\nu$, $1 \leqslant i \leqslant n$, and with respect to the same matrix decomposition, set

$$c = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n, \alpha_1, \ldots, \alpha_n, \alpha_1, \ldots, \alpha_n, \ldots, \alpha_1, \ldots, \alpha_n).$$

For each $1 \leqslant j \leqslant n$,

$$(s_j + t_{1,j} + t_{2,j} + \cdots + t_{\nu,j})\mathcal{A}(s_j + t_{1,j} + t_{2,j} + \cdots + t_{\nu,j}) \simeq \mathbb{M}_{\nu+1}(s_j\mathcal{A}s_j).$$

Since $\operatorname{diag}(\lambda_j, \alpha_j, \alpha_j, \ldots, \alpha_j)$ is a scalar matrix with trace zero with respect to this decomposition, once again we can invoke Lemma 3.5 of [14] to find $w_j \in (s_j + t_{1,j} + t_{2,j} + \cdots + t_{\nu,j})\mathcal{A}(s_j + t_{1,j} + t_{2,j} + \cdots + t_{\nu,j})$ so that $\|w_j\| \leqslant 2|\lambda_j|^{1/2} \leqslant 2\|b\|^{1/2} = 2\|a\|^{1/2}$ and $\operatorname{diag}(\lambda_j, \alpha_j, \alpha_j, \ldots, \alpha_j) = [w_j, w_j^*]$.

Thus $c = [w, w^*]$ where $w = w_1 + w_2 + \cdots + w_n$. Note that $\|w\| = \max \|w_j\| \leqslant 2\|a\|^{1/2}$, and

$$\|c - sbs\| < \max\{|\alpha_j| : 1 \leqslant j \leqslant n\} \leqslant \max\Big\{\Big|\frac{\lambda_j}{\nu}\Big| : 1 \leqslant j \leqslant n\Big\} \leqslant \frac{\|a\|}{\nu} = \frac{\|a\|}{\left[\frac{2^{m_0}}{n}\right] - 1}.$$

We conclude that

$$\begin{aligned}\|b - [u, u^*] - [w, w^*]\| &\leqslant \|rbr - [u, u^*]\| + \|sbs - [w, w^*]\| \\ &\leqslant \|rbr - b_0\| + \|sbs - c\| \leqslant \frac{\gamma\,\|a\|}{1-\gamma} + \frac{\|a\|}{\left[\frac{2^{m_0}}{n}\right] - 1} \\ &< \frac{\gamma\,\|a\|}{1-\gamma} + \gamma\,\|a\| = \gamma\,\|a\|\Big(\frac{2-\gamma}{1-\gamma}\Big),\end{aligned}$$

where $\|u\| \leqslant \sqrt{2}[((1+\gamma)/(1-\gamma))\|a\|]^{1/2}$, $\|w\| \leqslant 2\|a\|^{1/2}$.

Thus the theorem holds if we choose $\gamma > 0$ small enough so that $(1+\gamma)/(1-\gamma) < 2$ and $\gamma(2-\gamma)/(1-\gamma) < \varepsilon/\|a\|$. Clearly this is possible.

Finally, since $\varepsilon > 0$ was arbitrary, $b \in \overline{\Sigma_2 \mathfrak{sc}\mathcal{A}}$. ∎

In the next lemma, it will be convenient to adopt the notations of [47] with regards to projections, namely: if $p \in \mathfrak{P}(\mathcal{A})$, we denote the equivalence class of projections in $\mathcal{A}$ with representative p by $[p]$. For $p, q \in \mathfrak{P}(\mathcal{A})$, we write $[p] \leqslant [q]$ (respectively $[p] < [q]$) to mean that p is equivalent to a subprojection (respectively a proper subprojection) of q. A local addition on the set of such equivalence classes is defined as follows: $[p] + [q]$ is defined if there exist $p', q' \in \mathfrak{P}(\mathcal{A})$ with $p' \sim p$, $q' \sim q$ and $p'q' = 0$, in which case $[p] + [q] := [p' + q']$. For $m \in \mathbb{N}$, $m[p] = [p] + [p] + \cdots + [p]$ (m times), provided that this exists.

We shall also make use of the following numerical estimate. The routine verification thereof is left to the reader. Suppose $\sigma, \rho \geqslant 10$ are two integers satisfying $(\rho+1)/\sigma < 4/5$. If $\alpha \geqslant 5$, then

$$\frac{(2^\alpha+1)(\rho+1)+1}{(2^{\alpha+1})\sigma - (2^\alpha+1)(\rho+1)} < \frac{4}{5}\,.$$

For simple, unital AF C^*-algebras, the existence of a family of projections satisfying the conditions of the next lemma is due to T. Fack [14]. For the inductive limits to be considered in the next section, the result is due to K. Thomsen [42].

LEMMA 3.7. *Let $\mathcal{A}$ be a simple, unital C^*-algebra with real rank zero. Then there exist sequences $(p_n)_{n=1}^\infty$, $(q_n)_{n=1}^\infty$, $(r_n)_{n=1}^\infty$ in $\mathcal{A}$ satisfying:*

(i) $1 = p_1 + q_1 + r_1$;
(ii) $p_n \preceq q_n \preceq r_n$ *for all* $n \geqslant 1$;
(iii) $r_{n-1} = p_n + q_n$ *for all* $n \geqslant 2$;
(iv) *the* r_n*'s are all mutually orthogonal.*

Proof. The basic idea is as follows: first we shall partition the identity as a sum of three projections $1 = p_1 + q_1 + r_1$ of "approximately equal size" (as measured by their traces), satisfying $p_1 \preceq q_1 \preceq r_1$. To apply induction we assume that $p_1, p_2, \ldots, p_k, q_1, \ldots, q_k, r_1, \ldots, r_k$ have been constructed. We then partition r_k *almost* in half, as $r_k = p_{k+1} + q_{k+1}$, with $p_{k+1} \preceq q_{k+1}$ (very roughly speaking, $\tau(p_{k+1}) \approxeq ((2^\alpha-1)/2^\alpha)\tau(r_k), \tau(q_{k+1}) \approxeq ((2^\alpha+1)/2^\alpha)\tau(r_k)$ for sufficiently large $\alpha = \alpha(k)$). We shall then construct $r_{k+1} < 1 - \sum_{j=1}^{k} r_j$ with $\tau(r_{k+1}) > \tau(q_{k+1})$. This last step forces us to maintain both lower and upper bounds on $\tau(r_{k+1})$, and this is where the above numerical estimate comes into play.

First we use Zhang's Theorem to find projections $y_1, y' \in \mathcal{A}$ so that $1 = 1024[y_1] + [y']$ where $[y'] < [y_1]$. Choose e_1 with $[e_1] = [y'] + 4[y_1] < 5[y_1]$.

We then choose r_1 with $[r_1] = 350[y_1]$, and choose a projection $q_1 < 1 - r_1$ with $[q_1] = 340[y_1]$, whence $q_1 \prec r_1$, and let $p_1 = 1 - (q_1 + r_1)$ so that $[p_1] = 334[y_1] + [y'] < [q_1]$. Thus (i) is satisfied and (ii) is satisfied with $n = 1$.

Let $\rho_1 = 350, \sigma_1 = 670$. Write $s_1 = 1 - r_1$ so that $[s_1] = \sigma_1[y_1] + [e_1]$ where $[e_1] < 5[x]$. Observe that $\rho_1/(\rho_1+\sigma_1) = 350/1020 < 2/5$, $(\rho_1+1)/\sigma_1 = 351/670 < 4/5$, and $\sigma_1, \rho_1 > 10$.

Suppose that we have chosen $p_1, \ldots, p_k, q_1, \ldots, q_k, r_1, \ldots, r_k$ so that:

(a) $1 = p_1 + q_1 + r_1$;
(b) $p_j \preceq q_j \preceq r_j, 1 \leqslant j \leqslant k$;
(c) $r_{j-1} = p_j + q_j, 2 \leqslant j \leqslant k$;
(d) the r_j's are orthogonal, $1 \leqslant j \leqslant k$, and

(e) there exist $\rho_k, \sigma_k > 10$ satisfying $(\rho_k+1)/\sigma_k < 4/5$, $\rho_k/(\rho_k+\sigma_k) < 2/5$ and a projection y_k so that $[r_k] = \rho_k[y_k]$, and $s_k := 1 - \sum_{j=1}^{k} r_j$ satisfies $[s_k] = \sigma_k[y_k] + [e_k]$, where $[e_k] < 5[y_1]$.

(We have just seen that this is possible when $k = 1$, which begins our induction step. That (a) holds is therefore clear.) Then we claim that we can find p_{k+1}, q_{k+1}, and r_{k+1} which satisfy (a) through (e), which completes the induction, and thus the proof, since we only need satisfy (a) through (d)!

