J. OPERATOR THEORY
56:1(2006), 167–198

BELLMAN FUNCTIONS AND DIMENSIONLESS ESTIMATES OF LITTLEWOOD-PALEY TYPE

OLIVER DRAGIČEVIĆ and ALEXANDER VOLBERG

Communicated by Nikolai K. Nikolski

ABSTRACT. We use the Bellman function technique to obtain new Littlewood-Paley type estimates for classical as well as Gaussian spaces $L^p(\mathbb{R}^n)$. This is our principal result. The rest of the paper is purely methodological: it contains unified proofs of the known results. Namely, we present the uniform approach to proofs of the existence of dimension free bounds for corresponding Riesz transforms. Some of our results contain best to-date constants.

KEYWORDS: *Bellman functions, Riesz transforms, Ornstein-Uhlenbeck semigroup.*

MSC (2000): 42B20, 42B25, 44A15.

INTRODUCTION

Recall that the *Fourier transform* of a function $f \in L^1(\mathbb{R}^n)$ is given by

$$\widehat{f}(x) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} f(y)\, \mathrm{e}^{-\mathrm{i}\langle x, y\rangle}\, \mathrm{d}y\,.$$

We used the notation $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ belong to $\mathbb{R}^n$. For a *test function* f on $\mathbb{R}^n$ (say, belonging to C_c^∞ or the Schwartz class $\mathcal{S}$) we will denote by $\widetilde{f}$ its harmonic extension to the upper half-space $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$. Sometimes the "newly acquired" variable t will be labeled as x_{n+1}. We shall consider functions $f = (f_1, \ldots, f_M)$ with values in some $\mathbb{C}^M$, $M \in \mathbb{N}$. Here $\widetilde{f} = (\widetilde{f}_1, \ldots, \widetilde{f}_M)$. As usual, for $1 \leqslant p < \infty$ the Banach space $L^p(\mathbb{R}^n \to \mathbb{C}^M)$ is introduced, and the norm is given by

$$\|f\|_p = \Big(\int\limits_{\mathbb{R}^n} |f|^p \mathrm{d}m\Big)^{1/p} = \Big(\int\limits_{\mathbb{R}^n} \Big[\sum_{i=1}^M |f_i(x)|^2\Big]^{p/2} \mathrm{d}m(x)\Big)^{1/p}.$$

By $J\widetilde{f}$ we shall mean the *Jacobi matrix* of $\widetilde{f}$, which is defined as

$$J\widetilde{f} = \Big[\frac{\partial \widetilde{f}_i}{\partial x_j}\Big]_{\substack{i=1,\ldots,M \\ j=1,\ldots,n+1}}.$$

Furthermore, $\|\cdot\|_2$ will stand for the Hilbert-Schmidt norm on the space of matrices, unless specified otherwise.

Throughout the paper, p will be a number from $(1,\infty)$, while q will stand for its conjugate exponent. Let us denote $p^* = \max\{p,q\}$. We will most often encounter the factor

$$p^*-1=\left\{\begin{array}{ll} p-1 & p\geqslant 2\,, \\ (p-1)^{-1} & 1<p\leqslant 2\,. \end{array}\right.$$

We are ready to state our first main result.

THEOREM 0.1. *Let M,N,n be arbitrary natural numbers. Take test functions $f\in L^p(\mathbb{R}^n\to\mathbb{C}^M)$ and $g\in L^q(\mathbb{R}^n\to\mathbb{C}^N)$. Then*

$$2\int_0^\infty\int_{\mathbb{R}^n}\|J\widetilde{f}(x,t)\|_2\|J\widetilde{g}(x,t)\|_2\,t\,\mathrm{d}x\,\mathrm{d}t\leqslant(p^*-1)\|f\|_p\|g\|_q\,.$$

The proof will be presented in Section 1.

We believe this theorem displays a useful and rather general example of an inequality of the Littlewood-Paley type. In particular, its following corollary represents an inequality for the classical Riesz transforms that does not depend on the dimension, in other words, it represents Stein's theorem. First, we introduce the Riesz transforms.

Choose $k\in\{1,\dots,n\}$. The scalar *Riesz transform* R_k is defined on a test function f by

$$(R_kf)\widehat{\ }(x)=\mathrm{i}\frac{x_k}{\|x\|}\widehat{f}(x)\,.$$

For arbitrary functions $f\in L^p=L^p(\mathbb{R}^n)$ we extend this by density.

As mentioned above, Theorem 0.1 has an immediate corollary: these operators are bounded on $L^p(\mathbb{R}^n)$ for $p\in(1,\infty)$ with norms independent of the dimension n.

COROLLARY 0.2. *For every $n\in\mathbb{N}$ and every $f\in L^p$,*

$$\Big\|\Big(\sum_{i=1}^n|R_if|^2\Big)^{1/2}\Big\|_p\leqslant 2(p^*-1)\|f\|_p$$

and therefore also $\|R_k\|_{B(L^p)}\leqslant 2(p^*-1)$ for $k=1,\dots,n$.

As an example of how our technique can deliver results in non-standard settings, we treat in Section 3 spaces endowed with the Gaussian measure μ. We are able to obtain the analogues to the above Littlewood-Paley type estimates, which we formulated in Theorem 0.1. Here is our Littlewood-Paley dimensionless estimate for Gaussian case.

THEOREM 0.3. *For a function φ on $\mathbb{R}^n$ let the symbol $\widetilde{\varphi}$ stand for its extension to $\mathbb{R}^n_+$, generated by the Ornstein-Uhlenbeck operator. There is an absolute constant $C>0$*

such that for all test functions f, g and $1 < p < \infty$,

$$\int_0^\infty \int_{\mathbb{R}^n} \|J\widetilde{f}(x,t)\|_2 \|J\widetilde{g}(x,t)\|_2 \,\mathrm{d}\mu(x)\, t\,\mathrm{d}t \leqslant C\,(p^*-1)\|f\|_p\|g\|_q\,.$$

Consequently, we have the corollary, the famous result of Meyer-Pisier, representing the same effect of dimension free estimates for Riesz transforms as Stein's theorem (see Corollary 0.2), only now for Riesz transforms with respect to Gaussian measure. In the next corollary $\mathbf{R}_i$ are Gaussian Riesz transforms (see precise definitions in Section 3).

COROLLARY 0.4.

$$\Big\|\Big(\sum_{i=1}^n |\mathbf{R}_i f|^2\Big)^{1/2}\Big\|_p \leqslant C_p\|f\|_p$$

for some constants $C_p > 0$ that do not depend on the dimension n.

We will notice that Littlewood-Paley estimates in Theorems 0.1, 0.3 give us estimates of Riesz transforms in Corollaries 0.2, 0.4 with "a big margin" (see Remark 2.2). There will be a large extra positive term estimated at the same time. But we do not know how this extra positivity can be used.

Our Littlewood-Paley dimension free theorems allow applications in cases of other semigroup extensions. It is also demonstrated in Section 3 how closely our original problem of estimating Riesz transforms relates to the boundedness of particular spectral multipliers.

The result of Corollary 0.2 was also obtained by R. Bañuelos and G. Wang [3]. T. Iwaniec and G. Martin proved ([12], Theorem 1.5) that

$$\Big\|\Big(\sum_{i=1}^n |R_i f|^2\Big)^{1/2}\Big\|_p \leqslant \sqrt{2}\,H_p(1)\,, \tag{0.1}$$

where $H_p(1) = \|R_1 + \mathrm{i}R_2\|_p$ and R_1, R_2 are *planar* Riesz transforms. As far as we know, the factors $H_p(1)$ have not yet been computed. However, as it was also shown by [12],

$$\|R_k\|_p = \cot\Big(\frac{\pi}{2p^*}\Big)\,, \tag{0.2}$$

which immediately gives

$$H_p(1) \leqslant 2\cot\Big(\frac{\pi}{2p^*}\Big)\,. \tag{0.3}$$

From (0.1) we would then get the $2\sqrt{2}\cot\big(\frac{\pi}{2p^*}\big)$ estimate for the Riesz transforms on $\mathbb{R}^n$, which is worse than what we have in Corollary 0.2, provided that p^* is not large (it should, roughly, be smaller than 9.225). For other p's this is not the case. To summarize, these bounds, i.e. $2(p^*-1)$ for smaller p^* and $2\sqrt{2}\cot\big(\frac{\pi}{2p^*}\big)$ for larger p^* constitute the best estimates for $\Big\|\Big(\sum_{i=1}^n |R_i f|^2\Big)^{1/2}\Big\|_p$ that we know of.

Concerning the dimension free estimates of the Riesz transforms our paper does not contain anything new, it is purely methodological. But the approach we use can probably be extended to many other situations because its backbone, the Bellman function technique, is essentially dimension free. So the reader should consider the estimates of Stein and Meyer-Pisier type as the illustration of the method. But our dimension free Littlewood-Paley type estimates seem to be new. And, as we already mentioned, they contain a big positive extra term, which, if tamed, can give the final sharp constants.

REMARK 0.5. When $p = 1$ the Riesz transforms are not bounded. Instead, there are estimates of the weak type 1–1, but (so far) without dimension free constants. The best result of this kind that we are aware of is P. Janakiraman's [13]. He showed that the weak 1–1 constant is at most $c \log n$.

S.K. Pichorides [25], found the exact norm of Riesz transforms on L^p in one-dimensional case (Hilbert transform). This result was extended by T. Iwaniec and G. Martin [12] who proved that the same statement (contained in the estimate (0.2) above) holds for scalar Riesz transforms on arbitrary $\mathbb{R}^n$.

The fact that the norms of vector Riesz transforms can be bounded with estimates independent of the dimension n was first observed by E. Stein [29]. Probabilistic methods, applied by P.A. Meyer [20], were used to obtain the initial proof of this theorem in the Gaussian setting (the Ornstein-Uhlenbeck semigroup), whereas in [26] G. Pisier found an analytic proof. Later on, N. Arcozzi improved this result in his paper [1], where he also considered Riesz transforms on more general structures.

The question of estimating Riesz transforms in various (and sometimes very difficult) other situations has generated a lot of attention. We refer here to the articles of T. Coulhon, D. Müller and J. Zienkiewicz [7] and F. Lust-Piquard [19] for the results on Heisenberg groups, as well as to T. Coulhon and X.T. Duong [6], where Riesz transforms on rather general manifolds are considered. The works [18] and [17] of F. Lust-Piquard regard certain discrete analogues of Riesz transforms and Riesz transforms on Fock spaces, respectively. In [19], the same author also introduces Riesz transforms on Schatten classes and gives their dimension free bounds.

Our approach is different than any previous in that it uses the technique of Bellman functions.

The Bellman function approach can be viewed as an application of certain ideas from Bellman PDE in stochastic optimal control to harmonic analysis. By its nature it is very well suited to give dimension free estimates. It also usually gives estimates close to the sharp ones. However, in our application to Riesz transforms one uses the Bellman function in a new way, which allows self-improvement of the estimates.

The reader will observe that our approach in more general cases reduces the proof to the question of boundedness of certain spectral multipliers on L^p.

However, this appears to be the only true obstacle that we are to encounter. For this reason we believe that our technique is transformable within several other settings. The scope of generality will be the subject of further consideration.

We hope that the properties of the Bellman function could also be utilized in a way to obtain dimension free estimates of the weak type 1-1. So far this has eluded us.

Although dimensionless boundedness of Riesz transforms has been proven several times by now, the following results give, besides a new proof, in some cases also the best known estimates of their L^p-norms.

NEW FINDINGS. (i) We prove some Littlewood-Paley type estimates which imply dimensionless boundedness of Riesz transforms. These estimates seem to be new both in classical as well as Gaussian setting.

(ii) We show how the boundedness of Riesz transforms can be reduced to the boundedness of a certain spectral multiplier.

(ii) The Bellman function can provide a unified approach to many various Riesz transforms (provided that the corresponding spectral multiplier theorem is available). It tends to give quite good estimates in p (depending on how well the Bellman function is found).

(iii) There is a proof, due to D. Burkholder, of a weak type inequality for the martingale transform which contains a sort of a "Bellman function". This gives rise to the hope that our approach could also be useful for dimensionless weak type estimates.

0.1. FURTHER NOTATION AND PRELIMINARIES. There is another way to describe Riesz transforms. Let

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

be the classical Laplace operator on $\mathbb{R}^n$. For any test function φ we have

$$\langle \Delta\varphi, \varphi\rangle = \int_{\mathbb{R}^n} \sum_{i=1}^{n} \frac{\partial^2 \varphi}{\partial x_i^2}\, \overline{\varphi}\, \mathrm{d}x = -\int_{\mathbb{R}^n} \|\nabla\varphi\|^2\, \mathrm{d}x\,,$$

where ∇ denotes the gradient

$$\nabla = \Big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\Big)\,.$$

Thus Δ is an unbounded symmetric operator, whose domain can be taken to be the Schwartz class $\mathcal{S}$. One can show that Δ is closable and that its closure $\overline{\Delta}$ is a selfadjoint extension of Δ (it therefore being unique).

