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SPATIAL REPRESENTATION OF MINIMAL C^*-TENSOR PRODUCTS OVER ABELIAN C^*-ALGEBRAS

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ABSTRACT. We establish natural links between minimal C^*-tensor products of C^*-algebras over abelian C^*-algebras, whose definition is based on a natural decomposition in fields of C^*-algebras, and spatial W^*-tensor products of W^*-algebras over abelian W^*-algebras, defined up to natural $*$-isomorphism by using appropriate normal $*$-representations.

In particular, we obtain that if C is a unital, abelian C^*-algebra, A_1, A_2 are unital C^*-algebras over C and π_1, π_2 are non-degenerate $*$-representations of A_1 respectively A_2, which coincide on C, are separated by a type I von Neumann algebra with centre equal to the weak operator closure of the image of C and are faithful in a certain stronger sense, then the minimal C^*-tensor product of A_1 and A_2 over C can be identified with the C^*-algebra generated by the images $\pi_1(A_1)$ and $\pi_2(A_2)$ in the spatial W^*-tensor product of their weak operator closures with respect to the weak operator closure of the image of C.

KEYWORDS: *C^*-algebra, von Neumann algebra, tensor product, spatial representation.*

MSC (2000): 46L05, 46L06.

INTRODUCTION

For every C^*-algebra A, let $Z(A) = \{z \in A : az = za \text{ for all } a \in A\}$ be its *centre* and $M(A) = \{x \in A^{**} : Ax \cup xA \subset A\}$ its *multiplier algebra* (see e.g. 3.12 of [15], or 2.2 of [23]).

We recall that a $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ is called *non-degenerate* if for any $0 \neq \xi \in \mathcal{H}$ there is some $a \in A$ with $\pi(a)\xi \neq 0$, or equivalently, if the closed linear span $\mathcal{H}_e$ of $\pi(A)\mathcal{H}$ is equal to $\mathcal{H}$. To a given $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ we always can associate the non-degenerate $*$-representation $A \ni a \longmapsto \pi(a)|\mathcal{H}_e \in \mathcal{B}(\mathcal{H}_e)$. If A is unital and $\pi : A \to \mathcal{B}(\mathcal{H})$ is a non-degenerate $*$-representation, then π carries the unit 1_A of A to the identity map $1_{\mathcal{H}}$ on $\mathcal{H}$.

Every non-degenerate $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ extends to a unique unital $*$-representation $M(\pi) : M(A) \to \mathcal{B}(\mathcal{H})$, which is a $*$-isomorphism of $M(A)$ onto the C^*-subalgebra $\{T \in \mathcal{B}(\mathcal{H}) : \pi(A)T \cup T\pi(A) \subset \pi(A)\} \subset \mathcal{B}(\mathcal{H})$ whenever π is injective (see e.g. 3.12 of [15] or 2.2.11, 2.2.16, 2.2.17 in [23]). More precisely, $M(\pi)$ is the restriction to $M(A)$ of the normal extension $A^{**} \to \mathcal{B}(\mathcal{H})$ of π, so $\pi(A)$ and $M(\pi)(M(A))$ generate the same von Neumann algebra.

Let now C be a unital, abelian C^*-algebra and let Ω denote its Gelfand spectrum. If A is a C^*-algebra and $\iota : C \to Z(M(A))$ is an injective, unital $*$-homomorphism, then we say that (A, ι), or simply A if ι is clear from the context, is a *C^*-algebra over C*. In this case, for any non-degenerate $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$, the composition $\pi \circ \iota = M(\pi) \circ \iota$ can be considered.

If (A, ι) is a C^*-algebra over C, then

$$I_\iota(t) = \overline{\{\iota(c) : c \in C, c(t) = 0\}A}, \quad t \in \Omega \tag{0.1}$$

are closed two-sided ideals in A. We shall call them *Glimm ideals*. Let $\pi_{\iota,t}$ denote the canonical map $A \to A/I_\iota(t)$. Then we have $\bigcap\limits_{t\in\Omega} I_\iota(t) = \{0\}$, that is $\|a\| = \sup\limits_{t\in\Omega} \|\pi_{\iota,t}(a)\|$ for all $a \in A$ (see Remarks on page 232 in [7]). We notice that the functions

$$\Omega \ni t \longmapsto \|\pi_{\iota,t}(a)\|, \quad a \in A$$

are always upper semi-continuous (see Lemma 9 in [7] or Lemma 3.1 in [24] or Lemma 2.3 in [12]), but they are in general not continuous. If they are continuous, then (A, ι) will be called a *continuous C^*-algebra over C*.

C^*-tensor products of C^*-algebras over C were already considered by G.A. Elliott [5] and G.G. Kasparov ([11], 1.6), but a systematic study of such tensor products was undertaken only later by É. Blanchard [1], [2], B. Magajna [13] and T. Giordano and J. Mingo [6].

Let (A_1, ι_1) and (A_2, ι_2) be C^*-algebras over C and let us consider the $*$-homomorphisms

$$\pi_{\iota_1,t} \otimes \pi_{\iota_2,t} : A_1 \otimes A_2 \longrightarrow (A_1/I_{\iota_1}(t)) \otimes (A_2/I_{\iota_2}(t)), \quad t \in \Omega,$$

where $\otimes$ stands for the algebraic tensor product over $\mathbb{C}$. On every quotient $(A_1/I_{\iota_1}(t)) \otimes (A_2/I_{\iota_2}(t))$ there exists the least C^*-norm $\|\cdot\|_{\min}$ (see [22] or 6.4 in [14]) and

$$A_1 \otimes A_2 \ni a \longmapsto \|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|_{\min}$$

is a C^*-seminorm. Following É. Blanchard, the *minimal C^*-tensor product of A_1 and A_2 over C* is defined as the Hausdorff completion $A_1 \otimes_{C,\min} A_2$ of $A_1 \otimes A_2$ with respect to the C^*-seminorm

$$A_1 \otimes A_2 \ni a \longmapsto \|a\|_{C,\min} = \sup_{t\in\Omega} \|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|_{\min}, \tag{0.2}$$

that is the C^*-algebra obtained by the completion of the quotient $*$-algebra

$$(A_1 \otimes A_2)/\mathcal{J}_C \text{ with } \mathcal{J}_C = \{a \in A_1 \otimes A_2 : (\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a) = 0, t \in \Omega\}$$

relative to the C^*-norm induced by $\| \cdot \|_{C,\min}$.

On the other hand, spatial tensor products of W^*-algebras over abelian W^*-algebras were considered by Ş. Strătilă and L. Zsidó. They showed in Lemma 5.2 of [20] that if Z is an abelian W^*-algebra, M_1, M_2 are W^*-algebras and $\iota_1 : Z \longrightarrow Z(M_1)$, $\iota_2 : Z \longrightarrow Z(M_2)$ are injective unital, normal $*$-homomorphisms, then there exist injective unital, normal $*$-representations $\pi_1 : M_1 \longrightarrow \mathcal{B}(\mathcal{H})$, $\pi_2 : M_2 \longrightarrow \mathcal{B}(\mathcal{H})$ on the same Hilbert space $\mathcal{H}$, such that $\pi_1 \circ \iota_1 = \pi_2 \circ \iota_2$ and $\pi_1(M_1) \subset N, \pi_2(M_2) \subset N'$ for some type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre equal to $(\pi_j \circ \iota_j)(Z)$. On the other hand, according to Lemma 5.4 of [20], if $\rho_1 : M_1 \longrightarrow \mathcal{B}(\mathcal{K})$, $\rho_2 : M_2 \longrightarrow \mathcal{B}(\mathcal{K})$ are any injective unital normal $*$-representations such that $\rho_1 \circ \iota_1 = \rho_2 \circ \iota_2$ and $\rho_1(M_1) \subset R, \rho_2(M_2) \subset R'$ for some type I von Neumann algebra $R \subset \mathcal{B}(\mathcal{K})$ with centre equal to $(\rho_j \circ \iota_j)(Z)$, then there is a $*$-isomorphism

$$\Theta : \pi_1(M_1) \vee \pi_2(M_2) \longrightarrow \rho_1(M_1) \vee \rho_2(M_2)$$

satisfying

$$\Theta(\pi_1(x_1)\pi_2(x_2)) = \rho_1(x_1)\rho_2(x_2) \quad \text{for all } x_1 \in M_1, x_2 \in M_2.$$

In other words, the von Neumann algebra $\pi_1(M_1) \vee \pi_2(M_2)$ is unique up to canonical $*$-isomorphism. Since in the case $Z = \mathbb{C}$ it is $*$-isomorphic to the usual spatial tensor product (over $\mathbb{C}$) $M_1 \overline{\otimes} M_2$ (see Lemma 2 of [3]), it is natural to call it in the general case the *spatial W^*-tensor product of M_1 and M_2 over Z*.

