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GROUPOID MODELS FOR THE C^*-ALGEBRAS OF TOPOLOGICAL HIGHER-RANK GRAPHS

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ABSTRACT. We provide groupoid models for Toeplitz and Cuntz-Krieger algebras of topological higher-rank graphs. Extending groupoid models used in the theory of graph algebras and topological dynamical systems to our setting, we prove results on essential freeness and amenability of the groupoids which capture the existing theory, and extend results involving group crossed products of graph algebras.

KEYWORDS: *Topological graph, higher rank graph, groupoid, graph algebra, Cuntz-Krieger algebra.*

MSC (2000): Primary 46L05; Secondary 22A22.

1. INTRODUCTION

In recent years there has been significant interest in different generalizations of Cuntz-Krieger algebras, including the C^*-algebras of higher-rank graphs [12], [25], [26], [6], [30] and the C^*-algebras of topological graphs [10], [11]. In this article, we use groupoid models to explore a common approach to both generalizations.

We begin by introducing the notion of a topological higher-rank graph. Given a topological graph Λ of rank $k \in \mathbb{N}$, we define a topological path space X_Λ, which contains the finite paths of Λ together with paths which are infinite in some or all of their k dimensions. There is a natural action of Λ on X_Λ given by concatenation and removal of initial path segments, and using this we define a groupoid G_Λ, called the path groupoid of Λ, which has X_Λ as its unit space.

We identify a topological analogue of the finitely aligned condition of [26], [6], [18], [30] called compactly aligned. If Λ fails to be compactly aligned, then G_Λ fails to be a topological groupoid: the range and source maps are not continuous and the topology on G_Λ is not locally compact. However, if Λ is compactly aligned, then G_Λ is a locally compact topological groupoid which is r-discrete in the sense that the unit space $G_\Lambda^{(0)}$ is open in G_Λ. Furthermore, G_Λ admits a Haar system, and so we may define the full groupoid C^*-algebra $C^*(G_\Lambda)$, which we refer to as the Toeplitz algebra of Λ.

We identify a closed invariant subset $\partial\Lambda$ of $G_\Lambda^{(0)} = X_\Lambda$, called the boundary-path space. The boundary-path groupoid of Λ is the reduction $\mathcal{G}_\Lambda := G_\Lambda|_{\partial\Lambda}$: a locally compact r-discrete groupoid admitting a Haar system. The Cuntz-Krieger algebra of Λ is then defined to be the full groupoid C^*-algebra $C^*(\mathcal{G}_\Lambda)$. When Λ is a finitely aligned discrete k-graph or the finite-path space of a second-countable topological graph, we recover the usual Toeplitz and Cuntz-Krieger algebras of the graph.

We then consider an analogue of the Aperiodicity Condition used in [12]. We extend Proposition 4.5 of [12] and Proposition 7.2 of [6] to our setting, proving that a compactly aligned topological k-graph Λ satisfies the Aperiodicity Condition if and only if $\mathcal{G}_\Lambda$ is essentially free in the sense that the units with trivial isotropy are dense in $\mathcal{G}_\Lambda^{(0)}$.

Next, we address amenability of the boundary-path groupoid, showing that $\mathcal{G}_\Lambda$ is amenable if Λ is either a finitely aligned discrete k-graph, a topological 1-graph, or a *proper* topological k-graph *without sources*.

We end the article with a section on crossed products of topological k-graph algebras by coactions, extending Theorem 2.4 of [9] and Corollary 5.3 of [12]. To begin, given a topological k-graph Λ, a locally compact group A and a continuous functor $c : \Lambda \to A$, we define the notion of a skew-product topological k-graph $\Lambda \times_c A$.

If A is abelian, then there are induced actions α of the dual group $\widehat{A}$ on $C^*(G_\Lambda)$ and $C^*(\mathcal{G}_\Lambda)$, and we extend Corollary 5.3 of [12], proving that the crossed product C^*-algebras $C^*(G_\Lambda) \times_\alpha \widehat{A}$ and $C^*(\mathcal{G}_\Lambda) \times_\alpha \widehat{A}$ are isomorphic to $C^*(G_{\Lambda\times_c A})$ and $C^*(\mathcal{G}_{\Lambda\times_c A})$, respectively.

If A is discrete, there are induced coactions δ of A on $C^*(G_\Lambda)$ and $C^*(\mathcal{G}_\Lambda)$, and we extend Theorem 2.4 of [9], proving that $C^*(G_\Lambda) \times_\delta A \cong C^*(G_{\Lambda\times_c A})$ and $C^*(\mathcal{G}_\Lambda) \times_\delta A \cong C^*(\mathcal{G}_{\Lambda\times_c A})$.

2. TOPOLOGICAL HIGHER-RANK GRAPHS

DEFINITION 2.1. Given $k \in \mathbb{N}$, a *topological k-graph* is a pair (Λ, d) consisting of a small category $\Lambda = (\mathrm{Obj}(\Lambda), \mathrm{Mor}(\Lambda), r, s)$ and a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree map*, which satisfy the following:

(i) $\mathrm{Obj}(\Lambda)$, $\mathrm{Mor}(\Lambda)$ are second-countable locally compact Hausdorff spaces;

(ii) $r, s : \mathrm{Mor}(\Lambda) \to \mathrm{Obj}(\Lambda)$ are continuous and s is a local homeomorphism;

(iii) composition $\circ : \Lambda \times_c \Lambda \to \Lambda$ is continuous and open;

(iv) d is continuous, where $\mathbb{N}^k$ has the discrete topology;

(v) for all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$, there exists unique $(\xi, \eta) \in \Lambda \times_c \Lambda$ such that $\lambda = \xi\eta$, $d(\xi) = m$ and $d(\eta) = n$.

We refer to the morphisms of Λ as *paths* and to the objects of Λ as *vertices*. The codomain and domain maps from Λ are called the range and source maps, respectively.

NOTATION 2.2. For $m \in \mathbb{N}^k$, we write m_i for the ith coordinate of m. We use the partial ordering $\leqslant$ on $\mathbb{N}^k$ defined by $m \leqslant n \iff m_i \leqslant n_i$ for all $i \in \{1, \ldots, k\}$, so least upper bounds and greatest lower bounds are given by $(m \vee n)_i = \max\{m_i, n_i\}$ and $(m \wedge n)_i = \min\{m_i, n_i\}$, respectively. For $m \in \mathbb{N}^k$, define Λ^m to be the set $d^{-1}(\{m\})$ of paths of degree m. Define $\Lambda *_s \Lambda := \{(\lambda, \mu) \in \Lambda \times \Lambda \mid s(\lambda) = s(\mu)\}$, and for $U, V \subset \Lambda$ define $U *_s V := (U \times V) \cap (\Lambda *_s \Lambda)$. For $p, q \in \mathbb{N}^k$, $U \subset \Lambda^p$ and $V \subset \Lambda^q$, we write

$$U \vee V := U\Lambda^{(p\vee q)-p} \cap V\Lambda^{(p \vee q)-q}$$

for the set of *minimal common extensions* of paths from U and V. For $\lambda, \mu \in \Lambda$, we write

$$\Lambda^{\min}(\lambda, \mu) := \{(\alpha, \beta) \mid \lambda\alpha = \mu\beta, d(\lambda\alpha) = d(\lambda) \vee d(\mu)\}$$

for the set of pairs which give minimal common extensions of λ and μ; that is,

$$\Lambda^{\min}(\lambda, \mu) = \{(\alpha, \beta) \mid \lambda\alpha = \mu\beta \in \{\lambda\} \vee \{\mu\}\}.$$

DEFINITION 2.3. A topological k-graph (Λ, d) is *compactly aligned* if for all $p, q \in \mathbb{N}^k$ and compact $U \subset \Lambda^p$ and $V \subset \Lambda^q$, the set $U \vee V$ is compact.

REMARK 2.4. A discrete k-graph (Λ, d) is *finitely aligned* if for all $\lambda, \mu \in \Lambda$, the set $\Lambda^{\min}(\lambda, \mu)$ is finite; these discrete k-graphs form the scope of the higher-rank graph theory to date (see [26], [18], [6], [30]). As compactness is equivalent to finiteness for discrete topologies, it follows that a discrete k-graph is compactly aligned if and only if it is finitely aligned.

EXAMPLES 2.5. (i) Any higher-rank graph (as defined in [12], [26] etc.) may be regarded as a topological higher-rank graph with discrete topologies on the object and morphism sets.

(ii) Let E be a second-countable topological graph as defined in Definition 2.1 of [10]; that is, $E = (E^0, E^1, r, s)$ is a directed graph with E^0, E^1 second-countable locally compact Hausdorff spaces, $r, s : E^1 \to E^0$ continuous, and s a local homeomorphism. The free category generated by E, endowed with the relative topology inherited from the union of the product topologies, together with the length function $l(e_1 \cdots e_n) = n$, forms a topological 1-graph (E^*, l). Conversely, given a topological 1-graph (Λ, d), the quadruple $E_\Lambda := (\Lambda^0, \Lambda^1, r|_{\Lambda^1}, s|_{\Lambda^1})$ is a second-countable topological graph with $((E_\Lambda)^*, l) \cong (\Lambda, d)$.

(iii) Let (X, θ) be a singly generated dynamical system as defined in [2], [29]; that is, X is a second-countable locally compact Hausdorff space and θ is a local homeomorphism from an open subset $\mathrm{dom}(\theta)$ of X onto an open subset $\mathrm{ran}(\theta)$ of X. In Section 10.3 of [10], Katsura constructs a topological graph $E(X, \theta)$ by setting $E(X, \theta)^0 := X$ and $E(X, \theta)^1 := \mathrm{dom}(\theta)$, and for $x \in E(X, \theta)^1$, setting

$r(x) := x$ and $s(x) := \theta(x)$. So, as was done for a general topological graph in the previous example, we may form the topological 1-graph $(\Lambda(X,\theta), d) := (E(X,\theta)^*, l)$. The following example generalizes this construction.

(iv) Let X be a second-countable locally compact Hausdorff space. For $i = 1,\dots,k$, let θ_i be a local homeomorphism from an open subset $\mathrm{dom}(\theta_i)$ of X onto an open subset $\mathrm{ran}(\theta_i)$ of X, such that for all $i,j \in \{1,\dots,k\}$,

$$\mathrm{dom}(\theta_j\theta_i) = \theta_i^{-1}(\mathrm{ran}(\theta_i) \cap \mathrm{dom}(\theta_j)) = \theta_j^{-1}(\mathrm{ran}(\theta_j) \cap \mathrm{dom}(\theta_i)) = \mathrm{dom}(\theta_i\theta_j)$$

and for $x \in \mathrm{dom}(\theta_j\theta_i)$,

$$\theta_j(\theta_i(x)) = \theta_i(\theta_j(x)).$$

We define a topological k-graph $(\Lambda(X,\{\theta_i\}_{i=1}^k), d)$ by setting

$$\begin{aligned}
\mathrm{Obj}(\Lambda(X,\{\theta_i\})) &:= X, \\
\mathrm{Mor}(\Lambda(X,\{\theta_i\})) &:= \{(m,x) \in \mathbb{N}^k \times X \mid x \in \mathrm{dom}(\theta_1^{m_1}\theta_2^{m_2}\cdots\theta_k^{m_k})\} \\
&= \bigcup_{m\in\mathbb{N}^k} \{m\} \times \mathrm{dom}(\theta_1^{m_1}\cdots\theta_k^{m_k}), \\
r(m,x) := x, &\quad s(m,x) := \theta_1^{m_1}\cdots\theta_k^{m_k}(x),
\end{aligned}$$

and

$$(m,x)(n,\theta_1^{m_1}\cdots\theta_k^{m_k}(x)) := (m+n,x),$$

giving $\Lambda(X,\{\theta_i\}_{i=1}^k)$ the relative topology inherited from the product topology, and setting $d(m,x) := m$.