Choose $m_k \in \mathbb{N}$ so that $2^{m_k} > \sigma_k$. Using Zhang's Theorem, we can find projections $z_k, z'_k \in \mathcal{A}$ so that $z'_k \preceq z_k$ and $5[y_1] - [e_k] = 2^{m_k}[z_k] + [z'_k] > \sigma_k[z_k] + [z'_k]$. Now choose $\alpha_k \geqslant k+5$ so that

$$\frac{2^{\alpha_k}+1}{2^{\alpha_k+1}}\frac{\rho_k+1}{\sigma_k} < \frac{2}{5}. \tag{3.1}$$

This is possible because $(\sigma_k+1)/\sigma_k < 4/5$ and $(2^{\alpha_k}+1)/2^{\alpha_k+1}$ tends to $1/2$ as α_k tends to infinity.

A second application of Zhang's Theorem yields projections $y_{k+1}, y'_{k+1} \in \mathcal{A}$ so that $[y_k] = 2^{\alpha_k+1}[y_{k+1}] + [y'_{k+1}]$, where $[y'_{k+1}] \leqslant [y_{k+1}]$ and $[y'_{k+1}] < [z_k]$, whence $\sigma_k[y'_{k+1}] < 5[y_1] - [e_k]$.

Thus

$$\begin{aligned}
[r_k] &= \rho_k[y_k] = \rho_k(2^{\alpha_k+1}[y_{k+1}] + [y'_{k+1}]),\\
[s_k] &= \sigma_k[y_k] + [e_k] = \sigma_k(2^{\alpha_k+1}[y_{k+1}] + [y'_{k+1}]) + [e_k].
\end{aligned}$$

Choose $q_{k+1} < r_k$ with $[q_{k+1}] = \rho_k(2^{\alpha_k}+1)[y_{k+1}]$, and set $p_{k+1} = r_k - q_{k+1}$. Clearly (c) is satisfied with $j = k+1$. Now

$$[p_{k+1}] = \rho_k(2^{\alpha_k+1}[y_{k+1}] + [y'_{k+1}]) - \rho_k(2^{\alpha_k}+1)[y_{k+1}] = \rho_k((2^{\alpha_k}-1)[y_{k+1}] + [y'_{k+1}]).$$

Since $[y'_{k+1}] < [y_{k+1}]$, $p_{k+1} \preceq q_{k+1}$. Since $\rho_k/(\rho_k+\sigma_k) < 2/5$, we have $\sigma_k \geqslant \rho_k+1$, so that $(2^{\alpha_k}+1)(\rho_k+1) < 2^{\alpha_k+1}\sigma_k$. Thus we can choose $r_{k+1} < s_k$ with $[r_{k+1}] = (\rho_k+1)(2^{\alpha_k}+1)[y_{k+1}]$, which implies that r_{k+1} is orthogonal to all of the other r_j's, $1 \leqslant j \leqslant k$. That is, (d) holds for $j = k+1$. It is also clear that $q_{k+1} \preceq r_{k+1}$; thus (b) also holds for $j = k+1$.

Let $\rho_{k+1} = (2^{\alpha_k}+1)(\rho_k+1) > 10$, and set

$$\begin{aligned}
s_{k+1} &= 1 - \sum_{j=1}^{k+1} r_j = s_k - r_{k+1}\\
[s_{k+1}] &= \sigma_k(2^{\alpha_k+1}[y_{k+1}] + [y'_{k+1}]) + [e_k] - (\rho_k+1)(2^{\alpha_k}+1)[y_{k+1}]\\
&= (2^{\alpha_k+1}\sigma_k - (2^{\alpha_k}+1)(\rho_k+1))[y_{k+1}] + \sigma_k[y'_{k+1}] + [e_k].
\end{aligned}$$

If we choose e_{k+1} with $[e_{k+1}] = \sigma_k[y'_{k+1}] + e_k$, then we have $[e_{k+1}] < 5[y_1]$. Letting $\sigma_{k+1} = 2^{\alpha_k+1}\sigma_k - (2^{\alpha_k}+1)(\rho_k+1) \geqslant (2^{\alpha_k+1})(\rho_k+1) - (2^{\alpha_k}+1)(\rho_k +$

$1) = (2^{\alpha_k} - 1)(\rho_k + 1) > 10$ we have finally,

$$\frac{\rho_{k+1}}{\rho_{k+1} + \sigma_{k+1}} = \frac{(2^{\alpha_k} + 1)(\rho_k + 1)}{(2^{\alpha_k} + 1)(\rho_k + 1) + (2^{\alpha_k+1}\sigma_k - (2^{\alpha_k} + 1)(\rho_k + 1))}$$
$$= \frac{(2^{\alpha_k} + 1)}{2^{\alpha_k+1}} \frac{(\rho_k + 1)}{\sigma_k} < \frac{2}{5},$$

by our choice of α_k in Equation (3.1) above. Also,

$$\frac{\rho_{k+1} + 1}{\sigma_{k+1}} = \frac{(2^{\alpha_k} + 1)(\rho_k + 1) + 1}{2^{\alpha_{k+1}}\sigma_k - (2^{\alpha_k} + 1)(\rho_k + 1)} < \frac{4}{5},$$

as mentioned in the preamble to the lemma. This completes the induction step, and hence the proof. ∎

The following known result will prove useful.

LEMMA 3.8. *Let $\mathcal{A}$ be a simple, unital C^*-algebra and r be a non-zero projection in $\mathcal{A}$. Then every tracial state on $r\mathcal{A}r$ has an extension to a bounded trace on $\mathcal{A}$.*

A proof can be found in the paper [6] in the proof of Proposition 5.2, where it is shown that if $x \in r\mathcal{A}r$ and $x \in \mathcal{A}_0$, then $x \in (r\mathcal{A}r)_0$.

LEMMA 3.9. *Let $\mathcal{A}$ be a simple, unital C^*-algebra with real rank zero and satisfying* FCQ2. *Suppose that $\mathcal{A}$ has a unique tracial state τ. If $a \in \mathcal{A}_0$, then there exists $x_1, \dots, x_8 \in \mathcal{A}$ so that $a = \sum_{i=1}^{8} [x_i, x_i^*]$. That is, $a \in \Sigma_8\mathfrak{sc}(\mathcal{A})$. Thus $\mathfrak{sl}(\mathcal{A}) \subseteq \Sigma_{16}\mathfrak{c}(\mathcal{A})$.*

Proof. The proof is an adaptation of T. Fack's proof of Theorem 3.1 in [14], and of K. Thomsen's modification of it ([42], Theorem 1.8).

Choose projections $(p_n)_{n=1}^{\infty}$, $(q_n)_{n=1}^{\infty}$, $(r_n)_{n=1}^{\infty}$ satisfying the conditions of Lemma 3.7. We can assume without loss of generality that $\|a\| < 1$. By Lemma 3.4 of [14], there exist $u, v \in \mathcal{A}$ with $\|u\| \leqslant 2\|a\|^{1/2}$ and $\|v\| \leqslant 13\|a\|^{1/2}$ so that $a = [u, u^*] + [v, v^*] + a_1$, where $a_1 \in r_1\mathcal{A}r_1$ and $\|a_1\| < 3$. Now $a_1 \in \mathcal{A}_0$, so $a_1 \in (r_1\mathcal{A}r_1)_0$, by the comment preceding this lemma.

We recursively define elements $u(i, n) \in r_n\mathcal{A}r_n$, $i = 1, 2$, $a_n \in (r_n\mathcal{A}r_n)_0$, $v_n, w_n \in (r_n + r_{n+1})\mathcal{A}(r_n + r_{n+1})$ so that

$$a_n = [u(1, n), u(1, n)^*] + [u(2, n), u(2, n)^*] + [v_n, v_n^*] + [w_n, w_n^*] + a_{n+1},$$

where $\|a_n\| \leqslant 3/n$, $\|u(i, n)\| < 2\sqrt{3/n}$, $i = 1, 2$, $\|v_n\|, \|w_n\| < 1/2^n$, $n \in \mathbb{N}$.

Suppose that both $a_1, \dots, a_n$, $u(1, 1), \dots, u(1, n-1)$, $u(2, 1), \dots, u(2, n-1)$, $v_1, \dots, v_{n-1}$ and $w_1, \dots, w_{n-1}$ have been constructed.