The goal of this paper is to link the minimal C^*-tensor product with the spatial W^*-tensor product.

The first main result (Theorem 3.4) claims that if C is a unital abelian C^*-algebra, (A_1, ι_1) and (A_2, ι_2) are C^*-algebras over C and $\pi_j : A_j \longrightarrow \mathcal{B}(\mathcal{H})$, $j = 1, 2$, are non-degenerate $*$-representations such that

$$M(\pi_1) \circ \iota_1 = M(\pi_2) \circ \iota_2 \quad \text{and} \quad \pi_1(A_1) \subset N, \pi_2(A_2) \subset N' \tag{0.3}$$

for some type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $(M(\pi_j) \circ \iota_j)(C)''$, then there exists a $*$-representation of $A_1 \otimes_{C,\min} A_2$ on $\mathcal{H}$, which carries the canonical image $(a_1 \otimes a_2)/\mathcal{J}_C \in (A_1 \otimes A_2)/\mathcal{J}_C$ of any $a_1 \otimes a_2 \in A_1 \otimes A_2$ to $\pi_1(a_1)\pi_2(a_2)$. This $*$-representation is uniquely determined and we denote it by $\pi_1 \otimes_{C,\min} \pi_2$. Clearly, $\pi_1 \otimes_{C,\min} \pi_2$ maps the minimal C^*-tensor product $A_1 \otimes_{C,\min} A_2$ into the spatial W^*-tensor product $\pi_1(A_1)'' \vee \pi_2(A_2)''$ of $\pi_1(A_1)''$ and $\pi_2(A_2)''$ over $(\pi_j \circ \iota_j)(C)''$.

In Section 4 Glimm ideals are described in terms of a faithful spatial representation. As an application, $\mathcal{J}_C$ is characterized in terms of faithful non-degenerate $*$-representations $\pi_j : A_j \to \mathcal{B}(\mathcal{H})$ satisfying (0.3) (Corollary 4.6).

Finally, in Section 5 we first exhibit an example of faithful π_1 and π_2 for which $\pi_1 \otimes_{C,\min} \pi_2$ is not faithful (Proposition 5.2). Subsequently we prove criteria for faithful non-degenerate $*$-representations $\pi_j : A_j \to \mathcal{B}(\mathcal{H})$ satisfying (0.3) in order that $\pi_1 \otimes_{C,\min} \pi_2$ be faithful (Theorem 5.5). It will follow that if A_1, A_2 are unital and π_1, π_2 are faithful in a stronger sense, then $\pi_1 \otimes_{C,\min} \pi_2$ will be faithful, providing thus an identification of the minimal C^*-tensor product $A_1 \otimes_{C,\min} A_2$ with the C^*-subalgebra of the spatial W^*-tensor product $\pi_1(A_1)'' \vee \pi_2(A_2)''$ generated by the images $\pi_1(A_1)$ and $\pi_2(A_2)$ (Corollary 5.7).

For the basic facts concerning C^*-algebras and von Neumann algebras we send to the standard textbooks [4], [10], [14], [15], [16] and [19].

1. PRELIMINARIES RELATED WITH SPATIAL W^*-TENSOR PRODUCTS OVER ABELIAN W^*-ALGEBRAS

In Lemma 2.2 of [20], the commutation theorem of M. Tomita was extended to the frame of spatial W^*-tensor products over abelian W^*-subalgebras. The proof of this general commutative theorem is based on a careful analysis of the Z_h-submodule and Z-submodule of Ne, where N is a type I W^*-algebra with centre Z and e is an abelian projection in N, performed in Section 1 of [20]. In this section we recall certain facts concerning such submodules, completing them when our needs require this.

Let N be a type I von Neumann algebra with centre Z. If e is an abelian projection in N with central support $\mathrm{z}_N(e)$, then the map

$$Z\,\mathrm{z}_N(e) \ni z\,\mathrm{z}_N(e) \longmapsto z\,\mathrm{z}_N(e)e = ze \in eNe \tag{1.1}$$

is a $*$-isomorphism. For every $x \in N$, we denote the inverse image of exe in $Z\,\mathrm{z}_N(e)$ under this isomorphism by $\Phi_e(x)$. Then $\Phi_e : N \longrightarrow Z\,\mathrm{z}_N(e)$ is a normal positive Z-module mapping with $\Phi_e(1_N) = \mathrm{z}_N(e)$, uniquely defined by the equality

$$exe = \Phi_e(x)e, \quad x \in N \tag{1.2}$$

(see e.g. [8], [9]). Moreover, since (1.1) is isometric, we have

$$\|exe\| = \|\Phi_e(x)\|, \quad x \in N. \tag{1.3}$$

Furthermore, if $\mathrm{z}_N(e) = 1_N$, then Φ_e is a normal conditional expectation of N onto Z with support e.

The next three simple lemmas concerning abelian projections are variants of well known results. They are exposed here for further reference, for the convenience of the reader:

LEMMA 1.1. *Let N be a type* I *von Neumann algebra. If $f, p \in N$ are projections, $f \leqslant p$ and f is abelian, then there exists an abelian projection $e \in N$ such that*

$$f \leqslant e \leqslant p, \quad \mathrm{z}_N(e) = \mathrm{z}_N(p).$$

Proof. Let us first consider the case $f = 0$. Since N is of type I, so is pNp too. Let e be an abelian projection in pNp with central support one, that is $\mathrm{z}_{pNp}(e) =$

p. Since

$$exeye = e(pxep)(pyep) = e(pyep)(pxep) = eyexe, \quad x, y \in N,$$

e is an abelian projection also in N. Clearly, $e \leqslant p$ implies $\mathrm{z}_N(e) \leqslant \mathrm{z}_N(p)$. On the other hand, since $e \leqslant p\,\mathrm{z}_N(e)p \in Z(pNp)$ and $\mathrm{z}_{pNp}(e) = p$, we have $p \leqslant p\,\mathrm{z}_N(e)p = p\,\mathrm{z}_N(e) \leqslant \mathrm{z}_N(e)$. Consequently also the converse inequality $\mathrm{z}_N(p) \leqslant \mathrm{z}_N(e)$ holds.

The case of a general f can be reduced to the above treated case. Indeed, by the above part of the proof there is an abelian projection $e_0 \in N$ such that

$$e_0 \leqslant p - p\,\mathrm{z}_N(f), \quad \mathrm{z}_N(e_0) = \mathrm{z}_N(p - p\,\mathrm{z}_N(f)) = \mathrm{z}_N(p) - \mathrm{z}_N(f)$$

and then $e = f + e_0 \in N$ will be an abelian projection satisfying $f \leqslant e \leqslant p$ and $\mathrm{z}_N(e) = \mathrm{z}_N(p)$. ∎

LEMMA 1.2. *Let N be a type* I *von Neumann algebra. Then*

$$\|x\| = \sup\{\|xv\| : v \in N \textit{ partial isometry}, v^*v \leqslant e\}, \quad x \in N$$

holds for any abelian projection $e \in N$ with $\mathrm{z}_N(e) = 1_N$. On the other hand,

$$\|x\|^2 = \sup\{\|\Phi_e(x^*x)\| : e \in N \textit{ abelian projection}, \mathrm{z}_N(e) = 1_N\}, \quad x \in N.$$

Proof. First we prove that

$$\|x\| = \sup\{\|xf\| : f \in N \text{ abelian projection }\}, \quad x \in N. \tag{1.4}$$

For let $x \in N$ and $\varepsilon > 0$ be arbitrary. By the spectral theorem there exists a projection $p \in N$ commuting with x^*x such that

$$x^*xp \geqslant (\|x^*x\| - \varepsilon)p \quad \text{and} \quad x^*x(1_N - p) \leqslant (\|x^*x\| - \varepsilon)(1_N - p) \tag{1.5}$$

(see e.g. Corollary 2.21 of [19]). Note that $p \neq 0$, because $p = 0$ would imply $x^*x \leqslant \|x^*x\| - \varepsilon$, a contradiction. Since N is of type I, p majorizes a non-zero abelian projection $f \in N$ and (1.5) yields $fx^*xf = fx^*xpf \geqslant (\|x^*x\| - \varepsilon)f$. Consequently $\|xf\|^2 = \|fx^*xf\| \geqslant (\|x^*x\| - \varepsilon)\|f\| = \|x\|^2 - \varepsilon$.