3. THE PATH GROUPOID

We begin this section by associating a groupoid G_Λ to each topological k-graph (Λ, d). We first define the unit space $G_\Lambda^{(0)}$ as a space X_Λ of paths of Λ; the finite paths in X_Λ are characterized by the morphisms $\lambda \in \Lambda$, however we must also consider paths of Λ which are infinite in some or all of their k-dimensions. To do this, we first define appropriate rank-k path prototypes for each and every degree — finite, infinite and partially infinite — and then obtain X_Λ as the set of representations of the path prototypes. The morphisms of the category Λ are then in correspondence with the representations of those path prototypes which are finite in each of the k dimensions.

For $k \in \mathbb{N}$ and $m \in (\mathbb{N} \cup \{\infty\})^k$, define the topological k-graph $(\Omega_{k,m}, d)$ by giving the discrete topologies to the sets

$$\mathrm{Obj}(\Omega_{k,m}) := \{p \in \mathbb{N}^k \mid p \leqslant m\}$$

and

$$\mathrm{Mor}(\Omega_{k,m}) := \{(p,q) \in \mathbb{N}^k \times \mathbb{N}^k \mid p \leqslant q \leqslant m\},$$

and setting $r(p,q) := p$, $s(p,q) := q$, $(n,p) \circ (p,q) := (n,q)$ and $d(p,q) := q - p$.

Let (Λ_1, d_1) and (Λ_2, d_2) be topological k-graphs. A *graph morphism* between Λ_1 and Λ_2 is a continuous functor $x : \Lambda_1 \to \Lambda_2$ satisfying $d_2(f(\lambda)) = d_1(\lambda)$ for all $\lambda \in \Lambda_1$.

DEFINITION 3.1. Let (Λ, d) be a topological k-graph. We define

$$X_\Lambda := \bigcup_{m \in (\mathbb{N} \cup \{\infty\})^k} \{x : \Omega_{k,m} \to \Lambda \mid x \text{ is a graph morphism}\}.$$

We extend the range and degree maps to $x : \Omega_{k,m} \to \Lambda$ in X_Λ by setting $r(x) := x(0)$ and $d(x) := m$. For $v \in \Lambda^0$ we define $vX_\Lambda := \{x \in X_\Lambda \mid r(x) = v\}$.

NOTATION 3.2. For each $\lambda \in \Lambda$ there is a unique graph morphism $x_\lambda : \Omega_{k,d(\lambda)} \to \Lambda$ such that $x_\lambda(0, d(\lambda)) = \lambda$; in this sense, we may view Λ as a subset of X_Λ, and we refer to elements of X_Λ as paths. Indeed, for $\lambda \in \Lambda$ and $p, q \in \mathbb{N}^k$ with $0 \leqslant p \leqslant q \leqslant d(\lambda)$, we may write $\lambda(0, p)$, $\lambda(p, q)$ and $\lambda(q, d(\lambda))$ for the unique elements of Λ which satisfy $\lambda = \lambda(0, p)\lambda(p, q)\lambda(q, d(\lambda))$, $d(\lambda(0, p)) = p$, $d(\lambda(p, q)) = q - p$ and $d(\lambda(q, d(\lambda))) = d(\lambda) - q$.

Straightforward arguments give the following lemma.

LEMMA 3.3. *Let (Λ, d) be a topological k-graph. For $x \in X_\Lambda$, $m \in \mathbb{N}^k$ with $m \leqslant d(x)$, and $\lambda \in \Lambda$ with $s(\lambda) = r(x)$, there exist unique paths λx and $\sigma^m x$ in X_Λ satisfying $d(\lambda x) = d(\lambda) + d(x)$, $d(\sigma^m x) = d(x) - m$,*

$$(\lambda x)(0, p) = \begin{cases} \lambda(0, p) & \text{if } p \leqslant d(\lambda), \\ \lambda x(0, p - d(\lambda)) & \text{if } d(\lambda) \leqslant p \leqslant d(\lambda x), \end{cases}$$

and

$$(\sigma^m x)(0, p) = x(m, m + p) \quad \text{for } p \leqslant d(\sigma^m x).$$

DEFINITION 3.4. Let (Λ, d) be a topological k-graph. Define the *path groupoid* G_Λ to be the groupoid with object set $\mathrm{Obj}(G_\Lambda) := X_\Lambda$, morphism set

$$\begin{aligned} \mathrm{Mor}(G_\Lambda) &:= \{(\lambda x, d(\lambda) - d(\mu), \mu x) \in X_\Lambda \times \mathbb{Z}^k \times X_\Lambda \mid \\ &\qquad\qquad (\lambda, \mu) \in \Lambda *_s \Lambda,\ x \in X_\Lambda \text{ and } s(\lambda) = r(x)\} \\ &= \{(x, m, y) \in X_\Lambda \times \mathbb{Z}^k \times X_\Lambda \mid \text{ there exist } p, q \in \mathbb{N}^k \text{ such that} \\ &\qquad\qquad p \leqslant d(x),\ q \leqslant d(y),\ p - q = m \text{ and } \sigma^p x = \sigma^q y\}, \end{aligned}$$

range and source maps $r(x, m, y) := x$ and $s(x, m, y) := y$, composition

$$((x, m, y), (y, n, z)) \mapsto (x, m + n, z),$$

and inversion $(x, m, y) \mapsto (y, -m, x)$.

NOTATION 3.5. Let (Λ, d) be a topological k-graph. For $F \subset \Lambda *_s \Lambda$ and $m \in \mathbb{Z}^k$, define $Z(F, m) \subset G_\Lambda$ by

$$Z(F, m) := \{(\lambda x, d(\lambda) - d(\mu), \mu x) \in G_\Lambda \mid (\lambda, \mu) \in F, d(\lambda) - d(\mu) = m\}.$$

For $U \subset \Lambda$, define $Z(U) \subset G_\Lambda^{(0)}$ by

$$Z(U) := Z(U *_s U, 0) \cap Z(\Lambda^0 *_s \Lambda^0, 0).$$

PROPOSITION 3.6. *Let (Λ, d) be a topological k-graph. The family of sets of the form*

$$Z(U *_s V, m) \cap Z(F, m)^c,$$

where $m \in \mathbb{Z}^k$, $U, V \subset \Lambda$ are open and $F \subset \Lambda *_s \Lambda$ is compact, is a basis for a second-countable Hausdorff topology on G_Λ.

Proof. To see that the family of sets forms a basis, suppose

$$(x, m, y) \in (Z(U_1 *_s V_1, m) \cap Z(F_1, m)^c) \cap (Z(U_2 *_s V_2, m) \cap Z(F_2, m)^c),$$

where $m \in \mathbb{Z}^k$, $U_1, U_2, V_1, V_2 \subset \Lambda$ are open and $F_1, F_2 \subset \Lambda *_s \Lambda$ are compact. We then have the existence of $(\lambda, \mu) \in U_1 *_s V_1$, $(\xi, \eta) \in U_2 *_s V_2$ and $w, z \in X_\Lambda$ such that

$$(x, m, y) = (\lambda w, d(\lambda) - d(\mu), \mu w) = (\xi z, d(\xi) - d(\eta), \eta z).$$

Hence the pair w, z extend λ, ξ and μ, η to common paths x and y, respectively, and setting

$$\alpha := w(0, (d(\lambda) \vee d(\xi)) - d(\lambda)) = w(0, (d(\mu) \vee d(\eta)) - d(\mu))$$

and

$$\beta := z(0, (d(\lambda) \vee d(\xi)) - d(\xi)) = z(0, (d(\mu) \vee d(\eta)) - d(\eta)),$$

we have

$$\lambda\alpha = \xi\beta \quad \text{and} \quad \mu\alpha = \eta\beta.$$

Let $W_1 \subset \Lambda$ be an open neighbourhood of α such that $s|_{W_1}$ is a homeomorphism, and let $W_2 \subset \Lambda$ be an open neighbourhood of β such that $s|_{W_2}$ is a homeomorphism. Since composition is open, the sets $U_1W_1, V_1W_1, U_2W_2, V_2W_2 \subset \Lambda$ are open, and we have

$$(x, m, y) \in Z((U_1W_1 *_s V_1W_1) \cap (U_2W_2 *_s V_2W_2), m) \cap Z(F_1 \cup F_2, m)^c.$$

Furthermore, since $s|_{W_1}$ and $s|_{W_2}$ are homeomorphisms, it follows that

$$\begin{aligned} &Z((U_1W_1 *_s V_1W_1) \cap (U_2W_2 *_s V_2W_2), m) \cap Z(F_1 \cup F_2, m)^c \\ &\qquad \subset (Z(U_1 *_s V_1, m) \cap Z(F_1, m)^c) \cap (Z(U_2 *_s V_2) \cap Z(F_2, m)^c), \end{aligned}$$

as required.

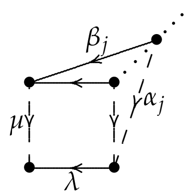
Second-countability is clear. It remains to show the topology is Hausdorff. Let (w, m, x) and (y, n, z) be distinct elements of G_Λ. If $m \neq n$, then $(w, m, x) \in Z(\Lambda *_s \Lambda, m)$, $(y, n, z) \in Z(\Lambda *_s \Lambda, n)$ and $Z(\Lambda *_s \Lambda, m) \cap Z(\Lambda *_s \Lambda, m) = \emptyset$, so we assume $m = n$. Furthermore, if $r(w) \neq r(y)$, then taking open neighbourhoods $U, V \subset \Lambda^0$ of $r(w)$ and $r(y)$, respectively, such that $U \cap V = \emptyset$, we have $(w, m, x) \in Z(U\Lambda *_s \Lambda, m)$, $(y, m, z) \in Z(V\Lambda *_s \Lambda, m)$ and $Z(U\Lambda *_s \Lambda, m) \cap Z(V\Lambda *_s \Lambda, m) = \emptyset$. A similar argument holds if $r(x) \neq r(z)$, so we assume $r(w) = r(y)$ and $r(x) = r(z)$.

We must have either $w \neq y$ or $x \neq z$, so assume $w \neq y$. Let $p \in \mathbb{N}^k$ be minimal with respect to the conditions: $p \leqslant d(w) \wedge d(y)$, $w(0,p) = y(0,p)$, and $w(0, p+e_i) \neq y(0, p+e_i)$ for some $i \in \{1, \ldots, k\}$. We must have either $d(w) \geqslant p + e_i$ or $d(y) \geqslant p + e_i$: If both $d(w) \geqslant p + e_i$ and $d(y) \geqslant p + e_i$, we can take open neighbourhoods $U, V \subset \Lambda^{p+e_i}$ of $w(0, p+e_i)$ and $y(0, p+e_i)$, respectively, such that $U \cap V = \emptyset$. Then $(w, m, x) \in Z(U\Lambda *_s \Lambda, m)$, $(y, m, z) \in Z(V\Lambda *_s \Lambda, m)$ and $Z(U\Lambda *_s \Lambda, m) \cap Z(V\Lambda *_s \Lambda, m) = \emptyset$, so we assume $d(w) \geqslant p + e_i$ and $d(y) \ngeqslant p + e_i$.

Express (w, m, x) as $(\lambda w', d(\lambda) - d(\mu), \mu w')$, where $d(\lambda) \geqslant p + e_i$, and let $U \subset \Lambda^{d(\lambda)}$ and $V \subset \Lambda^{d(\mu)}$ be relatively compact open neighbourhoods of λ and μ, respectively. Then $(w, m, x) \in Z(U *_s V, m)$, $(y, m, z) \in Z(\overline{U} *_s \overline{V}, m)^c$ and $Z(U *_s V, m) \cap Z(\overline{U} *_s \overline{V}, m)^c = \emptyset$, proving the topology is Hausdorff. ∎

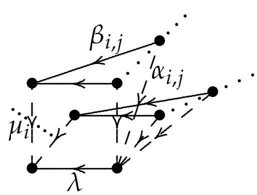
REMARK 3.7. If (Λ, d) is not compactly aligned, then the topology on G_Λ defined by Proposition 3.6 may not be locally compact, and, under this topology, G_Λ may not be a topological groupoid. To illustrate these two facts, we consider two 2-graphs which fail to be compactly aligned. We describe the 2-graphs in terms of their 1-skeletons as in Section 2 of [25].