Now $a_n \in (r_n\mathcal{A}r_n)_0$ and $r_n\mathcal{A}r_n$ is simple, unital and has real rank zero [5]. Furthermore, $r_n\mathcal{A}r_n$ satisfies FCQ2 by Lemma 3.5. Now, by Proposition 3.6, given $0 < \delta$ satisfying $13\sqrt{\delta} \leqslant 2^{-n}$, there exist $u(1, n), u(2, n) \in r_n\mathcal{A}r_n$ with $\|u(i, n)\| \leqslant 2\|a_n\|^{1/2} < 2\sqrt{3/n}$, $i = 1, 2$ so that

$$\|a_n - [u(1, n), u(1, n)^*] - [u(2, n), u(2, n)^*]\| < \delta.$$

If $z = a_n - [u(1,n), u(1,n)^*] - [u(2,n), u(2,n)^*]$, then $z \in (r_n \mathcal{A} r_n)_0$ and $\|z\| < \delta$. By Lemma 3.4 of [14], there exist $v_n, w_n \in (r_n + r_{n+1})\mathcal{A}(r_n + r_{n+1})$ such that $\|v_n\| \leqslant 2\|z\|^{1/2}$, $\|w_n\| \leqslant 13\|z\|^{1/2}$ for which $a_{n+1} := z - [v_n, v_n^*] - [w_n, w_n^*] \in r_{n+1}\mathcal{A}r_{n+1}$ and $\|a_{n+1}\| \leqslant 3\|z\|$.

It is not hard to see that since $(r_n \mathcal{A} r_n)_0 \subseteq ((r_n + r_{n+1})\mathcal{A}(r_n + r_{n+1}))_0$, we get $a_{n+1} \in ((r_n + r_{n+1})\mathcal{A}(r_n + r_{n+1}))_0$. But every tracial state of $r_{n+1}\mathcal{A}r_{n+1}$ extends to a positive bounded trace on $(r_n + r_{n+1})\mathcal{A}(r_n + r_{n+1})$. Thus $a_{n+1} \in (r_{n+1}\mathcal{A}r_{n+1})_0$. Moreover, $\|u(i,n)\| \leqslant 2\sqrt{3/n}$, $i = 1, 2$ while $\|v_n\|, \|w_n\| \leqslant 2^{-n}$. This completes the induction step.

Set $x_1 = u$, $x_2 = v$, $x_3 = \sum\limits_{n=1}^{\infty} u(1,n)$, $x_4 = \sum\limits_{n=1}^{\infty} u(2,n)$, $x_5 = \sum\limits_{i \text{ even}} v_i$, $x_6 = \sum\limits_{i \text{ odd}} v_i$, $x_7 = \sum\limits_{i \text{ even}} w_i$, and $x_8 = \sum\limits_{i \text{ odd}} w_i$. ∎

3.3. An examination of the proof shows that we may take $\|x_i\| \leqslant 13\|a\|^{1/2}$, $1 \leqslant i \leqslant 8$. We shall return to this later.

THEOREM 3.10. *Let $\mathcal{A}$ be a simple, unital C^*-algebra of real rank zero and satisfying* FCQ2. *Suppose that $\mathcal{A}$ has a unique tracial state τ. Then $\mathfrak{sl}(\mathcal{A}) = \Sigma_2\mathfrak{c}(\mathcal{A})$, and $\mathcal{A}_0 = \Sigma_4\mathfrak{sc}(\mathcal{A})$.*

Proof. The proof is an application of the commutator reduction argument, Theorem 2.3 above, and is modeled after the proof for AF C^*-algebras in Section 2. Since $\mathcal{A}$ is simple and of real rank zero, we can apply Zhang's Theorem 3.3 to find mutually orthogonal projections $q_1 \preceq q_2 \sim q_3 \sim \cdots \sim q_{33}$ so that $1 = \sum\limits_{i=1}^{33} q_i$.

Suppose $a \in \mathfrak{sl}(\mathcal{A})$ and let $a_{jk} = q_j a q_k$, $1 \leqslant j,k \leqslant 33$. If we let $v_k, w_{j,k}$ denote the operators from Lemma 2.1, then for any trace τ on $\mathcal{A}$, we get

$$\tau(a) = \sum_{k=1}^{33} \tau(a_{kk}) = \sum_{k=1}^{33} \tau(w_{k,33}^* a_{kk} w_{k,33}).$$

Since any trace on $q_{33}\mathcal{A}q_{33}$ extends to a trace on $\mathcal{A}$ by Lemma 3.8, it follows immediately that $\sum\limits_{k=1}^{33} w_{k,33}^* b_{kk} w_{k,33} \in \mathfrak{sl}(q_{33}\mathcal{B}q_{33})$.

But $q_{33}\mathcal{A}q_{33}$ is simple ([29], Theorem 3.2.8), is of real rank zero ([5]) and satisfies FCQ2. Furthermore, q_{33} serves as an identity for this algebra. By Lemma 3.9, every trace zero element of $q_{33}\mathcal{A}q_{33}$ lies in $\Sigma_{16}\mathfrak{c}(q_{33}\mathcal{A}q_{33})$.

By Theorem 2.3, $b \in \Sigma_2\mathfrak{c}(\mathcal{A})$. Since $b \in \mathfrak{sl}(\mathcal{A})$ was arbitrary, we are done. The second statement follows immediately from this, combined with Theorem 2.4. ∎

We thank D. Hadwin for pointing out that the previous theorem applies to the case of type II_1 factor von Neumann algebras:

COROLLARY 3.11. *Let $\mathcal{M}$ be a finite II_1 factor von Neumann algebra. Then every element of trace zero in $\mathcal{M}$ is a sum of two commutators, and every selfadjoint element of trace zero is a sum of four or fewer self-commutators.*

To the best of our knowledge, the best previous estimates belonged to T. Fack and P. de la Harpe [15], who showed that every trace zero element in such a factor is a sum of 10 or fewer commutators, and to C. Pearcy and D. Topping [32], who showed that in a certain class of II_1 factors known as type II_1 *factors of Wright*, every selfadjoint trace zero element is a *single* commutator, and hence, every trace zero element is a sum of two commutators.

4. APPROXIMATELY HOMOGENEOUS C^*-ALGEBRAS

4.1. In this section we shall be applying the commutator reduction argument, Theorem 2.3 to a class of approximately homogeneous (AH) C^*-algebras appearing in K. Thomsen's generalization of Fack's results [42]. Before defining that class, we shall need a definition, due to Nistor [31].

DEFINITION 4.1. Let (G, G_+) be an ordered group. We say that G has *large denominators* if for any $0 \leqslant a \in G$ and $n \in \mathbb{N}$, there exist $b \in G$ and $m \in \mathbb{N}$ so that $nb \leqslant a \leqslant mb$.

4.2. An element $\mathcal{A}$ of the class of algebras we wish to consider here is a unital C^*-algebra which is direct limit of algebras $\mathcal{A}_n$ of the form

$$\mathcal{A}_n \simeq \bigoplus_{j=1}^{k_n} \mathcal{C}(X_{n,j}) \otimes \mathbb{M}_{t(n,j)},$$

where $X_{n,j}$ is a compact, Hausdorff, connected space and $t(n,j) \in \mathbb{N}$ for all $1 \leqslant j \leqslant k_n$ and $n \geqslant 1$.

We shall say that the sequence $(\mathcal{A}_n)_n$ is of *bounded dimension* if there exists $d \in \mathbb{N}$ such that $\sup\limits_{n,j}\{\dim(X_{n,j})\} \leqslant d$, where $\dim(X_{n,j})$ refers to the *covering dimension* of $X_{n,j}$. We shall refer to $\mathcal{A}$ as being of bounded dimension d if it can be expressed as a limit whose building blocks have bounded dimension d. Without loss of generality, we may assume that each connecting homomorphism $\varphi_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$ is unital. (We refer the reader to [13] for details about covering dimension, and to [43] for more information about the K_0-groups appearing in the next statement.)

THEOREM 4.2 ([42], Theorem 1.8). *Let $\mathcal{A}$ be a unital* AH *C^*-algebra of the type considered in Section* 4.2. *Assume that $\mathcal{A}$ is of bounded dimension d and that $K_0(A)$ has large denominators.*

If $a \in \mathcal{A}_0$ then there exist $d+7$ elements $x_1, x_2, \ldots, x_{d+7} \in \mathcal{A}$ such that $a = \sum_{i=1}^{d+7} [x_i, x_i^*]$.

4.3. As pointed out by Thomsen, if $\mathcal{A}$ is an inductive limit as in Subsection 4.2 and $\mathcal{A}$ is simple, then $\mathcal{A}$ has large denominators unless $\mathcal{A}$ is finite-dimensional. Thus this result properly generalizes that of Fack.

