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SOME FORMULAE FOR NORMS OF ELEMENTARY OPERATORS

RICHARD M. TIMONEY

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ABSTRACT. We present a formula for the norm of an elementary operator on a C^*-algebra that seems to be new. The formula involves (matrix) numerical ranges and a kind of geometrical mean for positive matrices, the tracial geometric mean, which seems not to have been studied previously and has interesting properties. In addition, we characterise compactness of elementary operators.

KEYWORDS: *Tracial geometric mean, matrix numerical range, C^*-algebra.*

MSC (2000): 47B47, 46L07.

INTRODUCTION

We consider an elementary operator $Tx = \sum\limits_{j=1}^{\ell} a_j x b_j$ (with $x \in A$ a C^*-algebra and $a_j, b_j \in M(A)$, with $M(A)$ the multiplier algebra of A). We denote the class of elementary operators $T\colon A \to A$ by $\mathcal{E}\ell(A)$. Specifically, we address the question of finding a concrete formula for the operator norm $\|T\|$. This problem has been considered (at least implicitly) over a long period by several authors and there are solutions known under various special circumstances (generalised derivations [25], antiliminal by abelian C^*-algebras [5], for example). See [19] for a recent survey of the problem, or see Section 5.4 of [3] for a brief summary of its importance. One way to view the literature that relates to the problem is to separate two strands of problems. One strand concentrates on elementary operators of a rather special form (with $\ell \leqslant 2$) and the other (for arbitrary ℓ) has relied largely on dealing with the completely bounded norm $\|T\|_{\mathrm{cb}}$ and the Haagerup tensor norm estimate $\|T\|_{\mathrm{cb}} \leqslant \Big\| \sum\limits_{j=1}^{\ell} a_j \otimes b_j \Big\|_h$.

For special forms where $\ell \leqslant 2$, the case $\ell = 1$ is well understood (see [19]). There is a significant body of literature dealing with (inner) derivations $\delta_a(x) =$

$ax - xa$ and the estimate $\|\delta_a\| \leqslant 2\inf\|a - z\|$ with the infimum over z in the centre $Z(M(A))$ of $M(A)$ (see references in Section 4.1, Section 4.6 of [3] and [19]). In the case $A = \mathcal{B}(H)$ is the algebra of all bounded linear operators on a Hilbert space H (or $A = \mathcal{K}(H)$, the compacts) $Z(M(A))$ is just scalar multiples of the identity and [25] showed equality in this estimate for $\|\delta_a\|$. Subsequent work has generalised this equality to various classes of C^*-algebras but 3.2, 3.3 in [22] implies a characterisation of those A where equality always holds (those where all Glimm ideals of $M(A)$ are 3-primal). Moreover in case this condition is not true, then there is $a \in M(A)$ with $\|\delta_a\| \leqslant \sqrt{3} \inf_{z \in Z(M(A))} \|a - z\|$ (and further related work is to be found in [4], [21], [22], [23], [6]). An example of [7] shows that the condition on Glimm ideals of $M(A)$ is difficult to relate to the structure of the primitive ideal space of A, so that the results are perhaps most satisfactory in the unital case where $M(A) = A$. An alternative approach is given in 4.1.23 of [3] where it is shown that (for general A) $\|\delta_a\| = 2\inf\{\|a - z\| : z \in Z({}^{c}M(A))\}$ with ${}^{c}M(A)$ the bounded central closure of $M(A)$.

For generalised (inner) derivations $\delta_{a,b}(x) = ax - xb$, there are results that are rather less comprehensive than for δ_a. In particular [25] shows $\|\delta_{a,b}\| = \inf_{\lambda \in \mathbb{C}} \|a - \lambda\| + \|b - \lambda\|$ when $A = \mathcal{B}(H)$ and there is a result in terms of representations of ${}^{c}M(A)$ and $Z({}^{c}M(A))$ in 4.1.23 of [3]. One may invoke operator space methods and the fact that $\|\delta_{a,b}\| = \|\delta_{a,b}\|_{\mathrm{cb}}$ as another approach. The "obvious" estimate that arises from taking account of the centre is then $\|\delta_{a,b}\| \leqslant \|a \otimes 1 - 1 \otimes b\|_{Z,h}$ where $\|\cdot\|_{Z,h}$ is the central Haagerup tensor norm on $M(A) \otimes M(A)$. Results of [7] show that there is equality in this estimate (for all $a, b \in M(A)$) when all Glimm ideals of $M(A)$ are 5-primal, but that it is not sufficient for all Glimm ideals to be 3-primal.

The case of Jordan mappings $J_{a,b}(x) = axb + bxa$ was also the subject of several papers and a conjecture of M. Mathieu that $\|J_{a,b}\| \geqslant \|a\|\|b\|$ when $A = \mathcal{B}(H)$ (or more generally when A is a prime C^*-algebra) has recently been proved to be true. In [18] a weaker result $\|J_{a,b}\|_{\mathrm{cb}} \geqslant \|a\|\|b\|$ was shown and in [27] a different proof of this was given along with a proof of the conjecture using results from [26]. The conjecture was also shown slightly earlier in [10] by quite different methods.

Turning to progress on general elementary operators (where ℓ can be large) the most satisfactory progress to date deals with $\|T\|_{\mathrm{cb}}$ rather than $\|T\|$. It is shown in [5] that $\|T\|_{\mathrm{cb}} = \|T\|$ for all elementary operators T if and only if the C^*-algebra A is "antiliminal by abelian". It is shown in [24] that $\|T\|_{\mathrm{cb}} = \left\| \sum_{j=1}^{\ell} a_j \otimes b_j \right\|_{Z,h}$ for each elementary operator $Tx = \sum_{j=1}^{\ell} a_j x b_j$ on A if all Glimm ideals of $M(A)$ are primal, and [7] establishes the converse. In fact [7] also gives a necessary and sufficient condition for this equality when restricted to specific ℓ (all Glimm ideals of $M(A)$ should be N-primal for $N = \ell^2 + 1$).

Note that a result of [16] says that elementary operators can approximate arbitrary bounded linear operators on A which preserve all ideals of A (albeit in the strong operator topology). It follows that in many ways elementary operators must be typical, especially if A is simple. For general A, the special properties of elementary operators can be captured via the concepts of central bimodule homomorphisms used in Section 5.3 of [3].

In this work, we give slightly different approaches to our formula for the norm $\|T\|$ of an elementary operator T in Theorems 1.3 and 1.8 for the case $A = \mathcal{B}(H)$ and in Theorem 2.1 for the case of general A. The formulae involve the tracial geometric mean as defined in Definition 1.2 and we give some basic properties of this mean in Remarks 1.5. The formulae also involve matrix-valued numerical ranges considered in [26] but are independent of the different possible expressions for the elementary operator. We use the formulae to establish some bounds on the growth of k-norms of $T \in \mathcal{E}\ell(A)$ as k increases and exploit continuity in our formula to give a new proof of a result of [5] showing that there is no growth at all if A is antiliminal (Corollary 2.5).

The question of compactness of elementary operators has also been investigated under several circumstances (for example on the Calkin algebra — see references given in [3], p. 232) but a characterisation for general C^*-algebras has eluded proof. In Theorem 3.1 we characterise compactness of $T \in \mathcal{E}\ell(A)$ via the possibility of choosing compact a_j and b_j.

1. THE CASE $\mathcal{B}(H)$

For the moment we take $A = \mathcal{B}(H) = M(A)$ and establish a formula for $\|T\|$ in that case. Later we will extend the formula to general A.

Recall again the upper bound

$$\|T\| \leqslant \Big\| \sum_{j=1}^{\ell} a_j \otimes b_j \Big\|_h$$

in terms of the Haagerup norm $\|\cdot\|_h$ on $\mathcal{B}(H) \otimes \mathcal{B}(H)$ (see 5.4.7 of [3] for example). We know that equality holds in case the operators $a_i a_j^*$ commute and the operators $b_j^* b_k$ also commute ([26], Theorem 3.3 and Remark 2.5). In general the inequality is strict.

For $\eta, \xi \in H$ we use the notation $\eta \otimes \xi^*$ for the rank one operator on H with $(\eta \otimes \xi^*)(\theta) = \langle \theta, \xi \rangle \eta$.

LEMMA 1.1. *For $T \in \mathcal{E}\ell(\mathcal{B}(H))$, $Tx = \sum_{j=1}^{\ell} a_j x b_j$, we have*

$$\|T\| = \sup_{p_1, p_2} \Big\| \sum_{j=1}^{\ell} (p_1 a_j) \otimes (b_j p_2) \Big\|_h$$

where $p_1, p_2 \in \mathcal{B}(H)$ are rank one projections ($p_i^2 = p_i = p_i^*$ ($i = 1,2$)).

Proof. Let $p_1 = \xi \otimes \xi^*$ and $p_2 = \eta \otimes \eta^*$ be 1-dimensional projections (where $\eta, \xi \in H$ are unit vectors). We look at the operator

$$T_{p_1,p_2}(x) = \sum_{j=1}^{\ell} p_1 a_j x b_j p_2,$$

an operator with a 1-dimensional range. Specifically it is the operator

$$x \mapsto \langle (Tx)\eta, \xi\rangle \xi \otimes \eta^*$$

and thus almost a linear functional.

For this operator, $(p_1 a_i)(p_1 a_j)^*$ are commuting and so are $(b_i p_2)^*(b_j p_2)$. Thus,

$$\|T_{p_1,p_2}\| = \Big\| \sum_{j=1}^{\ell} (p_1 a_j) \otimes (b_j p_2) \Big\|_h$$

by the remarks above ([26], Theorem 3.3 and Remark 2.5). Alternatively, one can appeal to the fact that the norm of a linear functional is the same as its completely bounded norm, hence $\|T_{p_1,p_2}\| = \|T_{p_1,p_2}\|_{\mathrm{cb}} =$ the Haagerup tensor norm (by a result of Haagerup — see 5.4.7 of [3] for example).

Now, clearly

$$\begin{aligned}
\|T\| &= \sup\{\|Tx\| : x \in \mathcal{B}(H), \|x\| \leqslant 1\} \\
&= \sup\{\Re\langle (Tx)\eta, \xi\rangle : x \in \mathcal{B}(H), \|x\| \leqslant 1, \xi, \eta \in H, \|\xi\| = \|\eta\| = 1\} \\
&= \sup\{\Re\langle (T_{p_1,p_2}x)\eta, \xi\rangle : x, \xi, \eta \text{ as above}\} \leqslant \sup_{p_1,p_2} \|T_{p_1,p_2}\|.
\end{aligned}$$

Since $\|T_{p_1,p_2}\| \leqslant \|T\|$, the lemma follows. ∎

NOTATIONS. Following the ideas in [26], we introduce some notation relating to matrix numerical ranges.

For $\mathbf{b} = [b_1, b_2, \dots, b_\ell]^{\mathrm{t}}$ a column of operators $b_j \in \mathcal{B}(H)$ we consider the matrix of operators

$$Q(\mathbf{b}) = (b_i^* b_j)_{i,j=1}^{\ell} = [b_1, b_2, \dots, b_\ell]^*[b_1, b_2, \dots, b_\ell] \in M_\ell(\mathcal{B}(H)) \equiv \mathcal{B}(H^\ell)$$

and for $\eta \in H$

$$Q(\mathbf{b}, \eta) = (\langle b_i^* b_j \eta, \eta\rangle)_{i,j=1}^{\ell} = (\langle b_j \eta, b_i \eta\rangle)_{i,j=1}^{\ell} \in M_\ell(\mathbb{C}).$$

A matrix numerical range associated with $\mathbf{b}$ that was considered in [26] is

$$W_m(\mathbf{b}) = \{Q(\mathbf{b}, \eta)^{\mathrm{t}} : \eta \in H, \|\eta\| = 1\} \subset M_\ell(\mathbb{C}).$$

Then $W_{m,e}(\mathbf{b})$ denotes the set of matrices of maximal trace in the closure of $W_m(\mathbf{b})$. Recall that the maximal trace is in fact $\|\mathbf{b}\|^2 = \Big\| \sum_{j=1}^{\ell} b_j^* b_j \Big\|$.

