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GENERALIZED FREE AMALGAMATED PRODUCT OF C^*-ALGEBRAS

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ABSTRACT. We construct a generalized version for the free product of unital C^*-algebras $(A_i)_{i\in I}$ with amalgamation over a family of common unital subalgebras $(B_{ij})_{i,j\in I, i\neq j}$, starting from the group-analogue. When all the algebras are the same, we recover the free product with amalgamation over a common subalgebra. We reduce the problem to the study of minimal amalgams. We specialize to triangles of algebras and subalgebras, study freeness in this context, and give some examples of constructions of minimal amalgams derived from triangles of operator algebras.

KEYWORDS: *C^*-algebra, amalgam, free product, freeness with amalgamation, conditional expectation.*

MSC (2000): 46L05, 46L09, 20E06, 20E34.

1. INTRODUCTION

The von Neumann group algebra of a free product of groups, without and with amalgamation, was the starting point in Dan Voiculescu's free probability theory. Due to the success of the spatial theory of free products of C^*-algebras and von Neumann algebras he developed, we find it of interest to study other kinds of amalgams, starting from what happens in the category of groups.

In [7], [8] the generalized free product with amalgamated subgroups is introduced and studied extensively. Given a family of groups and subgroups $H_{ij} \subset G_i, \forall j, i \in I, j \neq i$ so that $H_{ij} \overset{\varphi_{ij}}{\simeq} H_{ji}, \forall i, j \in I, i \neq j$, and $\varphi_{ji} = \varphi_{ij}^{-1}$, the generalized free product G of the $G_i's$ with amalgamated subgroups H_{ij} is defined as follows. Take the free product of the $G_i's$ and factor it by the normal subgroup F generated by elements of the form

$$\varphi_{ij}(h_{ij})h_{ji}^{-1}, \quad h_{ij} \in H_{ij}, h_{ji} \in H_{ji} \quad i, j \in I, i \neq j.$$

The elements of the generalized free product of groups are words over the "alphabet" $\bigcup\limits_{i\in I} G_i$ such that $\cdots(g_i h_{ij})g_j\cdots = \cdots g_i(\varphi_{ij}(h_{ji})g_j)\cdots$. Unlike for the free product of groups, where the groups embed in their free product, it is not necessary the case that the groups G_i have isomorphic copies inside of G. When this is true we say that the generalized free product is realizable. Examples of collapsing families of groups can be found in Example 3.2 of [7], and Chapter II,11 of [6], and more recently in [14]. One striking sufficient condition for the existence of a realizable generalized free amalgamated product of a triangle of groups as the one bellow was given by John Stallings in [12]. He defined an

$$
\begin{tikzcd}
 & & G_1 & & \\
 & H_{12} \arrow[ur] \arrow[dl] & & H_{13} \arrow[ul] \arrow[dr] & \\
 & & K \arrow[ul] \arrow[ur] \arrow[d] & & \\
G_2 & & H_{23} \arrow[ll] \arrow[rr] & & G_3
\end{tikzcd}
$$

FIGURE 1. Triangle diagram of groups and injective morphisms

angle between the subgroups $H_{12}, H_{13} \subset G_1$ as follows. Consider the unique morphism $\phi : H_{12} *_K H_{13} \to G_1$. Let $2n = \inf\{|w| : w \in \ker\phi, w \neq e\}$ and define $\theta_1 = \frac{\pi}{n}$ to be the angle between the two subgroups. Consider the 3 angles involved in the triangle above. Stalling's theorem asserts that if $\theta_1 + \theta_2 + \theta_3 \leqslant \pi$ then the considered generalized free product is realizable. On the other hand, a famous example of a group that can be presented as the minimal amalgam of a triangle of finite groups and which does not satisfy this condition is Thompsons's infinite simple group G_{21} (see [3]). Generalized amalgams of groups lead to the theory of buildings [14], generalizing the tree amalgams of Serre [11]. Ramagge and Robertson [10] studied the reduced C^*-algebra and the group von Neumann algebra of such groups, producing examples of type III factors. Pull-back and push-out constructions with C^*-algebras are studied in [9]. To our knowledge, there is no systematic treatment of more complicated amalgams of C^*-algebras.

In this paper, we construct the C^*-algebraic version of the generalized free product for groups described above. We are mainly interested in the triangle amalgams, the simplest of the generalized free products, because of Stalling's theorem and the connection with the Thompson groups. The longstanding conjecture about the amenability of Thompson's group can be approached from the C^*-algebraic point of view, by considering the reduced C^*-algebra of the group. The finitely presented, infinite simple group G_{21} mentioned above is the minimal amalgam of a triangle of finite groups. Therefore, as we will see in Section 4, the group C^*-algebra is the minimal triangle amalgam of a triangle of finite dimensional C^*-algebras. This motivates our attention towards triangle amalgams. We show that up to a minimal amalgam, the generalized free product is a classical free product. This reduces the study of amalgams, and in particular the triangle amalgams, to the study of the minimal amalgam. As in the case of groups, the factors of the generalized free product do not necessarily embed in the product; due to too many relations, a complicated amalgam might collapse. When the factors do embed in the final amalgam, the amalgam is called realizable. We provide criteria for realization of triangle amalgams and study freeness up to the minimal amalgam.

The paper is organized as follows: in Section 2 we review the generalized free product in the category of groups, in Section 3 we review the construction of the full and reduced free product of unital C^*-algebras and in Section 4 we define the generalized free product of unital C^*-algebras. Section 5 studies the freeness in the sense of Voiculescu, in the presence of a realizable minimal amalgam and in Section 6 we give examples of realizable triangles of algebras.

2. GENERALIZED FREE PRODUCT OF GROUPS WITH AMALGAMATED SUBGROUPS

In this section we review the construction presented by H. Neumann in [7] and [8].

DEFINITION 2.1. Let $(G_i)_{i\in I}$ be a family of groups and for each $i \in I$ let $(H_{ij})_{j\in I, j\neq i}$ be a family of subgroups of G_i as above. The generalized group free product of the family $(G_i)_{i\in I}$ with amalgamated subgroups $(H_{ij})_{i,j\in I, i\neq j}$ is the unique (up to isomorphism) group G together with homomorphisms $\psi_i : G_i \to G$ satisfying the properties:

(i) the diagram in Figure 2 commutes for every $i, j \in I, i \neq j$;

$$\begin{tikzcd} H_{ij} \arrow[rr, "\psi_i"] \arrow[dr, "\varphi_{ij}"'] & & G \\ & H_{ji} \arrow[ur, "\psi_j"'] & \end{tikzcd}$$

FIGURE 2. Identification in G of isomorphic subgroups

(ii) given any group K and homomorphisms $\phi_i : G_i \to K$ such that the diagrams in Figure 3 commute for every $i, j \in I, i \neq j$, there exists a unique homomorphism $\Phi : G \to K$ so that the diagram in Figure 4 commutes for every $i \in I$.

If the maps ψ_i are injective, the amalgam is called *realizable*.

$$\begin{tikzcd} H_{ij} \arrow[rr, "\phi_i"] \arrow[dr, "\varphi_{ij}"'] & & K \\ & H_{ji} \arrow[ur, "\phi_j"'] & \end{tikzcd}$$

FIGURE 3. Identification in K of isomorphic subgroups

$$\begin{tikzcd} G_i \arrow[rr, "\phi_i"] \arrow[dr, "\psi_i"'] & & K \\ & G \arrow[ur, "\Phi"'] & \end{tikzcd}$$

FIGURE 4. Universal property

We will use the notation

$$G = *_{i\in I}(G_i, (H_{ij})_{j\neq i}).$$

When $G_i = \bigvee_{j\in I, j\neq i} H_{ij}$, we call G the *minimal generalized free product* or the *minimal amalgam of the family* $(G_i)_{i\in I}$.