If we wish to apply the same argument we used in Theorem 2.5 to this class of C^*-algebras, we are confronted with an obstacle, namely: the reduction argument requires us to find "small" projections q is $\mathcal{A}$ such that $\mathfrak{sl}(q\mathcal{A}q) \subseteq \Sigma_m \mathfrak{c}(q\mathcal{A}q)$ for some $m \geqslant 1$. When $\mathcal{A}$ was a simple, unital AF C^*-algebra, then so was $q\mathcal{A}q$, and so Fack's Theorem applied equally well to both $\mathcal{A}$ and $q\mathcal{A}q$, allowing us to choose $m = 14$. In the present case, it is no longer obvious that the fact that $\mathfrak{sl}(\mathcal{A}) \subseteq \Sigma_{14+2d}\mathfrak{c}(\mathcal{A})$ implies that $\mathfrak{sl}(q\mathcal{A}q) \subseteq \Sigma_m \mathfrak{c}(q\mathcal{A}q)$ for any $m \geqslant 1$. What is required is a more detailed analysis of Thomsen's proof, along with a couple of modifications which allow us to conclude that this does indeed hold.

A key to the proof (this is where Thomsen used the fact that $K_0(\mathcal{A})$ has "large denominators") is the existence of the families $\{p_n\}_n$, $\{q_n\}_n$, and $\{r_n\}_n$ of projections satisfying the conditions of Lemma 3.7. In this setting, the existence is provided by Lemma 1.7 of [42].

REMARK 4.3. The key observation that will allow us to extend Thomsen's result to our setting is that his proof of Theorem 4.2 actually shows something stronger, namely: suppose that $n_0 \geqslant 1$ and that $s = s^* = s^2 \in \mathcal{A}$ is a projection such that $s \geqslant r_j$ for all $j \geqslant n_0$. Suppose furthermore that $a = a^* \in (r_{n_0}\mathcal{A}r_{n_0})_0$. Then a is a sum of $d+5$ self-commutators from $s\mathcal{A}s$.

Thomsen's proof is a modification of the proof of Fack's result ([14], Theorem 3.1), and is the basis for Theorem 3.9 above. Rather than reproducing this proof yet again, we shall restrict ourselves to pointing out the relevant minor modifications. As such, the following comments refer to Theorem 1.8 of [42], and we maintain the notation used there.

When a and s are as above, it suffices to observe that in Thomsen's proof we can choose $u = v = 0$, $u(i,n) = 0$ if $n < n_0$, $i = 3, 4, \ldots, d+3$ and $v_i = w_i = 0$ if $n < n_0$.

Then $x_1 = u = 0, x_2 = v = 0$ both lie in $s\mathcal{A}s$ (trivially), and since $u(i-2, n) \in s\mathcal{A}s$ for all $n \geqslant n_0$, we get $x_i = \sum_{n=1}^{\infty} u(i-2,n) \in s\mathcal{A}s$, $3 \leqslant i \leqslant d+3$. We then let

$$x_{d+4} = \sum_{i \geqslant n_0, i \text{ even}} v_i\,, \qquad x_{d+6} = \sum_{i \geqslant n_0, i \text{ even}} w_i\,,$$
$$x_{d+5} = \sum_{i \geqslant n_0, i \text{ odd}} v_i\,, \qquad x_{d+6} = \sum_{i \geqslant n_0, i \text{ odd}} w_i\,,$$

and note that $v_i, w_i \in s\mathcal{A}s$ if $i \geqslant n_0$, so that $x_{d+4}, x_{d+5}, x_{d+6}, x_{d+7} \in s\mathcal{A}s$.

We now employ a technical device which will allow us to exploit this minor generalization.

LEMMA 4.4. *Let $\mathcal{A}$ be a unital* AH C^**-algebra of the type considered in Subsection* 4.2. *Suppose that $\mathcal{A}$ is of bounded dimension d and that $K_0(\mathcal{A})$ has large denominators. Let $1 < \mu \in \mathbb{N}$ be fixed. Given $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < 1/(2\mu+1)$, we can find an integer $\nu > \mu$ and two families $\{q_i\}_{i=0}^{\nu}$ and $\{p_j\}_{j=1}^{\lambda}$ of mutually orthogonal projections in $\mathcal{A}$ satisfying:*

(i) $1 = q_0 + q_1 + \cdots + q_\nu$;
(ii) $q_\nu \preceq q_{\nu-1} \preceq \cdots \preceq q_\mu \sim q_{\mu-1} \sim \cdots \sim q_1 \sim q_0$;
(iii) $\alpha_2 < \tau(q_0) < \beta_2$ *for all* $\tau \in \mathcal{T}(\mathcal{A})$;
(iv) $q_0 = p_1 + p_2 + \cdots + p_\lambda$;
(v) $p_\lambda \preceq p_{\lambda-1} \preceq \cdots \preceq p_1$;
(vi) $\alpha_1 < \tau(p_1) < \beta_1$ *for all* $\tau \in \mathcal{T}(\mathcal{A})$.

Proof. In essence, the proof of this result reduces to proving it in a sufficiently large finite-dimensional C^*-algebra, that is, in a direct sum of full matrix algebras, where it is simply an issue of ranks of projections, and then invoking the fact proven in [42] that $\mathcal{A}$ contains such an algebra.

As pointed out in p. 230 of [42], the assumption that $K_0(\mathcal{A})$ has large denominators is equivalent to the fact that for all $i \in \mathbb{N}$ and all minimal non-zero central projections $e_{il} \in \mathcal{A}_i$, we have

$$\lim_{j\to\infty}(\min\{\mathrm{rank}\varphi_{ji}(e_{il})_k : 1 \leqslant k \leqslant n_j : \mathrm{rank}\varphi_{ji}(e_{il})_k \neq 0\}) = \infty.$$

Given $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < 1$, we can find $M \in \mathbb{N}$ such that $M > \mu + 1$, so that $1/M < \min(\beta_1 - \alpha_1, \beta_2 - \alpha_2)$.

Let $i \in \mathbb{N}$. Fix $J > 0$ so that $j \geqslant J$ implies that

$$(\min\{\mathrm{rank}\varphi_{ji}(e_{il})_k : 1 \leqslant k \leqslant n_j : \mathrm{rank}\varphi_{ji}(e_{il})_k \neq 0\}) \geqslant M.$$

Since each φ_{ji} is unital, and since $1 = e_{i1} + e_{i2} + \cdots + e_{in_i}$, it follows that for any $k \in \{1, 2, \ldots, n_J\}$ there exists $l_k \in \{1, 2, \ldots, n_i\}$ so that $\varphi_{Ji}(e_{il_k})_k \neq 0$. From this it follows that $\mathrm{rank}\varphi_{Ji}(e_{il_k})_k \geqslant M$, whence $t(J,k) \geqslant M$ for all $1 \leqslant k \leqslant n_J$.

Thus $\mathcal{A}$ contains a finite-dimensional unital C^*-algebra $\mathcal{B}$ so that $\mathcal{B} \simeq^* \bigoplus_{k=1}^{n_J} \mathbb{M}_{r_k}$, where $r_k \geqslant M$ for all $1 \leqslant k \leqslant n_j$. Indeed, let $\mathcal{B} = \bigoplus_{k=1}^{n_J} 1_k \otimes \mathbb{M}_{t(J,k)}$ where 1_k denotes the constant function $1_k(x) = 1$ for all $x \in X_{J,k}$, $1 \leqslant k \leqslant n_j$.

For each such k, $1/r_k \leqslant 1/M < \min(\beta_1 - \alpha_1, \beta_2 - \alpha_2)$, and $(\mu+1)\beta_2 < 1$. Choose the largest integer ρ_k so that $\rho_k/r_k < \beta_2$. It is straightforward to verify that

$$\frac{\rho_k}{r_k} > \alpha_2 \quad \text{and} \quad (\mu+1)\rho_k < r_k\,.$$

Thus we can find $\mu+1$ mutually orthogonal projections $q_{0,k}, q_{1,k}, \ldots, q_{\mu,k} \in \mathbb{M}_{r_k}$ so that

$$\alpha_2 < \frac{\mathrm{rank} q_{j,k}}{r_k} = \frac{\rho_k}{r_k} < \beta_2, \quad 0 \leqslant j \leqslant \mu.$$

Note that $q_{i,k} \sim q_{j,k}$ for all $0 \leqslant i, j \leqslant \mu$, since their ranks agree.

Let $\gamma_k = (r_k - (\mu+1)\rho_k)$, $1 \leqslant k \leqslant n_J$, and set $\gamma = \max\{\gamma_1, \gamma_2, \ldots, \gamma_{n_J}\}$. If I_k denotes the identity operator in M_{r_k}, then we can write $I_k - \left(\sum_{j=0}^{\mu} q_{j,k}\right)$ as the sum of γ_k rank one projections $q_{\mu+1,k}, q_{\mu+2,k}, \ldots, q_{\mu+\gamma_k,k}$. For $\gamma_k < l \leqslant \gamma$, let $q_{\mu+l,k} = 0$, and consider

$$q_j = \sum_{k=1}^{n_J} q_{j,k}, \quad 0 \leqslant j \leqslant \gamma.$$

Since $q_{\mu+\gamma,k} \preceq q_{\mu+\gamma-1,k} \preceq \cdots \preceq q_{\mu,k} \sim q_{\mu-1,k} \sim \cdots \sim q_{1,k} \sim q_{0,k}$ for each k, it follows that

$$q_{\mu+\gamma} \preceq q_{\mu+\gamma-1} \preceq \cdots \preceq q_\mu \sim q_{\mu-1} \sim \cdots \sim q_1 \sim q_0.$$

Moreover, since $I_k = \sum_{j=0}^{\mu+\gamma_k} q_{j,k} = \sum_{j=0}^{\mu+\gamma} q_{j,k}$, it follows that $I = \sum_{j=0}^{\mu+\gamma} q_j$.