In [26] it is shown that there is equality in the Haagerup estimate for the norm of the elementary operator T,

$$\|T\| \leqslant \|\mathbf{a}\|\|\mathbf{b}\| = \sqrt{\Big\| \sum_{j=1}^{\ell} a_j a_j^* \Big\| \Big\| \sum_{j=1}^{\ell} b_j^* b_j \Big\|} \leqslant \frac{1}{2}(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2)$$

if and only if $W_{m,e}(\mathbf{a}^*) \cap W_{m,e}(\mathbf{b}) \neq \emptyset$ (for $\mathbf{a} = [a_1, a_2, \ldots, a_\ell] \in \mathcal{B}(H)^\ell$ a row matrix of operators $a_j \in \mathcal{B}(H)$ and $\mathbf{a}^* = [a_1^*, a_2^*, \ldots, a_\ell^*]^{\mathrm{t}}$ a column).

DEFINITION 1.2. For two positive semidefinite $\ell \times \ell$ matrices X and Y we define the *tracial geometric mean* of X and Y by

$$\mathrm{tgm}(X, Y) = \mathrm{trace} \sqrt{\sqrt{X} Y \sqrt{X}}$$

(where, of course, the square roots mean the positive semidefinite square roots).

Here is one version of our formula for the norm of an elementary operator.

THEOREM 1.3. *For $\mathbf{a} = [a_1, a_2, \ldots, a_\ell] \in \mathcal{B}(H)^\ell$ (a row matrix of operators $a_j \in \mathcal{B}(H)$) and $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^{\mathrm{t}} \in \mathcal{B}(H)^\ell$ a column, and $Tx = \sum_{j=1}^{\ell} a_j x b_j$ an elementary operator, we have*

$$\|T\| = \sup\{\mathrm{tgm}(Q(\mathbf{a}^*, \xi), Q(\mathbf{b}, \eta)) : \xi, \eta \in H, \|\xi\| = \|\eta\| = 1\}.$$

Proof. Note first that the result will follow from the lemma once we establish that

$$\|T_{p_1,p_2}\| = \Big\| \sum_{j=1}^{\ell} (p_1 a_j) \otimes (b_j p_2) \Big\|_h = \mathrm{trace} \sqrt{\sqrt{Q(\mathbf{a}^*, \xi)} Q(\mathbf{b}, \eta) \sqrt{Q(\mathbf{a}^*, \xi)}}$$

when $p_1 = \xi \otimes \xi^*$, $p_2 = \eta \otimes \eta^*$.

We now fix unit vectors $\xi, \eta \in H$. Notice that the matrix numerical range of the tuple $\mathbf{b} p_2 = [b_1 p_2, b_2 p_2, \ldots, b_\ell p_2]^{\mathrm{t}}$ consists of

$$\{|\lambda|^2 Q(\mathbf{b} p_2, \eta)^{\mathrm{t}} : \lambda \in \mathbb{C}, |\lambda| \leqslant 1\}$$

(if H is not one-dimensional; for the trivial one-dimensional case we only have $|\lambda| = 1$). Moreover $Q(\mathbf{b} p_2, \eta) = Q(\mathbf{b}, \eta)$ for $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^{\mathrm{t}}$. Similar remarks apply to the tuple $\mathbf{a}^*$ and $(p_1 \mathbf{a})^* = [p_1 a_1, p_1 a_2, \ldots, p_1 a_\ell]^*$. Thus the condition (from [26]) for equality in the Haagerup estimate

$$\|T_{p_1,p_2}\| \leqslant \frac{1}{2}(\|p_1 \mathbf{a}\|^2 + \|\mathbf{b} p_2\|^2)$$

is $Q((p_1 \mathbf{a})^*, \xi)^{\mathrm{t}} = Q(\mathbf{b} p_2, \eta)^{\mathrm{t}}$.

We now show how to rewrite $\sum_{j=1}^{\ell} (p_1 a_j) \otimes (b_j p_2)$ so as to get this equality condition satisfied. To simplify the notation, we assume now that $p_1 a_j = a_j$ and

$b_j p_2 = b_j$ for $1 \leqslant j \leqslant \ell$. Introduce the notation

$$\mathbf{a} \odot \mathbf{b} = \sum_{j=1}^{\ell} a_j \otimes b_j$$

for rows $\mathbf{a}$ and columns $\mathbf{b}$.

Suppose first that the tuples $(a_j)_{j=1}^{\ell}$ and $(b_j)_{j=1}^{\ell}$ are each linearly independent. Then the different ways to rewrite $\mathbf{a} \odot \mathbf{b}$ as a sum of the same type, without going outside the spans of the a_j and b_j (or equivalently sticking to linearly independent tuples) are all of the form

$$(\mathbf{a}\alpha) \odot (\alpha^{-1}\mathbf{b})$$

for invertible scalar matrices $\alpha \in M_\ell(\mathbb{C})$. We have

$$Q((\mathbf{a}\alpha)^*, \xi)^{\mathrm{t}} = \alpha^* Q(\mathbf{a}^*, \xi)^{\mathrm{t}} \alpha = \alpha^* (\langle a_i^* \xi, a_j^* \xi \rangle)_{i,j=1}^{\ell} \alpha$$

and

$$Q(\alpha^{-1}\mathbf{b}, \eta)^{\mathrm{t}} = \alpha^{-1} Q(\mathbf{b}, \eta)^{\mathrm{t}} (\alpha^{-1})^* = \alpha^{-1} (\langle b_i \eta, b_j \eta \rangle)_{i,j=1}^{\ell} (\alpha^{-1})^*.$$

For $Q(\mathbf{a}^*, \xi)$ and $Q(\mathbf{b}, \eta)$ invertible, we take $\alpha = \alpha_0 \alpha_1$,

$$\alpha_0 = \left(\sqrt{Q(\mathbf{a}^*, \xi)^{\mathrm{t}}}\right)^{-1}, \quad \alpha_1 = \left(\sqrt{Q(\mathbf{a}^*, \xi)^{\mathrm{t}}} Q(\mathbf{b}, \eta)^{\mathrm{t}} \sqrt{Q(\mathbf{a}^*, \xi)^{\mathrm{t}}}\right)^{1/4}.$$

The effect of α_0 is to make $Q((\mathbf{a}\alpha_0)^*, \xi)$ the identity matrix, and then α is designed so that we get an equality

$$(1.1) \qquad Q((\mathbf{a}\alpha)^*, \xi)^{\mathrm{t}} = Q(\alpha^{-1}\mathbf{b}, \eta)^{\mathrm{t}} = \left(\sqrt{\sqrt{Q(\mathbf{a}^*, \xi)} Q(\mathbf{b}, \eta) \sqrt{Q(\mathbf{a}^*, \xi)}}\right)^{\mathrm{t}}.$$

We have now rewritten $\mathbf{a} \odot \mathbf{b}$ to satisfy the criterion (from [26]) for equality in the Haagerup estimate for $\|T_{p_1,p_2}\|$ arising from $(\mathbf{a}\alpha) \odot (\alpha^{-1}\mathbf{b})$. (Explicitly, taking x to be a unitary operator in $\mathcal{B}(H)$ with $x((\alpha^{-1}\mathbf{b})_j \eta) = (\mathbf{a}\alpha)_j^* \xi$ for $1 \leqslant j \leqslant \ell$, we get $\langle T_{p_1,p_2}(x)\eta, \xi \rangle = \langle T(x)\eta, \xi \rangle =$ the trace of the matrix (1.1).) Thus $\|T_{p_1,p_2}\|$ is the trace of the matrix (1.1).

We have made a linear independence assumption and the calculation we made requires $Q(\mathbf{a}^*, \xi)$ and $Q(\mathbf{b}, \eta)$ to be invertible. Though we could perhaps manage without these assumptions and make use of various notions of generalised inverses, it is easier to use an approximation argument to deduce the general case. We embed $\mathcal{B}(H)$ in $\mathcal{B}(H \oplus H_0)$ where H_0 is an auxiliary Hilbert space of dimension ℓ. Then we can modify a_j so that $a_j^* \xi$ acquire small mutually orthogonal contributions in H_0. Similarly for b_j and $b_j \eta$. This will ensure the assumptions are valid and then we can take limits as the modifications of a_j and b_j tend to 0. ∎

EXAMPLE 1.4. If a_j^* have orthogonal ranges and b_j have orthogonal ranges, then $Q(\mathbf{a}^*, \xi)$ and $Q(\mathbf{b}, \eta)$ are diagonal matrices and the formula from Theorem 1.3 becomes

$$\|T\| = \sup \sum_{j=1}^{\ell} \|a_j^* \xi\| \|b_j \eta\|.$$

REMARKS 1.5. (i) In Theorem 1.3, we are considering the tracial geometric mean of two positive semidefinite matrices of the form $X = Q(\mathbf{a}^*, \xi)$ and $Y = Q(\mathbf{b}, \eta)$. Such matrices X and Y can be arbitrary elements of $M_\ell^+(\mathbb{C})$ (as long as the dimension of H is not smaller than ℓ).

For example, starting with $Y \in M_\ell^+(\mathbb{C})$ we could take $\eta_1, \eta_2, \ldots, \eta_\ell \in \mathbb{C}^\ell$ to be the rows of $\sqrt{Y^{\mathrm{t}}}$. Then take $b_j = \eta_j \otimes \eta^*$ (with η any unit vector in the ℓ-dimensional Hilbert space $\mathbb{C}^\ell$) to get $b_j \eta = \eta_j$ and

$$(\langle b_i \eta, b_j \eta \rangle)_{i,j=1}^{\ell} = Y^{\mathrm{t}} \Rightarrow Q(\mathbf{b}, \eta) = Y.$$

Similarly we can find $a_j = \eta \otimes \xi_j^*$ so that $X = Q(\mathbf{a}^*, \eta) = (\langle a_j^* \eta, a_i^* \eta \rangle)_{i,j=1}^{\ell}$ and for $p = p_1 = p_2 = \eta \otimes \eta^*$ the operator $T \in \mathcal{E}\ell(M_\ell)$ with $Tx = \sum_{j=1}^{\ell} a_j x b_j$ has $T = T_{p,p}$, $\|T\| = \mathrm{tgm}(X, Y)$.

(ii) A useful fact is that the eigenvalues of $\sqrt{X} Y \sqrt{X}$ are the same as those of $\sqrt{X}(\sqrt{X} Y) = XY$. (This well-known fact follows by conjugation if $\sqrt{X}$ is invertible.) So if $\lambda_i(XY)$ ($1 \leqslant i \leqslant \ell$) denote the eigenvalues of XY (arranged in non-increasing order, say), then

$$\mathrm{tgm}(X, Y) = \sum_{i=1}^{\ell} \sqrt{\lambda_i(XY)}.$$

(iii) The symmetry property $\mathrm{tgm}(X, Y) = \mathrm{tgm}(Y, X)$ follows because $\lambda_i(XY) = \lambda_i(YX)$. (It can also be established using $\|T\| = \|T^*\|$, where T^* means $T^*(x) = T(x^*)^*$, and (i).)