EXAMPLE 2.2. The amalgamated free product of groups contains isomorphic copies of each of its factors. For the triangle amalgam described above this is not true in general, and the following was proved to be a collapsing triangle of groups [6]:

$$\begin{aligned} G_1 &= \langle a, b | a^{-1}ba = b^2\rangle, \\ G_2 &= \langle a, c | c^{-1}ac = a^2\rangle, \\ G_3 &= \langle b, c | b^{-1}cb = c^2\rangle, \\ H_{ij} &= G_i \cap G_j \cong \mathbb{Z}, 1 \leqslant i, j \leqslant 3; \\ *_{i=1}^{3}(G_i, (H_{i,j})_{j\neq i}) &= \{e\}. \end{aligned}$$

REMARK 2.3. We will also use the following alternative description of the family of groups and subgroups that we will amalgamate: let G_i be groups for $i \in I$ and for each $i, j \in I, i \neq j$ let $H_{ij}\,(= H_{ji})$ be groups and $\varphi_{ij} : H_{ij} \to G_i$ be injective group homomorphisms. The groups H_{ij} have isomorphic copies inside the bigger groups G_i and G_j, and these copies will be identified. If we let $K_{ij} := \varphi_{ij}(H_{ij}) \overset{\varphi_{ji}\circ\varphi_{ij}^{-1}}{\simeq} K_{ji} := \varphi_{ji}(H_{ij})$, then the family of groups and subgroups $(G_i)_{i\in I}$, $(K_{ij})_{j\in I, j\neq i}$ is as in the previous definition. By the generalized free product of the family $(G_i)_{i\in I}$ with amalgamated subgroups $(H_{ij})_{j\in I, j\neq i}$ we will mean

$*_{i\in I}(G_i,(K_{ij})_{j\in I,j\neq i})$. Of course we can go both ways with the description of the families and we will use both notations alternatively in this paper.

The first criterion for the existence of the isomorphic copies of the groups G_i inside G refers to the reduced amalgam. For every $i \in I$, let H_i be the subgroup of G_i generated by the family $(H_{ij})_{j\in I,j\neq i}$, i.e. $H_i := \bigvee_{j\in I,j\neq i} H_{ij} \subseteq G_i$. The following reduction theorem is due to H. Neumann [7] and has also a nice presentation in Section 15 of [6]. We present the proof for consistency.

THEOREM 2.4. *With the above notations, the generalized free product*

$$G = *_{i\in I}(G_i,(H_{ij})_{j\neq i})$$

is realizable if and only if the generalized free product

$$H = *_{i\in I}(H_i,(H_{ij})_{j\neq i})$$

is realizable.

Proof. It is clear that if G is realizable, then each H_i will have isomorphic copies as subgroups of the isomorphic copies of G_i, hence H is obviously realizable. Suppose now that H is realizable. For each $i \in I$ we first take the free product $G_i *_{H_i} H$. Next consider

$$G = *_{H,i\in I}(G_i *_{H_i} H).$$

Each G_i has an isomorphic copy in G. Furthermore, since H is the amalgamated subgroup, and it is the realizable minimal amalgam, we have, keeping the same notation for the copies of G_i inside G: $G_i \cap G_j = G_i \cap G_j \cap H = H_i \cap H_j = H_{ij} = H_{ji} \subset G$. It follows that the $G_i's$ embed in $\bigvee_{i\in I} G_i \subset G$ such that the resulting diagrams concerning the subgroups commute, hence $*_{i\in I}(G_i,(H_{ij})_{j\neq i})$ is realizable. Notice that actually $G = \bigvee_{i\in I} G_i$. Indeed, a moment's thought suffices to see that an element $x \in G$ has the form

$$x = w_1(\{H_i : i \in I\})g_1w_2(\{H_i : i \in I\})g_2 \cdots w_n(\{H_i : i \in I\})g_n,$$

with $n \geqslant 1$ and where $g_k \in G_{i_k}, i_1 \neq i_2 \neq \cdots \neq i_n$ and $w_k(\{H_i : i \in I\})$ are words over the "alphabet" $\bigcup_{i\in I} H_i$ so that the letters adjacent to an element of G_{i_k} do not belong to $\bigcup_{j\in I,j\neq i_k} H_{i_k,j}$. But $w_k(\{H_i : i \in I\}) \in \bigvee_{i\in I} G_i$, hence $x \in \bigvee_{i\in I} G_i$. ∎

REMARK 2.5. We will be interested in this paper in a particular kind of family of groups, that can be represented on a triangle diagram. All arrows are injective group homomorphisms. The addition of the group H_{123} is just a necessary condition for the generalized free product to be realizable. Indeed, if $*_{i\in I}(G_i,(H_{ij})_{i\neq j})$, $I = \{1,2,3\}$, is realizable, then we may assume that the arrows are inclusions and that $G_i \cap G_j = H_{ij}$. Then $G_1 \cap G_2 \cap G_3 = H_{12} \cap H_{13} = H_{12} \cap H_{23} = H_{13} \cap H_{23} = H_{123}$. Therefore we will assume the triangles to be

$$\begin{tikzcd}
 & & G_1 & & \\
 & H_{12} \arrow[ur] \arrow[ddl] & & H_{13} \arrow[ul] \arrow[ddr] & \\
 & & H_{123} \arrow[ul] \arrow[ur] \arrow[d] & & \\
G_2 & & H_{23} \arrow[ll] \arrow[rr] & & G_3
\end{tikzcd}$$

FIGURE 5. Triangle diagram of groups and inclusions

fillable, i.e. the intersections of pairs of edge groups are isomorphic under the given family of maps:

$$H_{123} \simeq \varphi_{12}(H_{12}) \cap \varphi_{13}(H_{13})(\subset G_1) \overset{\varphi_{31}\circ\varphi_{13}^{-1}}{\simeq} \varphi_{32}(H_{23}) \cap \varphi_{31}(H_{13}) \subset G_3.$$

The last result of this section addresses the realization of triangles of groups as the one above.

THEOREM 2.6. *Given a minimal triangle of groups and injective groups homomorphisms as the previous one, each of the following is a sufficient condition for the triangle to be realizable:*

(i) *One of the vertex groups, G_3 say, is the free product of H_{23}, H_{13} with amalgamation over H_{123}, and with inclusion mappings $\varphi_{32}, \varphi_{13}$. Then*

$$*_{i=1}^{3}(G_i, (H_{ij})_{j\neq i}) = G_1 *_{H_{12}} G_2.$$

(ii) *Two of the groups G_i, G_1 and G_2 say, have the property that every element of H_{12} commutes with every element of H_{13} and of H_{23} $(i = 1, 2)$. In this case $*_{i=1}^{3}(G_i, (H_{ij})_{j\neq i})$ is a quotient of $G_1 *_{H_{12}} G_2$.*

This theorem and its proof can be found in Section 9 of [7]. The idea of the proof is to look at the way $H_{13} *_{H_{123}} H_{23}$ embeds into $G_1 *_{H_{12}} G_2$. This is the way to prove similar results for triangles of operator algebras.

3. FULL AND REDUCED PRODUCTS OF UNITAL C^*-ALGEBRAS

In this section we present a brief review on full and reduced free products of unital C^*-algebras with amalgamation over a common subalgebra. For more details on the subject the interested reader is encouraged to consult [16], [15], [13], [4], [1], [2].

DEFINITION 3.1. Given a family of unital C^*-algebras $(A_i)_{i\in I}$ (I is a set having at least two elements) with a common unital C^*-subalgebra B, and injective $*$-homomorphisms $\phi_i : B \to A_i, \forall i \in I$, the corresponding full amalgamated free product C^*-algebra is a unital C^*-algebra A, equipped with injective $*$-homomorphisms $\sigma_i : A_i \to A, \forall i \in I$ such that:

(i) $\sigma_i \circ \phi_i = \sigma_j \circ \phi_j$ for all $i, j \in I$ and A is generated by $\bigcup \sigma_i(A_i)$;

(ii) for any unital C^*-algebra D and any injective $*$-homomorphisms $\pi_i : A_i \to D$, $i \in I$, satisfying $\pi_i \circ \phi_i = \pi_j \circ \phi_j$ for all $i, j \in I$, there is a $*$-homomorphism $\pi : A \to D$ such that the next diagram commutes.

$$\begin{tikzcd}
A_i \arrow[rr, "\pi_i"] \arrow[dr, "\sigma_i"'] & & D \\
& A \arrow[ur, "\pi"'] &
\end{tikzcd}$$

FIGURE 6. Universal property

REMARK 3.2. If B is just the field of complex numbers, then the injectivity of the maps σ_i can be omitted from the definition and follows from the fact that the full free product has enough representations (see [16]). For the general case, injectivity cannot be omitted from the definition without changing the category of objects. If, for example, there exist conditional expectations $E_i : A_i \to B$ with faithful GNS representation (see Definition 3.4) for each $i \in I$, then injectivity follows using the properties of the reduced free product with amalgamation and the universal property of the full free product.