Since Δ was a negative operator, so is $\overline{\Delta}$. So there is a unique positive self-adjoint operator A with the property $A^2 = -\overline{\Delta}$. Symbolically, on $\mathcal{S}$ we have the formula

$$A = \sqrt{-\Delta}\,.$$

Then one can verify that

$$R_i = \frac{\partial}{\partial x_i} \circ A^{-1}\,.$$

Note that one can define Riesz transforms R_i in many other situations when we have an abstract Laplacian operator. For example, in this article we also define the Ornstein-Uhlenbeck Riesz transforms with the help of the Ornstein-Uhlenbeck second-order differential operator.

We will also use the operator of *Poisson extension* P to the upper half-plane $\mathbb{R}^n \times (0, \infty)$. By $P_t f(x)$ we mean the value of the harmonic extension of f at the point $(x, t) \in \mathbb{R}^n \times (0, \infty)$. In short, $P_t f(x) = \widetilde{f}(x, t)$, where $\widetilde{f}$ is defined on page 167. It is well known that

$$P_t = \mathrm{e}^{-tA}.$$

1. BILINEAR DIMENSIONLESS LITTLEWOOD-PALEY TYPE ESTIMATE

This section is devoted to proving the fundamental result of the paper, Theorem 0.1. We will need a few auxiliary results, which we will explain *en route*, as the need for them arises.

1.1. THE BELLMAN FUNCTION APPEARS. Take $p \in (1, \infty)$ and assume that M and N are natural numbers. Define

$$\Omega = \{(\zeta, \eta, Z, H) \in \mathbb{C}^M \times \mathbb{C}^N \times \mathbb{R} \times \mathbb{R} : |\zeta|^p < Z, |\eta|^q < H\}\,.$$

This is a convex domain in $\mathbb{C}^M \times \mathbb{C}^N \times \mathbb{R} \times \mathbb{R} \equiv \mathbb{R}^d$, $d = 2(M + N + 1)$.

THEOREM 1.1. *There is a function $B : \overline{\Omega} \to \mathbb{R}$, such that:*
(i) $0 \leqslant B(\zeta, \eta, Z, H) \leqslant (p^* - 1) Z^{1/p} H^{1/q}$ *everywhere on its domain;*
(ii)

$$B\Big(\frac{a_+ + a_-}{2}\Big) - \frac{B(a_+) + B(a_-)}{2} \geqslant \Big|\frac{\zeta_+ - \zeta_-}{2}\Big|\Big|\frac{\eta_+ - \eta_-}{2}\Big|$$

for any $a_\pm = (\zeta_\pm, \eta_\pm, Z_\pm, H_\pm) \in \overline{\Omega}$.

We can add smoothness to B by paying only a small price for it.

LEMMA 1.2. *If K is a compact subset of Ω and κ is a positive number, smaller than* $\min\{1,\, d(K, \partial\Omega)\}$, *then there exists a smooth function $B_{K,\kappa}$ on the neighbourhood $\Omega_\kappa := \{x \in \Omega : d(x, \partial\Omega) > \kappa\}$ of K, such that the second estimate above still holds. In that case, it takes the form*
(ii)′ $-\mathrm{d}^2 B_{K,\kappa}(\zeta, \eta, Z, H) \geqslant 2|\mathrm{d}\zeta||\mathrm{d}\eta|\,.$
The first estimate is perturbed only by the factor $1 + \kappa$.

We will deliver the proof of this theorem in Section 4.

This is our *Bellman function*. By the second property we mean that for any choice of $(\zeta,\eta,Z,H)\in\Omega$ and $(\alpha,\beta,\gamma,\delta)\in\mathbb{C}^M\times\mathbb{C}^N\times\mathbb{R}\times\mathbb{R}$ we have

$$\langle -\mathrm{d}^2B(\zeta,\eta,Z,H)\,[\alpha,\beta,\gamma,\delta]^{\mathrm{t}},\,[\alpha,\beta,\gamma,\delta]^{\mathrm{t}}\rangle \geqslant 2|\alpha||\beta|\,. \tag{1.1}$$

Naturally, the Hessian d^2B can be thought of as a real $d\times d$ matrix, whose upper left 2×2 submatrix, for instance, equals

$$\Big[\frac{\partial^2 B}{\partial\zeta_i^{(1)}\partial\zeta_j^{(1)}}\Big]_{i,j=1,2},$$

where $\zeta=(\zeta^{(1)},\dots,\zeta^{(M)})$ and $\zeta^{(1)}=\zeta_1^{(1)}+\mathrm{i}\,\zeta_2^{(1)}$.

In general we do not know of a formula, i.e. an algebraic expression, which would possess the properties of a Bellman function. The only exception is described in the next lemma.

LEMMA 1.3. *In case when $p=2$ one possible choice for B is*

$$B(\zeta,\eta,Z,H)=\sqrt{(Z-|\zeta|^2)(H-|\eta|^2)}\,.$$

Proof. The proof is given for the case $M=N=1$. The function obviously satisfies the first requirement. For the second one, take $\zeta=\zeta_1+\mathrm{i}\zeta_2$, $\eta=\eta_1+\mathrm{i}\eta_2$, Z, H so that $(\zeta,\eta,Z,H)\in\Omega$. Write

$$B=B(\zeta,\eta,Z,H)=\sqrt{(Z-\zeta_1^2-\zeta_2^2)(H-\eta_1^2-\eta_2^2)}\,.$$

Then

$$\begin{aligned}\nabla B&=\Big(\frac{\partial B}{\partial\zeta_1},\quad\frac{\partial B}{\partial\zeta_2},\quad\frac{\partial B}{\partial\eta_1},\quad\frac{\partial B}{\partial\eta_2},\quad\frac{\partial B}{\partial Z},\quad\frac{\partial B}{\partial H}\Big)\\&=\Big(-\frac{\zeta_1}{\sqrt{W}},\quad-\frac{\zeta_2}{\sqrt{W}},\quad-\eta_1\sqrt{W},\quad-\eta_2\sqrt{W},\quad\frac{1}{2\sqrt{W}},\quad\frac{\sqrt{W}}{2}\Big),\end{aligned}$$

where

$$W=\frac{Z-|\zeta|^2}{H-|\eta|^2}\,.$$

This gives rise to the equalities:

$$\begin{aligned}&\frac{\partial^2B}{\partial\zeta_i^2}=-(Z-\zeta_j^2)\,\frac{1}{BW}\quad\{i,j\}=\{1,2\}\,;\qquad&&\frac{\partial^2B}{\partial\zeta_1\partial\zeta_2}=-\zeta_1\zeta_2\,\frac{1}{BW}\,;\\&\frac{\partial^2B}{\partial\eta_j^2}=-(H-\eta_i^2)\,\frac{W}{B}\quad\{i,j\}=\{1,2\}\,;\qquad&&\frac{\partial^2B}{\partial\eta_1\partial\eta_2}=-\eta_1\eta_2\,\frac{W}{B}\,;\end{aligned}$$

$$\frac{\partial^2B}{\partial\zeta_i\partial\eta_j}=\frac{\zeta_i\eta_j}{B}\quad i,j\in\{1,2\}\,;$$

$$\frac{\partial^2 B}{\partial\zeta_i\partial Z}=\frac{\zeta_i}{2BW},\quad \frac{\partial^2 B}{\partial\zeta_i\partial H}=-\frac{\zeta_i}{2B}\quad i\in\{1,2\}\,;$$
$$\frac{\partial^2 B}{\partial\eta_j\partial Z}=-\frac{\eta_j}{2B},\quad \frac{\partial^2 B}{\partial\eta_j\partial H}=\frac{\eta_j W}{2B}\quad j\in\{1,2\}\,;$$

and

$$\frac{\partial^2 B}{\partial Z^2}=-\frac{1}{4BW}\quad \frac{\partial^2 B}{\partial H^2}=-\frac{W}{4B}\quad \frac{\partial^2 B}{\partial Z\partial H}=\frac{1}{4B}\,.$$

The expressions above enable us to form the Hessian $\mathrm{d}^2B=\mathrm{d}^2B(\zeta,\eta,Z,H)$. Note that every denominator contains B as a factor, therefore it is convenient to consider $B\cdot\mathrm{d}^2B$.

Take $(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma,\delta)\in\mathbb{R}^6$. Denote $v=[\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma,\delta]^{\mathrm{t}}$. We can write

$$\langle -B\cdot(\mathrm{d}^2B)v,\,v\rangle = a\delta^2+b\delta+c\,,$$

where a,b,c are appropriate functions of variables other than δ. Here $a>0$, therefore

$$\langle -B\cdot(\mathrm{d}^2B)v,\,v\rangle \geqslant c-\frac{b^2}{4a}\,.$$

But it turns out that

$$c-\frac{b^2}{4a}=|\alpha|^2(H-|\eta|^2)+|\beta|^2(Z-|\zeta|^2)\,,$$

hence obviously

$$\langle -B\cdot(\mathrm{d}^2B)v,\,v\rangle \geqslant 2|\alpha||\beta|B\,.\quad\blacksquare$$

Proof of Theorem 0.1. Choose test functions $f:\mathbb{R}^n\to\mathbb{C}^M$ and $g:\mathbb{R}^n\to\mathbb{C}^N$. Let us introduce the notation x_t for the element $(x,t)\in\mathbb{R}^{n+1}_+=\mathbb{R}^n\times(0,\infty)$. Define

$$\begin{array}{rccc} v: & \mathbb{R}^{n+1}_+ & \longrightarrow & \Omega\subset\mathbb{C}^M\times\mathbb{C}^N\times\mathbb{R}\times\mathbb{R}\\ & (x,t) & \longmapsto & (P_tf(x),\,P_tg(x),\,P_t|f|^p(x),\,P_t|g|^q(x))\,. \end{array}$$

Since the Poisson extension can be expressed as an integral against the Poisson kernel ([2], p. 6), applying Jensen's inequality shows that v is well defined, i.e. that it indeed maps into Ω.

Furthermore, let

$$\Delta_t:=\Delta+\frac{\partial^2}{\partial t^2}\,.$$

We are ready to get to the core of the proof.

Fix $0<\delta<1$ and an arbitrary compact set $M\subset\mathbb{R}^n\times(\delta,\infty)$. For $R>\frac12$ define $Q_R=(-R,R)^n\times(\delta,2R+\delta)$. Note that each Q_R contains the point $0_1=\{0\}^n\oplus\{1\}\in\mathbb{R}^n\times(0,\infty)$. Choose such R for which $M\subset Q_R$. The set $K:=v(\overline{Q_R})$ is compact and contained in Ω. Furthermore, take $0<\kappa<\min\{1,R^{-1},d(K,\partial\Omega)\}$ and let $B=B_{K,\kappa}$ be associated to K and κ as in Lemma 1.2. If $U:=v^{-1}(\Omega_\kappa)$, then the function $b:=B_{K,\kappa}\circ v$ maps $U\to[0,\infty)$. In particular, it is defined on Q_R.

Let G_{Q_R} be the Green function of the region Q_R. The Green formula says that

$$\int_{Q_R} b\,\Delta_t G_{Q_R}(\cdot,0_1) - \int_{Q_R} \Delta_t b\, G_{Q_R}(\cdot,0_1) = \int_{\partial Q_R} b\,\frac{\partial G_{Q_R}(\cdot,0_1)}{\partial n} - \int_{\partial Q_R} G_{Q_R}(\cdot,0_1)\,\frac{\partial b}{\partial n}\,.$$

Since $G_{Q_R}(\cdot,0_1) \equiv 0$ on ∂Q_R and since its Laplacian, multiplied by -1, acts like the Dirac function of 0_1 ([8], 2.2), this implies

$$-\int_{Q_R} \Delta_t b\, G_{Q_R}(\cdot,0_1) = b(0_1) + \int_{\partial Q_R} b\,\frac{\partial G_{Q_R}(\cdot,0_1)}{\partial n}\,.$$

Note that the normal derivative of G_{Q_R} on the boundary equals the Poisson kernel ([8], 2.2), again multiplied by -1, so it is negative. Since b is always positive, we get $-\int_{Q_R} \Delta_t b\, G_{Q_R}(\cdot,0_1) \leqslant b(0_1)$. The first property of B leads to the estimate $b(0_1) \leqslant (1+\kappa)(p^*-1)(P_1|f|^p(0))^{1/p}(P_1|g|^q(0))^{1/q}$ and so

$$\text{(1.2)}\quad -\int_{Q_R} \Delta_t b(x_t)\, G_{Q_R}(x_t,0_1)\,\mathrm{d}x_t \leqslant c_n(1+\kappa)(p^*-1)$$
$$\cdot\Big(\int_{\mathbb{R}^n} |f(y)|^p \frac{1}{(1+|y|^2)^{(n+1)/2}}\,\mathrm{d}y\Big)^{1/p}(\cdots|g(y)|^q\cdots)^{1/q}.$$

Here c_n is the normalizing factor for the Poisson kernel, like in p. 6 of [2].