(iv) $\mathrm{tgm}(X, Y)$ has some other desirable properties for geometric means. If $X = Y$, then $\mathrm{tgm}(X, Y) = \operatorname{trace} X$. If $X = \lambda I_\ell$ is a multiple of the identity matrix, then $\mathrm{tgm}(X, Y) = \sqrt{\lambda} \operatorname{trace} \sqrt{Y}$.

From the Haagerup estimate for $\|T_{p_1,p_2}\|$ and remark (i) above, we can see that

$$\mathrm{tgm}(X, Y) \leqslant \sqrt{(\operatorname{trace} X)(\operatorname{trace} Y)} \leqslant \frac{\operatorname{trace} X + \operatorname{trace} Y}{2}$$

holds. This is a tracial version of an arithmetic-geometric mean inequality.

According to a criterion established in [26] for overall equality in this estimate, equality will only hold if $X = Y$. It follows that

$$\mathrm{tgm}(X, Y) = \sqrt{(\operatorname{trace} X)(\operatorname{trace} Y)}$$

holds only when X and Y are linearly dependent.

(v) It is clear that if $u \in M_\ell(\mathbb{C})$ is unitary, then $\operatorname{tgm}(u^*Xu, u^*Yu) = \operatorname{tgm}(X, Y)$ and it follows from the earlier remark (ii) (or the proof of Theorem 1.3) that if $\alpha \in M_\ell$ is invertible then

$$\operatorname{tgm}(\alpha^* X \alpha, (\alpha^{-1})Y(\alpha^{-1})^*) = \operatorname{tgm}(X, Y).$$

(vi) It is easy to see that $\operatorname{tgm}(X, Y)$ is monotone in Y and by symmetry also in X. So if $X \leqslant X_1$ and $Y \leqslant Y_1$ then $\operatorname{tgm}(X, Y) \leqslant \operatorname{tgm}(X_1, Y_1)$.

(vii) There are various notions of geometric mean for two positive semidefinite matrices in the literature (see for example [2]). One of these is usually denoted $X\#Y$ and can be defined for the positive definite case by

$$X\#Y = \sqrt{X}(X^{-1/2}YX^{-1/2})^{1/2}\sqrt{X}.$$

In general $\operatorname{trace}(X\#Y) \leqslant \operatorname{tgm}(X, Y)$ and strict inequality is possible.

T. Ando has kindly provided the following proof of the inequality (private correspondence). Assuming X and Y are non-singular, by D4 of [2] we have $X\#Y = X^{1/2}UY^{1/2}$ for some unitary U. Hence

$$\operatorname{trace}(X\#Y) = \operatorname{trace}(UY^{1/2}X^{1/2})$$

is at most the trace class norm

$$\|Y^{1/2}X^{1/2}\|_1 = \operatorname{trace}(X^{1/2}Y^{1/2}Y^{1/2}X^{1/2})^{1/2} = \operatorname{tgm}(X, Y).$$

The case of general positive semidefinite X and Y follows by continuity.

To illustrate strict inequality, let

$$X = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}^2 = \begin{pmatrix} \frac{5}{4} & 1 \\ 1 & \frac{5}{4} \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$X\#Y = \begin{pmatrix} \frac{3}{2\sqrt{5}} & 0 \\ 0 & 0 \end{pmatrix}, \quad \operatorname{tgm}(X, Y) = \frac{\sqrt{5}}{2}.$$

(viii) If P is an orthogonal projection in M_ℓ (that is $P = P^2 = P^*$) and $Q = I_\ell - P$, we define a pinching map by $\mathcal{C}(X) = PXP + QXQ$ and then we have

$$\operatorname{tgm}(X, Y) \leqslant \operatorname{tgm}(\mathcal{C}(X), \mathcal{C}(Y)) = \operatorname{tgm}(PXP, PYP) + \operatorname{tgm}(QXQ, QYQ).$$

To verify this, we can assume (by a unitary change of basis) that P is the projection of $\mathbb{C}^\ell$ onto the first k coordinates ($0 \leqslant k \leqslant \ell$), in other words a diagonal matrix with k 1's and $\ell - k$ zeros. Choose $T \in \mathcal{E}\ell(M_\ell)$ as indicated in remark (i) so that $T = T_{p,p}$ and $\|T\| = \operatorname{tgm}(X, Y)$. Taking $T_1 x = \sum_{j=1}^{k} a_j x b_j$ and $T_2 x = \sum_{j=k+1}^{\ell} a_j x b_j$, we find

$$\|T\| \leqslant \|T_1\| + \|T_2\| = \operatorname{tgm}(PXP, PYP) + \operatorname{tgm}(QXQ, QYQ)$$

when we calculate $\|T_1\|$ and $\|T_2\|$ in a similar way.

(ix) The maps $X \mapsto \operatorname{tgm}(X, Y) = \operatorname{tgm}(Y, X)$ (with $Y \in M_\ell^+$ fixed) satisfy subadditivity

$$\operatorname{tgm}\Big(\sum_{j=1}^n X_j, Y\Big) \leqslant \sum_{j=1}^n \operatorname{tgm}(X_j, Y)$$

(This can be shown by using $\operatorname{tgm}(X, Y) = \operatorname{tgm}(\sqrt{Y}X\sqrt{Y}, I_\ell)$ to reduce to the case $Y = I_\ell$ and using a known property of the Schatten $1/2$ quasinorm — see Rotfel'd Theorem IV.2.14 in [9]). Hence, for $X_j, Y_k \in M_\ell^+$,

$$\operatorname{tgm}\Big(\sum_{j=1}^n X_j, \sum_{k=1}^m Y_k\Big) \leqslant \sum_{1\leqslant j\leqslant n, 1\leqslant k\leqslant m} \operatorname{tgm}(X_i, Y_j). \tag{1.2}$$

It follows by Cauchy-Schwarz that

$$\operatorname{tgm}\Big(\sum_{j=1}^n X_j, \sum_{k=1}^m Y_k\Big)^2 \leqslant mn \sum_{1\leqslant j\leqslant n, 1\leqslant k\leqslant m} \operatorname{tgm}(X_i, Y_j)^2. \tag{1.3}$$

(x) By operator concavity of the square root ([9], V.1.8, V.2.5), $X \mapsto \operatorname{tgm}(X, Y)$ is a concave function of X (for fixed Y).

LEMMA 1.6. *We can define a norm on the direct sum $H^\ell = H \oplus H \oplus \cdots \oplus H$ of ℓ copies of H by*

$$\|(\xi_1, \xi_2, \ldots, \xi_\ell)\|_{S1} = \operatorname{trace}\sqrt{(\langle \xi_i, \xi_j\rangle)_{i,j=1}^\ell}.$$

Proof. Fix a unit vector $\eta \in H$ and consider the element $x \in \mathcal{B}(H^\ell) \equiv M_\ell(\mathcal{B}(H))$ given by

$$x = \begin{pmatrix} \xi_1 \otimes \eta^* & \xi_2 \otimes \eta^* & \cdots & \xi_\ell \otimes \eta^* \\ & & & \\ & 0 & & \\ & & & \end{pmatrix}.$$

This depends linearly on $(\xi_1, \xi_2, \ldots, \xi_\ell) \in H^\ell$. Computing x^*x we find the block matrix $(\langle \xi_j, \xi_i\rangle \eta \otimes \eta^*)_{i,j=1}^\ell$. Since $\eta \otimes \eta^*$ is a self-adjoint projection, the square root of x^*x has entries that are scalar multiples of the projection, the scalars being the entries of the square root of $(\langle \xi_j, \xi_i\rangle)_{i,j=1}^\ell$. Since the projection has rank one, it follows that the trace class norm of x is the same as

$$\operatorname{trace}\sqrt{(\langle \xi_i, \xi_j\rangle)_{i,j=1}^\ell}.$$

Thus this expression gives a norm on H^ℓ. ∎

EXAMPLE 1.7. If $(\xi_1, \xi_2, \ldots, \xi_\ell) = (\lambda_1\xi, \lambda_2\xi, \ldots, \lambda_\ell\xi)$ (for a unit vector $\xi \in H$ and scalars λ_j), that is if the ξ_j are linearly dependent, then

$$\|(\xi_1, \xi_2, \ldots, \xi_\ell)\|_{S1} = \Big(\sum_{j=1}^\ell \|\xi_j\|^2\Big)^{1/2}.$$

On the other hand if the ξ_j are mutually orthogonal, $\|(\xi_1, \xi_2, \ldots, \xi_\ell)\|_{S1} = \sum_{j=1}^{\ell} \|\xi_j\|$.

NOTATION. For $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^{\mathrm{t}} \in \mathcal{B}(H)^\ell$, we may regard $\mathbf{b}$ as an operator from H to H^ℓ and then we denote by $\|\mathbf{b}\|_{S1}$ the operator norm of $\mathbf{b}$ as an operator from H to $(H^\ell, \|\cdot\|_{S1})$.

THEOREM 1.8. *For* $\mathbf{a} = [a_1, a_2, \ldots, a_\ell] \in \mathcal{B}(H)^\ell$, $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^{\mathrm{t}} \in \mathcal{B}(H)^\ell$ *and* $Tx = \sum_{j=1}^{\ell} a_j x b_j$, *we have*

$$\|T\| = \sup\left\{ \left\| \sqrt{Q(\mathbf{b}, \eta)^{\mathrm{t}}}\mathbf{a}^* \right\|_{S1} : \eta \in H, \|\eta\| = 1 \right\}.$$

Proof. We use Theorem 1.3 and show that, for a fixed unit vector $\eta \in H$

$$\sup\{\operatorname{tgm}(Q(\mathbf{a}^*, \xi), Q(\mathbf{b}, \eta)) : \xi \in H, \|\xi\| = 1\} = \left\| \sqrt{Q(\mathbf{b}, \eta)^{\mathrm{t}}}\mathbf{a}^* \right\|_{S1}.$$

It will be convenient to work with invertible $Q(\mathbf{b}, \eta)$ and it is sufficient to consider this case because of a perturbation argument. For example, consider $\mathcal{B}(H) \subset \mathcal{B}(H^{\ell+1})$ with the $\ell + 1$ copies of H numbered 0 to ℓ and $\mathcal{B}(H)$ included as the top left block in $\mathcal{B}(H^{\ell+1}) = M_{\ell+1}(\mathcal{B}(H))$. Let E_{0j} denote the operator

$$E_{0j}(\xi_0, \xi_1, \ldots, \xi_\ell) = (0, 0, \ldots, 0, \xi_0, 0, \ldots, 0)$$

with the ξ_0 in position j. We can replace b_j by $b_j + \varepsilon E_{0j}$ for $\varepsilon > 0$ small.

From Theorem 1.3 we know that for $Sx = (Tx)p_2 = (Tx)(\eta \otimes \eta^*)$

$$\|S\| = \sup\{\operatorname{tgm}(Q(\mathbf{a}^*, \xi), Q(\mathbf{b}, \eta)) : \xi \in H, \|\xi\| = 1\}.$$

We can rewrite Sx using $\mathbf{a} \odot \mathbf{b}p_2 = (\mathbf{a}\alpha) \odot (\alpha^{-1}\mathbf{b}p_2)$ with $\alpha = \sqrt{Q(\mathbf{b}, \eta)^{\mathrm{t}}}$ to transform to the case where $Q(\alpha^{-1}\mathbf{b}, \eta) = I_\ell$ is the $\ell \times \ell$ identity matrix.