Now let e be any abelian projection in N with $\mathrm{z}_N(e) = 1_N$. Let further $x \in N$ be arbitrary. Taking into account (1.4), $\|x\| = \sup\{\|xv\| : v \in N$ partial isometry, $v^*v \leqslant e\}$ will follow once we show that for every abelian projection $f \in N$ there exists a partial isometry $v \in N$ such that $v^*v \leqslant e$ and $\|xf\| \leqslant \|xv\|$.

But $\mathrm{z}_N(f) \leqslant 1_N = \mathrm{z}_N(e)$ implies the existence of a partial isometry $v \in N$ such that $vv^* = f, v^*v \leqslant e$ (see e.g. Proposition 4.10 of [19]). Then

$$\|xf\|^2 = \|xfx^*\| = \|xvv^*x^*\| = \|xv\|^2.$$

Finally, let $x \in N$ be arbitrary. Again by (1.4), $\|x\|^2 = \sup\{\|\Phi_e(x^*x)\| : e \in N$ abelian projection, $\mathrm{z}_N(e) = 1_N\}$ will follow once we show that for every abelian projection $f \in N$ there exists an abelian projection $e \in N$ with $\mathrm{z}_N(e) = 1_N$ such that $\|xf\|^2 \leqslant \|\Phi_e(x^*x)\|$.

But Lemma 1.1, applied with $p = 1_N$, implies the existence of an abelian projection $e \in N$ such that $f \leqslant e$ and $\mathrm{z}_N(e) = 1_N$. Then (1.2) yields

$$\|xf\|^2 \leqslant \|xe\|^2 = \|ex^*xe\| = \|\Phi_e(x^*x)e\| \leqslant \|\Phi_e(x^*x)\|. \quad ∎$$

LEMMA 1.3. *Let $N \subset \mathcal{B}(\mathcal{H})$ be a type* I *von Neumann algebra, e an abelian projection in N, and f an abelian projection in N'. Then ef is an abelian projection in $N \vee N'$ with $\mathrm{z}_{N\vee N'}(ef) = \mathrm{z}_N(e)\,\mathrm{z}_{N'}(f)$ and*

$$\Phi_{ef}(xy) = \Phi_e(x)\Phi_f(y), \quad x \in N, y \in N'.$$

Moreover, if $\mathrm{z}_N(e) = \mathrm{z}_{N'}(f)$, then

$$\Phi_e = \Phi_{ef}|_N \quad \textit{and} \quad \Phi_f = \Phi_{ef}|_{N'}.$$

Proof. Let us denote for convenience $Z = Z(N) = Z(N') = Z(N \vee N')$.

Clearly, $ef = fe$ is a projection in $N \vee N'$. Since, for every $x_1, x_2 \in N$ and $y_1, y_2 \in N'$,

$$\begin{aligned}(efx_1y_1ef)(efx_2y_2ef) &= (ex_1ex_2e)(fy_1fy_2f)\\ &= (ex_2ex_1e)(fy_2fy_1f) = (efx_2y_2ef)(efx_1y_1ef),\end{aligned}$$

ef is an abelian projection in $N \vee N'$.

If $p \in Z$ is a projection such that $ef \leqslant p$, then it follows successively:

$$\begin{aligned}&ey'f\xi = y'efp\xi = py'ef\xi \in p\mathcal{H} \quad \text{for all } y' \in N, \xi \in \mathcal{H}, \text{ i.e. } eN'f\mathcal{H} \subset p\mathcal{H};\\ &e\,\mathrm{z}_{N'}(f)\mathcal{H} \subset p\mathcal{H}, \quad \text{i.e. } \mathrm{z}_{N'}(f)e = e\,\mathrm{z}_{N'}(f) \leqslant p;\\ &\mathrm{z}_{N'}(f)ye\xi = ye\,\mathrm{z}_{N'}(f)\xi = ype\,\mathrm{z}_{N'}(f)\xi = py\,\mathrm{z}_{N'}(f)e\xi \in p\mathcal{H}, \quad y \in N, \xi \in \mathcal{H},\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{i.e. } \mathrm{z}_{N'}(f)Ne\mathcal{H} \subset p\mathcal{H};\\ &\mathrm{z}_{N'}(f)\,\mathrm{z}_N(e)\mathcal{H} \subset p\mathcal{H}, \quad \text{i.e. } \mathrm{z}_{N'}(f)\,\mathrm{z}_N(e) \leqslant p.\end{aligned}$$

Therefore $\mathrm{z}_{N'}(f)\,\mathrm{z}_N(e) \leqslant \mathrm{z}_{N\vee N'}(ef)$. But the converse inequality is trivial, so we actually have

$$\mathrm{z}_{N\vee N'}(ef) = \mathrm{z}_{N'}(f)\,\mathrm{z}_N(e). \tag{1.6}$$

Let $x \in N, y \in N'$ be arbitrary. According to (1.2), we deduce

$$efxyef = (exe)(fyf) = \Phi_e(x)e\Phi_f(y)f = \Phi_e(x)\Phi_f(y)ef.$$

Since, by (1.6), we have $\Phi_e(x)\Phi_f(y) \in Z\,\mathrm{z}_N(e)\,\mathrm{z}_{N'}(f) = Z\,\mathrm{z}_{N\vee N'}(ef)$, it follows that $\Phi_{ef}(xy) = \Phi_e(x)\Phi_f(y)$.

Assume now that $\mathrm{z}_N(e) = \mathrm{z}_{N'}(f) = \mathrm{z}_{N\vee N'}(ef)$. Then, for every $x \in N$, $efxef = (exe)f = \Phi_e(x)ef$ and $\Phi_e(x) \in Z\,\mathrm{z}_{N\vee N'}(ef)$ imply that $\Phi_{ef}(x) = \Phi_e(x)$. Therefore $\Phi_e = \Phi_{ef}|_N$. Similarly we deduce also $\Phi_f = \Phi_{ef}|_{N'}$. ∎

The following result concerning the structure of the Z-submodules of Ne, where N is a type I von Neumann algebra with centre Z and e is an abelian projection in N, will be used in the sequel:

LEMMA 1.4. *Let $N \subset \mathcal{B}(\mathcal{H})$ be a type* I *von Neumann algebra with centre Z, and $e \in N$ an abelian projection. If $X \subset Ne$ is a Z-submodule, then there is a unique projection $p \in N$ such that*

$$\overline{X}^s = pNe, \quad \mathrm{z}_N(p) \leqslant \mathrm{z}_N(e),$$

namely p is the orthogonal projection onto $\overline{\mathrm{lin}} X\mathcal{H}$ (the closed linear span of $\{x\xi : x \in X, \xi \in \mathcal{H}\}$). Moreover, if $X = Me$, where $Z \subset M \subset N$ is a von Neumann subalgebra, then

$$p \in M' \cap N, \quad e \leqslant p, \quad \mathrm{z}_N(e) = \mathrm{z}_N(p).$$

Proof. All the above statements, except those concerning central supports, were proved in 1.6 and 1.7 of [20]. For $\mathrm{z}_N(e) \geqslant \mathrm{z}_N(p)$, let $q \in Z$ be a projection majorizing e. Then $xe = xeq = qxe$ for every $x \in M$, so $q(xe\xi) = xe\xi$ for every $\xi \in \mathcal{H}$. Since p is the projection onto $\overline{\mathrm{lin}} Me\mathcal{H}$, it follows that $q \geqslant p$. ∎

We shall need also the following variant of Lemma 1.2 in [20], for which we have just to reproduce the proof of Lemma 1.2 in [20]:

LEMMA 1.5. *Let N be a type* I *von Neumann algebra with centre Z and $e \in N$ an abelian projection. For every $*$-subalgebra $B \subset N$ and $x \in \overline{Be}^s$, $\|x\| = 1$, we have*

$$x \in \overline{\{y \in BeZ_1^+ : \|y\| \leqslant 1\}}^s,$$

where Z_1^+ denotes the set of all elements $z \in Z$ with $0 \leqslant z \leqslant 1_N$.