One can show that

$$G_{Q_R}(x_t,y_s) = \frac{1}{R^{n-1}}G_{Q^1}\Big(\frac{x_{t-\delta}}{R},\frac{y_{s-\delta}}{R}\Big),$$

where $Q^1 = (-1,1)^n \times (0,2)$. We can also write $Q^1 = \frac{Q_R - 0_\delta}{R}$.

Since the vector $0_{R+\delta}$ is the center of the square Q_R and thus its image under v lies in Ω_κ, we can repeat the preceding calculation with $0_{R+\delta}$ in place of 0_1. In that case, (1.2) takes the form

$$\begin{aligned}
-\int_{Q_R} &\Delta_t b(x_t)\, G_{Q^1}\Big(\frac{x_{t-\delta}}{R},0_1\Big)\mathrm{d}x_t \\
&\leqslant b(0_{R+\delta}) \\
&\leqslant c_n(1+\kappa)(p^*-1) \\
&\quad\cdot\Big(\int_{\mathbb{R}^n} |f(y)|^p \frac{R^{n-1}(R+\delta)}{[(R+\delta)^2+|y|^2]^{(n+1)/2}}\,\mathrm{d}y\Big)^{1/p}(\cdots|g(y)|^q\cdots)^{1/q}.
\end{aligned}$$

Apply the Lagrange theorem for $G_{Q^1}(\cdot, 0_1)$ along the normal vector. Use again that this function is 0 on the boundary. We get that

$$\begin{aligned} G_{Q^1}\Big(\frac{x_{t-\delta}}{R}, 0_1\Big) &= G_{Q^1}\Big(\frac{x_{t-\delta}}{R}, 0_1\Big) - G_{Q^1}\Big(\frac{x_0}{R}, 0_1\Big) \\ &= \frac{\partial G_{Q^1}\big(\cdot, 0_1\big)}{\partial n}\Big(\frac{x_\vartheta}{R}\Big)\frac{t-\delta}{R}. \end{aligned}$$

Here ϑ is some value from the interval $(0, t-\delta)$. Moreover, since the integrand is positive and $M \subset Q_R$, we can estimate the integral on the left from below by the same integral, just over M instead of Q_R. Thus

$$\begin{aligned} -\int_M \Delta_t b(x_t)\,\frac{\partial G_{Q^1}(\cdot, 0_1)}{\partial n}\Big(\frac{x_\vartheta}{R}\Big)\,(t-\delta)\,\mathrm{d}x_t & \\ \leqslant c_n(1+\kappa)(p^*-1) & \\ \cdot\Big(\int_{\mathbb{R}^n} |f(y)|^p \frac{R^n(R+\delta)}{[(R+\delta)^2+|y|^2]^{(n+1)/2}}\,\mathrm{d}y\Big)^{1/p} & (\cdots|g(y)|^q\cdots)^{1/q}. \end{aligned}$$

Note that

$$\frac{R^n(R+\delta)}{[(R+\delta)^2+|y|^2]^{(n+1)/2}} \leqslant \frac{R^n(R+\delta)}{[(R+\delta)^2]^{(n+1)/2}} = \Big(\frac{R}{R+\delta}\Big)^n < 1$$

for every $y \in \mathbb{R}^n$, therefore

$$-\int_M \Delta_t b(x_t)\,\frac{\partial G_{Q^1}(\cdot, 0_1)}{\partial n}\Big(\frac{x_\vartheta}{R}\Big)\,(t-\delta)\,\mathrm{d}x_t \leqslant\; c_n(1+\kappa)(p^*-1)\|f\|_p\|g\|_q\,.$$

In order to proceed we need the following easy calculation. For the sake of simplifying the formulæ let us temporarily write x_{n+1} instead of the variable t.

LEMMA 1.4. *For every $x_t = (x,t) \in U$ and arbitrary smooth B, harmonic u, and $b = B \circ u$,*

$$\Delta_t b(x_t) = \sum_{i=1}^{n+1}\Big\langle \mathrm{d}^2B(u)\frac{\partial u}{\partial x_i}(x_t),\, \frac{\partial u}{\partial x_i}(x_t)\Big\rangle.$$

Proof. Denote $u_0 = u(x_t)$. The chain rule gives

$$\frac{\partial b}{\partial x_i}(x_t) = \Big\langle \nabla B(u_0),\, \frac{\partial u}{\partial x_i}(x_t)\Big\rangle_{\mathbb{R}^d} \qquad i = 1,\dots,n+1;$$

$$\Delta_t b(x_t) = \langle \nabla B(u_0),\, \Delta_t u(x_t)\rangle + \sum_{i=1}^{n+1}\Big\langle \mathrm{d}^2B(u_0)\frac{\partial u}{\partial x_i}(x_t),\, \frac{\partial u}{\partial x_i}(x_t)\Big\rangle.$$

So far the formulæ are true for arbitrary $B, u, b = B \circ u, x_t, u_0 = u(x_t)$.

Now we use that u is composed of harmonic functions, which forces the first term on the right to be zero. ∎

This links us to the Bellman function property (1.1). Namely,

$$-\sum_{i=1}^{n+1}\Big\langle \mathrm{d}^2B(v)\frac{\partial v}{\partial x_i},\,\frac{\partial v}{\partial x_i}\Big\rangle \geqslant 2\sum_{i=1}^{n+1}\Big|\frac{\partial}{\partial x_i}P_tg(x)\Big|\Big|\frac{\partial}{\partial x_i}P_tf(x)\Big|\,.$$

However, this inequality can be strengthened, i.e. it actually self-improves, thanks to the next useful result which will be proven in the Addendum.

LEMMA 1.5. *Suppose $\mathcal{H}$ is a finite-dimensional real Euclidean space, $\mathcal{H}_i$, $i=1,2$, are two mutually orthogonal subspaces of $\mathcal{H}$ and P_i are the corresponding orthogonal projections. Let T be a selfadjoint operator such that*

$$\langle Th,\,h\rangle \geqslant 2\|P_1h\|\,\|P_2h\| \tag{1.3}$$

for all $h\in\mathcal{H}$. Then there exists $\tau>0$, satisfying

$$\langle Th,\,h\rangle \geqslant \tau\|P_1h\|^2+\frac{1}{\tau}\,\|P_2h\|^2$$

again for all $h\in\mathcal{H}$.

This lemma quickly implies the following corollary.

COROLLARY 1.6. *Under the above assumptions, for any Hilbert-Schmidt operator L, acting from any space (not necessarily finite-dimensional) into $\mathcal{H}$, we have*

$$\mathrm{tr}(L^*TL)\geqslant 2\|P_1L\|_2\|P_2L\|_2\,,$$

where $\|\cdot\|_2$ stands for the Hilbert-Schmidt norm, as usual.

By applying this corollary to $T=-\mathrm{d}^2B(v)$ and $L=\nabla v(x_t)$, the latter understood as an operator $\mathbb{R}^{n+1}\to\mathbb{R}^d$, we find such $\tau=\tau(x,t)$ that we even have

$$\sum_{i=1}^{n+1}\Big\langle -\mathrm{d}^2B(v)\frac{\partial v}{\partial x_i}(x_t),\,\frac{\partial v}{\partial x_i}(x_t)\Big\rangle \geqslant 2\|J\widetilde{f}(x_t)\|_2\|J\widetilde{g}(x_t)\|_2\,.$$

To summarize, we proved that for fixed $\delta\in(0,1)$, fixed compact set M in $\mathbb{R}^n\times(\delta,\infty)$ and any $R>\frac12$ such that $M\subset Q_R$, we have

$$2\int_M\|J\widetilde{f}(x_t)\|_2\|J\widetilde{g}(x_t)\|_2\,\frac{\partial G_{Q^1}(\cdot,0_1)}{\partial n}\Big(\frac{x_\vartheta}{R}\Big)(t-\delta)\,\mathrm{d}x_t\leqslant c_n(1+\kappa)(p^*-1)\|f\|_p\|g\|_q\,.$$

It is only at this point we can send R to infinity. What remains is to persuade $\frac{\partial G_{Q^1}(\cdot,0_1)}{\partial n}\cdot(\frac{x_\vartheta}{R})$ to approach c_n in some sense while R grows. Since κ was chosen to be dominated by R^{-1}, the factor $1+\kappa$ disappears. The term $\frac{x_\vartheta}{R}$ will uniformly go to 0, because M was bounded. But 0 lies on the boundary of Q^1, where $\frac{\partial G_{Q^1}(\cdot,0_1)}{\partial n}$ equals the Poisson kernel. So when $R\to\infty$, it approaches the value of the Poisson kernel for the upper half-space $\mathbb{R}^{n+1}_+$, i.e. $p(0,0_1)$, which is exactly equal to c_n.

Since the resulting inequality is true for arbitrary M, we can have it with $\mathbb{R}^n \times (\delta, \infty)$ as the domain of integration. By dilating the variable t we get

$$2\int_0^\infty \int_{\mathbb{R}^n} \|J\widetilde{f}(x,t+\delta)\|_2 \|J\widetilde{g}(x,t+\delta)\|_2 \, t \, \mathrm{d}x \, \mathrm{d}t \leqslant (p^* - 1)\|f\|_p \|g\|_q \,.$$

Because δ can be arbitrarily small, this proves Theorem 0.1. ∎

REMARK 1.7. The fact that B is, in general, not smooth everywhere on Ω caused some technical difficulties in the preceding proof and called for additional care and few steps, such as mollification, passing to limits a few times etc., that would otherwise not be needed.

2. CLASSICAL RIESZ TRANSFORMS. PROOF OF COROLLARY 0.2

First, let us prove the following lemma.

LEMMA 2.1. *Let f and g be two test functions and let R_k, $1 \leqslant k \leqslant n$, be any Riesz transform. Then*

$$\langle g,\, R_k f\rangle = 4\int_0^\infty \langle AP_t g,\, \partial_k P_t f\rangle \, t \, \mathrm{d}t \,,$$

where $\partial_k = \frac{\partial}{\partial x_k}$.

Proof. Introduce

$$\varphi(t) = \langle P_t g,\, P_t R_k f\rangle \,.$$

Integration by parts gives

$$\varphi(0) = \int_0^\infty \varphi''(t) \, t \, \mathrm{d}t \quad \text{or} \quad \langle g,\, R_k f\rangle = \int_0^\infty \frac{\mathrm{d}^2}{\mathrm{d}t^2} \langle P_t g,\, P_t R_k f\rangle \, t \, \mathrm{d}t \,.$$

The right side is equal to

$$\int_0^\infty \langle P_t'' g,\, P_t R_k f\rangle \, t \, \mathrm{d}t + 2\int_0^\infty \langle P_t' g,\, P_t' R_k f\rangle \, t \, \mathrm{d}t + \int_0^\infty \langle P_t g,\, P_t'' R_k f\rangle \, t \, \mathrm{d}t.$$

By P_t' we mean $\frac{\mathrm{d}P_t}{\mathrm{d}t}$, of course. But then $P_t' = -AP_t$ and so $P_t'' = A^2 P_t$. Thus we can continue the line above by

$$\int_0^\infty (\langle A^2 P_t g,\, P_t R_k f\rangle + 2\langle AP_t g,\, AP_t R_k f\rangle + \langle P_t g,\, A^2 P_t R_k f\rangle) \, t \, \mathrm{d}t.$$

The operator A is symmetric with respect to the pairing $\langle\cdot,\,\cdot\rangle$, hence we get

$$4\int_0^\infty \langle AP_t g,\, AP_t R_k f\rangle\, t\,\mathrm{d}t\,.$$

Since A commutes with P_t and ∂_k, we see that $AP_tR_k = AP_t\partial_k A^{-1} = \partial_k P_t$, which yields the desired equality. ∎

Now

$$\Big\|\Big(\sum_{i=1}^n |R_i f|^2\Big)^{1/2}\Big\|_p = \Big[\int_{\mathbb{R}^n}\Big(\sum_{i=1}^n |R_i f|^2\Big)^{p/2}\mathrm{d}m\Big]^{1/p}$$
$$= \Big[\int_{\mathbb{R}^n} |(R_1 f,\ldots,R_n f)(x)|^p \mathrm{d}m(x)\Big]^{1/p} = \|\mathcal{R}f\|_{L^p(\mathbb{R}^n\to\mathbb{C}^n)}\,,$$

where $\mathcal{R}f = (R_1 f,\ldots,R_n f)$. By using duality, this is the same as $\sup\{|\langle\mathcal{R}f,\,g\rangle| : g\in L^q(\mathbb{R}^n\to\mathbb{C}^n),\ \|g\|_q = 1\}$. It follows from Lemma 2.1 that

$$|\langle\mathcal{R}f,\,g\rangle| = \Big|\sum_{i=1}^n \langle R_i f,\, g_i\rangle\Big|$$
$$= \Big|4\int_0^\infty\int_{\mathbb{R}^n}\sum_{i=1}^n \partial_i P_t f(x)\,\overline{AP_t g_i(x)}\,\mathrm{d}x\,t\,\mathrm{d}t\Big|$$
$$\leqslant 4\int_0^\infty\int_{\mathbb{R}^n}\Big(\sum_{i=1}^n \Big|\partial_i P_t f\Big|^2\Big)^{1/2}\Big(\sum_{i=1}^n\Big|\frac{\partial}{\partial t}P_t g_i\Big|^2\Big)^{1/2}\mathrm{d}x\,t\,\mathrm{d}t\,.$$

Finally, we use Theorem 0.1 for $M = 1$ and $N = n$. ∎

REMARK 2.2. At this point we can explain precisely what we meant by the "big margin" and "positive extra term" that were mentioned in the Introduction.