To obtain the projections $p_1, p_2, \ldots, p_\lambda$, we apply a similar argument. Since $1/r_k \leqslant 1/M < \min(\beta_1 - \alpha_1, \beta_2 - \alpha_2)$ and since $\beta_2 < \alpha_1$, we can choose a subprojection $p_{1,k}$ of $q_{0,k}$ so that

$$\alpha_1 < \frac{\mathrm{rank}\, p_{1,k}}{r_k} < \beta_1, \quad 1 \leqslant k \leqslant n_J.$$

Let $\lambda_k = 1 + (\mathrm{rank}\, q_{0,k} - \mathrm{rank}\, p_{1,k})$ for $1 \leqslant k \leqslant n_J$, and set $\lambda = \max\{\lambda_1, \lambda_2, \ldots, \lambda_{n_J}\}$. We can then write $q_{1,k} - p_{1,k}$ as a sum of $\lambda_k - 1$ rank one subprojections of $q_{0,k}$, say $\{p_{2,k}, p_{3,k}, \ldots, p_{\lambda_k,k}\}$. For $\lambda_k < l \leqslant \lambda$, we let $p_{l,k} = 0$ and consider

$$p_m = \sum_{k=1}^{n_J} p_{m,k}, \quad 1 \leqslant m \leqslant \lambda.$$

Since $p_{\lambda,k} \preceq p_{\lambda-1,k} \preceq \cdots \preceq p_{1,k}$ for all $1 \leqslant k \leqslant n_J$, it follows that $p_\lambda \preceq p_{\lambda-1} \preceq \cdots \preceq p_1$. Moreover, since $q_{0,k} - p_{1,k} = \sum_{j=2}^{\lambda_k} p_{j,k} = \sum_{j=2}^{\lambda} p_{j,k}$ for all k, we also have $q_0 = \sum_{m=1}^{\lambda} p_m$.

Finally, any trace $\tau \in \mathcal{T}(\mathcal{A})$ restricts to a trace on $\mathcal{B}$. Since these are convex combinations of the normalized (extremal) traces onto each factor $\mathbb{M}_{r_k}$, and since the normalized trace on $\mathbb{M}_{r_k}$ is unique, it follows that $\alpha_1 < \tau(p_1) < \beta_1$ and $\alpha_2 < \tau(q_0) < \beta_2$ for all $\tau \in \mathcal{T}(\mathcal{A})$. ∎

The following result must certainly be known. We state it for ease of reference:

LEMMA 4.5. *Suppose that $\mathcal{A}$ is a unital C^*-algebra which satisfies* FCQ2. *Suppose also that p, q and r are projections in $\mathcal{A}$ which satisfy:*

(i) $\tau(p) < \tau(r) < \tau(q)$ *for all* $\tau \in \mathcal{T}(\mathcal{A})$;

(ii) $p < q$.

Then $r \sim t$ where $p < t < q$.

Proof. Since $\tau(p) < \tau(r)$ for all $\tau \in \mathcal{T}(\mathcal{A})$, FCQ2 implies that $p \sim p_0$ for some projection $p_0 < r$. Thus $s_0 := r - p_0$ is a projection and $r = p_0 + s_0$. Now $\tau(s_0) = \tau(r - p_0) = \tau(r) - \tau(p_0) = \tau(r) - \tau(p) < \tau(q) - \tau(p) = \tau(q - p)$ for all $\tau \in \mathcal{T}(\mathcal{A})$. Again, by FCQ2, there exists a projection s so that $s_0 \sim s < q - p$.

But then p, s, p_0, s_0 are projections in $\mathcal{A}$ with $p \sim p_0$, $s \sim s_0$, $ps = sp = 0 = p_0 s_0 = s_0 p_0$. By Lemma 5.2.3 of [43], $p < p + s \sim p_0 + s_0 = r$, and $p + s < p + (q - p) = q$. Letting $t = p + s$ completes the proof. ∎

REMARK 4.6. If $\mathcal{A}$ be a unital AH C^*-algebra of the type considered in Subsection 4.2, and if $\mathcal{A}$ has bounded dimension d and $K_0(\mathcal{A})$ has large denominators, then $\mathcal{A}$ has *slow dimension growth* in the sense of Martin and Pasnicu [28]. By Theorem 3.7 of that paper along with the comments preceding that theorem, $\mathcal{A}$ satisfies FCQ2. This will be used in the following two results.

LEMMA 4.7. *Let $\mathcal{A}$ be a unital* AH *C^*-algebra of the type considered in Subsection* 4.2. *Suppose that $\mathcal{A}$ is of bounded dimension d and that $K_0(\mathcal{A})$ has large denominators. Then there exists $\nu > 100d$ and mutually orthogonal projections $\overline{q}_0, \overline{q}_1, \ldots, \overline{q}_\nu \in \mathcal{A}$ so that:*

(i) $1 = \sum_{k=0}^{\nu} \overline{q}_k$;

(ii) $\overline{q}_\nu \preceq \overline{q}_{\nu-1} \preceq \cdots \preceq \overline{q}_{100d} \sim \cdots \sim \overline{q}_1 \sim \overline{q}_0$;

(iii) *if $a = a^* \in \overline{q}_0 \mathcal{A} \overline{q}_0$ satisfies $\tau(a) = 0$ for all $\tau \in \mathcal{T}(\mathcal{A})$, then $a \in \Sigma_{d+7}\mathfrak{C}(\overline{q}_0 \mathcal{A} \overline{q}_0)$.*

Proof. Let $\mu = 200d$; $\alpha_2 = 1/(400d)$, $\beta_2 = 1/(300d)$. Choose $n_0 \in \mathbb{N}$ so that $1/2^{n_0 - 1} < \alpha_2$, and then choose $\alpha_1 = 1/(200 \cdot 2^{n_0})$, $\beta_1 = 1/(100 \cdot 2^{n_0})$.

We can then use Lemma 4.4 to find projections $\overline{q}_0, \overline{q}_1, \ldots, \overline{q}_\nu$ and $\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_\lambda$ satisfying the conditions of that lemma.

Using Lemma 1.7 of [42], we can choose projections that $\{p_n\}_{n=1}^\infty, \{q_n\}_{n=1}^\infty, \{r_n\}_{n=1}^\infty$ satisfying the conditions of Lemma 3.7. Note that Thomsen's construction of r_n implies that $\tau(r_n) \leqslant 1/2^n$ for all $n \geqslant 1$ and $\tau \in \mathcal{T}(\mathcal{A})$. Moreover, the fact that $1 = p_1 + q_1 + r_1$ and $p_1 \preceq q_1 \preceq r_1$ implies that $\tau(r_1) \geqslant 1/3$ for all $\tau \in \mathcal{T}(\mathcal{A})$. Since $p_n \preceq q_n \preceq r_n$ and $r_{n-1} = p_n + q_n$, $n \geqslant 2$, we also have

$$\frac{1}{2}\tau(r_{n-1}) \leqslant \tau(q_n) \leqslant \tau(r_n), \quad n \geqslant 2,$$

whence $(1/3)(1/2^{n-1}) \leqslant \tau(r_n) \leqslant 1/2^n, n \geqslant 1$. From this it follows that $\sum_{n=n_0}^{\infty} \tau(r_n) \leqslant \sum_{n=n_0}^{\infty} 1/2^n = 1/2^{n_0 - 1} < \alpha_2 < \tau(\overline{q}_0)$ for all $\tau \in \mathcal{T}(\mathcal{A})$.