Proof. Let $x \in \overline{Be}^s$ be such that $\|x\| = 1$. Consider a net

$$Be \ni b_\lambda e = x_\lambda \xrightarrow{s} x.$$

Then $\Phi_e(x_\lambda^* x_\lambda)^{1/2} \xrightarrow{s} \Phi_e(x^* x)^{1/2}$. Let $f, g : [0,\infty) \to [0,1]$ be functions such that

$$f(t) = 1 \quad \text{for } t \leqslant 1; \quad g(t) = 1 \quad \text{for } t \geqslant 1; \quad g(t) = tf(t) \quad \text{for all } t \in [0,\infty).$$

Since f is operator continuous, $Z_h \ni f(\Phi_e(x_\lambda^* x_\lambda)^{1/2}) \xrightarrow{s} f(\Phi_e(x^* x)^{1/2}) = 1_N$ and $\|f(\Phi_e(x_\lambda^* x_\lambda)^{1/2})\| \leqslant 1$ for all λ. Therefore $f(\Phi_e(x_\lambda^* x_\lambda)^{1/2}) x_\lambda \xrightarrow{s} x$ with

$$\begin{aligned}\|f(\Phi_e(x_\lambda^* x_\lambda)^{1/2}) x_\lambda\| &= \|\Phi_e(x_\lambda^* f(\Phi_e(x_\lambda^* x_\lambda)^{1/2})^2 x_\lambda)\| = \|f(\Phi_e(x_\lambda^* x_\lambda)^{1/2})^2 \Phi_e(x_\lambda^* x_\lambda)\| \\ &= \|f(\Phi_e(x_\lambda^* x_\lambda)^{1/2}) \Phi_e(x_\lambda^* x_\lambda)^{1/2}\|^2 = \|g(\Phi_e(x_\lambda^* x_\lambda)^{1/2})\|^2 \leqslant 1,\end{aligned}$$

and $f(\Phi_e(x_\lambda^* x_\lambda)^{1/2}) x_\lambda \in BeZ_1^+$ because $x_\lambda = b_\lambda e$, $\|f(\Phi_e(x_\lambda^* x_\lambda)^{1/2})\| \leqslant 1$. ∎

2. PRELIMINARIES RELATED WITH MINIMAL C^*-TENSOR PRODUCTS OVER ABELIAN C^*-ALGEBRAS

Let C be a unital, abelian C^*-algebra and let Ω denote its Gelfand spectrum. If (A, ι) is a C^*-algebra over C, then also $(M(A), \iota)$ is a C^*-algebra over C. To distinguish between the ideals defined by (0.1) for (A,ι) and for $(M(A),\iota)$, we shall keep the notation

$$I_\iota(t) = \overline{\{\iota(c) : c \in C, c(t) = 0\}A}, \quad t \in \Omega$$

for the ideals of A and shall set

$$\widetilde{I}_\iota(t) = \overline{\{\iota(c) : c \in C, c(t) = 0\}M(A)}, \quad t \in \Omega.$$

Similarly, we keep the notation $\pi_{\iota,t}$ for the canonical map $A \to A/I_\iota(t)$ and shall denote the canonical map $M(A) \to M(A)/\widetilde{I}_\iota(t)$ by $\widetilde{\pi}_{\iota,t}$.

The next proposition establishes a link between $I_\iota(t)$ and $\widetilde{I}_\iota(t)$, as well as between $\pi_{\iota,t}$ and $\widetilde{\pi}_{\iota,t}$ (cf. Lemma 3.4 of [24]):

PROPOSITION 2.1. *Let C be a unital, abelian C^*-algebra, Ω its Gelfand spectrum, and (A,ι) a C^*-algebra over C. Then:*

(i) $\pi_{\iota,t}(\iota(c)a) = c(t)\pi_{\iota,t}(a), t \in \Omega, c \in C, a \in A$;

(ii) $\|\pi_{\iota,t}(a)\| = \inf\limits_{c\in C, c(t)=1} \|\iota(c)a\| = \inf\limits_{c\in C, 0\leqslant c\leqslant 1_C, c(t)=1} \|\iota(c)a\|, t \in \Omega, a \in A$;

(iii) *for any $t \in \Omega$ we have*

$$I_\iota(t) = A \cap \widetilde{I}_\iota(t), \quad \|\pi_{\iota,t}(a)\| = \|\widetilde{\pi}_{\iota,t}(a)\|, a \in A.$$

Proof. (i) Since $\iota(c)a - c(t)a = (\iota(c) - c(t)1_{M(A)})a = \iota(c - c(t)1_C)a \in I_\iota(t)$, we have $\pi_{\iota,t}(\iota(c)a - c(t)a) = 0$.

(ii) Since $\|\pi_{\iota,t}\| \leqslant 1$, by the above proved (i) we have

$$\begin{aligned}\|\pi_{\iota,t}(a)\| &= \inf_{c\in C, c(t)=1} \|c(t)\pi_{\iota,t}(a)\| = \inf_{c\in C, c(t)=1} \|\pi_{\iota,t}(\iota(c)a)\| \leqslant \inf_{c\in C, c(t)=1} \|\iota(c)a\| \\ &\leqslant \inf_{c\in C, 0\leqslant c\leqslant 1_C, c(t)=1} \|\iota(c)a\|.\end{aligned}$$

For the converse inequalities, let $\varepsilon > 0$ be arbitrary. Since $\Big\{ \sum\limits_{j=1}^{n} \iota(c_j)a_j : c_j \in C, c_j(t) = 0, a_j \in A, n \in \mathbb{N}\Big\}$ is dense in $I_\iota(t)$ and $\|\pi_{\iota,t}(a)\| = \|a/I_\iota(t)\| = \inf\{\|a - y\| : y \in I_\iota(t)\}$, there exist $c_1, c_2, \ldots, c_n \in C$ and $a_1, a_2, \ldots, a_n \in A$ such that $c_j(t) = 0$ for all $j = 1, 2, \ldots, n$ and

$$\|\pi_{\iota,t}(a)\| \geqslant \Big\|a - \sum_{j=1}^{n} \iota(c_j)a_j\Big\| - \varepsilon$$

and then there is an open set $t \in V_0 \subset \Omega$ such that $s \in V_0 \Longrightarrow |c_j(s)| < \frac{\varepsilon}{n\|a_j\|}$ for all $1 \leqslant j \leqslant n$. By Urysohn's lemma, there is $c_0 \in C$ such that $0 \leqslant c_0 \leqslant 1_C, c_0(t) = 1$, and $c_0(s) = 0$ for every $s \in \Omega \setminus V_0$. Since $|(c_0c_j)(s)| = 0$ for $s \in \Omega \setminus V_0$ and $|(c_0c_j)(s)| \leqslant \dfrac{\varepsilon}{n\|a_j\|}$ for $s \in V_0$, we have for every $1 \leqslant j \leqslant n$

$\|\iota(c_0c_j)a_j\| \leqslant \|\iota(c_0c_j)\|\|a_j\| \leqslant \frac{\varepsilon}{n\|a_j\|}\|a_j\| = \frac{\varepsilon}{n}$. Therefore

$$\|\pi_{\iota,t}(a)\| + \varepsilon \geqslant \Big\|a - \sum_{j=1}^{n} \iota(c_j)a_j\Big\| \geqslant \Big\|\iota(c_0)a - \sum_{j=1}^{n} \iota(c_0c_j)a_j\Big\|$$

$$\geqslant \|\iota(c_0)a\| - \sum_{j=1}^{n} \|\iota(c_0c_j)a_j\| \geqslant \|\iota(c_0)a\| - \varepsilon,$$

so $\|\pi_{\iota,t}(a)\| + 2\varepsilon \geqslant \|\iota(c_0)a\| \geqslant \inf\limits_{c\in C, 0\leqslant c\leqslant 1_C, c(t)=1} \|\iota(c)a\|$.

(iii) Let $a \in A$ be arbitrary. Applying (ii) to $\pi_{\iota,t}(a)$ and to $\widetilde{\pi}_{\iota,t}(a)$, we get

$$\|\pi_{\iota,t}(a)\| = \inf_{c\in C, c(t)=1} \|\iota(c)a\| = \|\widetilde{\pi}_{\iota,t}(a)\|.$$

In particular, $a \in A \cap \widetilde{I}_\iota(t) \Longrightarrow a \in I_\iota(t)$, hence the inclusion $A \cap \widetilde{I}_\iota(t) \subset I_\iota(t)$ holds. Since the converse inclusion is trivial, we have $I_\iota(t) = A \cap \widetilde{I}_\iota(t)$. ∎

Proposition 2.1(iii) implies immediately:

COROLLARY 2.2. *Let C be a unital, abelian C^*-algebra, Ω its Gelfand spectrum, and (A, ι) a C^*-algebra over C. Then, for every $t \in \Omega$, the map*

$$\rho_{\iota,t} : A/I_\iota(t) \ni \pi_{\iota,t}(a) \longmapsto \widetilde{\pi}_{\iota,t}(a) \in M(A)/\widetilde{I}_\iota(t)$$

is a well defined injective $*$-homomorphism and the diagram

$$\begin{array}{ccc} A & \xrightarrow{\text{inclusion}} & M(A) \\ {\scriptstyle \pi_{\iota,t}}\downarrow & & \downarrow{\scriptstyle \widetilde{\pi}_{\iota,t}} \\ A/I_\iota(t) & \xrightarrow{\rho_{\iota,t}} & M(A)/\widetilde{I}_\iota(t) \end{array}$$

is commutative.