Note that the inequalities we use in the very last step of the proof above, i.e.

$$\Big(\sum_{i=1}^n \Big|\partial_i P_t f(x)\Big|^2\Big)^{1/2} \leqslant \|J\widetilde{f}(x,t)\|_2 \tag{2.1}$$

and

$$\Big(\sum_{i=1}^n \Big|\frac{\partial}{\partial t}P_t g_i(x)\Big|^2\Big)^{1/2} \leqslant \|J\widetilde{g}(x,t)\|_2\,, \tag{2.2}$$

are very rough. Namely, in the sums on the left-hand sides of (2.1) and (2.2) there are many terms "missing" in comparison to what we have on the right-hand sides. Explicitly, it is the ∂_t derivative of $P_t f(x)$ in (2.1) and for $\{P_t g_i(x) : i = 1,\ldots,n\}$ in (2.2) all their partial derivatives *but* ∂_t. It might well be that this "loss of information" accounts for the fact that the constant we get is not the optimal one. On the other hand, it suggests that if one could handle these extra terms more efficiently that would result in better norm estimates.

3. ESTIMATES FOR THE ORNSTEIN-UHLENBECK SEMIGROUP (GAUSSIAN CASE)

In this section we turn $\mathbb{R}^n$ into a probability space by endowing it with the canonical *Gaussian measure*

$$\mathrm{d}\mu(x) = (2\pi)^{-n/2}\,\mathrm{e}^{-\|x\|^2/2}\,\mathrm{d}x\,.$$

It is well known that the polynomials form a dense subspace of any $L^p(\mathbb{R}^n,\mu)$, $1 < p < \infty$. For reasons which will became apparent further on in this section, this (and not C_c^∞ as before) is going to be our family of test functions on the Gaussian space.

Since the formal adjoint of ∂_i now equals $\partial_i^* g = x_i g - \partial_i g$, the Laplacian Δ is not symmetric anymore. Instead, its symmetric analogue on Gaussian spaces is called *Ornstein-Uhlenbeck differential operator*, and is defined on polynomials as

$$\Delta_{\mathrm{OU}} := \Delta - \sum_{i=1}^{n} x_i\,\frac{\partial}{\partial x_i}\,.$$

By the same symbol we will denote its selfadjoint extension, whose domain $\mathrm{Dom}\,\Delta_{\mathrm{OU}}$ is the Gaussian Sobolev space $\mathcal{D}^{2,2}$ (see [14]). Since $-\Delta_{\mathrm{OU}}$ is also a positive operator, we can define its positive square root A, which has the space $\mathcal{D}^{1,2}$ as its domain. By definition constant functions lie in the kernel of A. Smooth functions with compact support belong to $\mathcal{D}^{1,2}$, see [14] again. The selfadjoint operator A generates an operator semigroup, denoted by

$$P_t = \mathrm{e}^{-tA}\,.$$

The main goal of this section is a new inequality of the Littlewood-Paley type. It is formulated in Theorem 0.3 and is analogous to the one that we have already demonstrated in the standard setting (Theorem 0.1). This time $\widetilde{F}$ shall mean the Δ_{OU}-generated extension of function F into the upper half-space $\mathbb{R}^n_+$, i.e. $\widetilde{F}(x,t) = (P_t F)(x)$.

As a consequence of this inequality we will deliver Corollary 0.4, which brings forward a new solution of the well-known problem about dimension free bounds of Riesz transforms associated to the Ornstein-Uhlenbeck operator Δ_{OU}.

Previously, Riesz transforms R_i were defined by $R_i := \frac{\partial}{\partial x_i} \circ A^{-1}$. In order to define A^{-1} in the Gaussian setting properly, we restrict ourselves to the orthogonal complement of the kernel of Δ_{OU}. This is a subspace of $L^2(\mathbb{R}^n,\mu)$ of co-dimension 1. It consists of functions f which are orthogonal to the constants on $\mathbb{R}^n$ and are thus of mean zero, i.e. $\int_{\mathbb{R}^n} f\,\mathrm{d}\mu = 0$. We denote by π_0 the orthogonal projection onto this subspace and define

$$\mathbf{R}_i := \frac{\partial}{\partial x_i} \circ A^{-1}\,\pi_0\,.$$

Our objective will be to estimate the vector-valued Riesz transforms, in other words, we will try to give L^p-estimates of the function

$$\mathbf{R}f = \Big(\sum_{i=1}^{n}|\mathbf{R}_i f|^2\Big)^{1/2}.$$

Thus Corollary 0.4 can be stated in more detail as follows:

If $1 < p < \infty$ *and* $f \in L^p(\mathbb{R}^n, \mathrm{d}\mu)$*, then*

$$\|\mathbf{R}f\|_p \leqslant C_p \|f\|_p\,,$$

for some constants $C_p > 0$ *that do not depend on the dimension* n*. One can take* $C_p = c_p + C(p^* - 1)\|\mathcal{O}\|_{L^p(\mu)\to L^p(\mu)}$*, where the constant* $C = 8\mathrm{e}^{1/\mathrm{e}}(1+\mathrm{e}^{-2})$ *is the same as in Theorem* 0.3*,* $\mathcal{O}$ *is a spectral multiplier associated with* Δ_{OU}*, and*

$$c_p = \begin{cases} 1 & p \geqslant 2\,,\\ O((p-1)^{-1/2}) & p \to 1\,.\end{cases}$$

It was first proved by P.A. Meyer [20] that for some $C_p > 0$ the inequality above holds. This was done by applying probabilistic methods. As mentioned in the introduction, G. Pisier [26] found an analytic proof of the same result, while subsequently several other results and improvements followed. N. Arcozzi [1] proved the inequality with $C_p = 2(p^* - 1)$, whereas L. Larsson-Cohn [16] showed that $p^* - 1$ is also the lower bound for the magnitude of the constants when $p^* \to 1$.

It is again our aim to use the Bellman function as in the previous section. However, there are difficulties which prevent us from reaching exactly the same result. The first of them is the non-commutativity of ∂_i and Δ_{OU}. Namely, one easily verifies that

$$[\Delta_{\mathrm{OU}}, \partial_i] = \Delta_{\mathrm{OU}}\partial_i - \partial_i\Delta_{\mathrm{OU}} = \partial_i\,. \tag{3.1}$$

This obstruction defies our efforts to restore a satisfactory version of Lemma 2.1, for we are not able to mimic the last step in its proof.

One way around this is to introduce a certain auxiliary operator $\mathcal{O}$.

In the classical case, Lemma 2.1 gave us

$$\langle g, R_k f\rangle = 4\int_0^\infty \langle AP_t g,\, \partial_k P_t f\rangle\, t\,\mathrm{d}t\,.$$

This time we are aiming for a more general formula, namely

$$\langle g, \mathbf{R}_k f\rangle = \int_0^\infty \langle AP_t g,\, \partial_k P_t \mathcal{O} f\rangle\, \psi(t)\,\mathrm{d}t\,, \tag{3.2}$$

where the operator $\mathcal{O}$ and the function ψ are yet to be determined.

SPECTRAL MULTIPLIERS. The eigenvalues of $-\Delta_{\mathrm{OU}}$ are precisely all natural numbers. The eigenvectors, corresponding to the eigenvalue $m \in \mathbb{N}$, are the *Hermite polynomials* h_α, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multiindex of *length* m, which means $|\alpha| := \alpha_1 + \cdots + \alpha_n = m$. If $n = 1$, then

$$h_m(x) = (-1)^m \mathrm{e}^{\,x^2/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} [\mathrm{e}^{-\,x^2/2}] \quad m \in \mathbb{N} \cup \{0\}\,.$$

One can easily see that

$$h_m' = m h_{m-1}\,. \tag{3.3}$$

For arbitrary n, we define $h_\alpha := \bigotimes_{i=1}^n h_{\alpha_i}$. That is, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we put

$$h_\alpha(x) := h_{\alpha_1}(x_1) \cdots h_{\alpha_n}(x_n) = \prod_{i=1}^n h_{\alpha_i}(x_i)\,.$$

We shall denote by $\mathcal{P}_m$ the orthogonal projection of $L^2(\mathbb{R}^n, \mathrm{d}\mu)$ onto the eigenspaces $H_m := \mathrm{Lin}\{h_\alpha : |\alpha| = m\}$.

The fact that $-\Delta_{\mathrm{OU}}$ is a self–adjoint operator implies that the polynomials h_α are mutually orthogonal, but it actually turns out that they form a complete orthogonal system in $L^2(\mathbb{R}^n, \mathrm{d}\mu)$.

Since $\Delta_{\mathrm{OU}} h_\alpha = -|\alpha|\, h_\alpha$, it follows that:

(i) $A h_\alpha = \sqrt{|\alpha|}\, h_\alpha$,

(ii) $P_t h_\alpha = \mathrm{e}^{-\sqrt{|\alpha|}t}\, h_\alpha$, and

(iii) $\partial_i h_\alpha = \alpha_i\, h_{\alpha_1} \otimes \cdots \otimes h_{\alpha_i - 1} \otimes \cdots \otimes h_{\alpha_n}$, thus

$$\partial_i : H_m \to H_{m-1}\,.$$

For the last equality we used (3.3).

This implies that if we test formula (3.2) for $f = h_\alpha$, we see that $\mathcal{O}$ must be a multiplier, more precisely, of the form

$$\mathcal{O} = \sum_{m \in \mathbb{N}} o_m\, \mathcal{P}_m$$

for a certain collection of scalars o_m, $m \in \mathbb{N}$.

Suppose $o_m = 1$ for all $m \in \mathbb{N}$ (i.e. $\mathcal{O} = I$). One can verify, after the same procedure of testing the formula on h_α, that ψ in this case must equal $\psi(t) = 2(\sin t + \sinh t)$. This function again does not enable us to continue the proof with the Bellman function, which once more explains why in the Gaussian setting (contrary to the classical one) there is no "suitable" formula of the form

$$\langle g, \mathbf{R}_k f\rangle = \int_0^\infty \langle A P_t g, \partial_k P_t f\rangle\, \psi(t)\, \mathrm{d}t$$

and similarly with the roles of f and g exchanged in the integral on the right. This hints that the generalization (3.2) might indeed be necessary.

It turns out that functions $\psi(t) = t\,\mathrm{e}^{-at}$, where $a > 0$ is arbitrary, suit our purpose. In this case computation returns

$$o_m = \frac{(\sqrt{m}+\sqrt{m-1}+a)^2}{\sqrt{m}\sqrt{m-1}}\,.$$

Of course, this is only true if $m \geqslant 2$. For $m = 1$ we define $o_1 := 0$, which leads to a slight modification of formula (3.2) into

$$\langle g, \mathbf{R}_k f\rangle = \langle g, \mathbf{R}_k \mathcal{P}_1 f\rangle + \int_0^\infty \langle AP_t g, \partial_k P_t \mathcal{O} f\rangle\, \psi(t)\,\mathrm{d}t\,. \tag{3.4}$$

3.1. PROOF OF COROLLARY 0.4 BY THE USE OF THE LITTLEWOOD-PALEY ESTIMATE IN THEOREM 0.3. We are going to utilize (3.4). First we estimate the linear part of the polynomial f. Write

$$\mathcal{P}_1 f = \sum_{j=1}^n c_j h_{e_j}\,,$$

where

$$e_j = \begin{array}{c} (0,\dots,0,1,0,\dots,0) \\ \uparrow \\ j\text{-th place} \end{array}.$$

Since $h_1(s) = s$ for $s \in \mathbb{R}$, observe that $h_j(x) = x_j$. Therefore $\mathbf{R}_j\mathcal{P}_1 f(x) \equiv c_j$, which means that $\|\mathbf{R}\mathcal{P}_1 f\|_p^2 = \sum_j |c_j|^2$. But this equals $\|\mathcal{P}_1 f\|_2^2$, since $\{h_\alpha : |\alpha| = 1\}$ are mutually orthogonal in $L^2(\mu)$.