Let (Λ_1, d) be the discrete topological 2-graph with 1-skeleton:



So $\lambda\alpha_j = \mu\beta_j$ for all $j \in \mathbb{N}$. The sequence $\langle(\lambda\alpha_j, (1,0), \alpha_j)\rangle_{j\in\mathbb{N}}$ converges to $(\lambda, (1,0), s(\lambda))$ in G_{Λ_1}, but $\langle r(\lambda\alpha_j, (1,0), \alpha_j)\rangle_{j\in\mathbb{N}} = \langle\lambda\alpha_j\rangle_{j\in\mathbb{N}}$ does not converge to $r(\lambda, (1,0), s(\lambda)) = \lambda$ in $G_{\Lambda_1}^{(0)}$ since $\{\lambda\} = Z(\lambda) \cap Z(\mu)^c$. Therefore the range map in G_{Λ_1} is not continuous.

Now, taking a family of copies of Λ_1 indexed by $\mathbb{N}$, and identifying the path λ from each, we obtain the 2-graph (Λ_2, d) with 1-skeleton:



So, $\lambda\alpha_{i,j} = \mu_i\beta_{i,j}$ for all $i, j \in \mathbb{N}$. We claim that the unit $\lambda \in G_{\Lambda_2}^{(0)}$ has no compact neighbourhood: First note that any neighbourhood of λ in $G_{\Lambda_2}^{(0)}$ contains a basis set of the form

$$Z(\lambda)\cap\bigcap_{k=1}^{n} Z(\mu_{i_k})^c,\quad \text{where } i_k\in\mathbb{N} \text{ for } k=1,\dots,n;$$

furthermore, given such a basis set, and choosing $l\in\mathbb{N}$ with $l\neq i_k$ for all $k\in\{1,\dots,n\}$, the family

$$\Big\{Z(\lambda)\cap\bigcap_{k=1}^{n} Z(\mu_{i_k})^c\cap Z(\mu_l)^c\Big\}\cup\{Z(\mu_l\beta_{l,j})\mid j\in\mathbb{N}\}$$

forms an infinite, disjoint, open cover which, consequently, has no finite subcover. Hence the topology on G_{Λ_2} is not locally compact.

The following two lemmas allow us to restrict the type of basis elements we will need to consider. We omit the proof of the first lemma.

LEMMA 3.8 (cf. Remark 5.5 of [6]). *Let (Λ,d) be a compactly aligned topological k-graph. For any relatively compact $U\subset\Lambda$ and compact $F\subset\Lambda$,*

$$Z(U)\cap Z(F)^c = Z(U)\cap Z(\overline{U}\vee F)^c.$$

Thus, since Λ is locally compact and since the source map in Λ is a local homeomorphism, we need only consider basis sets for $G_\Lambda^{(0)}$ of the form $Z(U)\cap Z(F)^c$, where $U\subset\Lambda$ is relatively compact and open, and $F\subset\Lambda$ is compact and satisfies $\mu\in F$ implies $\mu=\lambda\mu'$ for some $\lambda\in\overline{U}$.

LEMMA 3.9. *Let (Λ,d) be a compactly aligned topological k-graph. Let $p,q\in\mathbb{N}^k$, let $U\subset\Lambda^p$, $V\subset\Lambda^q$ be relatively compact open sets, and let $F\subset\Lambda *_s\Lambda$ be compact. There exists a compact set $F'\subset\Lambda *_s\Lambda$ such that*

$$Z(U *_s V, p-q)\cap Z(F,p-q)^c = Z(U *_s V, p-q)\cap Z(F',p-q)^c$$

and

$$(\xi,\eta)\in F' \text{ implies } (\xi,\eta)=(\lambda\alpha,\mu\alpha) \text{ for some } (\lambda,\mu)\in\overline{U}*_s\overline{V} \text{ and } \alpha\in\Lambda.$$

Proof. We can assume $d(\xi)-d(\eta)=m$ for all $(\xi,\eta)\in F$. Since F is compact, there exist $m^1,\dots,m^l\in\mathbb{N}^k$ such that each $m^j\geqslant p-q$ and

$$F=\bigcup_{j=1}^{l} F\cap(\Lambda^{m^j} *_s \Lambda^{m^j-(p-q)});$$

for $j=1,\dots,l$, define $F_j := F\cap(\Lambda^{m^j} *_s \Lambda^{m^j-(p-q)})$ and

$$\begin{aligned}F_j' := \{(\xi,\eta)\in\Lambda^{m^j\vee p} *_s \Lambda^{(m^j-p+q)\vee q} \mid (\xi(0,m^j),\eta(0,m^j-p+q))\in F_j,\\ (\xi(0,p),\eta(0,q))\in\overline{U}*_s\overline{V} \text{ and } \xi(p,m^j\vee p)=\eta(q,(m^j-p+q)\vee q)\}.\end{aligned}$$

Letting $P_1, P_2 : \Lambda *_s \Lambda \to \Lambda$ be the coordinate projections, we see that each F_j' is a closed subset of the compact set $(P_1(F_j) \vee \overline{U}) *_s (P_2(F_j) \vee \overline{V})$. Hence each F_j' is compact. Defining $F' := \bigcup_{j=1}^{l} F_j'$ completes the proof. ∎

DEFINITION 3.10. Let (Λ, d) be a topological k-graph. For $m, p, q \in \mathbb{N}^k$ with $p \leqslant q \leqslant m$, define the continuous map $\mathrm{Seg}^m_{(p,q)} : \Lambda^m \to \Lambda^{q-p}$ by $\mathrm{Seg}^m_{(p,q)}(\lambda) := \lambda(p,q)$.

DEFINITION 3.11. Let (Λ, d) be a topological k-graph. An infinite sequence of paths in Λ is *wandering* if for every compact set $F \subset \Lambda$, the sequence is eventually in $\Lambda \setminus F$; that is to say, the sequence visits any compact set at most finitely many times.

We have the following technical characterization of convergence in $G_\Lambda^{(0)}$ (cf. Remark 5.6 of [6] and page 653 of [17]).

PROPOSITION 3.12. *Let (Λ, d) be a compactly aligned topological k-graph, and let $\langle x_j \rangle_{j \in \mathbb{N}}$ and x be in $G_\Lambda^{(0)}$. Then*

$$\lim_{j \to \infty} x_j = x$$

if and only if the following two conditions hold:

(i) *for all $m \in \mathbb{N}^k$ with $m \leqslant d(x)$,*

$$\lim_{j \to \infty} x_j(0, m \wedge d(x_j)) = x(0, m);$$

(ii) *for all $i \in \{1, \ldots, k\}$ with $d(x)_i < \infty$ and for all $n \in \mathbb{N}^k$ with $n \leqslant d(x)$ and $n_i = d(x)_i$, if the set*

$$J(n,i) := \{j \in \mathbb{N} \mid d(x_j) \geqslant n + e_i\}$$

is infinite, then $\langle x_j(n, n+e_i) \rangle_{j \in J(n,i)}$ is wandering.

Proof. Assume $\lim_{j \to \infty} x_j = x$. For any $m \in \mathbb{N}^k$ with $m \leqslant d(x)$ and any open neighbourhood U of $x(0,m)$, we have $x \in Z(U)$. Hence $\langle x_j \rangle_{j \in \mathbb{N}}$ is eventually in $Z(U)$, it follows that $\langle x_j(0, m \wedge d(x_j)) \rangle_{j \in \mathbb{N}}$ is eventually in U, and Condition (i) holds.

To show that Condition (ii) holds, suppose $d(x) < \infty$ for some $i \in \{1, \ldots, k\}$, let $n \in \mathbb{N}^k$ satisfy $n \leqslant d(x)$ and $n_i = d(x)_i$, suppose $J(n,i) = \{j \in \mathbb{N} \mid d(x_j) \geqslant n + e_i\}$ is infinite, and let $W \subset \Lambda^{e_i}$ be compact; we show that $\langle x_j(n, n+e_i) \rangle_{j \in J(n,i)}$ is eventually in $\Lambda^{e_i} \setminus W$.

Let $U \subset \Lambda^n$ be a relatively compact open neighbourhood of $x(0,n)$. Then $x \in Z(U) \cap Z(\overline{U}W)^c$, and hence, eventually, so is $\langle x_j \rangle_{j \in \mathbb{N}}$. As $\langle x_j(0,n) \rangle_{j \in \mathbb{N}}$ is eventually in U, it follows that $\langle x_j(n, n+e_i) \rangle_{j \in J(n,i)}$ is eventually in $\Lambda^{e_i} \setminus W$, as required.

Conversely, assume Conditions (i) and (ii) hold. We argue by contradiction, so suppose there exist $m \in \mathbb{N}^k$, relatively compact open $U \subset \Lambda^m$ and compact $F \subset \Lambda$ such that

$$x \in Z(U) \cap Z(F)^c \tag{3.1}$$

and

$$x_j \notin Z(U) \cap Z(F)^c \quad \text{for infinitely many } j \in \mathbb{N}. \tag{3.2}$$

By Condition (i), we must have $x_j \in Z(U)$ eventually, so it follows that

$$x_j \in Z(F) \quad \text{for infinitely many } j \in \mathbb{N}. \tag{3.3}$$

Since F is compact, without loss of generality we assume $F \subset \Lambda^M$ for some $M \in \mathbb{N}^k$; by Lemma 3.8, we can assume $m \leqslant M$, retaining (3.1) through (3.3). (This is the only point in the proof which relies on (Λ, d) being compactly aligned.)

We claim that $M \nleqslant d(x)$; otherwise, (3.1) implies $x(0, M) \in \Lambda^M \setminus F$, which combines with (3.3) to contradict Condition (i). Thus $M \nleqslant d(x)$.

Define $N := M \wedge d(x)$. Then $N \leqslant d(x)$ and there exists $i \in \{1, \ldots, k\}$ such that $N_i = d(x)_i$ and $N + e_i \leqslant M$. We have that $J(N, i) = \{j \in \mathbb{N} \mid d(x_j) \geqslant N + e_i\}$ is infinite since it contains the infinite set $\{j \in \mathbb{N} \mid x_j \in Z(F)\}$. By (3.3), we also have $x_j(N, N + e_i)$ in the compact set $\mathrm{Seg}^M_{(N,N+e_i)}(F)$ for infinitely many $j \in J(N, i)$, which contradicts Condition (ii). Therefore $\langle x_j \rangle_{j \in \mathbb{N}}$ is eventually in $Z(U) \cap Z(F)^c$, and $\lim_{j \to \infty} x_j = x$. ∎

We deduce the following characterization of convergence in G_Λ.

PROPOSITION 3.13. *Let (Λ, d) be a compactly aligned topological k-graph, let $p, q \in \mathbb{N}^k$, and let the sequence $\langle (x_j, p - q, y_j) \rangle_{j \in \mathbb{N}}$ and point $(x, p - q, y)$ be contained in $Z(\Lambda^p *_s \Lambda^q, p - q)$. Then*

$$\lim_{j \to \infty} (x_j, p - q, y_j) = (x, p - q, y)$$

if and only if the following two conditions hold:

(i) *for all $m \in \mathbb{N}^k$,*

(a) $\lim_{j \to \infty} x_j(0, m \wedge d(x) \wedge d(x_j)) = x(0, m \wedge d(x))$ *and*

(b) $\lim_{j \to \infty} y_j(0, m \wedge d(y) \wedge d(y_j)) = y(0, m \wedge d(y))$;

(ii) *for all $i \in \{1, \ldots, k\}$ with $d(x)_i < \infty$ and for all $n \in \mathbb{N}^k$ with $p \leqslant n \leqslant d(x)$ and $n_i = d(x)_i$, if the set*

$$J(n, i) := \{j \in \mathbb{N} \mid d(x_j) \geqslant n + e_i\}$$

is infinite, then

$$\langle x_j(n, n + e_i) \rangle_{j \in J(n,i)}$$

is wandering.