To produce criteria for existence of generalized amalgamated free products, we need the following results on embedding of full and reduced amalgamated free product C^*-algebras. This first proposition works actually also for free products of non-unital C^*-algebras (see Proposition 2.4 of [1]).

PROPOSITION 3.3. *Suppose*

$$\begin{array}{ccccc}
\widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\
\uparrow & & \uparrow & & \uparrow \\
A & \hookleftarrow & D & \hookrightarrow & B
\end{array}$$

is a commuting diagram of inclusions of C^*-algebras. Let $\lambda : A *_D B \to \widetilde{A} *_{\widetilde{D}} \widetilde{B}$ be the resulting $*$-homomorphism of full free product C^*-algebras. Suppose there are conditional expectations $E_A : \widetilde{A} \to A$, $E_D : \widetilde{D} \to D$ and $E_B : \widetilde{B} \to B$ onto A, D and B, respectively, such that the following diagram commutes. Then λ is injective.

$$\begin{array}{ccccc} \widetilde{A} & \hookleftarrow & \widetilde{D} & \hookrightarrow & \widetilde{B} \\ \downarrow \mathrm{E}_A & & \downarrow \mathrm{E}_D & & \downarrow \mathrm{E}_B \\ A & \hookleftarrow & D & \hookrightarrow & B \end{array}$$

Suppose $\phi : A \to B$ is a conditional expectation and let $H = L^2(A, \phi)$ be the (right) Hilbert B-module obtained from A by separation and completion with respect to the norm $\|a\| = \|(a, a)_H\|^{1/2}$, where $(\cdot, \cdot)_H$ is the B-valued inner product, $(a_1, a_2)_H = \phi(a_1^* a_2)$. We denote the map $A \to H$ arising from the definition by $a \mapsto \widehat{a}$.

Let $\pi : A \to \mathcal{L}(H)$ denote the $*$-representation defined by $\pi(a)\widehat{b} = \widehat{ab}$, where $\mathcal{L}(H)$ is the C^*-algebra of all adjointable bounded B-module operators on H. Consider the specified element $\xi = \widehat{1}_A \in H$.

DEFINITION 3.4. With the above notations, we will call (π, H, ξ) the *GNS representation of* (A, ϕ) and write this as

$$(\pi, H, \xi) = \mathrm{GNS}(A, \phi).$$

We will say that the conditional expectation $\phi : A \to B$ has faithful GNS representation if π is a faithful representation.

REMARK 3.5. Note that the faithfulness of π is equivalent to the following condition:

$$\forall a \in A \setminus \{0\} \quad \exists x \in A \quad \phi(x^* a^* a x) \neq 0.$$

Let B be a unital C^*-algebra, let I be a set having at least two elements and for every $i \in I$ let A_i be a unital C^*-algebra containing a copy of B as a unital C^*-subalgebra. Voiculescu's construction of the reduced amalgamated free product (A, ϕ) starts with the Hilbert B-modules $(\pi_i, H_i, \xi_i) = \mathrm{GNS}(A_i, \phi_i)$. Next he constructs a Hilbert B-module H which turns out to coincide with $L^2(A, \phi)$. Letting $H_i^\circ = L^2(A_i, \phi_i) \ominus \xi B$, define

$$H = \xi B \oplus \bigoplus_{\substack{n \geqslant 1 \\ i_1, i_2, \ldots, i_n \in I \\ i_1 \neq i_2 \neq \cdots \neq i_n}} H_{i_1}^\circ \otimes_B H_{i_2}^\circ \otimes_B \cdots \otimes_B H_{i_n}^\circ.$$

Here $\xi = 1_B$ and ξB is the C^*-algebra B viewed as a Hilbert B-module with specified element ξ and all the tensor products are internal, arising from $*$-homomorphisms $P_i^\circ \pi_i|_B P_i^\circ$ from B to $\mathcal{L}(H_i^\circ)$ (P_i° is the projection onto H_i°). The Hilbert module H is called the *free product of the* H_i with respect to the specified vectors ξ_i, and is denoted by $(H, \xi) = *_{i \in I}(H_i, \xi_i)$. Next, we define the free product of the $*$-representations π_i. For $i \in I$ let

$$H(i) = \eta_i B \oplus \bigoplus_{\substack{n \geqslant 1 \\ i_1, i_2, \ldots, i_n \in I \\ i_1 \neq i_2 \neq \cdots \neq i_n \\ i_1 \neq i}} H_{i_1}^\circ \otimes_B H_{i_2}^\circ \otimes_B \cdots \otimes_B H_{i_n}^\circ,$$

with $\eta_i = 1_B$ and $\eta_i B$ a copy of the Hilbert B-module B, and let

$$V_i : H_i \otimes_B H(i) \to H$$

be the unitary defined as follows:

$$\xi_i \otimes \eta_i \overset{V_i}{\mapsto} \xi,$$
$$\zeta \otimes \eta_i \overset{V_i}{\mapsto} \zeta,$$
$$\xi_i \otimes (\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n) \overset{V_i}{\mapsto} \zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n,$$
$$\zeta \otimes (\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n) \overset{V_i}{\mapsto} \zeta \otimes \zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_n,$$

$\forall \zeta \in H_i^\circ, \forall \zeta_j \in H_{i_j}^\circ$, $i \neq i_1 \neq i_2 \neq \cdots \neq i_{n-1} \neq i_n$. Let $\lambda_i : A_i \to \mathcal{L}(H)$ be the $*$-homomorphism given by

$$\lambda_i(a) = V_i(\pi_i(a) \otimes 1)V_i^*.$$

Then A is defined to be the C^*-algebra generated by $\bigcup_{i \in I} \lambda_i(A_i)$, and $\phi : A \to B$ is the conditional expectation $\phi(\cdot) = (\xi, \cdot\xi)_H$. The pair (A, ϕ), together with the embeddings $\lambda_i : A_i \hookrightarrow A$ which restrict to the identity on B, is called the *reduced amalgamated free product of the* $(A_i, \phi_i)'s$ and is characterized by the following properties:

(i) $\forall i \in I \quad \phi|_{A_i} = \phi_i$;

(ii) the family $(A_i)_{i \in I}$ is free with respect to ϕ, i.e.

$$\phi(a_1 a_2 \cdots a_n) = 0 \quad \forall a_j \in A_{i_j} \cap \ker\phi_{i_j}, \text{ with } i_1 \neq i_2 \neq \cdots \neq i_n, n \geqslant 1;$$

(iii) A is generated by $\bigcup_{i \in I} \lambda(A_i)$;

(iv) the GNS representation of ϕ is faithful on A. We will write

$$(A, \phi) = *_{i \in I}(A_i, \phi_i).$$

As in the case of the tensor product, for the reduced free product we can construct canonical conditional expectations into the factors. The following proposition can be found in [2] as Lemma 1.1:

PROPOSITION 3.6. *Let $(A_i)_{i \in I}$ be a family of unital C^*-algebras having a common subalgebra $1 \in B \subset A_i, \forall i \in I$. Suppose there are conditional expectations $\phi_i : A \to B$, for all $i \in I$ with faithful* GNS *representation. If*

$$(A, \phi) = *_{i \in I}(A_i, \phi_i)$$

is the reduced amalgamated free product, then for every $i_0 \in I$ there is a canonical conditional expectation, $\psi_{i_0} : A \to A_{i_0}$ with the following properties:

(i) $\psi_{i_0}|_{A_i} = \phi_i$ *for every* $i \in I \setminus \{i_0\}$.

(ii) $\psi_{i_0}(a_1 a_2 \cdots a_n) = 0$ *whenever* $n \geqslant 2$ *and* $a_j \in A_{i_j} \cap \ker\phi_{i_j}$ *with* $i_1 \neq i_2 \neq \cdots \neq i_{n-1} \neq i_n$.

The last theorem we mention refers to the embedding of reduced free products of unital C^*-algebras and can be found also in [2] as Theorem 1.3.