Denote, as in [26],

$$\gamma(p) := \Big(\int_{\mathbb{R}} |t|^p\,\mathrm{d}\mu(t)\Big)^{1/p}\,.$$

Then

$$\|\mathbf{R}\mathcal{P}_1 f\|_p = \|\mathcal{P}_1 f\|_2 = \frac{\|\mathcal{P}_1 f\|_p}{\gamma(p)} \leqslant \frac{\gamma(p^*)}{\gamma(p)}\|f\|_p\,. \tag{3.5}$$

Note that, by the Stirling formula, $c_p := \frac{\gamma(p^*)}{\gamma(p)} = O((p-1)^{-1/2})$ as $p \to 1$. The coefficients $\gamma(p)$ appear in a similar role also in G. Pisier's article [26].

The more difficult part of the proof is to estimate the integral in the formula (3.4). We would like to show that

$$\int_0^\infty |\langle AP_t g, \nabla P_t \mathcal{O} f\rangle|\, \psi(t)\,\mathrm{d}t \leqslant C_p\|f\|_p$$

for some dimensionless constant C_p. Here g is any function from the unit sphere in $L^q(\mathbb{R}^n \to \mathbb{C}^n, \mathrm{d}\mu)$. This follows from Theorem 0.3, that is, our Littlewood-Paley-type inequality for the Gaussian setting. For that purpose apply it with $\mathcal{O}f$

in place of f (we can do that, since $\mathcal{O}$ clearly maps polynomials into polynomials). From here, from (3.4) and from (3.5) it follows, identically as in the classical setting (page 179), that

$$\|\mathbf{R}f\|_p \leqslant c_p\|f\|_p + C(p^*-1)\|\mathcal{O}f\|_p\,.$$

The coefficients o_m of our multiplier $\mathcal{O}$ can be written as $F(m^{-1/2})$ if $m \geqslant 2$, where F is a certain function, analytic in the neighborhood of the origin. This implies ([20], Theorem 3) dimensionless boundedness of $\mathcal{O}$ on $L^p(\mathbb{R}^n, \mathrm{d}\mu)$. Hereby the proof is wound up.

For further results and a detailed discussion of spectral multipliers and Riesz transforms in the setting of Ornstein-Uhlenbeck semigroup we refer to the series of articles by J. García-Cuerva, G. Mauceri, S. Meda, P. Sjögren and J.-L. Torrea [9], [10], [11].

3.2. PROOF OF THE ORNSTEIN-UHLENBECK-GENERATED LITTLEWOOD-PALEY ESTIMATE (THEOREM 0.3). Having prepared the ingredients, we can apply the same Bellman function technique as in the previous section. That is, we deal with the function

$$v : \mathbb{R}^n \times (0,\infty) \to \Omega$$

defined by

$$v(x,t) = (P_tf(x), P_tg(x), P_t|f|^p(x), P_t|g|^q(x))$$

and compose it with a Bellman function B (see Theorem 1.1) for

$$b = B \circ v\,.$$

REMARK 3.1. Henceforth we will assume that B is smooth everywhere on Ω. This will save us some technical complications and add clarity without changing the final outcome, for we already demonstrated how to treat the general (non-smooth) case in the proof of Theorem 0.1.

However, in our attempt of repeating the proof we encounter a few *problems*:

1. We need to explain why the function v is well defined, i.e. why it really maps into Ω. For that purpose, apply Jensen's inequality to the *Mehler formula* (see [28], for instance) for the Δ_{OU}-generated heat extension $H_tf = \mathrm{e}^{\Delta_{\mathrm{OU}}t}f$ of f:

$$H_tf(x) = \int_{\mathbb{R}^n} f(\mathrm{e}^{-t}x + y\sqrt{1-\mathrm{e}^{-2t}})\,\mathrm{d}\mu(y)\,.$$

We see that $|H_tf|^p \leqslant H_t|f|^p$ for any $p \geqslant 1$. By p. 180 of [20], with every $t>0$ there is associated a probability measure μ_t on $(0,\infty)$, such that $P_tf = \int_0^\infty H_sf\,\mathrm{d}\mu_t(s)$. Now Hölder's inequality implies that $v \in \Omega$, as desired.

In addition to that, one can show by direct calculation that

$$\|H_t f\|_p \leqslant \Big(\int_{\mathbb{R}^n} H_t |f|^p(x)\, \mathrm{d}\mu(x) \Big)^{1/p} \leqslant \|f\|_p \,.$$

Moreover, by applying H_t to a constant function we actually see that $\|H_t\|_{B(L^p(\mu))} = 1$. Consequently, if $p \geqslant 1$ then

$$\|P_t f\|_p \leqslant \int_0^\infty \|H_s f\|_p \,\mathrm{d}\mu_t(s) \leqslant \int_0^\infty \|f\|_p \,\mathrm{d}\mu_t(s) = \|f\|_p \,. \tag{3.6}$$

2. In the classical case, it were the expressions of the form

$$\Big\langle \mathrm{d}^2 B(v) \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_i} \Big\rangle$$

that linked the Jacobians of $P_t f$ and $P_t g$ with $\|f\|_p$ and $\|g\|_q$. In the proof of Lemma 1.4 we used that the components of v were harmonic functions, i.e. that they were in the kernel of the Laplacian on the upper half-space in $\mathbb{R}^{n+1}$. In order to mimic this reasoning in the Gaussian setting, we have to come up with an appropriate differential operator. Since $P_t = \mathrm{e}^{-At}$, it is obvious that

$$\frac{\partial^2}{\partial t^2} P_t = A^2 P_t$$

in the strong sense, which implies

$$\Big(\frac{\partial^2}{\partial t^2} + \Delta_{\mathrm{OU}} \Big) P_t = \Big(\frac{\partial^2}{\partial t^2} - A^2 \Big) P_t = 0\,,$$

hence the proper choice is given by

$$\Delta'_{\mathrm{OU}} = \frac{\partial^2}{\partial t^2} + \Delta_{\mathrm{OU}}\,.$$

By "proper" we of course mean what has just been explained, i.e. that now we have the analogue of Lemma 1.4:

$$\Delta'_{\mathrm{OU}} b(x,t) = \sum_{i=1}^{n+1} \Big\langle \mathrm{d}^2 B(v) \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_i} \Big\rangle \,. \tag{3.7}$$

The point here is that now we can repeat the reasoning from the classical setting without any modifications and obtain that

$$-\Delta'_{\mathrm{OU}} b(x,t) \geqslant 2 \|J\widetilde{f}\|_2 \|J\widetilde{g}\|_2 \,. \tag{3.8}$$

The only property we used for this, apart from the equality (3.7), was the characteristic estimate involving the Hessian of B.

3. The next step is to estimate the integral

$$- \int_{\mathbb{R}^{n+1}_+} \Delta'_{\mathrm{OU}} b(x,t)\, \psi(t)\, \mathrm{d}\mu(x)\, \mathrm{d}t \tag{3.9}$$

from above by $C(p)\|f\|_p\|g\|_q$, where $C(p)$ is some positive number. Recall that $\psi(t)$ was defined as $t\,\mathrm{e}^{-at}$ for some positive a. Eventually we want to take the limit $a \to 0$, which will obviously not affect the upper bound.

For $b \in \operatorname{Dom} \Delta_{\mathrm{OU}}$ we have

$$\int_{\mathbb{R}^n} \Delta_{\mathrm{OU}} b \,\mathrm{d}\mu = \langle \Delta_{\mathrm{OU}} b, \mathbf{1}\rangle = \langle b, \Delta_{\mathrm{OU}}\mathbf{1}\rangle = 0\,,$$

therefore the integral (3.9) reduces to

$$-\int_{\mathbb{R}^n}\int_0^\infty \frac{\partial^2 b}{\partial t^2}(x,t)\,\psi(t)\,\mathrm{d}t\,\mathrm{d}\mu(x)\,.$$

Fix $x \in \mathbb{R}^n$ and write $b_x(t) = b(x,t)$. We regard b_x as a function of one variable. Now, symbolically,

$$\begin{aligned}\int_0^\infty \frac{\partial^2 b}{\partial t^2}(x,t)\,\psi(t)\,\mathrm{d}t &= \int_0^\infty b_x''(t)\,\psi(t)\,\mathrm{d}t \\ &= \underbrace{b_x'(\infty)\psi(\infty)}_{\underline{\overline{\mathrm{I}}}} - \underbrace{b_x'(0)\psi(0)}_{\underline{\overline{\mathrm{II}}}} - \underbrace{b_x(\infty)\psi'(\infty)}_{\underline{\overline{\mathrm{III}}}} + \underbrace{b_x(0)\psi'(0)}_{\underline{\overline{\mathrm{IV}}}} \\ &\qquad + \underbrace{\int_0^\infty b_x(t)\,\psi''(t)\,\mathrm{d}t}_{\underline{\overline{\mathrm{V}}}}\,. \end{aligned} \tag{3.10}$$

We will in turn estimate the integrals $\int_{\mathbb{R}^n} \mathrm{d}\mu$ of each of the terms $\underline{\overline{\mathrm{I}}}$–$\underline{\overline{\mathrm{V}}}$.

We are going to borrow the function introduced by F. Nazarov and S. Treil in [21]. It is an intermediate but crucial step in building an explicit example of a Bellman function which satisfies all the properties of the function from the beginning of Subsection 1.1. Actually, not all the properties are attained, because their function gives a different (bigger) constant in the estimate for the upper bound. However, this does not seem to discomfort us too much, because we lose the $2(p^*-1)$ estimate in the continuation of the proof anyway. This loss happens when we come up with the estimate for our spectral multipliers.

Let us bring up the expression. It is defined in the domain Ω from page 172 by

$$Q(\zeta,\eta,Z,H) = 2(Z+H) - |\zeta|^p - |\eta|^q - \delta\widetilde{Q}(\zeta,\eta)\,, \tag{3.11}$$

where

$$\widetilde{Q}(\zeta,\eta) = \begin{cases} \dfrac{2}{p}|\zeta|^p + \Big(\dfrac{2}{q}-1\Big)|\eta|^q & \text{when } |\zeta|^p \geqslant |\eta|^q\,, \\ |\zeta|^2|\eta|^{2-q} & \text{when } |\zeta|^p \leqslant |\eta|^q\,. \end{cases}$$

If δ is small enough, this function satisfies:

(i) $0 \leqslant Q(\zeta,\eta,Z,H) \leqslant 2(Z+H)$,

(ii) $-\mathrm{d}^2 Q(\zeta,\eta,Z,H) \geqslant \frac{q(q-1)}{2}|\mathrm{d}\zeta||\mathrm{d}\eta|$,

everywhere in its domain. The first property is obvious, whereas the proof of the second one is presented in [21].

We want the constant appearing in the bound for the Hessian to be exactly 2, therefore we decide to introduce

$$B := \frac{4}{q(q-1)}\,Q = \frac{4(p-1)^2}{p}\,Q\,.$$

Furthermore, in order to obtain a function of ζ,η,Z,H, which is bounded from above by a constant multiple of $Z^{1/p}H^{1/q}$ rather than $Z+H$, we will later have to consider

$$\inf_{\lambda>0} B_\lambda(\zeta,\eta,Z,H)\,,$$

where

$$B_\lambda(\zeta,\eta,Z,H) = B(\lambda\zeta,\lambda^{-1}\eta,\lambda^p Z,\lambda^{-q}H)\,.$$

This B_λ will be our main tool for the time being. It will temporarily occupy the place of the "true" Bellman function. Accordingly, we denote

$$b_\lambda = B_\lambda \circ v\,.$$

The idea is to work with the concrete b_λ and then minimize the estimates over all $\lambda > 0$.

The reason for dealing with these special functions is that they seem to supply us with more information about the behaviour of their gradients than just the abstract functions, provided by the existence theorem (Theorem 1.1). We are going to need this information in order to justify some of our calculations of terms $\underline{\mathrm{I}}$–$\underline{\mathrm{IV}}$.