Now let C be a unital, abelian C^*-algebra with Gelfand spectrum Ω and let (A_1, ι_1), (A_2, ι_2) be C^*-algebras over C. For every $t \in \Omega$, Corollary 2.2 entails the existence of the injective $*$-homomorphisms $\rho_{\iota_1,t}$, $\rho_{\iota_2,t}$ and then the tensor product $*$-homomorphism

$$\rho_{\iota_1,t} \otimes_{\min} \rho_{\iota_2,t} : A_1/I_{\iota_1}(t) \otimes_{\min} A_2/I_{\iota_2}(t) \longrightarrow M(A_1)/\widetilde{I}_{\iota_1}(t) \otimes_{\min} M(A_2)/\widetilde{I}_{\iota_2}(t)$$

is injective, hence isometric , and the diagram

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{\text{inclusion}} & M(A_1) \otimes M(A_2) \\ {\scriptstyle \pi_{\iota_1,t}\otimes\pi_{\iota_2,t}}\downarrow & & \downarrow{\scriptstyle \widetilde{\pi}_{\iota_1,t}\otimes\widetilde{\pi}_{\iota_2,t}} \\ (A_1/I_{\iota_1}(t)) \otimes_{\min} (A_2/I_{\iota_2}(t)) & \xrightarrow{\rho_{\iota_1,t}\otimes_{\min}\rho_{\iota_2,t}} & (M(A_1)/\widetilde{I}_{\iota_1}(t)) \otimes_{\min} (M(A_2)/\widetilde{I}_{\iota_2}(t)) \end{array}$$

is commutative. Consequently:

COROLLARY 2.3. *Let C be a unital, abelian C^*-algebra with Gelfand spectrum Ω and let (A_1, ι_1), (A_2, ι_2) be C^*-algebras over C. Then, for every $t \in \Omega$,*

$$\|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|_{\min} = \|(\widetilde{\pi}_{\iota_1,t} \otimes \widetilde{\pi}_{\iota_2,t})(a)\|_{\min}, \quad a \in A_1 \otimes A_2.$$

As a consequence of the above corollary, we have

$$\sup_{t\in\Omega} \|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|_{\min} = \sup_{t\in\Omega} \|(\widetilde{\pi}_{\iota_1,t} \otimes \widetilde{\pi}_{\iota_2,t})(a)\|_{\min}, \quad a \in A_1 \otimes A_2,$$

hence the restriction of the C^*-seminorm

$$M(A_1) \otimes M(A_2) \ni x \longmapsto \sup_{t\in\Omega} \|(\widetilde{\pi}_{\iota_1,t} \otimes \widetilde{\pi}_{\iota_2,t})(x)\|_{\min}$$

to $A_1 \otimes A_2$ is equal to the C^*-seminorm

$$A_1 \otimes A_2 \ni a \longmapsto \sup_{t\in\Omega} \|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|_{\min}.$$

Therefore the C^*-seminorm (0.2) can be defined also by the formula

$$\|a\|_{C,\min} = \sup_{t\in\Omega} \|(\widetilde{\pi}_{\iota_1,t} \otimes \widetilde{\pi}_{\iota_2,t})(a)\|_{\min}\,, \quad a \in A_1 \otimes A_2.$$

Every bounded linear functional φ on a C^*-algebra A can be considered in the natural way a linear functional on A^{**}, hence also on $M(A) \subset A^{**}$: the obtained linear functional on $M(A)$, which will be still denoted by φ, is actually the strictly continuous extension of the original functional on $M(A)$ (for the strict topology see e.g. 2.3 of [23]).

The next result is slightly more general than Proposition 4.3.14 of [10], and can be deduced from Corollary 4.7 of [21]:

PROPOSITION 2.4. *Let C be a unital, abelian C^*-algebra, Ω its Gelfand spectrum, (A, ι) a C^*-algebra over C, and φ a state on A. Then, for every $t \in \Omega$, the conditions*
(i) $\varphi(\iota(c)a) = c(t)\varphi(a), c \in C, a \in A$;
(ii) $\varphi|_{I_\iota(t)} = 0$;
(iii) $\varphi(\iota(c)) = c(t), c \in C$;
are equivalent. Moreover, if φ is a pure state on A then the above conditions are satisfied for an appropriate $t \in \Omega$.

Proof. (i)⇒(ii) is obvious and (ii)⇒(iii) follows easily: any approximate unit $\{u_\lambda\}_\lambda$ for A is strictly convergent to $1_{M(A)}$ (see e.g. Lemma 2.3.3 of [23]) and the strict continuity of φ on $M(A)$ yields

$$\varphi(\iota(c - c(t)1_C)u_\lambda) \longrightarrow \varphi(\iota(c - c(t)1_C)) = \varphi(\iota(c)) - c(t), \quad c \in C.$$

Now let us assume that (iii) is satisfied and let $a \in A^+$, $\|a\| \leqslant 1$, be arbitrary. For $\varphi(a) = 0$ we have by the Schwarz inequality $\varphi(\iota(c)a) = 0 = c(t)\varphi(a), c \in C$, while for $\varphi(1_{M(A)} - a) = 0$ we deduce, again by the Schwarz inequality,

$$\varphi(\iota(c)a) = \varphi(\iota(c)) - \varphi(\iota(c)(1_{M(A)} - a)) = c(t) = c(t)\varphi(a), \quad c \in C.$$

On the other hand, if $\varphi(a) > 0$ and $\varphi(1_{M(A)} - a) > 0$ then $C \ni c \stackrel{\psi_1}{\longmapsto} \frac{1}{\varphi(a)}\varphi(\iota(\cdot)a)$, $C \ni c \stackrel{\psi_2}{\longmapsto} \frac{1}{\varphi(1_{M(A)}-a)}\varphi(\iota(\cdot)(1_{M(A)} - a))$ are states satisfying $\varphi \circ \iota = \varphi(a)\psi_1 + \varphi(1_{M(A)} - a)\psi_2$. Since $\varphi \circ \iota$ is by (iii) a character, hence a pure state, it follows that $\psi_1 = \psi_2 = \varphi \circ \iota$. Therefore

$$\varphi(\iota(c)a) = \varphi(a)\psi_1(c) = \varphi(a)\varphi(\iota(c)) = c(t)\varphi(a), \quad c \in C.$$

Finally, let us assume that φ is a pure state on A. Let $\pi_\varphi : A \to \mathcal{B}(\mathcal{H}_\varphi)$ denote the GNS representation associated to φ and let ξ_φ be its canonical cyclic vector. Then π_φ, hence also $M(\pi_\varphi)$ is irreducible and it follows that $M(\pi_\varphi)(\iota(C)) = \mathbb{C}1_{\mathcal{H}_\varphi}$. Therefore $(M(\pi_\varphi) \circ \iota)(c) = c(t)1_{\mathcal{H}_\varphi}, c \in C$ for some $t \in \Omega$ and we obtain

$$\varphi(\iota(c)) = (M(\pi_\varphi)(\iota(c))\xi_\varphi|\xi_\varphi) = c(t)(\xi_\varphi|\xi_\varphi) = c(t), \quad c \in C. \quad \blacksquare$$

$S(A)$ will denote the set of all states of the C^*-algebra A, while $P(A)$ will stand for the set of all pure states of A. If C and (A, ι) are as in Proposition 2.4, then we denote by $S_\iota(A)$ the set of all states φ of A for which $\varphi \circ \iota$ is a character on C. By Lemma 2.4, $P(A) \subset S_\iota(A)$.