THEOREM 3.7. *Let $B \subset \widetilde{B}$ be a (not necessarily unital) inclusion of unital C^*-algebras. Let I be a set and for each $i \in I$ suppose*

$$\begin{array}{ccccc} 1_{\widetilde{A}_i} \in & \widetilde{B} & \subset & \widetilde{A}_i \\ & \cup & & \cup \\ A_i \in & B & \subset & A \end{array}$$

are inclusions of C^*-algebras. Suppose that $\widetilde{\phi}_i : A_i \to B_i$ is a conditional expectation such that $\widetilde{\phi}_i(A_i) \subset B$ and assume that $\widetilde{\phi}_i$ and the restriction $\widetilde{\phi}_i|_{A_i}$ have faithful GNS *representations, for all $i \in I$. Let*

$$(\widetilde{A}, \widetilde{\phi}) = *_{i \in I}(\widetilde{A}_i \widetilde{\phi}_i),$$
$$(A, \phi) = *_{i \in I}(A_i, \widetilde{\phi}_i|_{A_i}),$$

be the reduced amalgamated free products of C^*-algebras. Then there is a unique $*$-homomorphism $k : A \to \widetilde{A}$ such that for every $i \in I$ the diagram

$$\begin{array}{ccc} \widetilde{A}_i & \hookrightarrow & \widetilde{A}_i \\ \cup & & \uparrow k \\ A_i & \hookrightarrow & A \end{array}$$

commutes, where the horizontal arrows are the inclusions arising from the free product construction. Moreover, k is necessarily injective.

4. GENERALIZED FREE PRODUCTS OF ALGEBRAS WITH AMALGAMATED SUBALGEBRAS

In this section we present the definition of the generalized amalgam in the category of unital algebras and C^*-algebras.

DEFINITION 4.1. Let $(A_i)_{i \in I}$ be a family of unital algebras for all $i \in I$, let $B_{ij} \subset A_i$ for all $j \in I, j \neq i$ be unital subalgebras such that $B_{ij} \overset{\phi_{ij}}{\simeq} B_{ji}$, $\phi_{ij}^{-1} = \phi_{ji}, j \in I, j \neq i$. The generalized free product of the family $(A_i)_{i \in I}$ with amalgamated subalgebras $(B_{ij})_{i,j \in I, i \neq j}$ is the unique unital algebra A (up to isomorphism) together with unital homomorphisms $\psi_i : A_i \to A$ satisfying the properties:

(i) the diagram in Figure 7 commutes for all $i, j \in I, i \neq j$ and A is generated by $\bigcup\limits_{i \in I} \psi_i(A_i)$.

(ii) for any unital algebra C and unital homomorphisms $\varphi_i : A_i \to C, i \in I$, that make the diagram in Figure 8 commute for all $i \in I$ there exists a unique

$$\begin{tikzcd} B_{ij} \arrow[rr, "\psi_i"] \arrow[dr, "\phi_{ij}"'] & & A \\ & B_{ji} \arrow[ur, "\psi_j"'] & \end{tikzcd}$$

FIGURE 7. Identification in A of isomorphic subalgebras

$$\begin{tikzcd} B_{ij} \arrow[rr, "\phi_i"] \arrow[dr, "\phi_{ij}"'] & & C \\ & B_{ji} \arrow[ur, "\phi_j"'] & \end{tikzcd}$$

FIGURE 8. Identification in C of isomorphic subalgebras

homomorphism $\Phi : A \to C$ making the diagrams in Figure 9 commute for all $i \in I$.

$$\begin{tikzcd} A_i \arrow[rr, "\phi_i"] \arrow[dr, "\psi_i"'] & & C \\ & A \arrow[ur, "\Phi"'] & \end{tikzcd}$$

FIGURE 9. Universal property

We will use the notation:

$$A = *_{i\in I}(A_i, (B_{ij})_{j\neq i}).$$

When $A_i = \bigvee_{j\in I, j\neq i} B_{ij}$, we call A the *minimal generalized free product* or the *minimal amalgam*.

REMARK 4.2. Similarly to case of groups, we do not expect the $*$-homomorphisms $\psi_i : A_i \to A$ to be always injective — passing from collapsing family of groups to their group algebras, we get collapsing families of algebras. When the ψ_i'-s are injective, we say that the generalized free product is realizable.

EXAMPLE 4.3. Suppose $I = \{1, 2, 3\}$ and $*_{i=1}^{3}(A_i, (B_{ij})_{j\neq i})$ is realizable. We can present the family of unital algebras $(A_i)_{i\in I}$, with unital subalgebras $(B_{ij})_{ij}$ as in the previous definition, in the form of a triangle of algebras. First, let us remark that, if $A = *_{i\in I}(A_i, (B_{ij})_{j\in I, j\neq i})$ is the realizable generalized free product, with $\psi_i : A_i \to A$ the corresponding injective homomorphisms, then

$$\bigcap_i (\psi_i(A_i)) = \psi_1(B_{12}) \cap \psi_2(B_{23}) = \psi_2(B_{23}) \cap \psi_3(B_{13}),$$

hence letting $D = \psi_1^{-1}\Big(\bigcap_{i\in I}(\psi_i(A_i))\Big)$, we have $D \subset B_{ij}$, for all $i, j \in I, i \neq j$. We get a triangle of algebras and inclusions, denoted $(A_1, A_2, A_3; B_{12}, B_{13}, B_{23}; D)$ — we will employ this notation also for triangles of groups or C^*-algebras.

$$\begin{tikzcd}
 & & A_1 & & \\
 & B_{12} \arrow[ur] \arrow[ddl] & & B_{13} \arrow[ul] \arrow[ddr] & \\
 & & B_{123} \arrow[ul] \arrow[ur] \arrow[d] & & \\
A_2 & & B_{23} \arrow[ll] \arrow[rr] & & A_3
\end{tikzcd}$$

FIGURE 10. Triangle diagram of unital C^*-algebras and inclusions

Therefore, for such families, a necessary condition for realization is that the intersections of the edge algebras in the vertex algebras have to be isomorphic and identified in the final amalgam. Such *triangles* are called *fillable*. We will consider only fillable triangles. As a D module, A is the quotient of the D-module which has as basis the set

$$B = \{a_1 a_2 \cdots a_n : n \in \mathbb{N}, a_j \in A_{i_j}, i_1 \neq i_2 \neq \cdots \neq i_n\}$$

by the submodule generated by relations of the form

$$\begin{aligned}
&a_1 \cdots a_{j-1}(\lambda a_j^{(0)} + \mu a_j^{(1)}) a_{j+i} \cdots a_n \\
&\qquad = \lambda a_1 \cdots a_{j-1} a_j^{(0)} a_{j+1} \cdots a_n + \mu a_1 \cdots a_{j-1} a_j^{(1)} a_{j+1} \cdots a_n \quad \text{where } \lambda, \mu \in \mathbb{C}; \\
&\quad a_j = 1 \Rightarrow a_1 \cdots a_n = a_1 \cdots a_{j-i} a_{j+i} \cdots a_n, \quad a_{j-1} \in A_k, a_j \in B_{kl}, a_{j+1} \in A_l \\
&\qquad\qquad \Rightarrow a_1 \cdots (a_{j-1} a_j) a_{j+1} \cdots a_n = a_1 \cdots a_{j-1}(\phi_{kl}(a_j) a_{j+1}) \cdots a_n
\end{aligned}$$

where $k, l \in I, k \neq l$ and $A_k \supseteq B_{kl} \overset{\phi_{kl}}{\simeq} B_{lk} \subseteq A_l$.

REMARK 4.4. If each of the A_i has an involution, then so does A in the obvious manner and, moreover, A is the universal object as in the previous definition in the category of unital $*$-algebras and unital $*$-homomorphisms.

The following proposition relates the generalized free products of groups and algebras. If G is a group, then its group algebra $\mathbb{C}[G]$ is the $*$-algebra with basis G, multiplication given by group multiplication and involution by $g^* = g^{-1}$.