But let us begin with the term $\underline{\mathrm{V}}$. Fix $\lambda > 0$.

$$\Big|\int_{\mathbb{R}^n} \underline{\mathrm{V}}\,\mathrm{d}\mu(x)\Big| = \Big|\int_0^\infty\int_{\mathbb{R}^n} b_\lambda(x,t)\,\mathrm{d}\mu(x)\psi''(t)\,\mathrm{d}t\Big| \leqslant \int_0^\infty\int_{\mathbb{R}^n} b_\lambda(x,t)\,\mathrm{d}\mu(x)|\psi''(t)|\,\mathrm{d}t\,.$$

By the first property of the function B we have

$$\int_{\mathbb{R}^n} b_\lambda(x,t)\,\mathrm{d}\mu(x) \leqslant \frac{8(p-1)^2}{p}\Big[\lambda^p\int_{\mathbb{R}^n} P_t|f|^p(x)\,\mathrm{d}\mu(x) + \lambda^{-q}\int_{\mathbb{R}^n} P_t|g|^q(x)\,\mathrm{d}\mu(x)\Big]\,.$$

The line (3.6) allows us to continue with

$$\leqslant \frac{8(p-1)^2}{p}(\lambda^p\|f\|_p^p + \lambda^{-q}\|g\|_q^q)\,.$$

Consequently,

$$\Big|\int_{\mathbb{R}^n} \underline{\mathrm{V}}\,\mathrm{d}\mu(x)\Big| \leqslant \frac{8(p-1)^2}{p}(\lambda^p\|f\|_p^p + \lambda^{-q}\|g\|_q^q)\int_0^\infty |\psi''(t)|\,\mathrm{d}t\,.$$

Recall that ψ was defined by $\psi(t) = t\mathrm{e}^{-at}$ for some $a > 0$. Hence, the integral on the right converges and is equal (independently of a) to

$$\int_0^\infty |s-2|\,\mathrm{e}^{-s}\,\mathrm{d}s = 2\mathrm{e}^{-2} + 1\,.$$

In order to estimate $\overline{\underline{\mathrm{IV}}}$ and $\overline{\underline{\mathrm{III}}}$, we have to consider the integral

$$\int_{\mathbb{R}^n} \lim_{t\to\omega} b_\lambda(x,t)\psi'(t)\,\mathrm{d}\mu(x)$$

for $\omega = 0$ and $\omega = \infty$. We have just proven that

$$\int_{\mathbb{R}^n} b_\lambda(x,t)\,\mathrm{d}\mu(x) \leqslant \frac{8(p-1)^2}{p}(\lambda^p\|f\|_p^p + \lambda^{-q}\|g\|_q^q)\,.$$

The expression on the right is also the bound of our integral when $\omega = 0$ (case $\overline{\underline{\mathrm{IV}}}$). However, in the special case when we actually know a formula for the Bellman function (Lemma 1.3) we have $b(x,0) = 0$. Thus the term $\overline{\underline{\mathrm{IV}}}$ is not the essential part of the estimate of the integral in (3.9).

On the other hand, $\lim\limits_{t\to\infty}\psi'(t) = 0$, therefore

$$\int_{\mathbb{R}^n} \overline{\underline{\mathrm{III}}}\,\mathrm{d}\mu(x) = 0\,.$$

Cases $\overline{\underline{\mathrm{II}}}$ and $\overline{\underline{\mathrm{I}}}$ are about estimating integrals

$$\int_{\mathbb{R}^n} \lim_{t\to\omega} \frac{\partial b_\lambda}{\partial t}(x,t)\psi(t)\mathrm{d}\mu(x) \tag{3.12}$$

where $\omega = 0$ and $\omega = \infty$, respectively. Since $\psi(\omega) = 0$ we suspect that these integrals might vanish as well. In order to conclude that, it suffices to know that $\frac{\partial b_\lambda}{\partial t}$ is not behaving too strangely.

The proof of Lemma 1.4 gave us

$$\begin{aligned}
-\frac{\partial b_\lambda}{\partial t}(x_0) &= -\Big\langle \nabla B_\lambda(v_0),\, \frac{\partial v}{\partial t}(x_0)\Big\rangle_{\mathbb{R}^d} \\
&= \Big\langle \frac{\partial B_\lambda}{\partial \zeta}(v_0),\, P_tAf(x)\Big\rangle_{\mathbb{R}^{2M}} + \Big\langle \frac{\partial B_\lambda}{\partial \eta}(v_0),\, P_tAg(x)\Big\rangle_{\mathbb{R}^{2N}} \\
&\quad + \frac{\partial B_\lambda}{\partial Z}(v_0)P_tA|f|^p(x) + \frac{\partial B_\lambda}{\partial H}(v_0)P_tA|g|^q(x)\,,
\end{aligned} \tag{3.13}$$

for $x_0 = (x,t)$ and $v_0 = v(x_0)$.

Observe that, since f is a polynomial, the generalized derivatives $\partial_i|f|^p$ exist and are equal to $p|f|^{p-2}\Re(\overline{f}\,\partial_i f)$, from where it follows that $|f|^p$ belongs to $\mathcal{D}^{1,2} = \mathrm{Dom}\,A$. In other words, $A|f|^p$ and $A|g|^q$ are well defined.

For the sake of simplicity we will omit writing

$$\frac{\partial B}{\partial \eta} = \Big(\frac{\partial B}{\partial \eta_1}, \dots, \frac{\partial B}{\partial \eta_M}\Big)\,.$$

Also note that $\frac{\partial B_\lambda}{\partial Z}$ and $\frac{\partial B_\lambda}{\partial H}$ are constant functions, by the formula (3.11). Thus the part of the integral (3.12), corresponding to the last term in (3.13), is, up to a multiplicative constant, bounded by

$$\limsup_{t\to\omega} \psi(t) \int\limits_{\mathbb{R}^n} P_t A|g|^q(x)\,\mathrm{d}\mu(x)\,.$$

Now,

$$\int\limits_{\mathbb{R}^n} P_t A|g|^q(x)\,\mathrm{d}\mu(x) = \langle P_t A|g|^q,\,\mathbf{1}\rangle = \langle P_t|g|^q,\,A\mathbf{1}\rangle = 0$$

for $\mathbf{1} \in \operatorname{Ker} A$. Of course, the same is true with $|f|^p$ in place of $|g|^q$.
What about the other terms? The part of (3.13), corresponding to η, gives

$$\limsup_{t\to\omega} \psi(t) \int\limits_{\mathbb{R}^n} \frac{\partial B}{\partial \eta}(v(x,t))\,P_t A g(x)\,\mathrm{d}\mu(x)\,. \tag{3.14}$$

We again use (3.11) to compute partial derivatives. In this case

$$\frac{\partial Q}{\partial \eta} = -q\eta|\eta|^{q-2} - \delta\frac{\partial \widetilde{Q}}{\partial \eta}\,,$$

where

$$\frac{\partial \widetilde{Q}}{\partial \eta} = \begin{cases} \Big(\dfrac{2}{q}-1\Big)q\eta|\eta|^{q-2} & \text{when } |\zeta|^p \geqslant |\eta|^q\,, \\ |\zeta|^2(2-q)\eta|\eta|^{-q} & \text{when } |\zeta|^p \leqslant |\eta|^q\,. \end{cases}$$

If $|\zeta|^p \leqslant |\eta|^q$, then

$$\Big|\frac{\partial \widetilde{Q}}{\partial \eta}\Big| \leqslant (2-q)|\eta|^{2(q/p)+1-q} = (2-q)|\eta|^{q-1}\,.$$

Thus there is a constant $M > 0$, such that

$$\Big|\frac{\partial B}{\partial \eta}\Big| \leqslant M|\eta|^{q-1}$$

everywhere in Ω. Therefore the absolute value of the integral in (3.14) is bounded from above by

$$M \int\limits_{\mathbb{R}^n} |P_t g(x)|^{q-1}|P_t A g(x)|\,\mathrm{d}\mu(x)\,,$$

which in turn admits estimate from the Hölder's inequality, i.e.

$$\leqslant M\|P_t g\|_{2(q-1)}^{q-1}\|P_t A g\|_2 \leqslant M\|g\|_{2(q-1)}^{q-1}\|Ag\|_2 < \infty\,.$$

Here we used (3.6). Consequently, the term (3.14) equals zero.

A similar proof works for the terms with the only remaining variable, ζ.

We conclude that

$$\int_{\mathbb{R}^n} \underline{\overline{\mathrm{I}}}\, \mathrm{d}\mu(x) = \int_{\mathbb{R}^n} \underline{\overline{\mathrm{II}}}\, \mathrm{d}\mu(x) = 0\,.$$

SUMMARY OF THE PROOF. Combining all our estimates, starting with (3.8), (3.10) and using the ones for the integrals $\underline{\overline{\mathrm{I}}} - \underline{\overline{\mathrm{V}}}$, we find that

$$\begin{aligned} 2 \int_{\mathbb{R}^{n+1}_+} \|J\widetilde{f}(x,t)\|_2 \|J\widetilde{g}(x,t)\|_2\, \psi(t)\, \mathrm{d}\mu(x)\, \mathrm{d}t &\leqslant - \int_{\mathbb{R}^{n+1}_+} \Delta'_{\mathrm{OU}} b_\lambda(x,t)\, \psi(t)\, \mathrm{d}\mu(x)\, \mathrm{d}t \\ &\leqslant C' \, \frac{(p-1)^2}{p} (\lambda^p \|f\|_p^p + \lambda^{-q} \|g\|_q^q) \end{aligned}$$

for all $\lambda > 0$, where $C' = 16(1 + \mathrm{e}^{-2})$. Taking the minimum in λ, using that $(p-1)^{1/p} \leqslant \mathrm{e}^{1/\mathrm{e}}$ and finally sending $a \to 0$ completes the proof of Theorem 0.3.

REMARK 3.2. Instead of Δ_{OU} we could run our proof on some other operators and still obtain dimensionless boundedness. From the considerations above it emerged that the properties we require are that these operators be positive (or negative) and of discrete spectrum $\{\lambda_m : m \in \mathbb{N}\}$. If the growth λ_m, $m \to \infty$, is of "proper" pace, then Meyer's theorem will probably work on multipliers of the form

$$\mathcal{O} = \sum_{m \in \mathbb{N}} o_m\, \mathcal{P}_m, \quad o_m = \frac{(\sqrt{\lambda_m} + \sqrt{\lambda_{m-1}} + a)^2}{\sqrt{\lambda_m}\sqrt{\lambda_{m-1}}}\,.$$

Namely, already the L^p-boundedness of such multipliers will be sufficient to run our Bellman function procedure.

4. THE EXISTENCE OF BELLMAN FUNCTION. PROOF OF THEOREM 1.1

First note that it suffices to consider the case when $M = N$. For in general, the Bellman function on $\Omega \subset \mathbb{C}^M \times \mathbb{C}^N \times \mathbb{R} \times \mathbb{R}$ can be defined as the restriction of the Bellman function on the domain in $\mathbb{C}^{\max\{M,N\}} \times \mathbb{C}^{\max\{M,N\}} \times \mathbb{R} \times \mathbb{R}$.

Each interval $I \subset \mathbb{R}$ gives rise to its *Haar function* h_I, defined by

$$h_I := \frac{\chi_{I_+} - \chi_{I_-}}{|I|^{1/2}}\,,$$

where I_- and I_+ denote the left and the right half of the interval I respectively, and χ_E stands for the characteristic function of the set E, as usual. Let $\mathcal{D}$ denote the standard family of dyadic intervals on the line. It is a well-known fact that the set $\{h_I : I \in \mathcal{D}\}$ forms an orthonormal basis of the space $L^2(\mathbb{R})$. Every complex-valued function with zero average, continuous and supported on one of the dyadic intervals, say on J, can be written as the sum of its Haar series: $f = \sum_I \langle f, h_I \rangle h_I$. The summation goes over $\mathcal{D}_J := \{I \in \mathcal{D} : I \subseteq J\}$. Consider the operator $T_\sigma f = \sum_I \sigma_I \langle f, h_I \rangle h_I$, where $\sigma = \{\sigma_I : I \in \mathcal{D}_J\}$ is any sequence of unimodular complex numbers.

We can extend this notions to $\mathbb{C}^M$. Define, for $j \in \{1, \dots, M\}$ and any interval $I \subset \mathbb{R}$, functions $h_I^j : \mathbb{R} \longrightarrow \mathbb{C}^M$ by

$$h_I^j = \underset{j\text{-th place}}{(0, \dots, 0, \underset{\uparrow}{h_I}, 0, \dots, 0)}\,.$$

The set $\{h_I^j : I \in \mathcal{D}_J,\ 1 \leqslant j \leqslant M\}$ is an orthonormal basis for the space $L^2(J \to \mathbb{C}^M)$.