After this transformation,

$$\|S\| = \sup\{\operatorname{trace} \sqrt{Q((\mathbf{a}\alpha)^*, \xi)} : \xi \in H, \|\xi\| = 1\} = \|(\mathbf{a}\alpha)^*\|_{S1}$$

and we have the result. ∎

LEMMA 1.9. *If* $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^{\mathrm{t}}$ *is an* ℓ*-tuple of elements of* $\mathcal{B}(H)$ $(\ell \geqslant 1)$*, then* $b_1, b_2, \ldots, b_\ell$ *are linearly independent if and only of there exist vectors* $\xi_1, \ldots, \xi_m \in H$ *such that*

$$\sum_{k=1}^{m} Q(\mathbf{b}, \xi_k)$$

is positive definite.

Proof. If the b_j are linearly dependent, so that $\sum_{j=1}^{\ell} \lambda_j b_j = 0$ for some scalars λ_j not all zero, then with $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_\ell]$, for each $\xi \in H$ we have

$$\lambda Q(\mathbf{b}, \xi)^{\mathrm{t}} \lambda^* = \sum_{j,k=1}^{\ell} \lambda_j \overline{\lambda}_k \langle b_j \xi, b_k \xi \rangle = \Big\| \sum_{j=1}^{\ell} \lambda_j b_j \xi \Big\|^2 = 0$$

and so each finite sum $\sum_{k=1}^{m} Q(\mathbf{b}, \xi_k)$ is singular.

Conversely, if each finite sum is singular, choose a sum $M = \sum_{k=1}^{m} Q(\mathbf{b}, \xi_k)$ of maximal rank among all such sums and a nonzero vector $\lambda \in \mathbb{C}^\ell$ with $M^{\mathrm{t}} \lambda^* = 0$. This λ^* must be in the kernel of each $Q(\mathbf{b}, \xi)^{\mathrm{t}}$ for $\xi \in H$ (as otherwise $Q(\mathbf{b}, \xi) + \sum_{k=1}^{m} Q(\mathbf{b}, \xi_k)$ would have larger rank). From the above calculation we get $\sum_{j=1}^{\ell} \lambda_j b_j \xi = 0$ for all ξ, hence $\sum_{j=1}^{\ell} \lambda_j b_j = 0$. ∎

We now characterise compactness of $T \in \mathcal{E}\ell(\mathcal{B}(H))$. Let $H_1 = \{\xi \in H : \|\xi\| \leqslant 1\}$ denote the closed unit ball of H (compact in the weak topology) and let $\mathcal{K}(H)$ denote the compact elements of $\mathcal{B}(H)$.

The equivalence (in the linearly independent case) of the first and last conditions in the following is known — see [13] or 5.3.26 of [3] and in the case $\ell = 1$ see [28].

THEOREM 1.10. *Let* $\mathbf{a} = [a_1, a_2, \ldots, a_\ell] \in \mathcal{B}(H)^\ell$, $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^{\mathrm{t}} \in \mathcal{B}(H)^\ell$ *and* $Tx = \sum_{j=1}^{\ell} a_j x b_j$. *Then the following are equivalent:*

(i) $T \colon \mathcal{B}(H) \to \mathcal{B}(H)$ *is compact.*

(ii) *For* $f_T \colon H_1 \times H_1 \to \mathbb{C}$ *given by*

$$f_T(\xi, \eta) = \mathrm{tgm}(Q(\mathbf{a}^*, \xi), Q(\mathbf{b}, \eta)),$$

the function f_T *is continuous in the product weak topology on* $H_1 \times H_1$.

(iii) *The same function* f_T *is continuous at each point of*

$$(H_1 \times \{0\}) \cup (\{0\} \times H_1)$$

in the product weak topology of $H_1 \times H_1$.

Assuming that a_j *(*$1 \leqslant j \leqslant \ell$*) are linearly independent and that* b_j *(*$1 \leqslant j \leqslant \ell$*) are also linearly independent, then these are also equivalent to:*

(iv) $a_j, b_j \in \mathcal{K}(H)$ *(for* $1 \leqslant j \leqslant \ell$*).*

Proof. It is clear that (ii) $\Rightarrow$ (iii). For the case of unit vectors ξ, η we have

$$f_T(\xi, \eta) = \|T_{p_1, p_2}\|$$

for $p_1 = \xi \otimes \xi^*$ and $p_2 = \eta \otimes \eta^*$, and this is also the norm of the linear functional $x \mapsto \langle (Tx)\eta, \xi\rangle$ by the proof of Lemma 1.1. We can see by homogeneity that for all (ξ, η), $f_T(\xi, \eta)$ is the norm of the linear functional $x \mapsto \langle (Tx)\eta, \xi\rangle$ and f_T does not depend on the representation of T. So it is sufficient to establish the equivalence of the conditions under the assumption that the a_j $(1 \leqslant j \leqslant \ell)$ and the b_j $(1 \leqslant j \leqslant \ell)$ are linearly independent.

(i) $\Rightarrow$ (ii): As the image $T(B_1)$ of the unit ball $B_1 = \{x \in \mathcal{K}(H) : \|x\| \leqslant 1\}$ is relatively compact, for each $\varepsilon > 0$ we can find a finite number of elements $x_1, x_2, \ldots, x_n \in T(B_1) \subset \mathcal{K}(H)$ so that

$$T(B_1) \subset \bigcup_{k=1}^{n} \{y \in \mathcal{K}(H) : \|y - x_k\| < \varepsilon\}.$$

Denote by $\omega^T_{\xi,\eta}$ the linear functional on $\mathcal{B}(H)$ given by $\omega^T_{\xi,\eta}(x) = \langle (Tx)\eta, \xi\rangle$ and recall that $\|\omega^T_{\xi,\eta}\| = f_T(\xi, \eta)$. The norm of $\omega^T_{\xi,\eta}$ is the same as the norm of its restriction to $\mathcal{K}(H)$ (by weak* continuity on $\mathcal{B}(H)$). Hence

$$\max_{1\leqslant k\leqslant n} |\langle x_k\eta, \xi\rangle| \leqslant f_T(\xi, \eta) \leqslant \Big(\max_{1\leqslant k\leqslant n} |\langle x_k\eta, \xi\rangle|\Big) + \varepsilon$$

and so we have $F_\varepsilon \colon H_1 \times H_1 \to \mathbb{R}$ given by $F_\varepsilon(\xi, \eta) = \max\limits_{1\leqslant k\leqslant n} |\langle x_k\eta, \xi\rangle|$ with

$$\sup_{(\xi,\eta)\in H_1\times H_1} |f_T(\xi, \eta) - F_\varepsilon(\xi, \eta)| < \varepsilon.$$

For $x = \theta_1 \otimes \theta_2^* \in \mathcal{B}(H)$ of rank 1 $(\theta_1, \theta_2 \in H)$, $(\xi, \eta) \mapsto \langle x\eta, \xi\rangle = \langle \eta, \theta_2\rangle\langle\theta_1, \xi\rangle$ is clearly continuous on $H_1 \times H_1$. So then is $(\xi, \eta) \mapsto \langle x\eta, \xi\rangle$ when x is of finite rank or when $x \in \mathcal{K}(H)$ by approximation. Hence F_ε is continuous on $H_1 \times H_1$. As a uniform limit of continuous functions, f_T must be continuous.

(iii) $\Rightarrow$ (iv): Using the linear independence assumption and Lemma 1.9 we can find $\eta_1, \eta_2, \ldots, \eta_n \in H$ so that $\sum_{k=1}^{n} Q(\mathbf{b}, \eta_k)$ is positive definite. We can scale the η_k so that $\eta_k \in H_1$ for $1 \leqslant k \leqslant n$.

Now, (weak) continuity of $f_T(\xi, \eta)$ at points $(0, \eta)$ implies continuity of

$$F_T(\xi, \eta) = \operatorname{trace}(Q(\mathbf{a}^*, \xi)Q(\mathbf{b}, \eta))$$

at the same points because

$$f_T(0, \eta) = F_T(0, \eta) = 0 \leqslant F_T(\xi, \eta) \leqslant \ell f_T(\xi, \eta)^2.$$

Thus we also have (weak) continuity at $\xi = 0$ of $\xi \mapsto \sum_{k=1}^{n} F_T(\xi, \eta_k)$. But as there is some $c > 0$ so that $\sum_{k=1}^{n} Q(\mathbf{b}, \eta_k) > cI_\ell$, we have

$$\operatorname{trace} Q(\mathbf{a}^*, \xi) \leqslant \frac{1}{c} \sum_{k=1}^{n} F_T(\xi, \eta_k) \to 0 \text{ as } \xi \to 0 \text{ weakly.}$$

This means that $\sum_{j=1}^{\ell} \langle a_j a_j^* \xi, \xi \rangle \to 0$ as $\xi \to 0$ weakly, or $\|a_j^* \xi\| \to 0$ as $\xi \to 0$ weakly. Hence each a_j^* maps weakly null bounded sequences to norm null sequences and a_j^* is compact. It follows that a_j is compact. A similar argument shows that each b_j is compact.

(iv) $\Rightarrow$ (i): This follows from Vala's theorem. ∎

2. GENERAL C^*-ALGEBRAS

We now extend some of the formulae to the case of $T \in \mathcal{E}\ell(A)$ with A an arbitrary C^*-algebra. For the remainder of this section, $Tx = \sum_{j=1}^{\ell} a_j x b_j$ for $x \in A$ with $a_j, b_j \in M(A)$.

By $P(A)$ we denote the set of pure states of A and for $\phi \in P(A)$ let $\pi_\phi \colon A \to \mathcal{B}(H_{\pi_\phi})$ denote the associated irreducible representation of A (arrived at by the GNS method). Observe that $\phi(x) = \langle \pi_\phi(x)\xi_\phi, \xi_\phi \rangle$ for ξ_ϕ the cyclic vector for the representation. $\widehat{A}$ denotes the (unitary equivalence classes of) irreducible representations of A and we write $\pi_1 \sim \pi_2$ to indicate unitary equivalence of representations. By $\phi_1 \sim \phi_2$ for pure states, we mean equivalence of the associated representations.

Let $T_\pi \colon \mathcal{B}(H_\pi) \to \mathcal{B}(H_\pi)$ be the elementary operator

$$T_\pi(y) = \sum_{j=1}^{\ell} \pi(a_j) y \pi(b_j),$$

using the fact that π extends to $M(A)$. It is known that $\|T\| = \sup_{\pi \in \widehat{A}} \|T_\pi\|$ (for example 5.3.12 in [3]).

For $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^{\mathrm{t}}$ a column of elements $b_j \in M(A)$ and $\phi \in P(A)$ we introduce the notation

$$Q(\mathbf{b}, \phi) = (\phi(b_i^* b_j))_{i,j=1}^{\ell} = \phi^{(\ell)}(Q(\mathbf{b})) \in M_\ell(\mathbb{C}).$$

Here $\phi^{(\ell)} \colon M_\ell(A) \to M_\ell(\mathbb{C})$ is given by $\phi^{(\ell)}((x_{ij})_{i,j=1}^{\ell}) = (\phi(x_{ij}))_{i,j=1}^{\ell}$.