PROPOSITION 4.5 (Group Algebras). *Let G be the generalized free product of the groups $(G_i)_{i\in I}$ with amalgamated subgroups $(H_{ij})_{i,j\in I, j\neq i}$. Then $\mathbb{C}[G]$ is the generalized free product of $(\mathbb{C}[G_i])_{i\in I}$ with amalgamation over $(\mathbb{C}[H_{ij}])_{i,j\in I, j\neq i}$.*

Proof. By the definition of the generalized free product of groups, there are group morphisms $f_i : G_i \to G$; they extend by linearity to $*$-algebra morphisms $f_i : \mathbb{C}[G_i] \to \mathbb{C}[G]$. To show that $\mathbb{C}[G]$ is the universal object described above, let B be a fixed unital $*$-algebra and let $\phi_i : \mathbb{C}[G_i] \to B$ be unital $*$-morphisms such that the diagrams from the definition, with $B_i = \mathbb{C}[G_i]$, etc., commute. A morphism $\phi : \mathbb{C}[G] \to B$ is completely determined by its restriction $\phi|_0 : G \to \mathcal{U}(B)$, where $\mathcal{U}(B)$ are the unitaries of B. Using the definition of G and the restrictions $\phi_i|_{G_i} : G_i \to \mathcal{U}(B)$, we obtain the desired restriction $\phi_0 : G \to \mathcal{U}(B)$. ∎

The generalized free product for unital C^*-algebras and the corresponding minimal amalgam is the universal object defined similarly to that for unital algebras.

DEFINITION 4.6. Let $(A_i)_{i\in I}$ be a family of unital C^*-algebras, and for all $i \in I$ let $B_{ij} \subset A_i$ for all $j \in I, j \neq i$ be unital C^*-subalgebras such that $B_{ij} \overset{\phi_{ij}}{\simeq} B_{ji}$, $\phi_{ij}^{-1} = \phi_{ji}, j \in I, j \neq i$. The generalized free product of the family $(A_i)_{i\in I}$ with amalgamated subalgebras $(B_{ij})_{i,j\in I, i\neq j}$ is the unique unital C^*-algebra A (up to C^*-isomorphism) together with unital $*$-homomorphisms $\psi_i : A_i \to A$ satisfying the properties:

(i) the diagram in Figure 11 commutes for all $i, j \in I, i \neq j$ and A is generated by $\bigcup_{i\in I} \psi_i(A_i)$ as a C^*-algebra.

$$\begin{tikzcd} B_{ij} \arrow[rr, "\psi_i"] \arrow[dr, "\phi_{ij}"'] & & A \\ & B_{ji} \arrow[ur, "\psi_j"'] & \end{tikzcd}$$

FIGURE 11. Identification in A of isomorphic C^*-subalgebras

(ii) for any unital C^*-algebra C and unital $*$-homomorphisms $\varphi_i : A_i \to C, i \in I$, that make the diagram in Figure 12 commute for all $i \in I$ there exists a unique $*$-homomorphism $\Phi : A \to C$ making the diagrams in Figure 13 commute for all $i \in I$.

We will use the notation

$$A = *_{i\in I}(A_i, (B_{ij})_{j\neq i}).$$

When $A_i = \bigvee_{j\in I, j\neq i} B_{ij}$, we call A the *minimal generalized free product* or the *minimal amalgam*.

$$\begin{tikzcd} B_{ij} \arrow[rr, "\phi_i"] \arrow[dr, "\phi_{ij}"'] & & C \\ & B_{ji} \arrow[ur, "\phi_j"'] & \end{tikzcd}$$

FIGURE 12. Identification in C of isomorphic C^*-subalgebras

$$\begin{tikzcd} A_i \arrow[rr, "\phi_i"] \arrow[dr, "\psi_i"'] & & C \\ & A \arrow[ur, "\Phi"'] & \end{tikzcd}$$

FIGURE 13. Universal property

REMARK 4.7. It is easy to see that such a C^*-algebra exists — just take the enveloping C^*-algebra of the $*$-generalized free product of the family $(A_i)_{i\in I}$. Unlike the free product, the $*$-homomorphisms $\psi_i : A_i \to A$ need not be injective — such an example relies on the examples with collapsing families of groups and the next proposition which gives the link between groups and C^*-algebras of groups.

REMARK 4.8. In what follows, all groups will be discrete. If G is such a group, we will denote by $C^*(G)$ the group C^*-algebra of G, and with $C^*_{\text{red}}(G)$ the reduced group C^*-algebra of G.

PROPOSITION 4.9 (Group C^*-algebras). *Let G be the generalized free product of the groups $(G_i)_{i\in I}$ with amalgamated subgroups $(H_{ij})_{i,j\in I, j\neq i}$. Then*

$$C^*(G) = *_{i\in I}(C^*(G_i); (C^*(H_{i,j}))_{i,j\in I, j\neq i}).$$

Proof. Using the above remark, the definition of the generalized free product, and, as in the previous proof, the fact that a unitary representation of the generalized free product of groups is determined by its restriction to the factors G_i , the proposition follows immediately. ∎

We address now the question of realization for generalized free products of algebras. For group C^*-algebras this amounts to the realization of the generalized free product of groups.

THEOREM 4.10. *Let $(G_i)_{i\in I}$ be groups and for each $i \in I$ and for each $j \in I, j \neq i$ let H_{ij} be a subgroup of G_i such that $H_{ij} \overset{\phi_{ij}}{\simeq} H_{ji}$, with $\phi_{ij}^{-1} = \phi_{ji}, \forall i, j \in I, i \neq j$. If the generalized free product of $(G_i)_{i\in I}$ with amalgamated subgroups $(H_{ij})_{i,j\in I, i\neq j}$ is realizable then:*

(i) *the generalized free product of the family of group algebras* $(\mathbb{C}[G_i])_{i\in I}$ *with amalgamated subalgebras* $(\mathbb{C}[H_{ij}])_{i,j\in I, i\neq j}$ *is realizable;*

(ii) *the generalized free product of the family of group C^*-algebras* $C^*(G_i)_{i\in I}$ *with amalgamated subalgebras* $C^*(H_{ij})_{i,j\in I,i\neq j}$ *is realizable.*

Proof. (i) From Proposition 4.5 we have $\mathbb{C}[G] = *_{i\in I}(\mathbb{C}[G_i], (\mathbb{C}[H_{ij}])_{j\neq i})$. The corresponding maps $f_i : \mathbb{C}[G_i] \to \mathbb{C}[G]$ are the extensions of the injective group homomorphism $\psi_i : G_i \to G$ from the definition of $*_{i\in I}(G_i, (H_{ij})_{j\neq i})$ and therefore they are also injective and realization is proved.

(ii) Use Proposition 4.9 and the fact that the inclusion $G_i \subset G$ induces an inclusion $C^*(G_i) \subset C^*(G)$. ∎

As in the case of groups, triangles of unital C^*-algebras are of special interest. They are the simplest non-trivial case of a generalized free product with amalgamation. Next we present a version of Theorem 2.6 for C^*-algebras.

THEOREM 4.11. *Let* $(A_1, A_2, A_3; B_{12}, B_{13}, B_{23}; B_{123})$ *be a minimal triangle of unital C^*-algebras (i.e. the vertex algebras are generated by the images of the edge algebras). Each of the following is a sufficient condition for the realization of the generalized free product with amalgamation* $*_{i=1}^{3}(A_i, (B_{ij})_{j\neq i})$*:*

(i) A_1 *is the full free product of* B_{12} *and* B_{13} *with amalgamation over* B_{123} *and there exist conditional expectations* $E_{12} : A_2 \to B_{12}, E_{13} : A_3 \to B_{13}$ *and* $E_{123} : B_{23} \to B_{123}$ *such that the diagram*

$$
\begin{array}{ccccc}
A_2 & \hookleftarrow & B_{23} & \hookrightarrow & A_3 \\
\downarrow \mathrm{E}_{12} & & \downarrow \mathrm{E}_{123} & & \downarrow \mathrm{E}_{13} \\
B_{12} & \hookleftarrow & B_{123} & \hookrightarrow & B_{13}
\end{array}
$$

commutes.