Take an arbitrary collection $\sigma = \{\sigma_I^j \in S^1 : I \in \mathcal{D}_J, 1 \leqslant j \leqslant M\}$. For $f \in L^2(J \to \mathbb{C}^M)$ define

$$T_\sigma f = \sum_{I \in \mathcal{D}_J, 1 \leqslant j \leqslant M} \sigma_I^j \langle f, h_I^j \rangle h_I^j\,.$$

The operators T_σ will be called *martingale transforms*. Note that $\langle f, h_I^j \rangle = \langle f_j, h_I \rangle$, where $f = (f_1, \dots, f_M)$. Finally, denote

$$\langle f, h_I \rangle h_I := \sum_{j=1}^M \langle f, h_I^j \rangle h_I^j = \begin{bmatrix} \langle f_1, h_I \rangle h_I \\ \vdots \\ \langle f_M, h_I \rangle h_I \end{bmatrix}.$$

We aim to utilize our "model" operator T_σ. The logic will be the following. First, let us find the sharp estimate of $\|T_\sigma\|_{L^p(\mathbb{C}) \to L^p(\mathbb{C})}$, i.e. the martingale transform for $M = 1$, in terms of p. This problem was solved by Burkholder. He found out in [5] that

$$\sup_\sigma \|T_\sigma\|_p = p^* - 1\,. \tag{4.1}$$

He proved (4.1) by constructing a function of two real variables (actually another Bellman function) with certain convexity and size properties. The reader is referred to the papers of Burkholder [5], [4] or the book of D. Stroock [30] to study his approach. In particular, on page 344 of [30] it is written about (4.1): "Quite recently Burkholder has discovered *the right argument*: (...) it is completely elementary. Unfortunately, it is also completely opaque. Indeed, his new argument is nothing but an elementary verification that he has got the right answer; it gives no hint about how he came to that answer". Further on: "for those who want to know the secret behind his proof, Burkholder has written an explanation in his article" [5]. Here is the Burkholder's function:

$$b(x,y) = p\Big(1 - \frac{1}{p^*}\Big)^{p-1} (|x| + |y|)^{p-1} (|y| - (p^* - 1)|x|)\,.$$

Actually stochastic Bellman PDE explains readily the way to write this function, and this is made, for example, in [31].

We want to use this Bellman function of Burkholder in solving our problem, but we are unable to do that. The reason is simple. The variables of the Burkholder's function stand for certain martingales, which are in his case related: one is subordinate to the other. And subordination exists in a sense a differential relation "of the first order". In our case we replace these variables not by martingales but by functions: the first is $R_j f$, the second is f. There is no subordination here. The only differential relationship between these two functions is

$$\frac{\partial}{\partial t}\widetilde{R_j f} = -\frac{\partial f}{\partial x_j}\,.$$

This is not a dominance relation, it does not say, for instance, that $|\nabla R_k f| \leqslant |\nabla f|$. That is exactly the obstacle to use Burkholder's function.

Of course, maybe there is a way around this. But we prefer another approach. It follows the approach in [24].

Idea: we formulate Burkholder's inequality in an equivalent (dual) form. The resulting inequality generates another Bellman function. This will be our B from Theorem 1.1.

We will use the following lemma due to Burkholder.

LEMMA 4.1. *Let $(\mathfrak{W}, \mathcal{F}, P)$ be a probability space, $\{\mathcal{F}_n : n \in \mathbb{N}\}$ a filtration in $\mathcal{F}$, and H a separable Hilbert space. Furthermore, let $(X_n, \mathcal{F}_n, P)$ and $(Y_n, \mathcal{F}_n, P)$ be H-valued martingales satisfying*

$$\|Y_n(\omega) - Y_{n-1}(\omega)\|_H \leqslant \|X_n(\omega) - X_{n-1}(\omega)\|_H$$

for all $n \in \mathbb{N}$ and almost every $\omega \in \mathfrak{W}$. Then for any $p \in (1, \infty)$

$$\|Y_n\|_{L^p(P,H)} \leqslant (p^* - 1)\|X_n\|_{L^p(P,H)}\,.$$

The constant $p^* - 1$ is sharp.

From the lemma we can easily obtain our next theorem. We will use $\langle f\rangle_I$ to denote $\frac{1}{|I|}\int\limits_I f(x)\mathrm{d}x$.

THEOREM 4.2. *Choose $J \in \mathcal{D}$ and $M \in \mathbb{N}$. Then, for any functions $f \in L^p(J \to \mathbb{C}^M)$ and $g \in L^q(J \to \mathbb{C}^M)$,*

$$\frac{1}{4|J|}\sum_{I \in \mathcal{D}, I \subseteq J} |\langle f\rangle_{I_+} - \langle f\rangle_{I_-}|\,|\langle g\rangle_{I_+} - \langle g\rangle_{I_-}|\,|I| \leqslant (p^* - 1)\langle |f|^p\rangle_J^{1/p}\langle |g|^q\rangle_J^{1/q}\,.$$

Proof. Take $H = \mathbb{C}^M$ and let $\mathcal{F}_n$ be the σ-algebra generated by dyadic subintervals of J with length 2^{-n}. For $\omega \in J$ define

$$X_n(\omega) := \sum_{I \in \mathcal{D}_J, |I| > 2^{-n}|J|} \langle f,\, h_I\rangle h_I(\omega)\,.$$

Choose a sequence of numbers $\sigma_I^j \in S^1$ and consider

$$Y_n(\omega) := \sum_{I\in\mathcal{D}_J, |I|>2^{-n}|J|} \sigma_I \langle f, h_I\rangle h_I(\omega)\,.$$

Here σ_I should be thought of as a vector $(\sigma_I^1, \dots, \sigma_I^M)$. It follows immediately from the construction that both $(X_n, \mathcal{F}_n, \mathrm{d}x)$ and $(Y_n, \mathcal{F}_n, \mathrm{d}x)$ are martingales. We have

$$X_{n+1} - X_n = \sum_{I\subset J, |I|=2^{-n}|J|} \langle f, h_I\rangle h_I\,, \quad Y_{n+1} - Y_n = \sum_{I\subset J, |I|=2^{-n}|J|} \sigma_I\langle f, h_I\rangle h_I\,.$$

Since the sums above contain functions whose supports have disjoint interiors, and $|\sigma_I^j| = 1$, these martingales satisfy the assumptions of Burkholder's lemma. Now $\|f\|_{L^p} = \lim_{n\to\infty} \|X_n\|_{L^p}$ and $\|T_\sigma f\|_{L^p} = \lim_{n\to\infty} \|Y_n\|_{L^p}$. Burkholder's lemma implies

$$\|T_\sigma f\|_{L^p} \leqslant (p^* - 1)\|f\|_{L^p} \tag{4.2}$$

for any sequence σ as above.

Let us reformulate (4.2) as $|\langle T_\sigma f, g\rangle| \leqslant (p^* - 1)\|f\|_{L^p}\|g\|_{L^q}$, where $g = (g_1, \dots, g_M) \in L^q(J \to \mathbb{C}^M)$. Definition of T_σ now implies

$$\frac{1}{|J|}\Big|\sum_{I\in\mathcal{D}_J} \sigma_I\langle f, h_I\rangle\overline{\langle g, h_I\rangle}\Big| \leqslant (p^* - 1)\langle |f|^p\rangle_J^{1/p}\langle |g|^q\rangle_J^{1/q}\,.$$

The expression under the summation sign actually means

$$\left\langle \begin{bmatrix} \sigma_I^1\langle f_1, h_I\rangle \\ \vdots \\ \sigma_I^M\langle f_M, h_I\rangle \end{bmatrix}, \begin{bmatrix} \langle g_1, h_I\rangle \\ \vdots \\ \langle g_M, h_I\rangle \end{bmatrix} \right\rangle_{\mathbb{C}^M}.$$

By the Cauchy-Schwartz inequality, it can be estimated as

$$\leqslant \|\langle f, h_I\rangle\|_{\mathbb{C}^M}\|\langle g, h_I\rangle\|_{\mathbb{C}^M}\,.$$

Moreover, we can choose the coefficients σ_I^j so that we actually get equality. Finally notice that

$$\langle f, h_I\rangle = \frac{\sqrt{|I|}}{2}(\langle f\rangle_{I_+} - \langle f\rangle_{I_-})$$

and the theorem follows. ∎

REMARK 4.3. When $M = 1$, the equality (4.2) is exactly equal to (4.1). But the point of the preceding theorem is that we need (4.2) for arbitrary M in order to prove Theorem 1.1, as we will see next. This stronger statement was not provided by nor does it seem to follow from (4.1), so the previous theorem was indeed necessary. However, it is also possible to derive (4.2) as a consequence of Theorem 3.2 in [5], but this theorem is in itself a corollary of Lemma 4.1.

Proof of Theorem 1.1. Fix $(\zeta,\eta,Z,H)\in\overline{\Omega}$. Consider all integrable functions $f,g:J\to\mathbb{C}^M$ such that $\zeta=\langle f\rangle_J$, $\eta=\langle g\rangle_J$, $Z=\langle|f|^p\rangle_J$ and $H=\langle|g|^q\rangle_J$ (it is not difficult to construct such functions). Let

$$B(\zeta,\eta,Z,H):=\frac{1}{4|J|}\sup\sum_{I\in\mathcal{D},I\subseteq J}|\langle f\rangle_{I_+}-\langle f\rangle_{I_-}|\,|\langle g\rangle_{I_+}-\langle g\rangle_{I_-}|\,|I|\,,$$

where the supremum is taken over all such f,g.

The supremum above clearly does not depend on the interval J. This observation helps to prove the property (ii) in Theorem 1.1.

Indeed, take $f_\pm:J_\pm\to\mathbb{C}^M$ such that $\langle f_\pm\rangle_{J_\pm}=\zeta_\pm$ and $\langle|f_\pm|^p\rangle_{J_\pm}=Z_\pm$. Define $f:J\to\mathbb{C}^M$ by $f(x):=f_\pm(x)$ if $x\in J_\pm$. In an analogous way we come up with g. Write $a=(\zeta,\eta,Z,H):=\frac{a_++a_-}{2}\in\overline{\Omega}$.

Note that:

(i) $\langle f\rangle_I=\langle f_\pm\rangle_I$ if $I\subseteq J_\pm$,
(ii) $\langle f\rangle_J=\zeta$ and $\langle|f|^p\rangle_J=Z$,

and similarly for g.

Now we can write

$$\begin{aligned}
B(a)&\geqslant\frac{1}{4|J|}\sum_{I\in\mathcal{D},I\subseteq J}|\langle f\rangle_{I_+}-\langle f\rangle_{I_-}|\,|\langle g\rangle_{I_+}-\langle g\rangle_{I_-}|\,|I|\\
&=\frac{1}{4|J|}\Big(\sum_{I\in\mathcal{D},I\subseteq J_+}+\sum_{I\in\mathcal{D},I\subseteq J_-}\Big)+\frac{1}{4|J|}|\langle f\rangle_{J_+}-\langle f\rangle_{J_-}|\,|\langle g\rangle_{J_+}-\langle g\rangle_{J_-}|\,|J|\\
&=\frac{1}{4|J|}\sum_{I\in\mathcal{D},I\subseteq J_+}|\langle f_+\rangle_{I_+}-\langle f_+\rangle_{I_-}|\,|\langle g_+\rangle_{I_+}-\langle g_+\rangle_{I_-}|\,|I|\\
&\quad+\frac{1}{4|J|}\sum_{I\in\mathcal{D},I\subseteq J_-}|\langle f_-\rangle_{I_+}-\langle f_-\rangle_{I_-}|\,|\langle g_-\rangle_{I_+}-\langle g_-\rangle_{I_-}|\,|I|\\
&\quad+\frac{1}{4}\,|\langle f_+\rangle_{J_+}-\langle f_-\rangle_{J_-}|\,|\langle g_+\rangle_{J_+}-\langle g_-\rangle_{J_-}|\,.
\end{aligned}$$

At this point we exploit the fact that the definition of B does not depend on the choice of the interval, as mentioned before. In particular, we can replace J by J_+ or J_-. Having done that, take the supremum of the expressions in the last line over all f_+ and f_- as above. This process clearly does not affect the free term, which can be rewritten using the definition of f_+ and f_-. We get exactly

$$\frac{B(a_+)+B(a_-)}{2}+\frac{1}{4}|\zeta_+-\zeta_-|\,|\eta_+-\eta_-|$$

which proves (1.1).

This establishes the second inequality of Theorem 1.1, whereas the first one, i.e. $0\leqslant B(\zeta,\eta,Z,H)\leqslant(p^*-1)Z^{1/p}H^{1/q}$, was proved in Theorem 4.2. ∎

Proof of Lemma 1.2. Finally, fix a compact K and $0<\varepsilon<1$ small enough for $d(K,\partial\Omega)>\sqrt{\varepsilon}$. Let S be the standard mollifier ([8], C. 4) on $\mathbb{R}^d\equiv\mathbb{C}^M\times\mathbb{C}^M\times$

$\mathbb{R}\times\mathbb{R}$. Set $S_\varepsilon(x)=\frac{1}{\varepsilon^d}S(\frac{x}{\varepsilon})$ and consider $B_\varepsilon=B*S_\varepsilon$. This function is smooth on $\Omega_\varepsilon:=\{a\in\Omega: d(a,\partial\Omega)>\varepsilon\}$. The size inequality can deteriorate on $\overline{\Omega}_{\sqrt{\varepsilon}}$ (and thus on K) at most by factor $1+\sqrt{\varepsilon}$, whereas the concavity inequality (1.1) does not change on Ω_ε, since $\int\limits_{\mathbb{R}^d} S_\varepsilon(x)\,\mathrm{d}x=1$. Moreover, we can use the Taylor formula to see that (1.1) is equivalent to the inequality (ii) from the theorem. ∎

5. ADDENDUM

Here we demonstrate Lemma 1.5.