THEOREM 2.1. *For $T \in \mathcal{E}\ell(A)$, $Tx = \sum_{j=1}^{\ell} a_j x b_j$, $\mathbf{a} = [a_1, a_2, \ldots, a_\ell] \in M(A)^\ell$ a row and $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^{\mathrm{t}}$ a column of elements of $M(A)$, we have*

$$\begin{aligned}\|T\| &= \sup\{\mathrm{tgm}(Q(\mathbf{a}^*, \phi_1), Q(\mathbf{b}, \phi_2)) : \phi_1, \phi_2 \in P(A), \phi_1 \sim \phi_2\} \\ &= \sup\left\{\left\|\sqrt{Q(\mathbf{b}, \phi)^{\mathrm{t}}}\,\pi_\phi(\mathbf{a}^*)\right\|_{S1} : \phi \in P(A)\right\}\end{aligned}$$

(where $\pi_\phi(\mathbf{a}^*)$ means $[\pi_\phi(a_1^*), \pi_\phi(a_2^*), \ldots, \pi_\phi(a_\ell^*)]^{\mathrm{t}}$).

Proof. This is immediate from the above remarks together with Theorems 1.3 and 1.8. ∎

NOTATIONS. We now consider the maps $T^{(k)}\colon M_k(A) \to M_k(A)$ on the space $M_k(A)$ of $k \times k$ matrices with entries in A, given by $T^{(k)}((x_{ij})_{i,j=1}^{\ell}) = (Tx_{ij})_{i,j=1}^{\ell}$. We use the canonical C^*-norms on $M_k(A)$ and the notation $\|T\|_k = \|T^{(k)}\|$. The completely bounded norm of T is $\|T\|_{\text{cb}} = \sup_k \|T\|_k$.

Let $F_k(A)$ denote the set of factorial states considered in [8]. They are those states ϕ of A that are convex combinations of at most k unitarily equivalent pure states, or those where the commutant $\pi_\phi(A)'$ of the GNS representation is a type I_n factor with $n \leqslant k$. For a pure state ψ on $M_k(A)$ there is a factorial state $\phi \in F_k(A)$ with

$$\psi\begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & x \end{pmatrix} = \phi(x)$$

for $x \in M(A)$. Conversely, if $\phi = \sum_{j=1}^{k} t_j\psi_j$ is a convex combination of $\psi_j \in P(A)$ ($t_j \geqslant 0$, $\sum_j t_j = 1$) and we take the irreducible representation $\pi\colon A \to \mathcal{B}(H_\pi)$ corresponding to ψ_1, then there are unit vectors $\xi_j \in H$ so that $\psi_j(x) = \langle x\xi_j, \xi_j\rangle$. The unit vector $(\sqrt{t_j}\xi_j)_{j=1}^k \in H^k$ gives rise to a vector state ψ for $M_k(A)$ (acting on H^k via $\pi^{(k)}$). This pure state ψ will relate to ϕ as above.

From 2.1(iii) in [8] we know that if ϕ is a proper convex combination of states in $P(A)$, not all of which are equivalent, then ϕ cannot be factorial. Relying on this, we can say that for $\phi_1, \phi_2 \in F_k(A)$, $(\phi_1 + \phi_2)/2$ is factorial if and only if the pure states in a convex combination making ϕ_1 are each unitarily equivalent to those in a convex combination making ϕ_2. We write $\phi_1 \asymp \phi_2$ to mean that $(\phi_1 + \phi_2)/2$ is factorial.

From Theorem 2.1, we can deduce the following.

COROLLARY 2.2. *For $T \in \mathcal{E}\ell(A)$, $Tx = \sum_{j=1}^{\ell} a_j x b_j$, $\mathbf{a} = [a_1, a_2, \ldots, a_\ell] \in M(A)^\ell$ a row and $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^{\mathrm{t}}$ a column of elements of $M(A)$, we have:*

$$\|T\|_k = \sup\{\operatorname{tgm}(Q(\mathbf{a}^*, \phi_1), Q(\mathbf{b}, \phi_2)) : \phi_1, \phi_2 \in F_k(A), \phi_1 \asymp \phi_2\}.$$

COROLLARY 2.3. *For $T \in \mathcal{E}\ell(A)$, $Tx = \sum_{j=1}^{\ell} a_j x b_j$, we have:*

$$\|T\|_k \leqslant \max(k, \sqrt{\ell})\|T\|, \quad \|T\|_{\text{cb}} \leqslant \sqrt{\ell}\|T\|.$$

Proof. It is well-known that (for any operator $T\colon A \to A$) $\|T\|_k \leqslant k\|T\|$. (See Exercise 3.10(ii) of [20].)

In [26], it was shown that $\|T\|_{\mathrm{cb}} = \|T\|_\ell$ and so we could deduce $\|T\|_k \leqslant \ell\|T\|$ for all k, but we seek the improved bound involving $\sqrt{\ell}$.

With $X = Q(\mathbf{a}^*, \phi_1)$, $Y = Q(\mathbf{b}, \phi_2)$, $\phi_1, \phi_2 \in P(A)$, $\phi_1 \sim \phi_2$, Theorem 2.1 tells us

$$\operatorname{tgm}(X, Y) = \sum_{j=1}^{\ell} \sqrt{\lambda_j(XY)} \leqslant \|T\| \Rightarrow \sum_{j=1}^{\ell} \lambda_j(XY) = \operatorname{trace}(XY) \leqslant \|T\|^2.$$

This latter is a convex condition on X and Y. Hence it remains true on replacing $\phi_1, \phi_2 \in P(A)$ by convex combinations. Hence for $\phi_1, \phi_2 \in F_k(A)$ with $\phi_1 \asymp \phi_2$,

$$\operatorname{trace} Q(\mathbf{a}^*, \phi_1) Q(\mathbf{b}, \phi_2) \leqslant \|T\|^2 \Rightarrow \operatorname{tgm}(Q(\mathbf{a}^*, \phi_1), Q(\mathbf{b}, \phi_2)) \leqslant \sqrt{\ell}\|T\|$$

(by the Cauchy Schwarz inequality for $\sum_j \sqrt{\lambda_j}$). Thus Corollary 2.2 implies $\|T\|_k \leqslant \sqrt{\ell}\|T\|$. ∎

EXAMPLE 2.4. (i) The well-known transpose example $T\colon M_n \to M_n$, with $Tx = x^{\mathrm{t}}$ has $\|T\| = 1$ while $\|T\|_{\mathrm{cb}} = \|T\|_n = n$. As an elementary operator $Tx = \sum_{i,j=1}^{n} e_{ij} x e_{ij}$ (with e_{ij} the matrix with 1 in the (i, j) place and zeros elsewhere) and has $\ell = n^2$. Thus the familiar estimate $\|T\|_k \leqslant k\|T\|$ cannot be improved in general.

For $Sx = e_{11} x^{\mathrm{t}} = \sum_{j=1}^{n} e_{1j} x e_{1j}$ the first row of the transpose, one can check that $\|S\|_n = \sqrt{n}$ (and $\|S\| = 1$) so that the $\sqrt{\ell}$ bound is also optimal in a sense.

For this S, it is true that $\|S\|_k = \sqrt{k}$ for $1 \leqslant k \leqslant n$, while for the transpose example $\|T\|_k = k$ ($1 \leqslant k \leqslant n$).

(ii) A natural question would be to describe the functions of k that can arise as $f(k) = \|T\|_k$ for $T \in \mathcal{E}\ell(A)$. This would mean finding further refinements of the estimates we have to pin down the possibilities for $f(k)$. For example $f(k_1 k_2) \leqslant k_1 f(k_2)$ follows because $T^{(k_1 k_2)} = (T^{(k_2)})^{(k_1)}$. It follows that, for $n \leqslant k$, $f(k+n) \leqslant f(2k) \leqslant 2f(k)$.

If $T_1 \in \mathcal{E}\ell(A_1)$ and $T_2 \in \mathcal{E}\ell(A_2)$ then $T_1 \oplus T_2 \in \mathcal{E}\ell(A_1 \oplus A_2)$ satisfies $\|T_1 \oplus T_2\|_k = \max(\|T_1\|_k, \|T_2\|_k)$. Hence we can, for example, combine the identity with the transpose example. Take $T_1 x = x$ and $T_2 x = x^{\mathrm{t}}/m$ on $M_n(\mathbb{C})$ (where $1 < m < n$) to produce $T = T_1 \oplus T_2$ where $f(k) = \|T\|_k = 1$ for $1 \leqslant k \leqslant m$, while $f(k) = k/m$ for $m+1 \leqslant k \leqslant n$. In examples of this type $f(k+1)/f(k)$ can be 1 and $(k+1)/k$ in different intervals.

We can use (1.2) to show that

$$\|T\|_{k+1} \leqslant \left(1 + \frac{2\sqrt{k}}{k+1}\right)\|T\|_k \tag{2.1}$$

holds for general $T \in \mathcal{E}\ell(A)$ and $k = 1, 2, \ldots$. Assume $\|T\|_k \leqslant 1$ and consider a pair of factorial states $\phi_i = \sum_{r=1}^{k+1} t_{ir}\phi_{ir}$ where $\sum_r t_{ir} = 1$, $t_{ir} \geqslant 0$, ϕ_{ir} are pure states that are all unitarily equivalent ($i = 1, 2$, $1 \leqslant r \leqslant k+1$). For at least one r, $t_{ir} \leqslant 1/(k+1)$ and we assume $t_{i1} \leqslant 1/(k+1)$ for $i = 1, 2$. Then write $\phi_i = t_{i1}\phi_{i1} + (1 - t_{i1})\psi_i$ for $\psi_i \in F_k(A)$ ($i = 1, 2$). Let $X = Q(\mathbf{a}^*, \phi_1)$, $Y = Q(\mathbf{b}, \phi_2)$, $X_1 = Q(\mathbf{a}^*, \phi_{1i})$, $X_2 = Q(\mathbf{a}^*, \psi_1)$, $Y_1 = Q(\mathbf{b}, \phi_{21})$, $Y_2 = Q(\mathbf{b}, \psi_2)$ so that $X = t_{11}X_1 + (1 - t_{11})X_2$ and $Y = t_{21}Y_1 + (1 - t_{21})Y_2$. Using (1.2) and $\operatorname{tgm}(X_i, Y_\alpha) \leqslant 1$ for $i, \alpha = 1, 2$ we get

$$\operatorname{tgm}(X, Y) \leqslant (\sqrt{t_{11}} + \sqrt{1 - t_{11}})(\sqrt{t_{21}} + \sqrt{1 - t_{21}}) \leqslant \left(\sqrt{\frac{1}{k+1}} + \sqrt{\frac{k}{k+1}} \right)^2.$$

By Corollary 2.2 we get (2.1).

The following result is shown in [5] but the proof relies in an essential way on Theorem 3.1 in [17] (concerning the case of prime C^*-algebras with zero socle).

COROLLARY 2.5. *If A is an antiliminal C^*-algebra and $T \in \mathcal{E}\ell(A)$, then $\|T\| = \|T\|_{\mathrm{cb}}$.*

Proof. It is known that for antiliminal C^*-algebras (or more generally for "antiliminal by abelian" ones [8]) every factorial state is a weak*-limit of pure states. To deduce the corollary directly from Corollary 2.2, we would need to know further that for $\phi_1, \phi_2 \in F_k(A)$ with $\phi_1 \asymp \phi_2$, we can find a net of pairs of pure states $\phi_{1,\alpha}, \phi_{2,\alpha} \in P(A)$ with $\lim_\alpha \phi_{1,\alpha} = \phi_1$, $\lim_\alpha \phi_{2,\alpha} = \phi_2$ and $\phi_{1,\alpha} \sim \phi_{2,\alpha}$ for each α.