As a corollary, we get the following formula for the minimal C^*-tensor product norm (see Sublemma 2.1 of [5]):

COROLLARY 2.5. *Let C be a unital, abelian C^*-algebra with Gelfand spectrum Ω and let (A_1, ι_1), (A_2, ι_2) be C^*-algebras over C. Then, for any $a \in A_1 \otimes A_2$,*

$$\|a\|^2_{C,\min} = \sup\Big\{\frac{(\varphi_1 \otimes \varphi_2)(b^*a^*ab)}{(\varphi_1 \otimes \varphi_2)(b^*b)} : \varphi_j \in P(A_j), j = 1,2, \varphi_1 \circ \iota_1 = \varphi_2 \circ \iota_2, \\ b \in A_1 \otimes A_2, (\varphi_1 \otimes \varphi_2)(b^*b) > 0\Big\}.$$

Proof. The well known formula for the spatial tensor product norm (see e.g. Corollary 3/4.20 of [21] or Lemma 4.7 in [12]) yields that $\|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|^2_{\min}$ is, for every $t \in \Omega$, the supremum of

$$\frac{(\psi_1 \otimes \psi_2)((\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(b^*a^*ab))}{(\psi_1 \otimes \psi_2)((\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(b^*b))} = \frac{((\psi_1 \circ \pi_{\iota_1,t}) \otimes (\psi_2 \circ \pi_{\iota_2,t}))(b^*a^*ab)}{((\psi_1 \circ \pi_{\iota_1,t}) \otimes (\psi_2 \circ \pi_{\iota_2,t}))(b^*b)} \tag{2.1}$$

over all $\psi_j \in P(A_j/I_{\iota_j}(t)), b \in A_1 \otimes A_2$ with $(\psi_1 \otimes \psi_2)((\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(b^*b)) > 0$. Thus $\|a\|^2_{C,\min}$ is the supremum of (2.1) over all $\psi_j \in P(A_j/I_{\iota_j}(t)), b \in A_1 \otimes A_2$ with $(\psi_1 \otimes \psi_2)((\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(b^*b)) > 0$ and all $t \in \Omega$. But, taking into account Proposition 2.4, it is easy to see that this supremum is equal to that one in the statement. $\blacksquare$

We can consider on the quotients $(A_1/I_{\iota_1}(t)) \otimes (A_2/I_{\iota_2}(t))$ also the greatest C^*-norm $\|\cdot\|_{\max}$ (see e.g. 6.3 of [14]) and define the C^*-seminorm

$$A_1 \otimes A_2 \ni a \longmapsto \|a\|_{C,\max} = \sup_{t\in\Omega} \|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|_{\max}.$$

Following É. Blanchard, the *maximal C^*-tensor product of A_1 and A_2 over C* is defined as the Hausdorff completion $A_1 \otimes_{C,\max} A_2$ of $A_1 \otimes A_2$ with respect to the above C^*-seminorm, that is the C^*-algebra obtained by the completion of the quotient $*$-algebra $(A_1 \otimes A_2)/\mathcal{J}_C$ relative to the C^*-norm induced by $\|\cdot\|_{C,\max}$.

The subscripts max and min for the seminorms $\|\cdot\|_{C,\max}$ and $\|\cdot\|_{C,\min}$ are explained by the following extremality properties proved by G.A. Elliott (see Sublemma 2.1 of [5]) and É. Blanchard (see Propositions 2.4 and 2.8 of [1]):

PROPOSITION 2.6. *Let C be a unital, abelian C^*-algebra and let (A_1, ι_1), (A_2, ι_2) be C^*-algebras over C. If $p(\cdot)$ is a C^*-seminorm on $A_1 \otimes A_2$, then:*

$$\mathcal{J}_C \subset \{a \in A_1 \otimes A_2 : p(a) = 0\} \Longrightarrow p(a) \leqslant \|a\|_{C,\max}, a \in A_1 \otimes A_2,$$
$$\mathcal{J}_C = \{a \in A_1 \otimes A_2 : p(a) = 0\} \Longrightarrow p(a) \geqslant \|a\|_{C,\min}, a \in A_1 \otimes A_2.$$

We recall that the algebraic tensor product $A_1 \otimes_C A_2$ is the quotient $*$-algebra $(A_1 \otimes A_2)/\mathcal{I}_C$, where $\mathcal{I}_C$ is the self-adjoint two-sided ideal of $A_1 \otimes A_2$ equal to the linear span

$$\operatorname{lin}(\{(\iota_1(c)a_1) \otimes a_2 - a_1 \otimes (\iota_2(c)a_2) : a_1 \in A_1, a_2 \in A_2, c \in C\}).$$

Since $\mathcal{I}_C$ is clearly contained in

$$\mathcal{J}_C = \{a \in A_1 \otimes A_2 : \|a\|_{C,\min} = 0\} = \{a \in A_1 \otimes A_2 : \|a\|_{C,\max} = 0\}$$

the seminorms $\|\cdot\|_{C,\min}$ and $\|\cdot\|_{C,\max}$ factorize to C^*-seminorms on $A_1 \otimes_C A_2$, still denoted by $\|\cdot\|_{C,\min}$ and $\|\cdot\|_{C,\max}$. These C^*-seminorms are not always C^*-norms, because in general $\mathcal{I}_C \neq \mathcal{J}_C$ (see Section 3 of [1]).

Nevertheless, according to Propositions 2.2 and 3.1 of [1] we have:

PROPOSITION 2.7. *Let C be a unital, abelian C^*-algebra and let (A_1, ι_1), (A_2, ι_2) be C^*-algebras over C. Then any C^*-seminorm on $A_1 \otimes A_2$, which vanishes on $\mathcal{I}_C$, will vanish on whole $\mathcal{J}_C$. Moreover, if (A_1, ι_1) or (A_2, ι_2) is continuous, then even $\mathcal{I}_C = \mathcal{J}_C$ holds.*

We remark that T. Giordano and J.A. Mingo studied the case when A_1, A_2 and C are von Neumann algebras and the mappings $c \mapsto \iota_1(c)$ and $c \mapsto \iota_2(c)$ are normal (see Section 3 of [6]). They showed that in this case, for given spatial representations $A_1 \subset \mathcal{B}(\mathcal{H})$ and $A_2 \subset \mathcal{B}(\mathcal{K})$, one gets a faithful representation of $A_1 \otimes_C A_2$ on the Hilbert space $\mathcal{H} \otimes_C \mathcal{K}$ constructed by J.-L. Sauvageot [17], such that $\|x\|_{C,\min}$ is the operator norm on $\mathcal{H} \otimes_C \mathcal{K}$ for all $x \in A_1 \otimes_C A_2$. In particular, $\|\cdot\|_{C,\min}$ is a norm on $A_1 \otimes_C A_2$, that is $\mathcal{I}_C = \mathcal{J}_C$. None the less, since in this case (A_1, ι_1) and (A_2, ι_2) are continuous (see Lemma 10 of [7]), the above equality follows also from Proposition 2.7.

A *proper C^*-algebra over C* is a C^*-algebra (A, ι) over C such that, for some faithful, unital $*$-representation $\pi : M(A) \longrightarrow \mathcal{B}(\mathcal{H})$, $(\pi \circ \iota)(C)$ is weak operator closed, i.e. $(\pi \circ \iota)(C) \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra. B. Magajna extended the above quoted result of Giordano and Mingo to the case when (A_1, ι_1) and

(A_2, ι_2) are proper C^*-algebras over C (see Section 3 of [13]). We notice that proper C^*-algebras over C are still continuous.

3. TENSOR PRODUCTS OF $*$-REPRESENTATIONS OVER ABELIAN C^*-ALGEBRAS

In this section we prove that if C is a unital, abelian C^*-algebra, (A_1, ι_1) and (A_2, ι_2) are C^*-algebras over C and $\pi_j : A_j \to \mathcal{B}(\mathcal{H})$, $j = 1, 2$, are non-degenerate $*$-representations such that

$$\pi_1 \circ \iota_1 = \pi_2 \circ \iota_2 \quad \text{and} \quad \pi_1(A_1) \subset N, \pi_2(A_2) \subset N'$$

for some type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $(\pi_j \circ \iota_j)(C)''$, then the $*$-homomorphism $\pi : A_1 \otimes A_2 \to \mathcal{B}(\mathcal{H})$ defined by

$$\pi(a_1 \otimes a_2) = \pi_1(a_1)\pi_2(a_2), \quad a_1 \in A_1, a_2 \in A_2,$$

can be factored through $A_1 \otimes_{C,\min} A_2$ and so gives rise to a $*$-representation $A_1 \otimes_{C,\min} A_2 \to \mathcal{B}(\mathcal{H})$, the C^*-tensor product over C of π_1 and π_2.