For suitable $d,m,n\in\mathbb{N}$ we will identify $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ with $\mathbb{R}^d$, $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Then we can think of T as of a matrix from $\mathbb{R}^{d,d}$. For $h\in\mathbb{R}^d$ denote $R=\|P_1h\|$, $r=\|P_2h\|$. We emphasize that R and r are not constants but will depend on h appearing in the context.

Since the assumption and the conclusion of the Lemma are homogenous inequalities with respect to h, it is equivalent to prove:

If $E\subset H$, then there is $\tau>0$ so that $E\subset E_\tau$,

where

$$E=E_T=\{h\in\mathbb{R}^d:\langle Th,h\rangle\leqslant 2\}\,,$$
$$H=\{h\in\mathbb{R}^d: Rr\leqslant 1\}\,,$$
$$E_\tau=\Big\{h\in\mathbb{R}^d:\tau R^2+\frac{1}{\tau}r^2\leqslant 2\Big\}\,.$$

Note that $E_\tau\subset H$ for every $\tau>0$. Since A is a positive matrix, E (more precisely, its boundary) is an ellipsoid. The geometrical shape of other two sets is also clear.

Denote $k:=d-(m+n)$. Suppose that we have the proof in case when $k=0$. Now take arbitrary natural numbers m,n,k. If $E\subset H$, then $E'\subset H'$, where E' and H' are images of E and H, respectively, under the orthogonal projection $\mathbb{R}^d\to\mathbb{R}^{m+n}\times\{0\}^k$. Since E' is again an ellipsoid, by assumption there is $\tau>0$ such that $E'\subset E'_\tau:=\big\{h\in\mathbb{R}^{m+n}\times\{0\}^k:\tau R^2+\frac{1}{\tau}r^2\leqslant 2\big\}$. It is clear that this τ also satisfies $E\subset E_\tau$. Hence it is enough to prove the lemma for $k=0$.

We may also assume that at some point equality is attained in (1.3). This implies there is $h\in\mathbb{R}^d$ for which $Rr=1$ and $\langle Th,h\rangle=2$. In other words, $h\in\partial E\cap\partial H$. For $\lambda=\sqrt{R^{-1}r}$ we have $\lambda R=\lambda^{-1}r=1$. The operator $\lambda I_{\mathbb{R}^m}\otimes\lambda^{-1}I_{\mathbb{R}^n}$ leaves H unchanged, whereas it maps E into some other ellipsoid, whose boundary intersects that of H in a point with $R=r=1$. Finally, there are rotations $U_m\in\mathrm{SO}(m)$ and $U_n\in\mathrm{SO}(n)$, such that $U_m\otimes U_n$ maps this point of intersection into $h_0:=(1,\underbrace{0,\ldots,0}_{m-1})\oplus(1,\underbrace{0,\ldots,0}_{n-1})\in\mathbb{R}^m\oplus\mathbb{R}^n=\mathbb{R}^d$.

To summarize, it suffices to solve the case when $k=0$ and ∂E intersects ∂H at h_0. Since in this case it is obvious that the only admissible τ is 1, our task is reduced to proving that E is contained in $E_1 = \sqrt{2}\,B^d =: B$, i.e. in the closed ball in $\mathbb{R}^d$, centered at 0 and with radius $\sqrt{2}$.

The intersection $\partial H \cap \partial B$ is the "torus" $\mathbb{T} = \{h \in \mathbb{R}^d : R = r = 1\} = S^{m-1} \times S^{n-1}$. Let $\mathcal{P}$ be the family of all 2-dimensional planes in $\mathbb{R}^d$ which pass through 0 and h_0. We would like to find a subfamily $\mathcal{P}'$ of planes that intersect with $\mathbb{T}$ in more than the obvious two points. Take $P \in \mathcal{P}$. There is $u \in h_0^\perp$ such that $\|u\|^2 = 2$ and $P = \mathrm{Lin}\{h_0, u\}$. Roughly speaking, this establishes a correspondence between $\mathcal{P}$ and a portion of the sphere $\partial B = \sqrt{2}S^{d-1}$. We can write $u = \{a\} \oplus b \oplus \{-a\} \oplus c$ for some $a \in \mathbb{R}$, $b \in \mathbb{R}^{m-1}$ and $c \in \mathbb{R}^{n-1}$. If $p = \lambda h_0 + \mu u \in P$ is to intersect $\mathbb{T}$ in a point, different than $\pm h_0$, then we must have

$$(\lambda + \mu a)^2 + \|\mu b\|^2 = (\lambda - \mu a)^2 + \|\mu c\|^2 = 1$$

for some λ, $\mu \in \mathbb{R}$, $\mu \neq 0$ (otherwise $p = \pm h_0$). After adding and subtracting equations and using that $2a^2 + \|b\|^2 + \|c\|^2 = \|u\|^2 = 2$, one can see that this set of equations is equivalent to

$$\begin{aligned} \lambda^2 + \mu^2 &= 1\,, \\ 4\lambda a + \mu(\|b\|^2 - \|c\|^2) &= 0\,. \end{aligned}$$

If $a = 0$ then we must have $\|b\| = \|c\|$. For $a \neq 0$ the system admits solutions

$$\lambda^2 = \frac{(1-a^2)^2 - \|b\|^2\|c\|^2}{(1+a^2)^2 - \|b\|^2\|c\|^2}\,, \quad \mu^2 = \frac{4a^2}{(1+a^2)^2 - \|b\|^2\|c\|^2}\,.$$

Note that

$$\|b\|\,\|c\| \leqslant \frac{\|b\|^2 + \|c\|^2}{2} = 1 - a^2 < 1 + a^2\,,$$

thus the denominators are always positive. Hence the solution does *not* exist (i.e. P does *not* belong to $\mathcal{P}'$) if and only if

$$a = 0 \;\wedge\; \|b\| \neq \|c\|\,.$$

This justifies employing identifications

$$\mathcal{P} \equiv \{u = u(a,b,c) \in \partial B : u \perp h_0\} \quad \text{and} \quad \mathcal{P}' \equiv \{u \in \mathcal{P} : a \neq 0 \;\vee\; \|b\| = \|c\|\}\,,$$

which in principle imply that the set $\mathcal{P}'$ is "dense" in $\mathcal{P}$.

We would like to show that $E \cap P \subset B \cap P$ or, equivalently, $\partial E \cap P \subset B \cap P$ for all $P \in \mathcal{P}'$. Since $\partial E \cap P$ is an ellipse, this simply follows from the fact that $E \subset H$ and that $\mathbb{T} \cap P$ contains at least four different points, as has just been shown.

Finally we note that the collection $\mathcal{P}'$ is sufficiently large to conclude that $E \subset B$, which had to be proven. ∎

Acknowledgements. We are very grateful to the referee for his careful reading of the text and many valuable suggestions.

REFERENCES

[1] N. ARCOZZI, Riesz transforms on compact Lie groups, spheres and Gauss space, *Ark. Mat.* **36**(1998), 201–231.

[2] D. BAKRY, The Riesz transforms associated with second order differential operators, in *Seminar on Stochastic Processes*, Progr. Probab., vol. 17, Birkhäuser, Boston 1989, pp. 1–43.

[3] R. BAÑUELOS, G. WANG, Sharp inequalities for martingales with applications to the Beurling-Ahlfors and Riesz transforms, *Duke Math. J.* **80**(1995), 575–600.

[4] D.L. BURKHOLDER, Boundary value problems and sharp inequalities for martingale transforms, *Ann. Probab.* **12**(1984), 647–702.

[5] D.L. BURKHOLDER, Explorations in martingale theory and its applications, in *Ecole d'Eté de Probabilités de Saint-Flour XIX – 1989*, Lecture Notes in Math., vol. 1464, Springer, Berlin 1991, pp. 1–66.

[6] T. COULHON, X.T. DUONG, Riesz transforms for $p > 2$, *C. R. Acad. Sci. Paris Sér. I Math.* **332**(2001), 1–6.

[7] T. COULHON, D. MÜLLER, J. ZIENKIEWICZ, About Riesz transforms on the Heisenberg groups, *Math. Ann.* **305**(1996), 369–379.

[8] L.C. EVANS, *Partial Differential Equations*, Amer. Math. Soc., Providence, RI 1998.

[9] J. GARCÍA-CUERVA, G. MAUCERI, S. MEDA, P. SJÖGREN, J.L. TORREA, Functional calculus for the Ornstein-Uhlenbeck operator, *J. Funct. Anal.* **183**(2001), 413–450.

[10] J. GARCÍA-CUERVA, G. MAUCERI, P. SJÖGREN, J.L. TORREA, Higher-order Riesz operators for the Ornstein-Uhlenbeck semigroup, *Potential Anal.* **10**(1999), 379–407.

[11] J. GARCÍA-CUERVA, G. MAUCERI, P. SJÖGREN, J.L. TORREA, Spectral multipliers for the Ornstein-Uhlenbeck semigroup, *J. Anal. Math.* **78**(1999), 281–305.

[12] T. IWANIEC, G. MARTIN, Riesz transforms and related singular integrals, *J. Reine Angew. Math.* **473**(1996), 25–57.

[13] P. JANAKIRAMAN, Weak-type estimates for singular integrals and Riesz transform, *Indiana Univ. Math. J.* **53**(2004), 533–555.

[14] S. JANSON, *Gaussian Hilbert Spaces*, Cambridge Univ. Press, Cambridge 1997.

[15] P.E. KOPP, *Martingales and Stochastic Integrals*, Cambridge Univ. Press, Cambridge 1984.

[16] L. LARSSON-COHN, On the constants in the Meyer inequality, *Monaths. Math.* **137**(2002), 51–56.

[17] F. LUST-PIQUARD, Riesz transforms on deformed Fock spaces, *Comm. Math. Phys.* **205**(1999), 519–549.

[18] F. LUST-PIQUARD, Dimension free estimates for discrete Riesz transforms on products of abelian groups, *Adv. Math.* **185**(2004), 289–327.

[19] F. LUST-PIQUARD, Riesz transforms on generalized Heisenberg groups and Riesz transforms associated to the CCR heat flow, *Publ. Mat.* **48**(2004), 309–333.

[20] P.A. MEYER, Transformations de Riesz pour les lois Gaussiennes, in *Seminaire de Probabilités XVIII*, Lecture Notes in Math., vol. 1059, Springer, Berlin-Heidelberg-New York 1984, pp. 179–193.

[21] F. NAZAROV, S. TREIL, The hunt for a Bellman function: applications to estimates of singular integral operators and to other classical problems of harmonic analysis, *St. Petersburg Math. J.* **8**(1997), 721–824.

[22] F. NAZAROV, A. VOLBERG, Heat extension of the Beurling operator and estimates for its norm, *St. Petersburg Math. J.* **15**(2004), 563–573.

[23] G.O. OKIKIOLU, *Aspects of the Theory of Bounded Integral Operators in L^p–Spaces*, Academic Press, London 1971.

[24] S. PETERMICHL, A. VOLBERG, Heating of the Beurling operator: weakly quasiregular maps on the plane are quasiregular, *Duke Math. J.* **112**(2002), 281–305.

[25] S.K. PICHORIDES, On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov, *Studia Math.* **44**(1972), 165–179.

[26] G. PISIER, Riesz transforms: a simpler analytic proof of P.A. Meyer's inequality, in *Séminaire de Probabilités XXII*, Lecture Notes in Math., vol. 1321, Sringer, Berlin-Heidelberg-New York 1988, pp. 485–501.

[27] W. RUDIN, *Functional Analysis*, McGraw-Hill, New York 1991.

[28] P. SJÖGREN, Operators associated with the Hermite semigroup — a survey, *J. Fourier Anal. Appl.* **3**(1997), Special Issue, 813–823.

[29] E.M. STEIN, Some results in harmonic analysis in $\mathbb{R}^n$ for $n \to \infty$, *Bull. Amer. Math. Soc.* **9**(1983), 71–73.

[30] D.W. STROOCK, *Probability Theory, an Analytic View*, Cambridge Univ. Press, Cambridge 1993.

[31] A. VOLBERG, Bellman approach to some problems in harmonic analysis, in *Séminaire aux équations dérives partielles*, vol. 20, Ecole Polytéchnique, Palaiseau 2002.

OLIVER DRAGIČEVIĆ, DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MI 48823, USA. *Current address*: INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS, UNIVERSITY OF LJUBLJANA, SLOVENIA
E-mail address: oliver.dragicevic@fmf.uni-lj.si

ALEXANDER VOLBERG, DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MI 48823, USA
E-mail address: volberg@math.msu.edu

Received October 21, 2004; revised March 21, 2005.