(ii) *There exist conditional expectations* $E^{B_{1i}}_{B_{123}} : B_{1i} \to B_{123}, i = 2, 3,$ *with faithful* GNS *representations,* A_3 *is the reduced free product of* $(B_{12}, E^{B_{12}}_{B_{123}})$ *and* $(B_{13}, E^{B_{13}}_{B_{123}})$ *with amalgamation over* B_{123} *and there are conditional expectations* $E^{A_i}_{B_{23}} : A_i \to B_{23}$ *such that* $E^{A_i}_{B_{23}}(B_{1i}) \subseteq B_{123}$ *and both* $E^{A_i}_{B_{23}}$ *and its restriction to* B_{1i} *have faithful* GNS *representations* $(i = 2, 3)$*.*

Proof. We show that the vertex algebras embed respectively into the full free product $A_2 *_{B_{23}} A_3$ for (i) and the reduced free product $(A_2 *_{B_{23}} A_3, E^{A_2}_{B_{23}} * E^{A_3}_{B_{23}})$ for (ii). This follows from Proposition 3.3 for (i) and from Proposition 3.7 for (ii). An immediate verification shows that the diagrams that need to commute do so, hence in both cases the generalized free product of the minimal triangle is realizable. ∎

For unital algebras, $*$-algebras and C^*-algebras, we can prove a similar reduction theorem as the one for groups.

THEOREM 4.12. *Let $(A_i)_{i\in I}$ be a family of unital ($*$-, C^*-) algebras. For each $i \in I$, let $B_{ij} \subset A_i, j \in I, j \neq i$ be unital ($*$-, C^*-) subalgebras such that $B_{ij} \overset{\phi_{ij}}{\simeq} B_{ji}$, with $\phi_{ij}^{-1} = \phi_{ji}, \forall i, j \in I, i \neq j$. Let $B_i := \bigvee_{j\in I, j\neq i} B_{ij} \subseteq A_i$. The following are equivalent:*

(i) *the minimal amalgam of the family $(B_i)_{i\in I}$ is realizable;*

(ii) *the generalized free product of $(A_i)_{i\in I}$ with amalgamated subalgebras $(B_{ij})_{i,j\in I, i\neq j}$ is realizable.*

Proof. The proof follows the lines of the reduction theorem for groups. The implication (ii)⇒(i) is trivial. For (i)⇒(ii), note that for each of the considered categories there exists a well defined free product with amalgamation. Denote by B the minimal realizable amalgam. In each of the cases consider the algebra

$$A_0 = *_{B, i\in I}(A_i *_{B_i} B).$$

We denote with the same A_i, B_{ij} the image of these algebras inside A_0. Note that in A_0 we have $A_i \cap A_j = B_{ij} = B_{ji}$. Then $A = \bigvee_{i\in I} A_i \subseteq A_0$ contains isomorphic copies of the factors A_i, hence the generalized free product of $(A_i)_{i\in I}$ is realizable. Even more is true. Since $B_i \subset A_i \subset A$ and $B = *_{i\in I}(B_i, (B_{ij})_{j\neq i})$, it follows that $B \subset A$. As in the case of groups, it is then easy to check that $A_0 = A$. For example, in the case of C^*-algebras, we have

$$\overline{(\mathrm{span}\{x_1x_2\cdots x_n : n \geqslant 1, x_k \in A_{i_k} *_{B_{i_k}} B, i_k \in I, i_1 \neq i_2 \neq \cdots \neq i_n\})}^{\|\cdot\|} = A_0.$$

But B is generated by all the B_i's, so the products $x_1x_2\cdots x_n$ all belong to A, which is the C^*-subalgebra generated by $(A_i)_{i\in I}$. Therefore

$$\begin{aligned} A &= \overline{(\mathrm{span}\{x_1x_2\cdots x_n : n \geqslant 1, x_k \in A_{i_k}, i_k \in I, i_1 \neq i_2 \neq \cdots \neq i_n\})}^{\|\cdot\|} \\ &= \overline{(\mathrm{span}\{x_1x_2\cdots x_n : n \geqslant 1, x_k \in A_{i_k} *_{B_{i_k}} B, i_k \in I, i_1 \neq i_2 \neq \cdots \neq i_n\})}^{\|\cdot\|} = A_0 \end{aligned}$$

and we are done. ∎

5. REDUCED FREE PRODUCTS IN THE PRESENCE OF THE MINIMAL AMALGAM

The next theorem shows that it is actually possible to have a "reduced" version of the generalized free product, if the minimal amalgam exists. We present a natural way to build a state that relates the generalized free product to the freeness concept introduced by Voiculescu.

THEOREM 5.1. *Let $(A_i)_{i\in I}$ be a family of unital C^*-algebras. For each $i \in I$ let $B_{ij} \subset A_i, j \in I, j \neq i$ be unital subalgebras such that $B_{ij} \overset{\varphi_{ij}}{\simeq} B_{ji}$, with $\varphi_{ij}^{-1} = \varphi_{ji}, \forall i, j \in I, i \neq j$. Let $B_i := \bigvee_{j\in I, j\neq i} B_{ij} \subset A_i$. Suppose that the minimal amalgam B is realizable and that there are conditional expectations $\phi_i : A_i \to B_i$, $\psi_i : B \to B_i, i \in I$, each* *having faithful* GNS *representation. Then there exists an algebra A and a conditional expectation $\phi : A \to B$ such that:*

(i) *for each $i \in I$, A there exists an injective $*$-homomorphism $\sigma_i : A_i \to A$ and A is generated by $\bigcup_{i\in I} \sigma_i(A_i)$;*

(ii) *$\phi|_{A_i} = \phi_i$ and the family $(\sigma_i(A_i))_{i\in I}$ is free with amalgamation over B in (A, ϕ).*

Proof. Fix $i \in I$ and consider the reduced free product $(A_i *_{B_i} B, \phi_i * \psi_i)$. Since both ϕ_i and ψ_i have faithful GNS representations, there exists a conditional expectation $E_i : A_i *_{B_i} B \to B$ like in Proposition 3.6. Let

$$\pi_i : A_i *_{B_i} B \to \mathcal{L}(L^2(A_i *_{B_i} B, E_i))$$

be the associated $*$-representation, as in Section 3. As described in [2], $\pi_i|_{A_i}$ is faithful. Let

$$(A, \phi) = *_{B,i\in I}(A_i *_{B_i} B, E_i).$$

This is done via the free product representation $\lambda = *_i\lambda_i$ as described in Section 3. Because $\pi_i|_{A_i}$ is faithful and $\lambda_i = V_i(\pi_i \otimes \mathrm{Id})V_i^*$ where V_i is the unitary from Section 3, we conclude that $\lambda_i|_{A_i}$ is also faithful. Furthermore, λ is precisely the GNS representation of A associated to ϕ. We have:

$$\begin{aligned} A &= \overline{\bigcup \lambda_i(A_i *_{B_i} B)}^{\|\cdot\|} \\ &= \overline{\operatorname{span}\{\lambda_{i_1}(x_1)\lambda_{i_2}(x_2)\cdots\lambda_{i_n}(x_n) : x_j \in A_{i_j} *_{B_{i_j}} B, i_1 \neq i_2 \neq \cdots \neq i_n, n \geqslant 1\}}^{\|\cdot\|} \\ &= \overline{\operatorname{span}\{\lambda_{i_1}(a_1)\lambda_{i_2}(a_2)\cdots\lambda_{i_n}(a_n) : x_j \in A_{i_j}, i_1 \neq i_2 \neq \cdots \neq i_n, n \geqslant 1\}}^{\|\cdot\|}, \end{aligned}$$

because the copy of B inside A is generated by $\bigcup_{i\in I} \lambda_i(B_i) \subset \bigcup_{i\in I} \lambda_i(A_i)$. Letting $\sigma_i = \lambda_i|_{A_i}$ we get (i). Remark that (ii) is just a consequence of the construction; inside A, the algebra generated by $\lambda(A_i)$ and $\lambda(B)$ is exactly $\lambda(A_i *_{B_i} B) = \lambda_i(A_i *_{B_i} B)$. ∎

REMARK 5.2. We can actually describe $L^2(A, \phi)$ using the description of $L^2(A_i *_{B_i} B, E_i), i \in I$, given in [2]. For $i \in I$, let $H_{i,1} = L^2(A_i, \phi_i)$ and $H_{i,2} = L^2(B, \psi_i)$; let $H^\circ_{i,k}$ denote the orthogonal complement with respect to B_i ($k = 1, 2$). From [2] we get

$$F_i := L^2(A_i *_{B_i} B, E_i) = B \oplus \bigoplus_{\substack{n \geqslant 1 \\ k_1, k_2, \ldots, k_n \in \{1,2\} \\ k_1 \neq k_2 \neq \cdots \neq k_n \neq 2}} H^\circ_{i,k_1} \otimes_{B_i} H^\circ_{i,k_2} \otimes_{B_i} \cdots \otimes_{B_i} H^\circ_{i,k_n} \otimes_{B_i} B.$$