Next,

$$\tau(\overline{p}_1) < \beta_1 = \frac{1}{100}\frac{1}{2^{n_0}} < \frac{1}{3}\frac{1}{2^{n_0-1}} \leqslant \tau(r_{n_0}) \leqslant \frac{1}{2^{n_0-1}} < \alpha_2 < \tau(\overline{q}_0) \quad \text{for all } \tau \in \mathcal{T}(\mathcal{A}).$$

By Lemma 4.5, $r_{n_0} \sim \overline{r}_{n_0}$ where $\overline{p}_1 < \overline{r}_{n_0} < \overline{q}_0$. Let $k \geqslant 1$ and suppose that we have fixed $\overline{r}_{n_0+i}$, $1 \leqslant i \leqslant k-1$. Then

$$\begin{aligned}\tau(r_{n_0+k}) &< \tau(\overline{q}_0) - \sum_{i=1}^{k-1} \tau(r_{n_0+i}) = \tau(\overline{q}_0) - \sum_{i=1}^{k-1} \tau(\overline{r}_{n_0+i}) \\ &= \tau\Big(\overline{q}_0 - \sum_{i=1}^{k-1} \overline{r}_{n_0+i}\Big) \quad \text{for all } \tau \in \mathcal{T}(\mathcal{A}),\end{aligned}$$

and hence by FCQ2, $r_{n_0+k} \sim \overline{r}_{n_0+k} < \overline{q}_0 - \sum\limits_{i=1}^{k-1} \overline{r}_{n_0+i}$.

The significance of this to us is that by replacing r_{n_0+k} by $\overline{r}_{n_0+k}$, $k \geqslant 0$, we may assume *a priori* that $\overline{q}_0 \geqslant r_{n_0+k}$, $k \geqslant 0$.

Suppose that $a = a^* \in \overline{q}_0\mathcal{A}\overline{q}_0$ and that $\tau(a) = 0$ for all $\tau \in \mathcal{T}(\mathcal{A})$. Now $\overline{q}_0 = \overline{p}_1 + \overline{p}_2 + \cdots + \overline{p}_\lambda$, where $\overline{p}_\lambda \preceq \overline{p}_{\lambda-1} \preceq \cdots \preceq \overline{p}_1$. By Lemma 1.1 and Lemma 1.2 of [42],

$$a = [x_1, y_1] + [x_2, y_2] + a_1,$$

where $a_1 = a_1^* \in \overline{p}_1\mathcal{A}\overline{p}_1$, $x_1, y_1, x_2, y_2 \in \overline{q}_0\mathcal{A}\overline{q}_0$. By Remark 4.3, $a_1 \in \Sigma_{d+5}\mathfrak{c}(\overline{q}_0\mathcal{A}\overline{q}_0)$. Thus $a \in \Sigma_{d+7}\mathfrak{c}(\overline{q}_0\mathcal{A}\overline{q}_0)$, completing the proof. ∎

THEOREM 4.8. *Let $\mathcal{A}$ be a unital* AH C^**-algebra of the type considered in Subsection* 4.2*. Assume that $\mathcal{A}$ has bounded dimension d, and that $K_0(\mathcal{A})$ has large denominators. Then every element in $\mathfrak{sl}(\mathcal{A})$ is a sum of two commutators, and every selfadjoint element in $\mathfrak{sl}(\mathcal{A})$ is a sum of at most four self-commutators.*

Proof. By Lemma 4.7, there exists $\nu > 100d$ as well as mutually orthogonal projections $\overline{q}_0, \overline{q}_1, \ldots, \overline{q}_\nu \in \mathcal{A}$ so that:

(a) $1 = \sum\limits_{k=0}^{\nu} \overline{q}_k$;
(b) $\overline{q}_\nu \preceq \overline{q}_{\nu-1} \preceq \cdots \preceq \overline{q}_{100d} \sim \cdots \sim \overline{q}_1 \sim \overline{q}_0$;
(c) if $a = a^* \in \overline{q}_1\mathcal{A}\overline{q}_1$ satisfies $\tau(a) = 0$ for all $\tau \in \mathcal{T}(\mathcal{A})$, then $a \in \Sigma_{d+7}\mathfrak{c}(\overline{q}_1\mathcal{A}\overline{q}_1)$.

This tells us that $a \in \mathfrak{sl}(\overline{q}\mathcal{A}\overline{q})$ implies that $a \in \Sigma_{2d+14}\mathfrak{c}(\overline{q}\mathcal{A}\overline{q})$. The remainder of the argument is similar to that used in Theorem 2.5 and in Theorem 3.10. ∎

4.4. A number of algebras which were originally defined by other means have been shown to be isomorphic to inductive limits of the type we are considering here. Among these are the Bunce-Deddens algebras [16] and the irrational rotation algebras [12] which have been shown to be expressible as inductive limits of the above type with $X_{n,j} = \mathbb{T}$, the unit circle in $\mathbb{C}$, for all n and j. The same is true of the crossed product C^*-algebra $\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$, where $\varphi : X \to X$ is a minimal homeomorphism of the Cantor set X ([37] combined with a result of Elliott;

see Theorem VIII.7.5 of [9]). All of these are examples of simple C^*-algebras. As well, all unital AF C^*-algebras $\mathcal{A}$ for which $K_0(\mathcal{A})$ has large denominators — occurring by setting each $X_{n,j}$ to be a singleton for all n and j — whether or not these are simple.

COROLLARY 4.9. *Let $\mathcal{A}$ be one of the following algebras. Then every element of $\mathfrak{sl}(\mathcal{A})$ is a sum of at most two commutators, and each self-adjoint element of $\mathfrak{sl}(\mathcal{A})$ is a sum of at most four self-commutators in:*

(i) *Bunce-Deddens algebras.*

(ii) *The irrational rotation algebras.*

(iii) *The crossed product C^*-algebra $\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$, where $\varphi : X \to X$ is a minimal homeomorphism of the Cantor set X.*

(iv) *Unital* AF *C^*-algebras for which K_0 has large denominators. In particular, this is true for all infinite dimensional* UHF *C^*-algebras.*

5. AN APPLICATION

5.1. In this section we shall combine the above results along with those of [25] to show that in a large number of C^*-algebras, every element can be expressed as a linear combination of a relatively small number of projections. Given a C^*-algebra $\mathcal{A}$, we shall denote by $\mathfrak{N}^{(2)}(\mathcal{A})$ the set $\{n \in \mathcal{A} : n^2 = 0\}$.

In [25] the following results were proven:

THEOREM 5.1. *Let $\mathcal{A}$ be a unital C^*-algebra with mutually orthogonal projections p_1, p_2, p_3 satisfying $1 = p_1 + p_2 + p_3$ and $p_i \preceq 1 - p_i$, $1 \leqslant i \leqslant 3$. Then:*

(i) $\mathfrak{c}(\mathcal{A}) \subseteq \Sigma_{21}\mathfrak{N}^{(2)}(\mathcal{A})$;

(ii) $\mathfrak{c}(\mathcal{A}) \subseteq \Sigma_{84}\mathbb{C}\mathfrak{P}(\mathcal{A})$;

As pointed out in [25], if $p_1 \sim p_2$, then it is possible to improve the estimates on the number of nilpotents of order two as well as the number of projections occurring above. Unfortunately, the matrix r appearing in Theorem 3.5(ii) of that paper is missing the $[x_{33}, y_{33}]$ coordinate, and the estimate of “13" nilpotents of order two and “52" projections given there must be adjusted to “14" nilpotents and “56" projections, as the following argument shows.

Given $x, y \in \mathcal{A}$, $1 = p_1 + p_2 + p_3$ where the p_i's are mutually orthogonal projections, let $x_{ij} = p_i x p_j$, $y_{ij} = p_i y p_j$ and consider $r = [r_{ij}] := [[x_{ij}], [y_{ij}]]$. As is shown in the proof of Lemma 3.4 of [25], we can find an element $m_0 \in \Sigma_6\mathfrak{N}^{(2)}$ and elements $t_{ij} \in p_i\mathcal{A}p_j$, $1 \leqslant i \neq j \leqslant 3$ so that

$$r - m_0 = \begin{bmatrix} [x_{11}, y_{11}] & t_{12} & t_{13} \\ t_{21} & [x_{22}, y_{22}] & t_{23} \\ t_{31} & t_{32} & [x_{33}, y_{33}] \end{bmatrix}.$$

But then

$$r - m_0 = \begin{bmatrix} x_{11}y_{11} & x_{11} & 0 \\ -y_{11}x_{11}y_{11} & -y_{11}x_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} x_{22}y_{22} & x_{22} & 0 \\ -y_{22}x_{22}y_{22} & -y_{22}x_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & -y_{33}x_{33} & y_{33} \\ 0 & -x_{33}y_{33}x_{33} & x_{33}y_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & y_{33}x_{33} & -y_{33}x_{33} \\ 0 & y_{33}x_{33} & -y_{33}x_{33} \end{bmatrix}$$

$$+ \begin{bmatrix} -w & -w & 0 \\ w & w & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & u & t_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ v & 0 & -y_{33} + y_{33}x_{33} + t_{23} \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t_{31} & t_{32} + x_{33}y_{33}x_{33} - y_{33}x_{33} & 0 \end{bmatrix},$$

where $w = y_{11}x_{11} + x_{22}y_{22}$, $u = t_{12} + w - x_{11} - x_{22}$, and $v = t_{21} + y_{11}x_{11}y_{11} + y_{22}x_{22}y_{22} - w$.