However, consideration of the proof in 11.2.3 of [11] reveals that this further fact is true. From the Kadison transitivity theorem, we can choose a pure state ϕ, unitary $u_{ij} \in \widetilde{A}$ = the unitisation of A and $0 \leqslant t_{ij} \leqslant 1$ ($i = 1, 2$, $1 \leqslant j \leqslant k$) so that

$$\phi_i(x) = \sum_{j=1}^{k} t_{ij}\phi(u_{ij}^* x u_{ij}), \quad \sum_{j=1}^{k} t_{ij} = 1 \quad (i = 1, 2).$$

The state ϕ can then be approximated by a net ψ_α of states where there exists an irreducible representation $\pi_\alpha \colon A \to \mathcal{B}(H_{\pi_\alpha})$ with ψ_α vanishing on the inverse image $\pi_\alpha^{-1}(\mathcal{K}(H_{\pi_\alpha}))$ of the compacts ([11], 11.2.2). By 11.2.1 in [11], the two states

$$\psi_{i\alpha}(x) = \sum_{j=1}^{k} t_{ij}\psi_\alpha(u_{ij}^* x u_{ij}) \quad (i = 1, 2)$$

can be approximated by vector states on $\mathcal{B}(H_{\pi_\alpha})$ composed with π_α, hence by equivalent pure states of A. ∎

3. COMPACT ELEMENTARY OPERATORS

We now characterise compact elementary operators on a C^*-algebra A. For the case $A = \mathcal{B}(H)$, Theorem 1.10 extends known results somewhat. For prime C^*-algebras A, 5.3.26 of [3] gives a characterisation but a similar result for general A seems not to have been established up to now. In addition, we characterise weakly compact elements of $\mathcal{E}\ell(\mathcal{B}(H))$ in a similar way to Theorem 1.10.

Compactness of elements a of a C^*-algebra A (in terms of a being in the closure of the socle, or weak compactness of the operators of left or right multiplication by a, or compactness of the elementary operator $x \mapsto axa$) has been studied by several authors and references can be found in [3], p. 36. The set $\mathcal{K}(A)$ of compact elements of A is a closed ideal in A.

We introduce the notion of the $\mathcal{K}(A)$ topology on the space $E(A) = \{\psi : \psi = \lambda\phi, \phi \in P(A), 0 \leqslant \lambda \leqslant 1\}$ of multiples of pure states (a subset of the dual of A). This means the topology of pointwise convergence on $\mathcal{K}(A)$, so that a net $(\psi_\alpha)_\alpha$ converges to $\psi \in E(A)$ if and only if $\lim_\alpha \psi_\alpha(x) = \psi(x)$ for each $x \in \mathcal{K}(A)$. If $\mathcal{K}(A)$ is small (zero for example) then this will be a very coarse topology on $E(A)$ (even the trivial topology if $\mathcal{K}(A) = 0$). By $R(A)$ we denote the subset of the product $E(A) \times E(A)$ consisting of pairs $(\lambda_1\phi_1, \lambda_2\phi_2)$ with $\phi_1 \sim \phi_2$ unitarily equivalent pure states (and $0 \leqslant \lambda_1, \lambda_2 \leqslant 1$).

When $A = \mathcal{K}(H)$, there is a surjection $\mu\colon H_1 \to E(A)$ given by $\mu(\xi) = \omega_\xi$. This is continuous (from the weak topology on H_1 to the $\mathcal{K}(A)$ topology on $E(A)$). Neighbourhoods of $\omega_\xi \in E(A)$ contain finite intersections of neighbourhoods of the form $N = \{\omega_\eta : |\omega_\eta(x) - \omega_\xi(x)| < 1\}$ with $x = \theta_1 \otimes \theta_2^*$ of rank one ($\theta_1, \theta_2 \in H$). Since $\mu^{-1}(N) = \{\eta \in H_1 : |\langle \eta, \theta_2\rangle\langle\theta_1, \eta\rangle - \langle \xi, \theta_2\rangle\langle\theta_1, \xi\rangle| < 1\}$, continuity of μ is clear. μ is also surjective and the inverse image of ω_ξ consists of multiples $\zeta\xi$ with $\zeta \in \mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. As the quotient space of H_1 by the action of $\mathbb{T}$ is compact in the quotient topology and $E(A)$ is Hausdorff, it follows that μ induces a homeomorphism from the quotient space $H_1/\mathbb{T}$ to $E(A) = E(\mathcal{K}(H))$.

In the case $A = \mathcal{B}(H)$ (with H infinite dimensional) the elements of $E(\mathcal{B}(H))$ not in $E(\mathcal{K}(H))$ are the nonzero multiples $\lambda\phi$ of pure states ϕ of $\mathcal{B}(H)$ vanishing on $\mathcal{K}(H)$, and so they are not separated from 0 in the $\mathcal{K}(\mathcal{B}(H))$ topology of $E(\mathcal{B}(H))$.

For the case of a general C^*-algebra A, we can also analyse the topology on $E(A)$. Pure states $P(A)$ of A are either pure states of $\mathcal{K}(A)$ or vanish on $\mathcal{K}(A)$, so that $P(A) = P(\mathcal{K}(A)) \cup P(A/\mathcal{K}(A))$ is a disjoint union. Note that multiples of states in $P(A/\mathcal{K}(A))$ are in the closure of 0 in the $\mathcal{K}(A)$ topology on $E(A)$. Similarly irreducible representations $\pi\colon A \to \mathcal{B}(H_\pi)$ of A are either irreducible when restricted to $\mathcal{K}(A)$ or may be regarded as irreducible representations of the quotient. We will use the notation $\widehat{A}$ in a *slightly ambiguous way* as also standing for a specific set of representatives of the unitary equivalence classes of irreducibles.

We have then $\widehat{A} = \widehat{\mathcal{K}}(A) \cup (A/\mathcal{K}(A))^{\widehat{}}$ (where we use $\widehat{\mathcal{K}}(A)$ for $(\mathcal{K}(A))^{\widehat{}}$). For $\pi \in \widehat{\mathcal{K}}(A)$ we have $\pi(\mathcal{K}(A)) = \mathcal{K}(H_\pi)$.

It is known that $\mathcal{K}(A)$ is a direct sum of algebras $\mathcal{K}(H_i)$ of compact operators on Hilbert spaces H_i (for $i \in I =$ some index set). This is known because $\mathcal{K}(A)$ is a C^*-algebra of compact elements and we can apply Theorem 8.2 of [1] (or a fact now known to be equivalent shown in Theorem 8.3 of [15]). It follows that the H_i are the H_π with $\pi \in \widehat{\mathcal{K}}(A)$. Also for $\lambda\phi \in E(A)$ with $\lambda \neq 0$ and where $\phi \in P(\mathcal{K}(A))$, π_ϕ is (unitarily equivalent to) some $\pi \in \widehat{\mathcal{K}}(A)$. Hence there is a nonzero vector ξ in the unit ball of H_π so that $\lambda\phi(x) = \omega_\xi(\pi(x))$. By considering $\mathcal{K}(H_\pi)$ as contained in the direct sum $\mathcal{K}(A)$ we see that there exist neighbourhoods of $\lambda\phi$ in $E(A)$ that consist entirely of functionals $\psi \in E(\mathcal{K}(H_\pi)) \setminus \{0\}$. So the complement of the closure of 0 in $E(A)$ is homeomorphic to the disjoint union of $E(\mathcal{K}(H_\pi)) \setminus \{0\}$ (for $\pi \in \widehat{\mathcal{K}}(A)$). Moreover each $E(\mathcal{K}(H_\pi)) \setminus \{0\}$ is homeomorphic to $((H_\pi)_1 \setminus \{0\})/\mathbb{T}$ (for $\pi \in \widehat{\mathcal{K}}(A)$). On the other hand neighbourhoods of 0 in $E(A)$ contain all but finitely many of $E(\mathcal{K}(H_\pi))$ for $\pi \in \widehat{\mathcal{K}}(A)$ (and intersect the remaining $E(\mathcal{K}(H_\pi))$ in neighbourhoods of 0). This is the case because a basic neighbourhood of $\psi \in E(A)$ is an intersection of neighbourhoods of the form $N_{\psi,x} = \{\psi_1 \in E(A) : |\psi_1(x) - \psi(x)| < 1\}$ with $x \in \mathcal{K}(A)$. Any such x can be approximated in norm by $x_0 \in \bigoplus_{j=1}^{n} \mathcal{K}(H_{\pi_j})$, a finitely supported element of the direct sum making up $\mathcal{K}(A)$. Choosing x_0 so that $\|x - x_0\| < 1/4$ we get $N_{\psi,2x_0} \subset N_{\psi,x}$ and the smaller neighbourhood places a restriction only on finitely many $E(\mathcal{K}(H_\pi))$.

For $(\phi,\psi) \in R(A)$, if ϕ is outside the closure $\overline{\{0\}}$ of 0 in the $\mathcal{K}(A)$ topology on $E(A)$ then ψ is either 0 or also outside $\overline{\{0\}}$. We can say that $R(A)$ is the union of $R(\mathcal{K}(A))$ and a set contained in $\overline{\{0\}} \times \overline{\{0\}}$.

THEOREM 3.1. *Let A be a C^*-algebra and $T \in \mathcal{E}\ell(A)$. Then the following are equivalent for T:*

(i) *T is compact.*

(ii) *If $Tx = \sum_{j=1}^{\ell} a_j x b_j$ for $a_j, b_j \in M(A)$, and $f_T \colon R(A) \to \mathbb{R}$ is defined as*

$$f_T(\phi_1,\phi_2) = \operatorname{tgm}(Q(\mathbf{a}^*,\phi_1), Q(\mathbf{b},\phi_2)),$$

then the function f_T is continuous on $R(A)$.

(iii) *T can be expressed as $Tx = \sum_{j=1}^{\ell} a_j x b_j$ for a_j and b_j compact elements of A ($1 \leqslant j \leqslant \ell$).*

Proof. (i) $\Rightarrow$ (ii): Let $Tx = \sum_{j=1}^{\ell} a_j x b_j$.

Compactness of T implies compactness of its double transpose (again $x \mapsto \sum_{j=1}^{\ell} a_j x b_j$) and of its restriction T_{π_a} to the atomic part $\prod_{\pi \in \widehat{A}} \mathcal{B}(H_\pi)$ of the double dual of A (where the same formula $x \mapsto \sum_{j=1}^{\ell} a_j x b_j$ holds modulo identifying elements of $M(A)$ with their images under the reduced atomic representation $\pi_a = \bigoplus_{\pi \in \widehat{A}} \pi$). Let $T_\pi \in \mathcal{E}\ell(\mathcal{B}(H_\pi))$ be the elementary operator arising from T and π. As a restriction of T_{π_a}, T_π is compact. Moreover, for each $\varepsilon > 0$, we have $\{\pi \in \widehat{A} : \|T_\pi\| > \varepsilon\}$ finite. This is a consequence of 5.3.17 in [3], but we offer a self-contained argument.

If not we could find a sequence of unit norm elements $x_n \in \mathcal{B}(H_{\pi_n})$ with $\pi_n \in \widehat{A}$ all distinct and $\|T_{\pi_n}(x_n)\| > \varepsilon$. Inside $\prod_{\pi \in \widehat{A}} \mathcal{B}(H_\pi)$ we have then an infinite dimensional space spanned by the x_n on which T restricts to a topological linear isomorphism, contradicting compactness of T.

As each T_π ($\pi \in \widehat{A}$) is compact, we have $T_\pi(\mathcal{B}(H_\pi)) \subset \mathcal{K}(H_\pi)$ by Theorem 1.10. Putting these facts together, we get $T(A) \subset \mathcal{K}(A)$ (via 1.2.30(g) in [3]).