LEMMA 3.1. *Let $N \subset \mathcal{B}(\mathcal{H})$ be a type* I *von Neumann algebra of centre Z, $Z \subset M_1 \subset N$, $Z \subset M_2 \subset N'$ von Neumann subalgebras, $B_1 \subset M_1$, $B_2 \subset M_2$ s-dense $*$-subalgebras, and e, f abelian projections in N, N', respectively. Let further $p \in M_1' \cap N$ and $q \in M_2' \cap N'$ be the projections such that*

$$\overline{M_1 e}^s = pNe, \quad e \leqslant p, \quad \mathrm{z}_N(e) = \mathrm{z}_N(p),$$
$$\overline{M_2 f}^s = qN'f, \quad f \leqslant q, \quad \mathrm{z}_{N'}(f) = \mathrm{z}_{N'}(q)$$

(such p, q exist and are unique by Lemma 1.4*). Then:*

(i) *ef is an abelian projection of central support pq in $pq(N \vee N')pq$;*
(ii) $\overline{(M_1 \vee M_2)ef}^s = pq(N \vee N')ef$;
(iii) *for every $x \in N \vee N'$, we have*

$$\|xpq\| = \sup\{\|xy\| : y \in \mathrm{lin}(B_1 B_2) ef Z_1^+, \|y\| \leqslant 1\}.$$

Proof. (i) By Lemma 1.3, ef is an abelian projection in $N \vee N'$. Since $ef \leqslant pq$, it is an abelian projection also in $pq(N \vee N')pq$.

On the other hand, since the centre of the reduced algebra $pq(N \vee N')pq$ is equal to $pqZ(N \vee N') = pqZ$, the central support $\mathrm{z}_{pq(N\vee N')pq}(ef)$ is of the form pqz_0 for some projection $z_0 \in Z$. Now, taking into account Lemma 1.3, we deduce successively:

$$ef \leqslant \mathrm{z}_{pq(N\vee N')pq}(ef) = pqz_0 \leqslant z_0,$$
$$pq \leqslant \mathrm{z}_N(p)\,\mathrm{z}_{N'}(q) = \mathrm{z}_N(e)\,\mathrm{z}_{N'}(f) = \mathrm{z}_{N\vee N'}(ef) \leqslant z_0,$$
$$pq = pqz_0 = \mathrm{z}_{pq(N\vee N')pq}(ef).$$

(ii) Since

$$x_1x_2ef = x_1ex_2f = px_1eqx_2f = pqx_1x_2ef, \quad x_1 \in M_1, x_2 \in M_2,$$

we have $\overline{(M_1 \vee M_2)ef}^s \subset pq(N \vee N')ef$.

To prove the reverse inclusion, let $y \in N$, $y' \in N'$ be arbitrary. Then $pye \in \overline{M_1 e}^s$ and $qy'f \in \overline{M_2 f}^s$, so by Lemma 1.5 there exist nets $\{a_\lambda e\}_\lambda \subset M_1 e$ and $\{b_\mu f\}_\mu \subset M_2 f$ such that

$$a_\lambda e \xrightarrow{s} pye \quad \text{and} \quad \|a_\lambda e\| \leqslant \|pye\| \quad \text{for every } \lambda,$$
$$b_\mu f \xrightarrow{s} qy'f \quad \text{and} \quad \|b_\mu f\| \leqslant \|qy'f\| \quad \text{for every } \mu.$$

It follows that $a_\lambda b_\mu ef \xrightarrow[\lambda,\mu]{s} pqyy'ef$, hence $pqyy'ef \in \overline{(M_1 \vee M_2)ef}^s$.

(iii) Let $x \in N \vee N'$ be arbitrary.

According to (i), ef is an abelian projection of central support pq in the type I von Neumann algebra $pq(N \vee N')pq$. Thus Lemma 1.2 and (ii) yield

$$\begin{aligned}\|xpq\|^2 &= \|pqx^*xpq\| \\ &= \sup\{\|pqx^*xv\| : v \in pq(N \vee N')pq \text{ partial isometry}, v^*v \leqslant ef\} \\ &\leqslant \|xpq\| \sup\{\|xv\| : v \in pq(N \vee N')pq \text{ partial isometry}, v^*v \leqslant ef\},\end{aligned}$$

so

$$\begin{aligned}\|xpq\| &= \sup\{\|xv\| : v \in pq(N \vee N')pq \text{ partial isometry}, v^*v \leqslant ef\} \\ &= \sup\{\|xv\| : v \in pq(N \vee N')pq \text{ partial isometry}\} \\ &= \sup\{\|xy\| : y \in pq(N \vee N')ef, \|y\| \leqslant 1\} \\ &= \sup\{\|xy\| : y \in \overline{(M_1 \vee M_2)ef}^s, \|y\| \leqslant 1\}.\end{aligned}$$

Since $\operatorname{lin}(B_1B_2)$ is a $*$-subalgebra of $N \vee N'$ and $\overline{\operatorname{lin}(B_1B_2)ef}^s = \overline{\operatorname{lin}(M_1M_2)ef}^s = \overline{(M_1 \vee M_2)ef}^s$ Lemma 1.5 entails that $\{y \in \overline{(M_1 \vee M_2)ef}^s : \|y\| \leqslant 1\} = \overline{\{y \in \operatorname{lin}(B_1B_2)efZ_1^+ : \|y\| \leqslant 1\}}^s$. Consequently

$$\begin{aligned}\|xpq\| &= \sup\{\|xy\| : y \in \overline{(M_1 \vee M_2)ef}^s, \|y\| \leqslant 1\} \\ &= \sup\{\|xy\| : y \in \operatorname{lin}(B_1B_2)efZ_1^+, \|y\| \leqslant 1\}. \quad \blacksquare\end{aligned}$$

LEMMA 3.2. *Let C be a unital, abelian C^*-algebra with Gelfand spectrum Ω and let $(A_1, \iota_1), (A_2, \iota_2)$ be C^*-algebras over C. Let further $\pi_j : A_j \to \mathcal{B}(\mathcal{H})$, $j = 1, 2$, be non-degenerate $*$-representations, such that*

$$M(\pi_1) \circ \iota_1 = M(\pi_2) \circ \iota_2 \quad \text{and} \quad \pi_1(A_1) \subset N, \pi_2(A_2) \subset N'$$

for a type I *von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $Z = (M(\pi_j) \circ \iota_j)(C)''$, $\widetilde{\Omega}$ the Gelfand spectrum of Z, and $\pi : A_1 \otimes A_2 \to \mathcal{B}(\mathcal{H})$ the $*$-homomorphism defined by*

$$\pi(a_1 \otimes a_2) = \pi_1(a_1)\pi_2(a_2), \quad a_1 \in A_1, a_2 \in A_2.$$

If $p \in \pi_1(A_1)' \cap N$, $q \in \pi_2(A_2)' \cap N'$ are projections such that

$$pNe = \overline{\pi_1(A_1)e}^s, \quad qN'f = \overline{\pi_2(A_2)f}^s$$

for some abelian projections $e \in N$ and $f \in N'$ satisfying

$$e \leqslant p, \mathrm{z}_N(e) = \mathrm{z}_N(p), \quad f \leqslant q, \mathrm{z}_{N'}(f) = \mathrm{z}_{N'}(q),$$

then, denoting $z_0 = \mathrm{z}_{N \vee N'}(ef) = \mathrm{z}_N(e)\,\mathrm{z}_{N'}(f)$, we have for all $a \in A_1 \otimes A_2$:

$$\begin{aligned}
&\|\pi(a)pq\| \\
&= \sup\{\chi(z)(\chi \circ \Phi_{ef} \circ \pi)(b^*a^*ab)^{1/2} : b \in A_1 \otimes A_2, z \in Z_1^+, \chi \in \widetilde{\Omega} \\
&\qquad\qquad \|\pi(b)efz\| \leqslant 1\} && (3.1)\\
&= \sup\{\chi(z)((\chi \circ \Phi_{ez_0} \circ \pi_1) \otimes (\chi \circ \Phi_{fz_0} \circ \pi_2))(b^*a^*ab)^{1/2} : \\
&\qquad\qquad b \in A_1 \otimes A_2, z \in Z_1^+, \chi \in \widetilde{\Omega}, \|\pi(b)efz\| \leqslant 1\} && (3.2)\\
&\leqslant \sup_{t \in \Omega} \|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|_{\min}. && (3.3)
\end{aligned}$$

Proof. We notice that the equality $\mathrm{z}_{N \vee N'}(ef) = \mathrm{z}_N(e)\,\mathrm{z}_{N'}(f)$ in the definition of z_0 holds by Lemma 1.3.