REMARK 3.14. Condition (ii) of Proposition 3.13 is equivalent to the following condition stated in terms of y and the y_j:

(ii') *for all* $i \in \{1, \dots, k\}$ *with* $d(y)_i < \infty$ *and for all* $n \in \mathbb{N}^k$ *with* $q \leqslant n \leqslant d(y)$ *and* $n_i = d(y)_i$*, if the set*

$$J(n,i) = \{j \in \mathbb{N} \mid d(y_j) \geqslant n + e_i\}$$

is infinite, then

$$\langle y_j(n, n+e_i) \rangle_{j \in J(n,i)}$$

is wandering.

We now deduce that for compactly aligned (Λ, d), the groupoid G_Λ has a locally compact topology; recall from Remark 3.7 that if (Λ, d) fails to be compactly aligned, then the topology on G_Λ may not be locally compact.

PROPOSITION 3.15. *Let* (Λ, d) *be a compactly aligned topological k-graph. For* $p, q \in \mathbb{N}^k$ *and compact sets* $U \subset \Lambda^p$ *and* $V \subset \Lambda^q$*, the set* $Z(U *_s V, p-q)$ *is compact.*

Proof. Let $\langle (\lambda_j x_j, p-q, \mu_j x_j) \rangle_{j \in \mathbb{N}}$ be a sequence in $Z(U *_s V, p-q)$ with each $(\lambda_j, \mu_j) \in U *_s V$. Since $U *_s V$ is compact, there exists an infinite set $I_0 \subset \mathbb{N}$ such that $\langle (\lambda_j, \mu_j) \rangle_{j \in I_0}$ converges to some $(\lambda, \mu) \in U *_s V$. We construct an element $z \in X_\Lambda$ and an infinite set $I \subset I_0$ such that $\langle (\lambda_j x_j, p-q, \mu_j x_j) \rangle_{j \in I}$ converges to $(\lambda z, p-q, \mu z)$.

Define $f : \mathbb{N} \to \{1, \dots, k\}$ by $f(j) = j (\operatorname{mod} k) + 1$, and iteratively construct a sequence $\langle z_j \rangle_{j \in \mathbb{N}}$ in Λ as follows: First, set $z_0 := s(\lambda) = s(\mu)$. Let $l \in \mathbb{N}$, and suppose $z_0, \dots, z_l \in \Lambda$ and infinite sets $I_l \subset \cdots \subset I_0 \subset \mathbb{N}$ have been defined and satisfy:

$$s(z_i) = r(z_{i+1}) \quad \text{for all } 0 \leqslant i \leqslant l-1, \tag{3.4}$$

$$d(x_j) \geqslant d(z_0 \cdots z_l) \quad \text{for all } j \in I_l, \tag{3.5}$$

and

$$\lim_{j \in I_l} (\lambda_j x_j(0, d(z_0 \cdots z_l)), \mu_j x_j(0, d(z_0 \cdots z_l))) = (\lambda z_0 \cdots z_l, \mu z_0 \cdots z_l). \tag{3.6}$$

One of the following two properties must hold:

(1) There exists a compact set $W_{l+1} \subset \Lambda^{e_{f(l+1)}}$ and an infinite set $I'_{l+1} \subset I_l$ such that $d(x_j) \geqslant d(z_0 \cdots z_l) + e_{f(l+1)}$ for all $j \in I'_{l+1}$, and the sequence $\langle x_j(d(z_0 \cdots z_l), d(z_0 \cdots z_l) + e_{f(l+1)}) \rangle_{j \in I'_{l+1}}$ is contained in W_{l+1}.

(2) The set $J_{l+1} := \{j \in I_l \mid d(x_j) \geqslant d(z_0 \cdots z_l) + e_{f(l+1)}\}$ is either

(-) finite, or

(-) infinite and the sequence $\langle x_j(d(z_0 \cdots z_l), d(z_0 \cdots z_l) + e_{f(l+1)}) \rangle_{j \in J_{l+1}}$ is wandering.

If (1) holds, then fix compact W_{l+1} and infinite I'_{l+1} satisfying the conditions of (1); as W_{l+1} is compact, fix an infinite subset $I_{l+1} \subset I'_{l+1}$ such that the sequence

$\langle x_j(d(z_0\cdots z_l), d(z_0\cdots z_l)+e_{f(l+1)})\rangle_{j\in I_{l+1}}$ converges in W_{l+1}, and define

$$z_{l+1} := \lim_{j\in I_{l+1}} x_j(d(z_0\cdots z_l), d(z_0\cdots z_l)+e_{f(l+1)}).$$

If (2) holds, then set $z_{l+1} := s(z_l)$ and $I_{l+1} := I_l$.

There exists a unique path $z \in X_\Lambda$ such that

$$d(z) = \lim_{l\to\infty} d(z_0\cdots z_l) \quad \text{and} \quad z(0, d(z_0\cdots z_l)) = z_0\cdots z_l \quad \text{for all } l \in \mathbb{N}.$$

We also have for all $l \in \mathbb{N}$,

$$\lim_{j\in I_l}(\lambda_j x_j(0, d(z_0\cdots z_l)), \mu_j x_j(0, d(z_0\cdots z_l))) = (\lambda z_0\cdots z_l, \mu z_0\cdots z_l) \tag{3.7}$$

and

$$\begin{aligned}&\text{if } J_{l+1} := \{j \in I_l \mid d(x_j) \geqslant d(z_0\cdots z_l)+e_{f(l+1)}\} \text{ is infinite}\\ &\text{and } \langle x_j(d(z_0\cdots z_l), d(z_0\cdots z_l)+e_{f(l+1)})\rangle_{j\in J_{l+1}} \text{ is not wandering,}\\ &\text{then } z_{l+1} \in \Lambda^{e_{f(l+1)}}; \text{ otherwise } z_{l+1} = s(z_l).\end{aligned} \tag{3.8}$$

Define an infinite set $I = \{j_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ by choosing any $j_0 \in I_0$, and, after $j_0, \ldots, j_l$ have been set, choosing $j_{l+1} \in I_{l+1}$ such that $j_{l+1} \geqslant j_l$.

We claim that Conditions (i) and (ii) of Proposition 3.13 hold for the sequence $\langle(\lambda_j x_j, p-q, \mu_j x_j)\rangle_{j\in I}$ and point $(\lambda z, p-q, \mu z)$.

Condition (i) of Proposition 3.13 follows from (3.7). Suppose for contradiction that Condition (ii) of Proposition 3.13 does not hold, so there exist $i \in \{1, \ldots, k\}$ and $n \in \mathbb{N}^k$ such that $n \leqslant d(z)$, $n_i = d(z)_i$,

$$J(p+n, i) := \{j \in I \mid d(\lambda_j x_j) \geqslant p+n+e_i\}$$

is infinite, and the sequence $\langle(\lambda_j x_j)(p+n, p+n+e_i)\rangle_{j\in J(p+n,i)}$ is not wandering. Then there exists a compact set $V_1 \subset \Lambda^{e_i}$ such that V_1 contains infinitely many elements of $\langle(\lambda_j x_j)(p+n, p+n+e_i)\rangle_{j\in J(p+n,i)}$.

Let $L \in \mathbb{N}$ be the smallest number such that $d(z_0\cdots z_L) \geqslant n$ and $f(L+1) = i$, so $d(z)_i = n_i = d(z_0\cdots z_L)_i$. Let $V_2 \subset \Lambda^{d(z_0\cdots z_L)-n}$ be a compact neighbourhood of $z(n, d(z_0\cdots z_L))$. Then $\langle x_j(n, d(z_0\cdots z_L))\rangle_{j\in J(p+n,i)}$ is eventually in V_2 by Condition (i).

Since Λ is compactly aligned and $(d(z_0\cdots z_L)-n)_i = 0$, it follows that $V_1 \vee V_2$ is a compact subset of $\Lambda^{d(z_0\cdots z_L)-n+e_i}$. Furthermore, $V_1 \vee V_2$ contains infinitely many elements of $\langle x_j(n, d(z_0\cdots z_L)+e_i)\rangle_{j\in J(p+n,i)}$. Since $J(p+n,i)$ is contained in the set J_{L+1} of (3.8), we have that $\mathrm{Seg}^{d(z_0\cdots z_L)-n+e_i}_{(d(z_0\cdots z_L)-n, d(z_0\cdots z_L)-n+e_i)}(V_1 \vee V_2)$ is a compact subset of Λ^{e_i} which contains infinitely many elements of the sequence $\langle x_j(d(z_0\cdots z_L), d(z_0\cdots z_L)+e_i)\rangle_{j\in J_{L+1}}$. By (3.8), we then have $z_{L+1} \in \Lambda^{e_i}$, which contradicts $d(z_0\cdots z_L)_i = d(z)_i$. Therefore Condition (ii) of Proposition 3.13 holds, and Proposition 3.13 implies

$$\lim_{j\in I}(\lambda_j x_j, p-q, \mu_j x_j) = (\lambda z, p-q, \mu z),$$

completing the proof. ∎

THEOREM 3.16. *Let (Λ, d) be a compactly aligned topological k-graph. Then G_Λ is a locally compact r-discrete topological groupoid admitting a Haar system consisting of counting measures.*

Proof. Local compactness of G_Λ follows from Proposition 3.15. Straightforward applications of Proposition 3.13 give continuity of composition and inversion. Therefore G_Λ is a locally compact topological groupoid.

To show that G_Λ is r-discrete and admits a Haar system, by Proposition I.2.8 of [28] it suffices to show that $r : G_\Lambda \to G_\Lambda^{(0)}$ is a local homeomorphism. Fixing $(\lambda x, d(\lambda) - d(\mu), \mu x) \in G_\Lambda$ and choosing open neighbourhoods $U \subset \Lambda^{d(\lambda)}$ and $V \subset \Lambda^{d(\mu)}$ of λ and μ, respectively, such that $s|_U$ and $s|_V$ are homeomorphisms, one checks that $r|_{Z(U *_s V, d(\lambda) - d(\mu))}$ is a homeomorphism. Therefore G_Λ is r-discrete and admits a Haar system, and by Lemma I.2.7 of [28] we can choose the Haar system to comprise counting measures. ∎

EXAMPLES 3.17. (i) Let E be a discrete directed graph, and recall the construction of the topological 1-graph (E^*, l) from Example 2.5(ii). In [17], Paterson defines an inverse semigroup S_E^{Pat} and an action of S_E^{Pat} on the path space X_{E^*}. He then defines the topological groupoid H_E^{Pat} as the groupoid of germs of the action. Comparing G_{E^*} with the description of H_E^{Pat} given in Theorem 1 of [17] and comparing the topological structures of G_{E^*} and H_E^{Pat} given in Proposition 3.6 of this article and Proposition 3 of [17], respectively, we see that G_{E^*} and H_E^{Pat} are isomorphic as topological groupoids.

(ii) Given a finitely aligned discrete k-graph, the authors of [6] define and study an r-discrete groupoid G_Λ^{FMY} (see Section 6 of [6]). Comparing G_Λ with the description of G_Λ^{FMY} given in Remark 6.2 of [6] and comparing the topological structures of G_Λ and G_Λ^{FMY} given in Proposition 3.6 of this article and Remark 6.4 of [6], respectively, we see that G_Λ and G_Λ^{FMY} are isomorphic as topological groupoids.

4. THE BOUNDARY-PATH GROUPOID

Given a compactly aligned topological k-graph (Λ, d), we now identify a closed invariant subset $\partial\Lambda$ of $X_\Lambda = G_\Lambda^{(0)}$, and define our boundary-path groupoid $\mathcal{G}_\Lambda$ as the reduction $G_\Lambda|_{\partial\Lambda}$.