For $x \in A$ and $(\phi, \psi) \in R(A)$ we define $|x(\phi, \psi)|$ as follows. Choose $\pi \in \widehat{A}$ and $\xi, \eta \in (H_\pi)_1$ with $\phi(x) = \omega_\xi(\pi(x))$, $\psi(x) = \omega_\eta(\pi(x))$, and set $|x(\phi, \psi)| = |\langle \pi(x)\eta, \xi \rangle|$.

Then continuity of f_T can be established using $\|T\| = \sup\{|y(\phi, \psi)| : y \in T(A_1), (\phi, \psi) \in R(A)\}$ (where A_1 is the unit ball of A) and an argument similar to the one used in the proof of (i) $\Rightarrow$ (ii) of Theorem 1.10. Choose $y_1, y_2, \ldots, y_n \in T(B_1) \subset \mathcal{K}(A)$ so that each $y \in T(B_1)$ has $\|y - y_k\| < \varepsilon$ for some $1 \leqslant k \leqslant n$. $f_T(\phi, \psi)$ is approximately $\max_k |y_k(\phi, \psi)|$.

To finish, we need to know that for $z \in \mathcal{K}(A)$ the map

$$(\phi, \psi) \mapsto |z(\phi, \psi)| \colon R(A) \to \mathbb{R}$$

is continuous. We can approximate z in norm by a finitely supported element $z' = \sum_{j=1}^{n} z_j \in \bigoplus_{j=1}^{n} \mathcal{K}(H_{\pi_j})$ in the direct sum making up $\mathcal{K}(A)$. Then $|z'(\phi, \psi)| = \sum_{j=1}^{n} |z_j(\phi, \psi)|$. The continuity of $(\phi, \psi) \mapsto |z_j(\phi, \psi)|$ can be shown by noting first that the map is zero on those parts of $R(A)$ coming from pairs of pure states with GNS representation different from π_j. Consider the weakest topology τ on $R(A)$ so that the maps $(\phi, \psi) \mapsto (\phi(z'), \psi(z''))\colon R(A) \to \mathbb{C}^2$ are continuous for $z', z'' \in \mathcal{K}(H_{\pi_j})$. Note that τ is weaker than the usual topology on $R(A)$. This space $(R(A), \tau)$ is $R(\mathcal{K}(H_{\pi_j}))$ plus points that cannot be separated from $(0,0)$, and it becomes $R(\mathcal{K}(H_{\pi_j})) = E(\mathcal{K}(H_{\pi_j})) \times E(\mathcal{K}(H_{\pi_j}))$ if we identify these points with $(0,0)$. We saw in the proof of Theorem 1.10 that $(\eta, \xi) \mapsto \langle z_j \eta, \xi \rangle$ is continuous on $(H_\pi)_1 \times (H_\pi)_1$ and it follows then that $(\eta, \xi) \mapsto |\langle z_j\eta, \xi\rangle|$ is continuous on $(H_\pi)_1/\mathbb{T} \times (H_\pi)_1/\mathbb{T}$. Hence $(\omega_\xi, \omega_\eta) \mapsto |\langle z_j\eta, \xi\rangle|$ is continuous on $E(\mathcal{K}(H_{\pi_j})) \times E(\mathcal{K}(H_{\pi_j}))$ and $(\phi, \psi) \mapsto |z_j(\phi, \psi)|$ is continuous on $(R(A), \tau)$, thus on $R(A)$.

(ii) $\Rightarrow$ (iii): Let $T \in \mathcal{E}\ell(A)$ and write $Tx = \sum_{j=1}^{\ell} a_{0j} x b_{0j}$ for some $a_{0j}, b_{0j} \in M(A)$. Our aim is to show that there is an alternative way to write T with a_{0j} and b_{0j} replaced by compact elements of A.

First, note that continuity of f_T implies that $f_T(\phi, \psi) = 0$ when $(\phi, \psi) \in R(A/\mathcal{K}(A))$. Since $\|T\|_\pi = \sup\{f_T(\phi, \psi) : (\phi, \psi) \in R(\mathcal{B}(H_\pi))\}$, it follows that $T_\pi = 0$ for $\pi \in (A/\mathcal{K}(A))\widehat{}$. Also note that for each $\varepsilon > 0$, we have $\{\pi \in \widehat{\mathcal{K}(A)} : \|T_\pi\| > \varepsilon\}$ finite. This follows from continuity of f_T at $(0, 0)$ and earlier remarks about neighbourhoods of 0 in $E(A)$.

Let S denote the restriction of T to $\mathcal{K}(A)$ and observe that $S \in \mathcal{E}\ell(\mathcal{K}(A))$ and S has a representation as a sum of ℓ terms.

The function f_S is the restriction of f_T to $R(\mathcal{K}(A))$ and the function f_{S_π} of Theorem 1.10 on $(H_\pi)_1 \times (H_\pi)_1$ is $f_{S_\pi}(\xi, \eta) = f_T(\omega_\xi \circ \pi, \omega_\eta \circ \pi)$. Therefore f_{S_π} is continuous by the relationships between the topologies concerned. Thus each S_π is compact for $\pi \in \widehat{\mathcal{K}}(A)$.

By Theorem 1.10 or 5.3.26 in [3], each S_π can be represented as $S_\pi(x) = \sum_{j=1}^{\ell} a_{\pi j} x b_{\pi j}$ with $a_{\pi j}, b_{\pi j} \in \mathcal{K}(H_\pi)$. We can moreover arrange that

$$\|S_\pi\|_{\mathrm{cb}} = \Big\| \sum_{j=1}^{\ell} a_{\pi j} \otimes b_{\pi j} \Big\|_h = \Big\| \sum_{j=1}^{\ell} a_{\pi j} a_{\pi j}^* \Big\| = \Big\| \sum_{j=1}^{\ell} b_{\pi j}^* b_{\pi j} \Big\|$$

using the Haagerup's theorem ([3], 5.4.7) and the fact that the infimum defining the Haagerup norm of a tensor can be realised without increasing the length of its representation or the span of either $\{a_{\pi j} : 1 \leqslant j \leqslant \ell\}$ or $\{b_{\pi j} : 1 \leqslant j \leqslant \ell\}$ ([12], Proposition 9.2.6). By Corollary 2.3, $\|S_\pi\|_{\mathrm{cb}} \leqslant \sqrt{\ell}\|S_\pi\|$ and so

$$\max(\|a_{\pi j}\|^2, \|b_{\pi j}\|^2) \leqslant \sqrt{\ell}\|S_\pi\|.$$

It follows that we can define $a_j, b_j \in \bigoplus_{\pi \in \widehat{\mathcal{K}}(A)} \mathcal{K}(H_\pi) = \mathcal{K}(A)$ by $a_j = (a_{\pi j})_{\pi \in \widehat{\mathcal{K}}(A)}$ and $b_j = (b_{\pi j})_{\pi \in \widehat{\mathcal{K}}(A)}$. Then we have $Sx = \sum_{j=1}^{\ell} a_j x b_j$ for $x \in \mathcal{K}(A)$ since $\pi(Sx) = S_\pi(\pi(x)) = \sum_{j=1}^{\ell} a_{\pi j} \pi(x) b_{\pi j} = \pi\Big(\sum_{j=1}^{\ell} a_j x b_j\Big)$ for $\pi \in \widehat{\mathcal{K}}(A)$.

Let $T_1 \in \mathcal{E}\ell(A)$ be $T_1 x = \sum_{j=1}^{\ell} a_j x b_j$. We have $T_1 x = Sx = Tx$ for $x \in \mathcal{K}(A)$.

Finally we show $T_1 = T$ to complete the proof. Let $x \in A$. For any $\pi \in (A/\mathcal{K}(A))\widehat{}$ we have $\pi(Tx) = 0 = \pi(T_1 x)$. For $\pi \in \widehat{\mathcal{K}}(A)$, $\pi(\mathcal{K}(A)) = \mathcal{K}(H_\pi)$.

Since $(T_1)_\pi = T_\pi = S_\pi$ on $\mathcal{K}(H_\pi)$, a density argument shows $(T_1)_\pi = T_\pi$. Thus $\pi(Tx) = T_\pi(\pi(x)) = (T_1)_\pi(\pi(x)) = \pi(T_1 x)$ for $\pi \in \widehat{\mathcal{K}}(A)$ also.

(iii) $\Rightarrow$ (i): If a_j and b_j are each compact ($1 \leqslant j \leqslant \ell$) then T is compact by Vala's theorem (as remarked in the proof of 5.3.26 in [3]). ∎

We now consider weak compactness of elementary operators on $\mathcal{B}(H)$ (for H infinite dimensional). Our arguments will involve the matrix valued essential numerical ranges $Q(\mathbf{a}^*, \phi)$ and $Q(\mathbf{b}, \phi)$ where ϕ is a state of the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$. A fact we rely upon is that all states ϕ of the Calkin algebra (or states of $\mathcal{B}(H)$ vanishing on $\mathcal{K}(H)$) are weak*-limits of vector states $\omega_\xi(x) = \langle x\xi, \xi\rangle$ (for unit vectors $\xi \in H$) ([11], 11.2.1). That is, for any state ϕ vanishing on $\mathcal{K}(H)$ there is a net ω_{ξ_α} so that $\phi(x) = \lim_\alpha \omega_{\xi_\alpha}(x)$ for all $x \in \mathcal{B}(H)$. Taking x to be a rank one operator $\eta^* \otimes \eta$ we see that $0 = \phi(\eta^* \otimes \eta) = \lim_\alpha |\langle \xi_\alpha, \eta\rangle|^2$ and so $\xi_\alpha \to 0$ weakly in H.

LEMMA 3.2. *If $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^{\mathrm{t}}$ is an ℓ-tuple of elements of $\mathcal{B}(H)$ ($\ell \geqslant 1$), then $b_1, b_2, \ldots, b_\ell$ are linearly independent modulo $\mathcal{K}(H)$ if and only if there exists a weakly null net $(\xi_\alpha)_\alpha$ of unit vectors in H such that*

$$\lim_\alpha Q(\mathbf{b}, \xi_\alpha)$$

is positive definite.

Proof. We assume that the ℓ-tuple of elements $b_j + \mathcal{K}(H)$ ($1 \leqslant j \leqslant \ell$) of the Calkin algebra are linearly independent. We apply Lemma 1.9, by temporarily representing $\mathcal{B}(H)/\mathcal{K}(H)$ as an algebra of operators on some Hilbert space. Normalising the vectors we obtain from Lemma 1.9 so that the sum of the squares of their norms is 1, we see that we have a state ϕ of $\mathcal{B}(H)/\mathcal{K}(H)$ so that $Q(\mathbf{b}, \phi)$ is positive definite.

Applying the above remarks about states of the Calkin algebra we deduce the existence of the net $(\xi_\alpha)_\alpha$ as stated.