By construction, we know the structure of $L^2(A, \phi)$ is (with F_i° the orthogonal complement with respect to B)

$$L^2(A, \phi) = B \oplus \bigoplus_{\substack{n \geqslant 1 \\ i_1, i_2, \ldots, i_n \in I \\ i_1 \neq i_2 \neq \cdots \neq i_n \neq 2}} F^\circ_{i_1} \otimes_B F^\circ_{i_2} \otimes_B \cdots \otimes_B F^\circ_{i_n}$$

$$= B \oplus \bigoplus_{\substack{n \geqslant 1 \\ i_1, i_2, \ldots, i_n \in I \\ i_1 \neq i_2 \neq \cdots \neq i_n \neq 2}} \bigoplus_{\substack{m_{i_1} \geqslant 1, m_{i_2} \geqslant 1, \ldots, m_n \geqslant 1 \\ k_{(i_p,1)}, k_{(i_p,2)}, \ldots, k_{(i_p,m_p)} \in \{1,2\}, \\ k_{(i_p,1)} \neq k_{(i_p,2)} \neq \cdots \neq k_{(i_p,m_p)} \neq 2, \\ p = \overline{1,n}}} H^\circ_{i_1,k_{(i_1,1)}} \otimes_{B_{i_1}} H^\circ_{i_1,k_{(i_1,2)}} \otimes_{B_{i_1}}$$

$$\cdots \otimes_{B_{i_1}} H^\circ_{i_1,k_{(i_1,m_1)}} \otimes_{B_{i_1}} B \otimes_B H^\circ_{i_2,k_{(i_2,1)}} \otimes_{B_{i_2}} H^\circ_{i_2,k_{(i_2,2)}} \otimes_{B_{i_2}}$$

$$\cdots \otimes_{B_{i_n}} H^\circ_{i_{n-1},k_{(i_{n-1},m_{n-1})}} \otimes_{B_{i_{n-1}}} B \otimes_B H^\circ_{i_n,k_{(i_n,1)}} \otimes_{B_{i_n}} \cdots \otimes_{B_{i_n}} H^\circ_{i_n,k_{(i_n,m_n)}} \otimes_{B_{i_n}} B.$$

This description of the Fock space associated to ϕ, and hence of the kernel of ϕ, corresponds to the following intuitive decomposition of a word over the alphabet $\bigcup_{i \in I} A_i$ in the generalized free product with amalgamations. If $w = a_1 a_2 \cdots a_n$, with $n \geqslant 1$ and $a_j \in A_{k_j}, i_1 \neq k_2 \neq \cdots \neq k_n$, then $a_j^\circ = a_j - \phi_{k_j}(a_j) \in A^\circ_{k_j} \subset H^\circ_{k_j,1}$, so between elements from the A_i^s with 0 expectation there will be elements from B, hence

$$w \in B + \sum_{\substack{m \geqslant 1 \\ i_1, i_2, \ldots, i_m \in I}} A^\circ_{i_1} B(i_1, i_2) A^\circ_{i_2} B(i_2, i_3) \cdots B(i_{m-1}, i_m) A^\circ_{i_m},$$

where if $s \neq r$ then $B(r,s)$ is the set of words over $\bigcup_{i \in I} B_i$ that do not start with a "letter" in B_r and do not end with a "letter" from B_s and $B(r,r) = B^{\circ,r} \subset H^\circ_{r,2}$.

6. EXAMPLES OF TRIANGLES OF ALGEBRAS

In this section we conclude our paper with 3 examples of realizable minimal amalgams, sharing some common features. One of this is that they all appear as the generalized free product of a family $(D_i, (D_{ij})_{j \neq i})_{i \in I}$, where $I = \{1,2,3\}$. The relations among the algebras can be schematically described using a triangle diagram of algebras and injective $*$-homomorphisms, represented by arrows:

$$\begin{tikzcd}
& & D_1 & & \\
& D_{12} \arrow[ur] \arrow[ddl] & & D_{13} \arrow[ul] \arrow[ddr] & \\
& & D_{123} \arrow[ul] \arrow[ur] \arrow[d] & & \\
D_2 & & D_{23} \arrow[ll] \arrow[rr] & & D_3
\end{tikzcd}$$

FIGURE 14. Triangle diagram of unital C^*-algebras and inclusions

EXAMPLE 6.1. The first example presents the free amalgamated product of 3 unital C^*-algebras as the result of the minimal amalgam of a triangle of unital C^*-algebras. For this, let A_1, A_2, A_2 be unital C^*-algebras and let B be a common unital C^*-subalgebra. Let

$$A = *_{B,i} A_i.$$

Let $D_i = A_j *_B A_k \subset A$, for $i, j, k \in \{1, 2, 3\}, i \neq j \neq k \neq i$. With $D_{ij} := D_i \cap D_j = A_k$,$i, j, k \in \{1, 2, 3\}, i \neq j \neq k \neq i$ we get $D = B$ and it is an easy exercise to check that

$$A = *_{i=1}^{3} (D_i, (D_{ij})_{j \neq i}).$$

Suppose $\phi_i : A_i \to B$ are conditional expectations with faithful GNS representations. Then the triangle embeds also in the reduced amalgamated free product $(A, \phi) = *_i (A_i, \phi_i)$. By Proposition 3.6, there exist conditional expectations which reverse the arrows of the triangle. Moreover, if we let $E_{ij} : D_i \to D_{ij}$ be this maps, then $E_{ij}|_{D_{ik}} = \phi_k$ for all $i_1 \leqslant i \neq j \neq k \leqslant 2$. We could think of (A, ϕ) as the reduced version of the minimal amalgam of the triangle.

EXAMPLE 6.2. The second example presents the tensor product of 3 unital $*$-algebras (C^*-algebras) as the minimal amalgam of a triangle. Let A_1, A_2, A_3 be unital $*$-algebras and let

$$A = A_1 \otimes A_2 \otimes A_3.$$

Choose $D_i = A_j \otimes A_k$, $D_{ij} = A_k = D_{ji}$ where $i, j, k \in \{1, 2, 3\}, i \neq j \neq k \neq i$. Automatically we need $D = \mathbb{C}$. The injective homomorphisms from $D_{ij} = A_k$ into D_i are the obvious ones, sending A_k into its corresponding component in D_i. Note that the entire family embeds naturally into A, and A is generated by the images of D_i, hence

$$A = A_1 \otimes A_2 \otimes A_3 = *_{i=1}^{3} (D_i, (D_{ij})_{j \neq i}).$$

Suppose now that $\phi_i : A_i \to \mathbb{C}$ are states. We can then define conditional expectations $E_{ij} :_i \to D_{ij}$, letting $E_{ij}(a_k \otimes a_j) = \phi_k(a_k) a_j$. Then $E_{ij}|_{D_{ik}} = \phi_k$.

EXAMPLE 6.3. Denote by $M_n := M_n(\mathbb{C})$ the algebra of n by n matrices of complex numbers. The previous example shows the trivial way to get $M_8 \simeq M_2 \otimes M_2 \otimes M_2$ as the minimal amalgam of a triangle. In this example we show that there is another way of getting M_8 as the minimal amalgam of a triangle with the same "vertices" as above. Let $D_i = M_2 \otimes M_2$ for $i = 1, 2, 3$. Consider the following two selfadjoint unitaries :

$$u = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It is worth mentioning that u and v are the only biunitaries in M_4 which are also permutation matrices; a matrix $w = (w_b^a)_j^i \in M_n \otimes M_k$ is called a *biunitary* if it is a unitary matrix and if the matrix defined by $((w_1)^a_b)^i_j = (w^b_a)^i_j$, the block-transpose of w, is also a unitary (see [5] for more details on biunitary permutation matrices). Since u, v are biunitaries, the following two are non-degenerated commuting squares:

$$\begin{array}{ccc} u(M_2 \otimes I_2)u & \subset & M_2 \otimes M_2 \\ \cup & & \cup \\ \mathbb{C} & \subset & I_2 \otimes M_2 \end{array}, \quad \begin{array}{ccc} v(M_2 \otimes I_2)v & \subset & M_2 \otimes M_2 \\ \cup & & \cup \\ \mathbb{C} & \subset & I_2 \otimes M_2 \end{array}.$$