DEFINITION 4.1. Let (Λ, d) be a topological k-graph and let $V \subset \Lambda^0$. A set $E \subset V\Lambda$ is *exhaustive* for V if for all $\lambda \in V\Lambda$ there exists $\mu \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. For $v \in \Lambda^0$, let $v\mathcal{CE}(\Lambda)$ denote the set of all compact sets $E \subset \Lambda$ such that $r(E)$ is a neighbourhood of v and E is exhaustive for $r(E)$.

DEFINITION 4.2. Let (Λ, d) be a topological k-graph. A path $x \in X_\Lambda$ is called a *boundary path* if for all $m \in \mathbb{N}^k$ with $m \leqslant d(x)$, and for all $E \in x(m)\mathcal{CE}(\Lambda)$, there exists $\lambda \in E$ such that $x(m, m + d(\lambda)) = \lambda$. We write $\partial\Lambda$ for the set of all boundary paths in X_Λ. For $v \in \Lambda^0$ and $V \subset \Lambda^0$, we define $v\partial\Lambda = \{x \in \partial\Lambda \mid r(x) = v\}$ and $V(\partial\Lambda) = \{x \in \partial\Lambda \mid r(x) \in V\}$.

PROPOSITION 4.3. *Let (Λ, d) be a topological k-graph. Then $v\partial\Lambda$ is nonempty for all $v \in \Lambda^0$.*

Proof. We construct a path $x \in v\partial\Lambda$. Define $f : \mathbb{N} \to \{1, \ldots, k\}$ by $f(j) := j (\operatorname{mod} k) + 1$. If $v\Lambda^{e_{f(1)}} \neq \emptyset$, then choose $\lambda_1 \in v\Lambda^{e_{f(1)}}$, otherwise set $\lambda_1 := v$. Once $\lambda_1, \ldots, \lambda_l$ have been defined, choose $\lambda_{l+1} \in s(\lambda_l)\Lambda^{e_{f(l+1)}}$ if $s(\lambda_l)\Lambda^{e_{f(l+1)}} \neq \emptyset$, otherwise set $\lambda_{l+1} := s(\lambda_l)$. There exists $x \in X_\Lambda$ such that $d(x) = \lim_{j \in \mathbb{N}} d(\lambda_1 \cdots \lambda_j)$ and $x(0, d(\lambda_1 \cdots \lambda_j)) = \lambda_1 \cdots \lambda_j$ for all $j \in \mathbb{N}$.

To show $x \in \partial\Lambda$, let $m \in \mathbb{N}^k$ satisfy $m \leqslant d(x)$ and let $E \in x(m)\mathcal{CE}(\Lambda)$. Since E is exhaustive, for each $n \in \mathbb{N}^k$ with $m \leqslant n \leqslant d(x)$, there exists $\mu_n \in E$ such that $\Lambda^{\min}(x(m, n), \mu_n) \neq \emptyset$. We will show there exists $N \in \mathbb{N}^k$ such that $d(\mu_N) \leqslant N - m$; for this N we then have $x(m, m + d(\mu_N)) = \mu_N$, as required.

Since E is compact, it follows that $\{d(\mu_n) \mid m \leqslant n \leqslant d(x)\}$ is finite. If $d(x)_i = \infty$ for all $i \in \{1, \ldots, k\}$, then choosing $N := \bigvee\{m + d(\mu_n) \mid m \leqslant n \leqslant d(x)\}$ will do. So suppose there exists $i \in \{1, \ldots, k\}$ such that $d(x)_i < \infty$.

Define $I := \{i \in \{1, \ldots, k\} \mid d(x)_i < \infty\}$ and

$$p := \left(\bigvee\{m + d(\mu_n) \mid m \leqslant n \leqslant d(x)\}\right) \wedge d(x). \tag{4.1}$$

For each $i \in I$, let $l_i \in \mathbb{N}$ be the smallest number such that $f(l_i) = i$ and $s(\lambda_{l_{i-1}})\Lambda^{e_i} = \emptyset$, so $s(\lambda_j)\Lambda^{e_i} = \emptyset$ for all $j \geqslant l_i$. Let $L \in \mathbb{N}$ be the smallest number such that

$$L \geqslant \max_{i \in I} l_i \quad \text{and} \quad d(\lambda_1 \cdots \lambda_L) \geqslant p. \tag{4.2}$$

Define $N := d(\lambda_1 \cdots \lambda_L)$. Suppose for contradiction that $d(\mu_N) \not\leqslant N - m$, so there exists $i \in \{1, \ldots, k\}$ such that $d(\mu_N)_i > (N - m)_i$. Then (4.1) and (4.2) imply $N_i = d(x)_i$. Thus $d(x)_i < \infty$, and it follows from (4.2) that $s(\lambda_L)\Lambda^{e_i} = \emptyset$. But $\Lambda^{\min}(x(m, N), \mu_N) \neq \emptyset$ and for any $(\alpha, \beta) \in \Lambda^{\min}(x(m, N), \mu_N)$ we must have $d(\alpha)_i > 0$, contradicting $s(\lambda_L)\Lambda^{e_i} = \emptyset$. Therefore $d(\mu_N) \leqslant N - m$, so $x(m, m + d(\mu_N)) = \mu_N$ and $x \in v\partial\Lambda$. ∎

PROPOSITION 4.4. *Let (Λ, d) be a topological k-graph. Then $\partial\Lambda$ is closed in $G_\Lambda^{(0)}$.*

Proof. Let $\langle x_j \rangle_{j \in \mathbb{N}}$ be a sequence in $\partial\Lambda$ converging to some $x \in X_\Lambda$. Suppose for contradiction that $x \notin \partial\Lambda$, so there exists $m \in \mathbb{N}^k$, $m \leqslant d(x)$, and $E \in x(m)\mathcal{CE}(\Lambda)$ such that $x(m, p) \notin E$ for all $p \in \mathbb{N}^k$ with $m \leqslant p \leqslant d(x)$.

Let $U \subset \Lambda^m$ be a relatively compact open neighbourhood of $x(0, m)$ such that $s(U) \subset r(E)$. Then $x \in Z(U) \cap Z(\overline{U}E)^c$, so there exists $J \in \mathbb{N}$ such that $x_j \in Z(U) \cap Z(\overline{U}E)^c$ whenever $j \geqslant J$. But then for $j \geqslant J$ and $p \in \mathbb{N}^k$ with

$m \leqslant p \leqslant d(x_j)$, we have $x_j(0,p) \notin \overline{U}E$, which implies $x_j(m,p) \notin E$, contradicting $x_j \in \partial\Lambda$ and $E \in x_j(m)\mathcal{CE}(\Lambda)$. Hence $x \in \partial\Lambda$, and $\partial\Lambda$ is closed. ∎

To prove $\partial\Lambda$ is an invariant subset of $G_\Lambda^{(0)}$, we first need a definition and a lemma.

DEFINITION 4.5. Let (Λ, d) be a topological k-graph. For $E, F \subset \Lambda$, define the *minimal extenders of E by F* to be the set

$$\operatorname{Ext}(E;F) = \bigcup_{\lambda\in E}\bigcup_{\mu\in F}\{\alpha \in \Lambda \mid (\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu) \text{ for some } \beta \in \Lambda\}.$$

If E is a singleton set $E = \{\lambda\}$, we write $\operatorname{Ext}(\lambda;F)$ for $\operatorname{Ext}(\{\lambda\};F)$.

The proof of the following lemma is straightforward.

LEMMA 4.6 (cf. Lemma C.5 of [26]). *Let (Λ, d) be a compactly aligned topological k-graph, let $v \in \Lambda^0$ and $\lambda \in v\Lambda$, and suppose $E \in v\mathcal{CE}(\Lambda)$. Then for any compact neighbourhood $U \subset \Lambda^{d(\lambda)}$ of λ, $\operatorname{Ext}(U;E) \in s(\lambda)\mathcal{CE}(\Lambda)$.*

PROPOSITION 4.7. *Let (Λ, d) be a compactly aligned topological k-graph. For $x \in \partial\Lambda$, $m \in \mathbb{N}^k$ with $m \leqslant d(x)$, and $\lambda \in \Lambda r(x)$, we have $\sigma^m x, \lambda x \in \partial\Lambda$. Hence $\partial\Lambda$ is an invariant subset of $G_\Lambda^{(0)}$.*

Proof. To see that $\sigma^m x \in \partial\Lambda$, let $n \in \mathbb{N}^k$ satisfy $n \leqslant d(\sigma^m x)$, and let $E \in (\sigma^m x)(n)\mathcal{CE}(\Lambda)$. Then $E \in x(m+n)\mathcal{CE}(\Lambda)$, so there exists $\mu \in E$ such that $(\sigma^m x)(n, n+d(\mu)) = x(m+n, m+n+d(\mu)) = \mu$, as required.

Now let $n \in \mathbb{N}^k$ satisfy $n \leqslant d(\lambda x)$, and let $E \in (\lambda x)(n)\mathcal{CE}(\Lambda)$. Define $\xi := x(n, n \vee d(\lambda))$ and let $U \subset \Lambda^{d(\xi)}$ be a relatively compact open neighbourhood of ξ such that $s|_U$ is a homeomorphism. Then by Lemma 4.6 we have

$$\operatorname{Ext}(\overline{U};E) \in (\lambda x)(n \vee d(\lambda))\mathcal{CE}(\Lambda) = x((n \vee d(\lambda)) - d(\lambda))\mathcal{CE}(\Lambda),$$

so there exists $\alpha \in \operatorname{Ext}(\overline{U};E)$ such that

$$x((n \vee d(\lambda)) - d(\lambda), (n \vee d(\lambda)) - d(\lambda) + d(\alpha)) = \alpha.$$

Since $s(\xi) = r(\alpha)$ and $s|_U$ is a homeomorphism, we have $\xi\alpha = \mu\beta$ for some $\mu \in E$ and $\beta \in \Lambda$. Hence $(\lambda x)(n, n+d(\mu)) = \mu$, giving $\lambda x \in \partial\Lambda$. ∎

DEFINITION 4.8. Let (Λ, d) be a compactly aligned topological k-graph. The Propositions 4.3, 4.4 and 4.7 imply that $\partial\Lambda$ is a nonempty closed invariant subset of $G_\Lambda^{(0)}$, and we define the *boundary-path groupoid* $\mathcal{G}_\Lambda$ to be the reduction $\mathcal{G}_\Lambda := G_\Lambda|_{\partial\Lambda}$. Then $\mathcal{G}_\Lambda$ is a locally compact r-discrete groupoid admitting a Haar system consisting of counting measures.

NOTATION 4.9. To distinguish basis sets of $\mathcal{G}_\Lambda$ from those of G_Λ, for $F \subset \Lambda *_s \Lambda$, $m \in \mathbb{Z}^k$ and $U \subset \Lambda$, define

$$\mathcal{Z}(F,m) = Z(F,m) \cap \mathcal{G}_\Lambda$$

and

$$\mathcal{Z}(U) = Z(U) \cap \mathcal{G}_\Lambda = \mathcal{Z}(U *_s U, 0) \cap \mathcal{Z}(\Lambda^0 *_s \Lambda^0, 0).$$

EXAMPLES 4.10. (i) Recalling Example 3.17(i), for a discrete directed graph E, Paterson [17] defines an r-discrete groupoid H_E^{Pat}, and we saw that G_{E^*} and H_E^{Pat} are isomorphic. Paterson then identifies a closed invariant subset X^{Pat} of $(H_E^{\mathrm{Pat}})^{(0)}$ (see paragraph preceding Proposition 5 of [17]) and studies the reduction of H_E^{Pat} by X^{Pat} (see Theorem 2 of [17]). In the setting of directed graphs, it is straightforward to see that our boundary paths are precisely the elements of Paterson's set X^{Pat}, so it follows that $\mathcal{G}_{E^*}$ and $H_E^{\mathrm{Pat}}|_{X^{\mathrm{Pat}}}$ are isomorphic as topological groupoids.