For the converse, given a weakly null net $(\xi_\alpha)_\alpha$ of unit vectors as in the statement, we can pass to a subnet and assume that the vector states ω_{ξ_α} converge weak* to some state ϕ of $\mathcal{B}(H)$. One can easily see that ϕ vanishes on rank one operators and so on $\mathcal{K}(H)$. Note that $Q(\mathbf{b}, \phi) = \lim_\alpha Q(\mathbf{b}, \omega_{\xi_\alpha})$ is positive definite. If a linear combination $\sum_{j=1}^{\ell} \lambda_j b_j \in \mathcal{K}(H)$, then for $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_\ell]$ we can compute

$$0 = \phi\Big(\Big(\sum_{j=1}^{\ell} \lambda_j b_j\Big)^* \Big(\sum_{j=1}^{\ell} \lambda_j b_j\Big)\Big) = \lambda^* Q(\mathbf{b}, \phi)^{\mathrm{t}} \lambda = 0$$

and so $\lambda = 0$. ∎

THEOREM 3.3. *Let* $\mathbf{a} = [a_1, a_2, \ldots, a_\ell] \in \mathcal{B}(H)^\ell$, $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^{\mathrm{t}} \in \mathcal{B}(H)^\ell$ *and* $Tx = \sum_{j=1}^{\ell} a_j x b_j$. *Then the following are equivalent:*

(i) $T\colon \mathcal{B}(H) \to \mathcal{B}(H)$ *is weakly compact.*

(ii) $T\colon \mathcal{K}(H) \to \mathcal{K}(H)$ *is weakly compact.*

(iii) *For* $f_T\colon H_1 \times H_1 \to \mathbb{C}$ *given by*

$$f_T(\xi, \eta) = \mathrm{tgm}(Q(\mathbf{a}^*, \xi), Q(\mathbf{b}, \eta)),$$

the function f_T *is continuous at* $(0, 0)$ *(in the product weak topology on* $H_1 \times H_1$*, with* H_1 *the closed unit ball of* H *as before).*

(iv) *The function* $g_T(\xi) = f_T(\xi, \xi)$ *on* H_1 *is continuous at 0.*

(v) *There exist* $c_1, c_2, \ldots, c_\ell \in \mathcal{K}(H)$, $d_1, d_2, \ldots, d_\ell \in \mathcal{B}(H)$ *and* $0 \leqslant m \leqslant \ell$ *so that*

$$Tx = \sum_{j=1}^{m} c_j x d_j + \sum_{j=m+1}^{\ell} d_j x c_j\,.$$

Proof. In the case when H is finite dimensional, the statements are all true about every $T \in \mathcal{E}\ell(\mathcal{B}(H))$. So we assume that H is infinite dimensional.

(i) $\Rightarrow$ (ii) is clear by restriction of T.

(ii) $\Rightarrow$ (i) is clear because the elementary operator on $\mathcal{B}(H)$ is the double transpose of the operator on $\mathcal{K}(H)$.

(i) $\Rightarrow$ (iii): If f_T is not continuous at $(0, 0)$ there exist weakly null nets $(\xi_\alpha)_\alpha$ and $(\eta_\alpha)_\alpha$ in H_1 so that $f_T(\xi_\alpha, \eta_\alpha)$ does not converge to 0. Taking a subnet we can assume that $\|\xi_\alpha\|$ and $\|\eta_\alpha\|$ are both bounded away from 0 and then we can normalise them to be unit vectors. Passing to subnets again we assume $\lim_\alpha f_T(\xi_\alpha, \eta_\alpha)$ exists and is nonzero. Passing to further subnets we can assume that $\omega_{\xi_\alpha} \to \phi_1$ and $\omega_{\eta_\alpha} \to \phi_2$ for states ϕ_1 and ϕ_2 of $\mathcal{B}(H)$ vanishing on $\mathcal{K}(H)$. Now

$$\mathrm{tgm}(Q(\mathbf{a}^*, \phi_1), Q(\mathbf{b}, \phi_2)) \neq 0.$$

The state $\phi = (\phi_1 + \phi_2)/2$ satisfies

$$\mathrm{tgm}(Q(\mathbf{a}^*, \phi), Q(\mathbf{b}, \phi)) \geqslant \frac{1}{2}\,\mathrm{tgm}(Q(\mathbf{a}^*, \phi_1), Q(\mathbf{b}, \phi_2)) > 0.$$

As all states on the Calkin algebra are weak*-limits of pure states (by 11.2.4 of [11] and Theorem 3.3 of [14]), there must be a pure state ψ of $\mathcal{B}(H)/\mathcal{K}(H)$ so that

$$\mathrm{tgm}(Q(\mathbf{a}^*, \psi), Q(\mathbf{b}, \psi)) \neq 0.$$

Thus the operator induced by T on the Calkin algebra has nonzero norm by Theorem 2.1.

Consider an arbitrary $x \in \mathcal{B}(H)$ (of norm $\|x\| \leqslant 1$). Take the net of all finite rank projections P on H ordered by range inclusion. Then the net $x_P = (1 - P)x(1 - P)$ converges to 0 in the weak*-topology on $\mathcal{B}(H)$ and so $T(x_P) \to 0$ in the weak*-topology (by weak*-continuity of $T \in \mathcal{E}\ell(\mathcal{B}(H))$). By weak compactness of T, a subnet of $T(x_P)$ must converge weakly to 0.

Let $\pi = \pi_\psi \colon \mathcal{B}(H)/\mathcal{K}(H) \to \mathcal{B}(H_\pi)$ be the irreducible representation determined by the pure state ψ and let θ_ψ be the cyclic vector for the representation. By the proofs of Lemma 1.1 and Theorem 1.3, we can see that $\mathrm{tgm}(Q(\mathbf{a}^*,\psi), Q(\mathbf{b},\psi))$ is the norm of the linear functional $y \mapsto \langle (T^\pi y)\theta_\psi, \theta_\psi\rangle$. Choose $y \in \mathcal{B}(H_\pi)$ so that $\langle (T^\pi y)\theta_\psi, \theta_\psi\rangle \neq 0$ and $x \in \mathcal{B}(H)$ so that $\pi(x) = y$. Then $\psi(Tx) \neq 0$. Since $x - x_P \in \mathcal{K}(H)$, it follows that $Tx - Tx_P \in \mathcal{K}(H)$ and so $|\psi(Tx_P)| = |\psi(Tx)| > 0$. This contradicts any subnet $Tx_P \to 0$ weakly.

(iii) $\Rightarrow$ (iv) is clear.

(iv) $\Rightarrow$ (i): Consider a net $(x_\alpha)_\alpha$ in the unit ball of $\mathcal{B}(H)$. It has a subnet converging weak* to some x_0. We call the subnet $(x_\alpha)_\alpha$ again and aim to show $Tx_\alpha \to Tx_0$ weakly.

We denote the dual spaces of $\mathcal{K}(H)$ by $\mathcal{K}(H)'$ and of $\mathcal{B}(H)$ by $\mathcal{B}(H)'$. Since $\mathcal{K}(H)$ is an M-ideal in $\mathcal{B}(H)$, $\mathcal{B}(H)'$ is an ℓ^1 direct sum of those functionals vanishing on $\mathcal{K}(H)$ (or $(\mathcal{B}(H)/\mathcal{K}(H))'$) and a complement which is the canonical image of $\mathcal{K}(H)'$ in its double dual $\mathcal{K}(H)''' = \mathcal{B}(H)'$. Thus every functional γ on $\mathcal{B}(H)$ is a sum of (singular and normal) functionals $\gamma_s \in (\mathcal{B}(H)/\mathcal{K}(H))'$ and $\gamma_n \in \mathcal{K}(H)'$. We know $\lim_\alpha \gamma_n(Tx_\alpha) = \gamma_n(Tx_0)$ by weak*-continuity of T and so we concentrate on establishing $\lim_\alpha \gamma_s(Tx_\alpha) = \gamma_s(Tx_0)$.

As γ_s can be expressed as a linear combination of 4 states, it is enough to deal with the case when $\gamma_s = \phi$ is a state of the Calkin algebra. But in that case, we know we can express ϕ as a weak*-limit $\phi(y) = \lim_\beta \omega_{\xi_\beta}(y)$ $(y \in \mathcal{B}(H))$ for a weakly null net $(\xi_\beta)_\beta$ of unit vectors in H. The norm of the functional $x \mapsto \phi(Tx)$ is

$$\mathrm{tgm}(Q(\mathbf{a}^*,\phi), Q(\mathbf{b},\phi)) = \lim_\beta \mathrm{tgm}(Q(\mathbf{a}^*,\omega_{\xi_\beta}), Q(\mathbf{b},\omega_{\xi_\beta})) = \lim_\beta f_T(\xi_\beta,\xi_\beta) = 0$$

by (iv). Hence $\phi(Tx_\alpha) = \phi(Tx_0) = 0$ for all α.

(iii) $\Rightarrow$ (v): Consider a maximal subset of $\{a_1, a_2, \ldots, a_\ell\}$ which is linearly independent modulo $\mathcal{K}(H)$. By renumbering, we can assume this maximal set is $a_1, a_2, \ldots, a_m$ for $0 \leqslant m \leqslant \ell$. (If $m = 0$, all the a_j are compact.) We can then express a_j for $m+1 \leqslant j \leqslant \ell$ as a compact c_j plus a linear combination of $a_1, a_2, \ldots, a_m$. This allows us to write T in the form

$$Tx = \sum_{j=1}^m a_j x b_j' + \sum_{j=m+1}^{\ell} c_j x b_j = T_1 x + T_2 x.$$

It is easy to see that f_{T_2} is continuous at points $\{0\} \times H_1$. As $f_T(\xi,\eta)$ is the norm of the functional $x \mapsto \langle (Tx)\eta, \xi\rangle$, it follows that $f_{T_1}(\eta,\xi) \leqslant f_T(\eta,\xi) + f_{T_2}(\eta,\xi)$ and is continuous at $(0,0) \in H_1 \times H_1$.

Thus it is sufficient to consider the case where $m = \ell$ and the a_j are linearly independent modulo $\mathcal{K}(H)$ and to show that each b_j is compact in this case. By Lemma 3.2 there is a weakly null net $(\xi_\alpha)_\alpha$ in H_1 so that $\lim_\alpha Q(\mathbf{a}^*,\xi_\alpha)$ is positive definite. Thus for α large there is $\epsilon > 0$ so that $Q(\mathbf{a}^*,\xi_\alpha) > \varepsilon I_\ell$. For any net $(\eta_\beta)_\beta$

in H_1 which is weakly null, we can order pairs (α, β) via $(\alpha_1, \beta_1) \leqslant (\alpha_2, \beta_2) \iff \alpha_1 \leqslant \alpha_2$ and $\beta_1 \leqslant \beta_2$ and thereby create a net $((\xi_\alpha, \eta_\beta))_{(\alpha,\beta)}$ which converges to $(0,0)$ in $H_1 \times H_1$. However

$$f_T(\xi_\alpha, \eta_\beta) = \mathrm{tgm}(Q(\mathbf{a}^*, \xi_\alpha), Q(\mathbf{b}, \eta_\beta)) \geqslant \sqrt{\varepsilon}\sqrt{\mathrm{trace}\, Q(\mathbf{b}, \eta_\beta)}$$

and it follows that $\lim_\beta \|b_j \eta_\beta\| = 0$ for each j. Thus each b_j is compact in this case.

(v) $\Rightarrow$ (iii): is easy to verify by writing

$$Tx = \sum_{j=1}^{m} c_j x d_j + \sum_{j=m+1}^{\ell} d_j x c_j = T_1 x + T_2 x,$$

and using $f_T \leqslant f_{T_1} + f_{T_2}$. It is easy to show that f_{T_1} is continuous at points of $\{0\} \times H_1$ and f_{T_2} is continuous at points of $H_1 \times \{0\}$. ∎

A natural question which remains unresolved is whether an analogue of Theorem 3.3 holds for weakly compact elementary operators on (general) C^*-algebras. There are several related results established in Section 5.3 of [3].

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RICHARD M. TIMONEY, SCHOOL OF MATHEMATICS, TRINITY COLLEGE, DUBLIN 2, IRELAND

E-mail address: richardt@maths.tcd.ie

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