Thus every commutator is a sum of 14 nilpotents of order two. In fact, through tedious but elementary bookkeeping, one can show that if $\|x\|, \|y\| \leqslant 1$, then each of the nilpotents appearing above is bounded in norm by 16. The argument from the above cited paper shows that every such nilpotent of order two appearing above is a linear combination of at most 4 projections, and that the coefficients of the projections appearing for each such nilpotent may be bounded in magnitude by 8. Thus if $\|x\|, \|y\| \leqslant 1$, then $[x, y]$ can be expressed as a linear combination of 56 projections, and the magnitude of the projections may be bounded by 8. The point is not that the estimates are particularly low, but rather that they exist, and can be explicitly computed.

As an immediate corollary to this, we obtain:

COROLLARY 5.2. *Suppose that $\mathcal{A}$ is a unital, simple C^*-algebra of real rank zero with a unique tracial state τ. Assume that $\mathcal{A}$ satisfies* FCQ2. *Then $\mathfrak{sl}(\mathcal{A}) = \Sigma_{28}\mathfrak{N}^{(2)}(\mathcal{A})$, and*

$$\mathcal{A} = \Sigma_{113}\mathbb{C}\mathfrak{P}(\mathcal{A}).$$

That is, every element of $\mathcal{A}$ can be written as a linear combination of 113 *or fewer projections in $\mathcal{A}$.*

Proof. We can use Zhang's Theorem 3.3 to write $1 = \sum_{i=1}^{9} q_i$ where $q_9 \preceq q_8 \sim q_7 \sim \cdots \sim q_1$ and $q_i q_j = 0$ if $1 \leqslant i \neq j \leqslant 9$. Let $p_1 = q_1 + q_2 + q_3$, $p_2 = q_4 + q_5 + q_6$, $p_3 = q_7 + q_8 + q_9$. Then $p_1 \sim p_2$ and $p_i \preceq 1 - p_i$, $1 \leqslant i \leqslant 3$. By Theorem 5.1 above, $\mathfrak{c}(\mathcal{A}) \subseteq \Sigma_{56}\mathbb{C}\mathfrak{P}(\mathcal{A}) \cap \Sigma_{14}\mathfrak{N}^{(2)}(\mathcal{A})$.

By Theorem 3.10, $\mathfrak{sl}(\mathcal{A}) = \Sigma_2\mathfrak{c}(\mathcal{A}) \subseteq \Sigma_{112}\mathbb{C}\mathfrak{P}(\mathcal{A}) \cap \Sigma_{28}\mathfrak{N}^{(2)}(\mathcal{A})$. Since $\mathfrak{N}^{(2)}(\mathcal{A}) \subseteq \mathfrak{sl}(\mathcal{A})$, the reverse inclusion also holds. Finally, if $a \in \mathcal{A}$ is arbitrary, then $a_0 := a - \tau(a)1 \in \mathfrak{sl}(\mathcal{A})$, and so $a = a_0 + \tau(a)1 \in \Sigma_{113}\mathbb{C}\mathfrak{P}(\mathcal{A})$. ∎

REMARK 5.3. Suppose that in Corollary 5.2 we know that $\|a\| \leqslant 1$. Then $|\tau(a)| \leqslant \|a\| \leqslant 1$, and so $\|a_0\| \leqslant 2$. Recall from the remarks of Subsection 3.3 that we can write $\mathrm{Re}(a_0) = \sum_{i=1}^{8} [x_i, x_i^*]$ where $\|x_i\| \leqslant 13\|a_0\|^{1/2}$, $1 \leqslant i \leqslant 8$, and that a similar result holds for the imaginary part of a_0.

In fact, another tedious but routine calculation based upon Lemmas 2.1 and 2.2 as well as Theorem 2.3 and 3.9 reveals that if $b \in \mathfrak{sl}(\mathcal{A})$ and $\|b\| \leqslant 1$, then we can write $b = [d, z] + [s, v]$ where $\|d\| \leqslant 100$, $\|z\| \leqslant (5577) + 64 \max \|\tau_{d_i, d_j}^{-1}\| \leqslant 5961$, $\|s\| \leqslant 356{,}928$ and $\|v\| \leqslant 1$. Thus we can write $a_0 = \alpha_1[d_0, z_0] + \alpha_2[s_0, v_0]$ where $\alpha_1 = (200(5961)) = 1{,}192{,}200$, $\alpha_2 = (2(356{,}928)) = 713{,}856$, and also $\|d_0\|, \|z_0\|, \|s_0\|, \|v_0\| \leqslant 1$.

Combining this with the comments following Theorem 5.1, we see that $[d_0, z_0]$ and $[s_0, v_0]$ can be expressed as linear combinations of 56 projections each, and that in each case, the coefficients appearing in that linear combination may be bounded by 8. Thus a can be written as a linear combination of 113 projections, and the coefficients of those projections may be bounded by $8(\max\{\alpha_1, \alpha_2\}) = 9{,}537{,}600$! The reader may well imagine that we make no claims whatsoever as to the sharpness of these estimates — only that they exist, and that they are useful.

For example, one simple yet interesting consequence of this fact is that if $\mathfrak{X}$ is a Banach space and $\varphi : \mathcal{A} \to \mathfrak{X}$ is a linear map, then φ is continuous if and only if $\|\varphi\|_{\mathfrak{P}(\mathcal{A})} := \sup\{\|\varphi(p)\| : p \in \mathfrak{P}(\mathcal{A})\} < \infty$.

COROLLARY 5.4. *Suppose that $\mathcal{A}$ is a unital, simple C^*-algebra of real rank zero with no tracial states. Then there exists a constant $m_0 \geqslant 1$ so that $\mathcal{A} = \Sigma_{56m_0}\mathbb{C}\mathfrak{P}(\mathcal{A}) = \Sigma_{14m_0}\mathfrak{N}^{(2)}(\mathcal{A})$.*

Proof. By [35], there exists a constant m_0 so that $\mathcal{A} = \Sigma_{m_0}\mathfrak{c}(\mathcal{A})$. The existence of projections p_1, p_2, p_3 satisfying $1 = p_1 + p_2 + p_3$ and $p_i \preceq (1 - p_i), 1 \leqslant i \leqslant 3$ follows as in the previous corollary. Thus $\mathcal{A} = \Sigma_{56\, m_0}\mathbb{C}\mathfrak{P}(\mathcal{A}) = \Sigma_{14m_0}\mathfrak{N}^{(2)}(\mathcal{A})$, by Theorem 5.1 and the comments following it. ∎

As the above proof demonstrates, the result holds more generally in any unital C^*-algebra which does not admit a tracial state, and for which projections satisfying the conditions of Theorem 5.1 can be found.

Even when we can not find projections satisfying the conditions of that theorem, all is not lost.

PROPOSITION 5.5. *Suppose that $1 \in \mathcal{A}$ is a simple C^*-algebra which does not admit a tracial state. If $\mathcal{A}$ has a non-trivial projection, then:*

(i) *$\mathcal{A}$ is the linear span of its projections, and therefore*

(ii) *there exist $n, k \in \mathbb{N}$ so that for all $a \in \mathcal{A}$ with $\|a\| \leqslant 1$, and for all $\varepsilon > 0$, there exist projections $p_1, p_2, \ldots, p_n \in \mathfrak{P}(\mathcal{A})$, $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ with $|\lambda_i| \leqslant k$, $1 \leqslant i \leqslant n$*

such that

$$\left\| a - \sum_{i=1}^{n} \lambda_i p_i \right\| < \varepsilon.$$

In particular, $\Sigma_n \mathbb{C}\mathfrak{P}(\mathcal{A})$ *is dense in* $\mathcal{A}$.

Proof. (i) By Proposition 4.3 of [26], $\bigcup_{m=1}^{\infty} \Sigma_m \mathbb{C}\mathfrak{P}(\mathcal{A})$ is either contained in $\mathbb{C}1$ or it contains $[\mathcal{A}, \mathcal{A}] := \operatorname{span} \mathfrak{c}(\mathcal{A})$. In the first case, clearly 1 is the only non-zero projection in $\mathcal{A}$. In the second case, Pop's Theorem [35] implies that $[\mathcal{A}, \mathcal{A}] = \mathcal{A}$, whence $\mathcal{A} = \bigcup_{m=1}^{\infty} \Sigma_m \mathbb{C}\mathfrak{P}(\mathcal{A})$; i.e. $\mathcal{A} = \operatorname{span} \mathfrak{P}(\mathcal{A})$.