(ii) Let (Λ, d) be a finitely aligned discrete k-graph. As discussed in Example 3.17(ii), the authors of [6] associate a topological groupoid G_Λ^{FMY} to Λ, and this groupoid is isomorphic to our path groupoid G_Λ. In [6], the authors identify a set of boundary paths $\partial\Lambda$ of Λ as a closed invariant subset of $(G_\Lambda^{\mathrm{FMY}})^{(0)}$ and study the reduction $G_\Lambda^{\mathrm{FMY}}|_{\partial\Lambda}$. Since our definition of a boundary path corresponds to Definition 5.10 of [6] in the setting of finitely aligned discrete k-graphs, it follows that $\mathcal{G}_\Lambda$ and $G_\Lambda^{\mathrm{FMY}}|_{\partial\Lambda}$ are isomorphic as topological groupoids.

(iii) Given a singly generated dynamical system (X, θ), we formed a topological 1-graph $(\Lambda(X,\theta), d)$ (see Example 2.5(iii)). In [29], Renault defines a topological groupoid $G(X,\theta) \subset X \times \mathbb{Z} \times X$ by setting

$$G(X,\theta) = \{(x, m-n, y) \mid x \in \mathrm{dom}(\theta^m), y \in \mathrm{dom}(\theta^n), \theta^m(x) = \theta^n(y)\},$$

with the usual groupoid structure, and defining basis sets

$$\mathcal{U}(U; m, n; V) = \{(x, m-n, y) \mid (x,y) \in U \times V, \theta^m(x) = \theta^n(y)\}$$

where $U \subset \mathrm{dom}(\theta^m)$, $V \subset \mathrm{dom}(\theta^n)$ are open sets on which, respectively, θ^m and θ^n are injective (see Section 2 of [29] for details; see also [4] for the same construction with X compact and θ surjective).

The boundary paths of $\Lambda(X,\theta)$ can be identified with X, and under this identification, σ is intertwined with θ. One can then show that the groupoid and topological structures on $\mathcal{G}_{\Lambda(X,\theta)}$ and $G(X,\theta)$ are equivalent, hence the two topological groupoids are isomorphic.

5. APERIODICITY IN TOPOLOGICAL HIGHER-RANK GRAPHS AND ESSENTIAL FREENESS OF BOUNDARY-PATH GROUPOIDS

In this section we consider an analogue of the Aperiodicity Condition used in [12] and [6]. Using the condition, we extend Proposition 4.5 of [12] and Proposition 7.2 of [6] to our setting.

DEFINITION 5.1. Let (Λ, d) be a topological k-graph. A boundary path $x \in \partial\Lambda$ is *aperiodic* if

$$\text{for all } p, q \in \mathbb{N}^k \text{ with } p, q \leqslant d(x),\ p \neq q \text{ implies } \sigma^p x \neq \sigma^q x. \tag{5.1}$$

Recall that a topological groupoid Γ is *essentially free* if the set of units with trivial isotropy is dense in $\Gamma^{(0)}$; that is, $\overline{\{x \in \Gamma^{(0)} \mid x\Gamma x = \{x\}\}} = \Gamma^{(0)}$.

THEOREM 5.2. *Let (Λ, d) be a compactly aligned topological k-graph. Then $\mathcal{G}_\Lambda$ is essentially free if and only if*

(A) *for all nonempty open $V \subset \Lambda^0$, there exists an aperiodic path $x \in V(\partial\Lambda)$.*

To prove the theorem, we need the following two lemmas.

LEMMA 5.3. *Let (Λ, d) be a compactly aligned topological k-graph. A boundary-path $x \in \partial\Lambda$ is aperiodic if and only if its associated isotropy group in $\mathcal{G}_\Lambda$ is trivial.*

Proof. The lemma follows from the equivalence: For $m \in \mathbb{Z}^k$, the triple (x, m, x) is an element of $\mathcal{G}_\Lambda$ if and only if there exist $p, q \in \mathbb{N}^k$ such that $p, q \leqslant d(x)$, $p - q = m$ and $\sigma^p x = \sigma^q x$. ∎

LEMMA 5.4. *Let (Λ, d) be a topological k-graph. For any aperiodic $x \in \partial\Lambda$ and $\lambda \in \Lambda r(x)$, λx is aperiodic.*

Proof. Arguing by contrapositive, suppose that λx is not aperiodic, so there exists $p, q \in \mathbb{N}^k$ such that $p, q \leqslant d(\lambda x)$, $p \neq q$ and $\sigma^p(\lambda x) \neq \sigma^q(\lambda x)$. It follows that

$$(d(\lambda) + p) \wedge d(\lambda x) \neq (d(\lambda) + q) \wedge d(\lambda x)$$

and

$$\sigma^{(d(\lambda)+p)\wedge d(\lambda x)}(\lambda x) = \sigma^{(d(\lambda)+q)\wedge d(\lambda x)}(\lambda x).$$

Thus we have $p \wedge d(x) \neq q \wedge d(x)$ and

$$\sigma^{p\wedge d(x)} x = \sigma^{(d(\lambda)+p)\wedge d(\lambda x)}(\lambda x) = \sigma^{(d(\lambda)+q)\wedge d(\lambda x)}(\lambda x) = \sigma^{q\wedge d(x)} x,$$

proving x is not aperiodic. ∎

Proof of Theorem 5.2. First assume that $\mathcal{G}_\Lambda$ is essentially free, and let $V \subset \Lambda^0$ be nonempty and open. By Lemma 4.3, $V(\partial\Lambda)$ is nonempty, so the open set $Z(V)$ is nonempty in $\mathcal{G}_\Lambda^{(0)}$. Therefore, there exists $x \in Z(V)$ with trivial isotropy, and Lemma 5.3 implies that $x \in V(\partial\Lambda)$ is aperiodic. Hence (Λ, d) satisfies Condition (A).

Conversely, assume that Condition (A) holds. Fix $x \in \mathcal{G}_\Lambda^{(0)}$ and let $\mathcal{Z}(U) \cap \mathcal{Z}(F)^c$ be a basis set containing x. There exists $\lambda \in U$ such that $x(0, d(\lambda)) = \lambda$; we can assume $U \subset \Lambda^{d(\lambda)}$, U is relatively compact and open, $s|_U$ is a homeomorphism, and, by Lemma 3.8, every $\mu \in F$ has the form $\mu = \xi\mu'$ for some $\xi \in \overline{U}$.

The set $\{d(\mu) \mid \mu \in F\}$ is finite, and $d(\lambda) \leqslant d(\mu)$ for all $\mu \in F$, so we define the compact set

$$E := \bigcup_{m \in \{d(\mu) \mid \mu \in F\}} \mathrm{Seg}^m_{(d(\lambda),m)}(F \cap \Lambda^m).$$

We know that $(s|_{\overline{U}})^{-1}(s(\lambda)) = \{\lambda\}$, so for $\nu \in E$, if $r(\nu) = s(\lambda)$, then $\lambda\nu \in F$. It follows that if there exists $\nu \in E$ such that $x(d(\lambda), d(\lambda) + d(\nu)) = \nu$, then $x(0, d(\lambda\nu)) \in F$, which contradicts $x \in \mathcal{Z}(F)^c$. Since x is a boundary path, we deduce that $E \notin s(\lambda)\mathcal{CE}(\Lambda)$.

Now, E must fail to be an element of $s(\lambda)\mathcal{CE}(\Lambda)$ on account of one of two reasons: either $r(E)$ is not a neighbourhood of $s(\lambda)$, or E is not exhaustive for $r(E)$. In either case, there exists $\eta \in s(U)\Lambda$ such that $\Lambda^{\min}(\eta, \nu) = \emptyset$ for all $\nu \in E$.

We claim there exists a neighbourhood $W \subset \Lambda^{d(\eta)}$ of η with $\Lambda^{\min}(\eta', \nu) = \emptyset$ for all $\eta' \in W$ and $\nu \in E$: Suppose for contradiction that there exist sequences $\langle \eta_j \rangle_{j \in \mathbb{N}} \subset \Lambda^{d(\eta)}$ and $\langle \nu_j \rangle_{j \in \mathbb{N}} \subset E$ such that $\lim_{j \in \mathbb{N}} \eta_j = \eta$ and $\Lambda^{\min}(\eta_j, \nu_j) \neq \emptyset$ for each $j \in \mathbb{N}$. Since E is compact, we can assume $\langle \nu_j \rangle_{j \in \mathbb{N}}$ converges to some $\nu \in E$.

Let $\langle (\alpha_j, \beta_j) \rangle_{j \in \mathbb{N}}$ be a sequence such that $(\alpha_j, \beta_j) \in \Lambda^{\min}(\eta_j, \nu_j)$ for each $j \in \mathbb{N}$. We can assume $d(\nu_j) = d(\nu)$ for all $j \in \mathbb{N}$, and hence $d(\alpha_j) = (d(\eta) \vee d(\nu)) - d(\eta)$ and $d(\beta_j) = (d(\eta) \vee d(\nu)) - d(\nu)$ for all $j \in \mathbb{N}$.

Taking a compact neighbourhood $Y \subset \Lambda^{d(\eta)}$ of η, we can assume $\langle \eta_j \rangle_{j \in \mathbb{N}} \subset Y$. Since Λ is compactly aligned, it follows that $Y \vee E$ is compact and contains the sequence $\langle \eta_j \alpha_j \rangle_{j \in \mathbb{N}} = \langle \nu_j \beta_j \rangle_{j \in \mathbb{N}}$. Hence there exists a convergent subsequence $\langle \eta_j \alpha_j \rangle_{j \in J}$ contained in $\Lambda^{d(\eta) \vee d(\nu)}$. We then have $\lim_{j \in J} \eta_j \alpha_j = \eta\alpha$ and $\lim_{j \in J} \nu_j \beta_j = \nu\beta$ for some $\alpha, \beta \in \Lambda$. It follows that $(\alpha, \beta) \in \Lambda^{\min}(\eta, \nu)$; a contradiction. Therefore there exists a neighbourhood $W \subset \Lambda^{d(\eta)}$ of η such that $\Lambda^{\min}(\eta', \nu) = \emptyset$ for all $\eta' \in W$ and $\nu \in E$. We can assume that $r(W) \subset s(U)$.

Condition (A) gives an aperiodic path $z \in s(W)(\partial\Lambda)$. Since $r(W) \subset s(U)$, there exist $\lambda' \in U$ and $\eta' \in W$ such that $s(\lambda') = r(\eta')$ and $s(\eta') = r(z)$. Proposition 4.7 and Lemma 5.4 imply $\lambda'\eta' z$ is an aperiodic boundary path. Taking into account that $\Lambda^{\min}(\eta', \nu) = \emptyset$ for all $\nu \in E$, it follows that $\lambda'\eta' z \in \mathcal{Z}(U) \cap \mathcal{Z}(F)^c$. Lemma 5.3 now gives the result. ∎

6. AMENABILITY OF THE BOUNDARY-PATH GROUPOID

In this section we prove amenability of the boundary-path groupoid under certain conditions on the topological k-graph. Rather than detail the characterizations of groupoid amenability here, we refer the reader to Chapter 2 of [1].

Recall that a locally compact groupoid Γ is *proper* if $(r, s) : \Gamma \to \Gamma^{(0)} \times \Gamma^{(0)}$, defined by $\gamma \mapsto (r(\gamma), s(\gamma))$, is a proper mapping; that is, if the inverse image of any compact set from $\Gamma^{(0)} \times \Gamma^{(0)}$ is compact.

The next proposition is standard in groupoid theory; its proof involves, for the most part, tracing through definitions.

PROPOSITION 6.1. *Let Γ be a locally compact proper groupoid admitting a Haar system. Then Γ is amenable in the sense of Definition* 2.2.7 *of* [1].

PROPOSITION 6.2. *Let (Λ, d) be a topological k-graph. If either*
(i) $k = 1$, *or*
(ii) (Λ, d) *is a finitely aligned discrete k-graph,*
then $\mathcal{G}_\Lambda$ is amenable.