The non-degeneracy means that,with $D_{12} = u(M_2 \otimes I_2)u, D_{23} = v(M_2 \otimes I_2)v$ and $D_{13} = I_2 \otimes M_2$, the following holds:

$$M_2 \otimes M_2 = \operatorname{span}\{AB : A \in D_{12}, B \in D_{13}\} = \operatorname{span}\{BA : A \in D_{12}, B \in D_{13}\},$$

and also

$$M_2 \otimes M_2 = \operatorname{span}\{AB : A \in D_{23}, B \in D_{13}\} = \operatorname{span}\{BA : A \in D_{23}, B \in D_{13}\}.$$

Even more is true in commuting squares: let τ_n be the normalized trace in M_n and define $E_0 : M_2 \otimes M_2 \to I_2 \otimes M_2$, $E_0(a \otimes b) = \tau_2(a)(I_2 \otimes b)$; $E_u : M_2 \otimes M_2 \to D_{12}$, $E_u(a \otimes b) = \tau_2(b)u(a \otimes I_2)u$ and finally $E_v : M_2 \otimes M_2 \to D_{23}$, $E_v(a \otimes b) = \tau_2(b)v(a \otimes I_2)v$. Then $E_u(I_2 \otimes b) = \tau_2(b) = E_v(I_2 \otimes b)$. Let $e(i,j)$ be the matrix units of M_2 and let $e_u(i,j) = u(e(i,j) \otimes I_2)u$, $e_v(i,j) = v(e(i,j) \otimes I_2)v$ and $e_0(i,j) = I_2 \otimes e(i,j)$. One checks that we also have

$$M_2 \otimes M_2 = \operatorname{span}\{AB : A \in D_{12}, B \in D_{23}\} = \operatorname{span}\{BA : A \in D_{12}, B \in D_{23}\},$$

and that $D_{12} \cap D_{23} = \mathbb{C}$. However, $E_u(v(a \otimes I_2)v) = a_{11}e_u(1,1) + a_{22}e_u(2,2)$, so that the commuting squares condition fails for this pair of algebras. We want to find the minimal amalgam. For this purpose note the relations in Table 1, where $\sigma = (12) \in S_2$.

TABLE 1. Commutation relations

$e_u(i,i)x_v = x_v e_u(i,i)$,	for all $i \in I, x_v \in D_{23}$
$e_u(i,i)e_0(k,k) = e_0(k,k)e_u(i,i)$	for all i, k
$e_u(i,j)e_0(k,l) = e_0(k,l)e_u(\sigma(i),\sigma(j))$,	for all i, j, k, l;
$e_v(i,i)x_0 = x_0 e_v(i,i)$,	for all $i \in I, x_0 \in D_{13}$
$e_v(i,i)e_u(k,k) = e_u(k,k)e_v(i,i)$	for all i, k
$e_v(i,j)e_u(k,l) = e_u(k,l)e_v(\sigma(i),\sigma(j))$,	for all i, j, k, l;
$e_0(i,i)x_u = x_u e_0(i,i)$,	for all $i \in I, x_v \in D_{12}$
$e_0(i,i)e_v(k,k) = e_v(k,k)e_0(i,i)$	for all i, k
$e_0(i,j)e_v(k,l) = e_v(k,l)e_0(\sigma(i),\sigma(j))$,	for all i, j, k, l.

As a vector space, because of these commutation relations (see Example 4.3), the minimal amalgam A will be generated by the non-zero elements

$$\{e_u(i,j)e_v(k,l)e_0(r,s) : i, j, k, l, r, s \in I\}.$$

The products of minimal projections generate in the amalgam 8 minimal projections, $\{e_u(i,i)e_v(k,k)e_0(r,r) : i,k,r = 1,2\}$, which add up to one; the partial isometries can then be constructed and a complete system of matrix units for A can be found. Even more, the "vertices" $M_2 \otimes M_2$ embed in this algebra in such a way that the resulting diagrams commute. Let us check it for D_2. Define $\phi_2 : D_2 \to A$ by

$$\phi(a \otimes b) = \phi_2\Big(\sum_{i,j,k,l=1}^{2} x(i,j,k,l)e_u(i,j)e_v(k,l)\Big) = \sum_{i,j,k,l=1}^{2} x(i,j,k,l)e_u(i,j) \cdot e_v(k,l).$$

A straightforward computation shows the one-to-one correspondence between $\{e_u(i,j)e_v(k,l) : 1 \leqslant i,j,k,l \leqslant 2\}$ and $\{e_4(\alpha,\beta) : 1 \leqslant \alpha,\beta \leqslant 4\}$, the matrix units of M_4. This shows that ϕ_2 is a well defined linear map. It is easy to check that it preserves conjugation. We will proceed to prove injectivity. Suppose $\phi_2\Big(\sum_{i,j,k,l=1}^{2} x(i,j,k,l)e_u(i,j)e_v(k,l)\Big) = 0$. Then

$$\sum_{i,j,k,l=1}^{2} x(i,j,k,l)e_u(i,j) \cdot e_v(k,l) \cdot 1 = 0 \quad \text{in } A. \tag{6.1}$$

Suppose there are i_0, j_0, k_0, l_0 such that $x(i_0, j_0, k_0, l_0) \neq 0$. Multiply (6.1) by the left with $e_u(i_0, i)$ and by the right with $e_u(j, j_0)$ to get

$$\sum_{i,j,k,l=1}^{2} x(i,j,k,l)(e_u(i_0,i)e_u(i,j)e_u(j,j_0)) \cdot e_v(\sigma(k),\sigma(l)) \cdot 1 = 0$$

or

$$\sum_{k,l=1}^{2} x(i_0,j_0,k,l)e_u(i_0,j_0) \cdot e_v(\sigma(k),\sigma(l)) \cdot 1 = 0. \tag{6.2}$$

Multiply again from the left with $e_u(j_0, i_0)$ to get

$$\sum_{k,l=1}^{2} x(i_0,j_0,k,l)e_u(j_0,j_0) \cdot e_v(\sigma(k),\sigma(l)) \cdot 1 = 0,$$

hence

$$\sum_{k,l=1}^{2} x(i_0,j_0,k,l)e_u(i_0,j_0) \cdot e_v(k,l) \cdot 1 = 0.$$

Still in (6.2) multiply from the right with $e_u(j_0, i_0)$, use the fact that $\sigma^2 = \text{id}$ to get

$$\sum_{k,l=1}^{2} x(i_0,j_0,k,l)e_u(i_0,i_0) \cdot e_v(k,l) \cdot 1 = 0.$$

Multiply in both (6.2) and the last equation from the left with $e_v(\sigma(k_0), \sigma(k))$ and from the right with $e_v(l, l_0)$ and we get:

$$e_u(i_0,i_0) \cdot e_v(k_0,k_0) \cdot 1 = 0, \quad e_u(j_0,j_0) \cdot e_v(k_0,k_0) \cdot 1 = 0,$$

and similarly for l_0. Even if $i_0 = j_0$ or $k_0 = l_0$ we will still get $e_u(i,i)e_v(k,k) = 0$ for $1 \leqslant i,k \leqslant 2$. Adding them up we get $1 = 0$ in the amalgam, a contradiction, since the amalgam contains at least $\mathbb{C}$. So ϕ_2 is injective. One can define similar maps $\phi_1 : D_1 \to A, \phi_3 : D_3 \to A$, which turn out to be injective. It is also clear that $\phi_1|_{D_{12}} = \phi_2|_{D_{12}}, \phi_1|_{D_{13}} = \phi_3|_{D_{13}}$, and $\phi_2|_{D_{23}} = \phi_3|_{D_{23}}$. Hence $A \simeq M_8$ is the realizable minimal amalgam of the triangle of algebras.

REMARK 6.4. One can generalize the previous example to arbitrary unitaries and finite dimensional vertex algebras. With the hypothesis that the commuting squares of algebras appearing in the construction are non-degenerated (i.e. the vertex algebras are the span of the product of the edge algebras) and that the edge algebras intersect along the complex numbers, one can see that the result of the amalgam will always be the span of the product of the edge algebras.

REMARK 6.5. The reduction theorems presented in the paper point to the fact that the minimal amalgam plays the key role. We are still looking for a good sufficient condition for such a minimal amalgam to exist.

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