(ii) Let $0 \leqslant r \in \mathbb{N}$. Let us denote by $(r\overline{\mathbb{D}})$ the set $\{z \in \mathbb{C} : |z| \leqslant r\}$. Thus for $m \geqslant 1$,

$$\Sigma_m(r\overline{\mathbb{D}})\mathfrak{P}(\mathcal{A}) = \Big\{ \sum_{i=1}^{m} \lambda_i p_i : p_i \in \mathfrak{P}(\mathcal{A}), \lambda_i \in \mathbb{C}, |\lambda_i| \leqslant r, 1 \leqslant i \leqslant m \Big\}.$$

By part (i), $\mathcal{A}$ is spanned by its projections. Thus, given $a \in \mathcal{A}$, $a = \sum_{i=1}^{m} \lambda_i p_i \in \Sigma_m(r\overline{\mathbb{D}})\mathfrak{P}(\mathcal{A})$ for some $m \in \mathbb{N}$ and for $|r| = [\max\{|\lambda_i| + 1 : 1 \leqslant i \leqslant m\}]$. It follows that

$$\mathcal{A} = \bigcup_{n=1}^{\infty} \bigcup_{r=1}^{\infty} \Sigma_n(r\overline{\mathbb{D}})\mathfrak{P}(\mathcal{A}) = \bigcup_{n=1}^{\infty} \bigcup_{r=1}^{\infty} \overline{(\Sigma_n(r\overline{\mathbb{D}})\mathfrak{P}(\mathcal{A}))}.$$

Since $\mathcal{A}$ is a complete metric space, the Baire Category Theorem tells us that it is not the countable union of nowhere dense sets. Thus there exist r_0, n_0 so that $\overline{\Sigma_{n_0}(r_0\overline{\mathbb{D}})\mathfrak{P}(\mathcal{A})}$ has interior.

A routine calculation shows that $\overline{\Sigma_{2n_0}(r\mathbb{D})\mathfrak{P}(\mathcal{A})}$ contains a disk $\mathcal{A}_\varepsilon := \{a \in \mathcal{A} : \|a\| < \varepsilon\}$ centered at the origin, and hence it contains $\overline{\mathcal{A}}_{\varepsilon/2}$. But then

$$\mathcal{A}_1 \subseteq \overline{\Sigma_{2n_0}\Big(\frac{2r_0}{\varepsilon}\mathbb{D}\Big)\mathfrak{P}(\mathcal{A})}.$$

Setting $n = 2n_0$, $k = [2r_0/\varepsilon] + 1$, we see that this is equivalent to the first statement of part (ii). The last statement is a simple consequence of this. ∎

We emphasize that the conclusion of part (ii) holds whenever a C^*-algebra $\mathcal{A}$ is linearly spanned by its projections.

5.2. In relation to closed sets, it was shown in [26] that a closed subspace $\mathfrak{L}$ of $\mathcal{A}$ which is invariant under conjugation by unitaries in the connected component of the identity is necessarily a *Lie ideal*; that is, given $m \in \mathfrak{L}$ and $x \in \mathcal{A}$, $[m, x] \in \mathfrak{L}$.

Since Proposition 4.3 of [26] is really a statement about Lie ideals in simple, unital C^*-algebras, it follows that the conclusion of Proposition 5.5 (ii) holds if we replace projections by any unitarily invariant subset of $\mathcal{A}$ which is not contained in $\mathbb{C}1$.

For example, we can take $x \in \mathcal{A}$ to be any non-scalar element and set $\mathcal{U}(x) = \{u^*xu : u \in \mathcal{A} \text{ unitary}\}$. Then the same argument shows that there exists $n \in \mathbb{N}$ such that

$$\mathcal{A} = \overline{\Sigma_n \mathbb{C}\mathcal{U}(x)},$$

and that there exists some control over the magnitude of the coefficients in the linear combination in terms of the norm of the element that we are approximating.

COROLLARY 5.6. *Let $\mathcal{A}$ be a unital* AH *C^*-algebra of the type considered in Subsection* 4.2. *Assume that $\mathcal{A}$ is of bounded dimension d and that $K_0(A)$ has large denominators. Then:*

(i) $\mathfrak{sl}(\mathcal{A}) = \Sigma_{28}\mathfrak{N}^{(2)}(\mathcal{A})$; *and*

(ii) *if the projections span a dense subset of $\mathcal{A}$ and $\mathcal{A}$ has exactly m extremal tracial states for some $1 \leqslant m < \infty$, then $\mathcal{A} = \Sigma_{112+m}\mathbb{C}\mathfrak{P}(\mathcal{A})$.*

Proof. (i) Recall from the proof of Lemma 4.4 that given $M > 0$, $\mathcal{A}$ contains a finite dimensional unital C^*-subalgebra $\mathcal{B} \simeq \bigoplus_{k=1}^{n} \mathbb{M}_{r_k}$ where $r_k \geqslant M$ for all $1 \leqslant k \leqslant n$. In particular, with $M \geqslant 3$, it is straightforward to find mutually orthogonal projections p_i, $i = 1,2,3$ in $\mathcal{B} \subseteq \mathcal{A}$ satisfying $1 = p_1 + p_2 + p_3$, $p_1 \sim p_2$, and $p_i \preceq 1 - p_i$ for each i. By Theorem 4.8,

$$\mathfrak{sl}(\mathcal{A}) \subseteq \Sigma_2\mathfrak{c}(\mathcal{A}).$$

By Theorem 5.1, $\mathfrak{sl}(\mathcal{A}) = \Sigma_{28}\mathfrak{N}^{(2)}(\mathcal{A})$.

(ii) Using (i) and Theorem 5.1 above, we see that $\mathfrak{c}(\mathcal{A}) \subseteq \Sigma_{56}\mathbb{C}\mathfrak{P}(\mathcal{A})$, and hence $\mathfrak{sl}(\mathcal{A}) \subseteq \Sigma_{112}\mathbb{C}\mathfrak{P}(\mathcal{A})$ by Theorem 4.8. Now, $\mathfrak{sl}(\mathcal{A}) = \bigcap\{\ker\tau : \tau \text{ an extremal tracial state}\}$ is a closed subspace of $\mathcal{A}$ of codimension m. Since $\operatorname{span}\mathfrak{P}(\mathcal{A}) \supseteq \mathfrak{sl}(\mathcal{A})$ is the sum of a finite dimensional space and a closed subspace of $\mathcal{A}$, then $\operatorname{span}\mathfrak{P}(\mathcal{A})$ is closed, i.e. $\operatorname{span}\mathfrak{P}(\mathcal{A}) = \mathcal{A}$.

The remainder of the argument follows exactly as in Theorem 4.11(ii) of [25]. That is, we let $\pi : \mathcal{A} \to \mathcal{A}/\mathfrak{sl}(\mathcal{A})$ be the canonical quotient map between Banach spaces. Choose $x_1, \ldots, x_m \in \mathcal{A}$ so that $\{\pi(x_i)\}_{i=1}^m$ is a basis for $\mathcal{A}/\mathfrak{sl}(\mathcal{A})$. Choose projections $\{q_j\}_{j=1}^s \subseteq \mathcal{A}$ so that $\{x_i\}_{i=1}^m \subseteq \operatorname{span}\{q_j\}_{j=1}^s$. Since $\{\pi(q_j)\}_{j=1}^s$ spans $\mathcal{A}/\mathfrak{sl}(\mathcal{A})$, we can find m linearly independent vectors in this set — which we may relabel $\{\pi(q_j)\}_{j=1}^m$.

Given $a \in \mathcal{A}$, write $a = \sum_{j=1}^{m} \lambda_j q_j + b$, $b \in \mathfrak{sl}(\mathcal{A})$. Then $a \in \Sigma_{112+m}\mathbb{C}\mathfrak{P}(\mathcal{A})$. ∎

REMARK 5.7. We point out that even when $1 \in \mathcal{A}$ is a simple C^*-algebra inductive limit as in Subsection 4.2 with bounded dimension d and large denominators, the linear span of the projections in $\mathcal{A}$ need not be dense. For example, in [17], K.R. Goodearl shows that one can choose X a non-empty, separable, compact Hausdorff space (not totally disconnected) and $\mathcal{A}_n \simeq \mathbb{M}_{k_n}(\mathcal{C}(X))$ for appropriate k_n, for which it is possible to fix the embedding φ_n of $\mathcal{A}_n$ into $\mathcal{A}_{n+1}$ so that:

(i) $1 \in \mathcal{A}$ is simple, and
(ii) the span of the projections in $\mathcal{A}$ is not dense, and $\mathcal{A}$ is of real rank 1.

In particular, one can choose $X = \mathbb{T}$, so that $\mathcal{A}$ is a limit circle algebra.

Finally, in relation to the results of this section we mention that N. Kataoka [24] has shown that if $\mathcal{J}$ is a closed, ideal in a properly infinite (or stable) C^*-algebra $\mathcal{A}$, then every element of $\mathcal{J}$ is a sum of nilpotents of order two in $\mathcal{J}$. No bound on the number of terms required to express an element of $\mathcal{J}$ as a sum of such nilpotents is given.

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