Proof. Suppose $k = 1$. Then Theorem 5.2 of [31] implies $C^*(\mathcal{G}_\Lambda)$ is isomorphic to the Cuntz-Pimsner algebra $\mathcal{O}(E_\Lambda)$ (see Section 5 of [31]). Thus, Proposition 6.1 of [10] implies $C^*(\mathcal{G}_\Lambda)$ is nuclear, and since $\mathcal{G}_\Lambda$ is r-discrete, Corollary 6.2.14 of [1] and Theorem 3.3.7 of [1] imply $\mathcal{G}_\Lambda$ is amenable.

Now suppose (Λ, d) is a finitely aligned discrete k-graph. By Theorem 6.13 of [6], $C^*(\mathcal{G}_\Lambda) \cong C^*(\Lambda)$, where $C^*(\Lambda)$ is defined in Remark 3.9 of [6]. We then deduce $C^*(\mathcal{G}_\Lambda)$ is nuclear by Proposition 8.1 of [30]. Since $\mathcal{G}_\Lambda$ is r-discrete, Corollary 6.2.14 of [1] and Theorem 3.3.7 of [1] imply $\mathcal{G}_\Lambda$ is amenable. ∎

DEFINITION 6.3. Let (Λ, d) be a topological k-graph and let $v \in \Lambda^0$. Then v is said to be a *source* if $v\Lambda^{e_i} = \emptyset$ for some $i \in \{1, \ldots, k\}$, and v is said to be a *sink* if $\Lambda^{e_i} v = \emptyset$ for some $i \in \{1, \ldots, k\}$.

The following definition generalizes the row-finite condition on discrete higher-rank graphs.

DEFINITION 6.4. A topological k-graph is *proper* if, for all $m \in \mathbb{N}^k$, $r|_{\Lambda^m}$ is a proper map; that is, if, for all $m \in \mathbb{N}^k$ and compact $U \subset \Lambda^0$, $U\Lambda^m$ is compact.

REMARK 6.5. It's straightforward to see that any proper topological k-graph is compactly aligned.

LEMMA 6.6. *Let (Λ, d) be a proper topological k-graph without sources. Then $d(x) = (\infty, \ldots, \infty)$ for all $x \in \partial\Lambda$.*

Proof. For any $v \in \Lambda^0$, compact neighbourhood $U \subset \Lambda^0$ of v and $i \in \{1, \ldots, k\}$, we have $U\Lambda^{e_i} \in v\mathcal{CE}(\Lambda)$. Therefore, given $x \in \partial\Lambda$, $m \leqslant d(x)$ and $i \in \{1, \ldots, k\}$, it follows that $x(m, m + e_i) \in x(m)\Lambda^{e_i}$, which can only occur if $d(x) = (\infty, \ldots, \infty)$. ∎

LEMMA 6.7. *Let (Λ, d) be a proper topological k-graph. The groupoid $H_\Lambda = \mathcal{Z}(\Lambda *_s \Lambda, 0)$, comprising all elements in $\mathcal{G}_\Lambda$ of the form $(x, 0, y)$, is amenable.*

Proof. For $m \in \mathbb{N}^k$, define

$$H(m) := \bigcup_{n \leqslant m} \mathcal{Z}(\Lambda^n *_s \Lambda^n, 0).$$

Then each $H(m)$ is a subgroupoid of $\mathcal{G}_\Lambda$, each has $\mathcal{G}_\Lambda^{(0)}$ as its unit space, and for $m \leqslant n$, $H(m)$ is both an open and closed subgroupoid of $H(n)$. Hence we have a direct system of groupoids $\{H(m) \mid m \in \mathbb{N}^k\}$ with direct-limit groupoid $H_\Lambda := \bigcup_{m \in \mathbb{N}^k} H(m) = \mathcal{Z}(\Lambda *_s \Lambda, 0)$.

We claim that for each $m \in \mathbb{N}^k$, $H(m)$ is a proper groupoid: To see this, let $W \subset \mathcal{G}_\Lambda^{(0)} \times \mathcal{G}_\Lambda^{(0)}$ be compact. There exist compact sets $U_i, V_j \subset \Lambda^0$, for $i = 1, \ldots, l_1$ and $j = 1, \ldots, l_2$, such that $W \subset \bigcup_{i=1}^{l_1} \bigcup_{j=1}^{l_2} \mathcal{Z}(U_i) \times \mathcal{Z}(V_j)$. We then have

$$
\begin{aligned}
(6.1) \qquad (r,s)^{-1}\Big(\bigcup_{i=1}^{l_1}\bigcup_{j=1}^{l_2} \mathcal{Z}(U_i) \times \mathcal{Z}(V_j)\Big)& \\
&= \Big\{(x,0,y) \in H(m) \mid x \in \bigcup_{i=1}^{l_1} \mathcal{Z}(U_i), y \in \bigcup_{j=1}^{l_2} \mathcal{Z}(V_j)\Big\} \\
&= \bigcup_{n \leqslant m}\bigcup_{i=1}^{l_1}\bigcup_{j=1}^{l_2} \mathcal{Z}(U_i\Lambda^n *_s V_j\Lambda^n, 0),
\end{aligned}
$$

which is compact since the $U_i\Lambda^n, V_j\Lambda^n$ are compact on account of Λ being proper. Since $(r,s)^{-1}(W)$ is a closed subset of (6.1), it follows that $(r,s)^{-1}(W)$ is compact. Thus each $H(m)$ is a proper groupoid, and Proposition 6.1 implies each $H(m)$ is amenable. By Proposition 5.3.37 of [1], the direct limit H_Λ is amenable. ∎

THEOREM 6.8. *Let (Λ, d) be a proper topological k-graph without sources. Then $\mathcal{G}_\Lambda$ is amenable.*

Proof. Let $c : \mathcal{G}_\Lambda \to \mathbb{Z}^k$ be the continuous functor given by $c(x,m,y) = m$. We will show that the skew-product groupoid $\mathcal{G}_\Lambda(c)$ is amenable; the result will then follow from Proposition II.3.8 of [28].

We identify the unit space $\mathcal{G}_\Lambda(c)^{(0)}$ with $\mathcal{G}_\Lambda^{(0)} \times \mathbb{Z}^k$, and for each $m \in \mathbb{Z}^k$, define $U_m := \mathcal{G}_\Lambda^{(0)} \times \{m\}$. Each $\mathcal{G}_\Lambda(c)|_{U_m}$ is isomorphic to H_Λ, so by Lemma 6.7, each $\mathcal{G}_\Lambda(c)|_{U_m}$ is amenable.

For $m \in \mathbb{Z}^k$, define $H_m := \mathcal{G}_\Lambda(c)|_{[U_m]}$, where $[U_m] := s(r^{-1}(U_m))$ is the saturation of U_m. By Example 2.7 of [15], $\mathcal{G}_\Lambda(c)|_{U_m}$ is equivalent to H_m, so it follows from Theorem 2.2.17 of [1] that each H_m is amenable. We also have $[U_m] \subset [U_n]$ whenever $m \leqslant n$, so, defining a cofinal sequence $\langle m_j \rangle_{j \in \mathbb{N}}$ in $\mathbb{N}^k$, we have $\bigcup_{j \in \mathbb{N}} [U_{m_j}] = \mathcal{G}_\Lambda(c)^{(0)}$.

For each $j \in \mathbb{N}$, amenability gives $C^*(H_{m_j}) = C^*_{\mathrm{red}}(H_{m_j})$, and since $[U_{m_j}]$ is open in $\mathcal{G}_\Lambda(c)^{(0)}$, there exists a homomorphism $\pi_j : C^*(H_{m_j}) \to C^*_{\mathrm{red}}(\mathcal{G}_\Lambda(c))$ defined by the inclusion $C_c(H_{m_j}) \to C_c(\mathcal{G}_\Lambda(c))$. Furthermore, amenability implies that each $C^*(H_{m_j})$ is nuclear, so it follows that each image $\pi_j(C^*(H_{m_j}))$ is nuclear.

We also have $\pi_j(C^*(H_{m_j})) \subset \pi_j(C^*(H_{m_{j+1}}))$ for each $j \in \mathbb{N}$, and

$$\overline{\bigcup_{j\in\mathbb{N}} \pi_j(H_{m_j})} = C^*_{\mathrm{red}}(\mathcal{G}_\Lambda(c)).$$

Therefore Theorem 2.3.9 of [14] implies $C^*_{\mathrm{red}}(\mathcal{G}_\Lambda(c))$ is nuclear, and it follows from Corollary 6.2.14(ii) of [1] and Theorem 3.3.7 of [1] that $\mathcal{G}_\Lambda(c)$ is amenable. Since $\mathbb{Z}^k$ is amenable, it follows from Proposition II.3.8 of [28] that $\mathcal{G}_\Lambda$ is amenable. ∎

7. C^*-ALGEBRAS OF TOPOLOGICAL HIGHER-RANK GRAPHS

EXAMPLES 7.1. (i) Let (Λ, d) be a finitely aligned discrete k-graph. It follows from Examples 3.17(ii) and 4.10(ii) together with Theorem 6.9 of [6] and Theorem 6.13 of [6] that $C^*(G_\Lambda) \cong \mathcal{T}C^*(\Lambda)$ and $C^*(\mathcal{G}_\Lambda) \cong C^*(\Lambda)$, where $\mathcal{T}C^*(\Lambda)$ and $C^*(\Lambda)$ are defined and studied in [24] and [26], respectively. This example includes the Toeplitz and Cuntz-Krieger algebras of arbitrary directed graphs as studied in [8], [3], [7], [17], [27], [5], [23] (among others).

(ii) As we saw in Example 2.5(ii), there is a one-to-one correspondence between topological 1-graphs and second-countable topological graphs. Given a topological 1-graph (Λ, d) with corresponding topological graph E_Λ, Theorem 5.1 of [31] and Theorem 5.2 of [31] say that $C^*(G_\Lambda) \cong \mathcal{T}(E_\Lambda)$ and $C^*(\mathcal{G}_\Lambda) \cong \mathcal{O}(E_\Lambda)$, where $\mathcal{T}(E_\Lambda)$ and $\mathcal{O}(E_\Lambda)$ are, respectively, the Toeplitz and Cuntz-Krieger algebras of the topological graph E_Λ, as defined in [10].

(iii) Consider a second-countable locally compact Hausdorff space X and a family $\{\theta_i\}_{i=1}^k$ of commuting homeomorphisms of X onto itself. There is an induced action Θ of $\mathbb{Z}^k$ on $C_0(X)$ defined by

$$\Theta_m(f)(x) = f(\theta_1^{m_1} \cdots \theta_k^{m_k}(x)),$$

with universal crossed product $(C_0(X) \times_\Theta \mathbb{Z}^k, j_{C_0(X)}, j_{\mathbb{Z}^k})$.

Recall the topological k-graph $(\Lambda(X, \{\theta\}_{i=1}^k), d)$ defined in Example 2.5(iv). We have $C^*(\mathcal{G}_{\Lambda(X,\{\theta_i\})}) \cong C_0(X) \times_\Theta \mathbb{Z}^k$; there are a number of ways to see this: for example, one can show that $\mathcal{G}_{\Lambda(X,\{\theta_i\})}$ is isomorphic to the transformation groupoid $X \times \mathbb{Z}^k$ whose C^*-algebra is isomorphic to the crossed-product C^*-algebra.

The above examples allow us to coherently make the following definition.

DEFINITION 7.2. Let (Λ, d) be a compactly aligned topological k-graph. We define the *Toeplitz algebra of* Λ to be the full groupoid C^*-algebra $C^*(G_\Lambda)$, and we define the *Cuntz-Krieger algebra of* Λ to be the full groupoid C^*-algebra $C^*(\mathcal{G}_\Lambda)$.

8. SKEW-PRODUCT TOPOLOGICAL HIGHER-RANK GRAPHS AND CROSSED PRODUCTS BY COACTIONS

In this section we extend the definition of a skew-product k-graph to topological k-graphs, and show that the associated groupoids can be realized as skew-product groupoids, extending Theorem 5.2 of [12]. We can then realize certain crossed product C^*-algebras as topological higher-rank graph C^*-algebras, building on Corollary 5.3 of [12] and Theorem 2.4 of [9].

Given a topological groupoid Γ, a locally compact group A and a continuous functor $b : \Gamma \to A$, we denote the *skew product of* Γ *by* b as $\Gamma(b)$; that is, $\Gamma(b)$ is the locally compact groupoid obtained by defining on $\Gamma \times A$ the multiplication $(x, g)(y, gb(x)) := (xy, g)$ and the inverse $(x, g)^{-1} := (x^{-1}, gb(x))$. In our setting, the groupoid Γ is r-discrete and admits a Haar system, and it follows that the same is true for $\Gamma(b)$.

DEFINITION 8.1. Let (Λ, d) be a topological k-graph, let A be a locally compact group, and let $c : \Lambda \to A$ be a continuous functor. Define $\Lambda \times_c A$ to be the category with object and morphism sets

$$\mathrm{Obj}(\Lambda \times_c A) = \mathrm{Obj}(\Lambda) \times A \quad \text{and} \quad \mathrm{Mor}(\Lambda \times_c A) = \mathrm{Mor}(\Lambda) \times A,$$

range and source maps $r(\lambda, a) = (r(\lambda), a)$ and $s(\lambda, a) = (s(\lambda), ac(\lambda))$, and composition $(\lambda, a)(\mu, ac(\lambda)) = (\lambda\mu, a)$.

Define a functor $d : \Lambda \times_c A \to \mathbb{N}^k$ by $d(\lambda, a) = d(\lambda)$. Then, giving the object and morphism sets their product topologies, the pair $(\Lambda \times_c A, d)$ is a topological k-graph, called the *skew-product of* (Λ, d) *by* c.

LEMMA 8.2. *If* (Λ, d) *is compactly aligned, then so is* $(\Lambda \times_c A, d)$.

Proof. Let $p, q \in \mathbb{N}^k$, and let $E \subset (\Lambda \times_c A)^p$ and $F \subset (\Lambda \times_c A)^q$ be compact. Let $P_\Lambda : \Lambda \times_c A \to \Lambda$ and $P_A : \Lambda \times_c A \to A$ be the coordinate maps $P_\Lambda(\lambda, a) = \lambda$ and $P_A(\lambda, a) = a$. We then see that $E \vee F$ is compact since it is a closed subset of the compact set $(P_\Lambda(E) \vee P_\Lambda(F)) \times (P_A(E) \cap P_A(F))$. ∎

The proof of the following lemma is straightforward.

LEMMA 8.3. *Let* (Λ, d) *be a compactly aligned topological k-graph, let* A *be a locally compact topological group, and let* $c : \Lambda \to A$ *be a continuous functor. Then there is a continuous functor* $\widetilde{c} : G_\Lambda \to A$ *defined by*

$$\widetilde{c}(\lambda x, d(\lambda) - d(\mu), \mu x) = c(\lambda)c(\mu)^{-1}.$$

PROPOSITION 8.4. *Let* (Λ, d) *be a compactly aligned topological k-graph, let* A *be a locally compact topological group, and let* $c : \Lambda \to A$ *be a continuous functor. Then, with the notation of Lemma* 8.3,

$$G_\Lambda(\widetilde{c}) \cong G_{\Lambda \times_c A}.$$

Furthermore, denoting the restriction of $\widetilde{c}$ to $\mathcal{G}_\Lambda = G_\Lambda|_{\partial\Lambda}$ again by $\widetilde{c}$, we have

$$\mathcal{G}_\Lambda(\widetilde{c}) \cong \mathcal{G}_{\Lambda\times_c A}.$$

Proof. We first define a functor $\phi : G_\Lambda(\widetilde{c}) \to G_{\Lambda\times_c A}$. For $(x,a) \in (G_\Lambda(\widetilde{c}))^{(0)}$, define a path $\phi(x,a) : \Omega_{k,d(x)} \to \Lambda \times_c A$ in $X_{\Lambda\times_c A}$ by

$$(\phi(x,a))(m,n) = (x(m,n), ac(x(0,m))) \quad \text{for } m \leqslant n \leqslant d(x).$$

Now let $((\lambda x, d(\lambda)-d(\mu), \mu x), a) \in G_\Lambda(\widetilde{c})$ and define

$$\begin{aligned}\phi((\lambda x, d(\lambda)-d(\mu), \mu x), a) &= (\phi(\lambda x, a), d(\lambda)-d(\mu), \phi(\mu x, a\widetilde{c}(\lambda x, d(\lambda)-d(\mu), \mu x)))\\ &= (\phi(\lambda x, a), d(\lambda)-d(\mu), \phi(\mu x, ac(\lambda)c(\mu)^{-1})).\end{aligned}$$

Straightforward but lengthy calculations then show that $\phi : G_\Lambda(\widetilde{c}) \to G_{\Lambda\times_c A}$ is a bijective continuous functor with continuous inverse, and the first part of the proposition follows.

We now show that $\mathcal{G}_\Lambda(\widetilde{c}) \cong \mathcal{G}_{\Lambda\times_c A}$. We have $\mathcal{G}_\Lambda(\widetilde{c}) = G_\Lambda(\widetilde{c})|_{(\partial\Lambda)\times A}$ by definition, so it suffices to show that $\phi((\partial\Lambda)\times A) = \partial(\Lambda\times_c A)$.

First fix $\phi(x,a) \in \partial(\Lambda\times_c A)$; we show that $(x,a) \in (\partial\Lambda)\times A$. Let $m \in \mathbb{N}^k$ satisfy $m \leqslant d(x)$, and let $E \in x(m)\mathcal{CE}(\Lambda)$. Choosing any compact neighbourhood B of $ac(x(0,m))$, we have $E \times B \in (\phi(x,a)(m))\mathcal{CE}(\Lambda\times_c A)$, so there exists $(\lambda, b) \in E\times B$ such that $\phi(x,a)(m, m+d(\lambda)) = (\lambda, b)$. We then have $x(m, m+d(\lambda)) = \lambda \in E$, giving $\partial(\Lambda\times_c A) \subset \phi((\partial\Lambda)\times A)$.

On the other hand, fix $(x,a) \in (\partial\Lambda)\times A$, let $m \in \mathbb{N}^k$ satisfy $m \leqslant d(\phi(x,a)) = d(x)$, and let $E \in (\phi(x,a)(m))\mathcal{CE}(\Lambda\times_c A)$.

Let $\{\mathcal{B}_j\}_{j\in\mathbb{N}}$ be a neighbourhood basis for $\phi(x,a)(m) = (x(m), ac(x(0,m)))$ such that each $\mathcal{B}_j \subset r(E)$ and $\overline{\mathcal{B}}_{j+1} \subset \mathcal{B}_j$. For each $j \in \mathbb{N}$, define

$$F_j := \overline{\mathcal{B}}_j(\Lambda\times_c A) \cap E = \{(\lambda, b) \in E \mid r(\lambda, b) \in \overline{\mathcal{B}}_j\},$$

so $F_j \in (\phi(x,a)(m))\mathcal{CE}(\Lambda\times_c A)$.

Let $P_\Lambda : \Lambda\times_c A \to \Lambda$ denote the coordinate map. For each $j \in \mathbb{N}$, we have $P_\Lambda(F_j) \in x(m)\mathcal{CE}(\Lambda)$, so there exists $(\lambda_j, b_j) \in F_j$ such that $x(m, m+d(\lambda_j)) = \lambda_j$. Since each $r(\lambda_j, b_j) \in \mathcal{B}_j$, it follows that

$$\lim_{j\in\mathbb{N}} (r(\lambda_j), b_j) = (x(m), ac(x(0,m))). \tag{8.1}$$

Since $\langle(\lambda_j, b_j)\rangle_{j\in\mathbb{N}}$ is contained in the compact set E, there exists a convergent subsequence $\langle(\lambda_j, b_j)\rangle_{j\in J}$ with limit $(\lambda, b) \in E$. We then have

$$\begin{aligned}(r(\lambda), b) = r(\lambda, b) &= r\Big(\lim_{j\in J}(\lambda_j, b_j)\Big)\\ &= \lim_{j\in\mathbb{N}} r(\lambda_j, b_j) = (x(m), ac(x(0,m))) \quad \text{by (8.1)},\end{aligned}$$

hence $b = ac(x(0,m))$. Furthermore,

$$\lambda = \lim_{j\in J}\lambda_j = \lim_{j\in J} x(m, m+d(\lambda_j)) = x(m, m+d(\lambda)),$$

and it follows that $\phi(x,a)(m, m+d(\lambda)) = (\lambda, b) \in E$, and $\phi(x,a) \in \partial(\Lambda \times_c A)$.

Therefore $\phi((\partial\Lambda) \times A) = \partial(\Lambda \times_c A)$, so ϕ restricts to an isomorphism from $\mathcal{G}_\Lambda(\widetilde{c})$ onto $\mathcal{G}_{\Lambda\times_c A}$, completing the proof. ∎

NOTATION 8.5. For a continuous functor b from a locally compact groupoid G with continuous Haar system to a locally compact abelian group A, we denote by $\alpha(b)$ the action of the dual group $\widehat{A}$ on $C^*(G)$ defined by Proposition II.5.1 of [28]. On the other hand, for a continuous functor b from a locally compact r-discrete groupoid G with Haar system to a discrete group A, we denote by $\delta(b)$ the coaction of A on $C^*(G)$ defined by Lemma 4.2 of [9].

REMARK 8.6. Our Theorem 8.8 concerns coactions of discrete groups (see [16], [20], [9], [23], for example). There is, however, much literature on the theory of coactions of locally compact groups on C^*-algebras (see [13], [21], [19], [22], for example). Our reliance on groupoid theory in the proofs of Theorem 8.7 and Theorem 8.8 has meant that our theorems only address actions of locally compact abelian groups and coactions of discrete groups. It is possible that Theorem 8.8 holds in the generality of locally compact groups; consequently, Theorem 8.7 would follow as the abelian case.

THEOREM 8.7. *Let (Λ, d) be a compactly aligned topological k-graph, let A be a locally compact abelian group, and let $c : \Lambda \to A$ be a continuous functor. Then, with the notation of Lemma* 8.3 *and Notation* 8.5,

$$C^*(G_\Lambda) \times_{\alpha(\widetilde{c})} \widehat{A} \cong C^*(G_{\Lambda\times_c A}).$$

Furthermore, denoting the restriction of $\widetilde{c}$ to $\mathcal{G}_\Lambda$ again by $\widetilde{c}$, we have

$$C^*(\mathcal{G}_\Lambda) \times_{\alpha(\widetilde{c})} \widehat{A} \cong C^*(\mathcal{G}_{\Lambda\times_c A}).$$

Proof. Both parts of of the theorem are achieved in two steps, using Theorem II.5.7 of [28] and Proposition 8.4. ∎

THEOREM 8.8. *Let (Λ, d) be a compactly aligned topological k-graph, let A be a discrete group and let $c : \Lambda \to A$ be a continuous functor. Then, with the notation of Lemma* 8.3 *and Notation* 8.5,

$$C^*(G_\Lambda) \times_{\delta(\widetilde{c})} A \cong C^*(G_{\Lambda\times_c A}).$$

Furthermore, denoting the restriction of $\widetilde{c}$ to $\mathcal{G}_\Lambda$ again by $\widetilde{c}$, we have

$$C^*(\mathcal{G}_\Lambda) \times_{\delta(\widetilde{c})} A \cong C^*(\mathcal{G}_{\Lambda\times_c A}).$$

Proof. The theorem follows from Theorem 4.3 of [9], Theorem 6.2 of [9] and Proposition 8.4. ∎

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