Set

$$M_j = \pi_j(A_j)'' = \overline{\pi_j(A_j)}^s, \quad j = 1, 2.$$

Applying Lemma 3.1(iii) with $B_j = \pi_j(A_j), j = 1, 2$, we obtain for every $x \in N \vee N'$:

$$\begin{aligned}
\|xpq\| &= \sup\{\|xy\| : y \in \mathrm{lin}(\pi_1(A_1)\pi_2(A_2))efZ_1^+, \|y\| \leqslant 1\} \\
&= \sup\{\|xy\| : y \in \pi(A_1 \otimes A_2)efZ_1^+, \|y\| \leqslant 1\} \\
&= \sup\{\|x\pi(b)efz\| : b \in A_1 \otimes A_2, z \in Z_1^+, \|\pi(b)efz\| \leqslant 1\}.
\end{aligned}$$

Let $a \in A_1 \otimes A_2$ be arbitrary. Using the above equality with $x = \pi(a)$, as well as (1.3), we deduce (3.1):

$$\begin{aligned}
\|\pi(a)pq\|^2 &= \sup\{\|\pi(ab)efz\|^2 : b \in A_1 \otimes A_2, z \in Z_1^+, \|\pi(b)efz\| \leqslant 1\} \\
&= \sup\{\|efz^2\pi(b^*a^*ab)ef\| : b \in A_1 \otimes A_2, z \in Z_1^+, \|\pi(b)efz\| \leqslant 1\} \\
&= \sup\{\|\Phi_{ef}(z^2\pi(b^*a^*ab))\| : b \in A_1 \otimes A_2, z \in Z_1^+, \|\pi(b)efz\| \leqslant 1\} \\
&= \sup\{\|z^2(\Phi_{ef} \circ \pi)(b^*a^*ab)\| : b \in A_1 \otimes A_2, z \in Z_1^+, \|\pi(b)efz\| \leqslant 1\} \\
&= \sup\{\chi(z)^2(\chi \circ \Phi_{ef} \circ \pi)(b^*a^*ab) : b \in A_1 \otimes A_2, z \in Z_1^+, \chi \in \widetilde{\Omega} \\
&\qquad\qquad \|\pi(b)efz\| \leqslant 1\}.
\end{aligned}$$

By Lemma 1.3, we have for every $\chi \in \widetilde{\Omega}$ and $a_1 \in A_1, a_2 \in A_2$:

$$\begin{aligned}
(\chi \circ \Phi_{ef} \circ \pi)(a_1 \otimes a_2) &= \chi(\Phi_{efz_0}(\pi_1(a_1)\pi_2(a_2))) = \chi(\Phi_{ez_0}(\pi_1(a_1))\Phi_{fz_0}(\pi_2(a_2))) \\
&= (\chi \circ \Phi_{ez_0} \circ \pi_1)(a_1)(\chi \circ \Phi_{fz_0} \circ \pi_2)(a_2) \\
&= ((\chi \circ \Phi_{ez_0} \circ \pi_1) \otimes (\chi \circ \Phi_{fz_0} \circ \pi_2))(a_1 \otimes a_2).
\end{aligned}$$

Therefore

$$\chi \circ \Phi_{ef} \circ \pi = (\chi \circ \Phi_{ez_0} \circ \pi_1) \otimes (\chi \circ \Phi_{fz_0} \circ \pi_2), \quad \chi \in \widetilde{\Omega} \tag{3.4}$$

and (3.2) follows.

According to Corollary 2.3, for the proof of (3.3) we can assume without loss of generality that both A_1 and A_2 are unital. (3.3) will follow once we show that, for every $b \in A_1 \otimes A_2$, $z \in Z_1^+$ and $\chi \in \widetilde{\Omega}$ with $\|\pi(b)efz\| \leqslant 1$,

$$\chi(z)^2((\chi \circ \Phi_{ez_0} \circ \pi_1) \otimes (\chi \circ \Phi_{fz_0} \circ \pi_2))(b^*a^*ab) \leqslant \sup_{t\in\Omega} \|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|^2_{\min}. \tag{3.5}$$

If $\chi(z_0) = 0$, then $\chi \circ \Phi_{ez_0} \circ \pi_1 = \chi \circ \Phi_{fz_0} \circ \pi_2 = 0$ and (3.5) holds trivially. Therefore we shall assume in the sequel that $\chi(z_0) \neq 0$. Since $\chi(z_0)\chi(z_0) = \chi(z_0^2) = \chi(z_0)$, then $\chi(z_0) = 1$.

Let us denote, for convenience,

$$\varphi_1 = \chi \circ \Phi_{ez_0} \circ \pi_1, \quad \varphi_2 = \chi \circ \Phi_{fz_0} \circ \pi_2.$$

φ_1 and φ_2 are positive linear functionals and $\|\varphi_j\| = \varphi_j(1_{A_j}) = \chi(z_0) = 1$, so they are states. Furthermore, since

$$(\varphi_j \circ \iota_j)(c) = \chi(z_0(\pi_j \circ \iota_j)(c)) = \chi(z_0)\chi((\pi_j \circ \iota_j)(c)) = (\chi \circ \pi_j \circ \iota_j)(c), \quad c \in C,$$

$\varphi_1 \circ \iota_1 = \chi \circ \pi_j \circ \iota_j = \varphi_2 \circ \iota_2$ is a multiplicative state on C, that is a character $t_\chi \in \Omega$.

We claim that φ_1 vanishes on $I_{\iota_1}(t_\chi)$. Indeed, for every $c \in C, c(t_\chi) = 0$, and $a_1 \in A_1$, $\varphi_1(\iota_1(c)a_1) = \chi((\pi_1 \circ \iota_1)(c)\Phi_{ez_0}(\pi_1(a_1))) = c(t_\chi)\varphi_1(a_1) = 0$. Consequently there exists a state ψ_1 on $A_1/I_{\iota_1}(t_\chi)$ such that $\varphi_1 = \psi_1 \circ \pi_{\iota_1,t_\chi}$. Similarly, φ_2 vanishes on $I_{\iota_2}(t_\chi)$ and so $\varphi_2 = \psi_2 \circ \pi_{\iota_2,t_\chi}$ for some state ψ_2 on $A_2/I_{\iota_2}(t_\chi)$. Then $\varphi_1 \otimes \varphi_2$ factors by the tensor product state $\psi_1 \otimes_{\min} \psi_2$ on $(A_1/I_{\iota_1}(t_\chi)) \otimes_{\min} (A_2/I_{\iota_2}(t_\chi))$:

$$\varphi_1 \otimes \varphi_2 = (\psi_1 \otimes_{\min} \psi_2) \circ (\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi}). \tag{3.6}$$

Now, the norm of the positive linear functional $\theta = \chi(z)^2(\psi_1 \otimes_{\min} \psi_2)((\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi})(b)^* \cdot (\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi})(b))$ on $(A_1/I_{\iota_1}(t_\chi)) \otimes_{\min} (A_2/I_{\iota_2}(t_\chi))$ is $\leqslant 1$. Indeed, since $\|\theta\|$ is equal to the value of θ in the unit of $(A_1/I_{\iota_1}(t_\chi)) \otimes_{\min} (A_2/I_{\iota_2}(t_\chi))$, by (3.6) and (3.4) we obtain:

$$\begin{aligned}\|\theta\| &= \chi(z)^2(\psi_1 \otimes_{\min} \psi_2)((\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi})(b^*b)) \\ &= \chi(z)^2(\varphi_1 \otimes \varphi_2)(b^*b) = \chi(z)^2(\chi \circ \Phi_{ef} \circ \pi)(b^*b) \\ &= \chi(\Phi_{ef}(z^2\pi(b^*b))) = \chi(\Phi_{ef}(zef\pi(b)^*\pi(b)efz)) \leqslant \|\pi(b)efz\|^2 \leqslant 1.\end{aligned}$$

Thus, by (3.6),

$$\begin{aligned}&\chi(z)^2((\chi \circ \Phi_{ez_0} \circ \pi_1) \otimes (\chi \circ \Phi_{fz_0} \circ \pi_2))(b^*a^*ab) \\ &\quad = \chi(z)^2(\varphi_1 \otimes \varphi_2)(b^*a^*ab) = \chi(z)^2((\psi_1 \otimes_{\min} \psi_2) \circ (\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi}))(b^*a^*ab) \\ &\quad = \theta((\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi})(a^*a)) \leqslant \|(\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi})(a^*a)\|_{\min} = \|(\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi})(a)\|^2_{\min}\end{aligned}$$

and (3.5) follows. ∎

LEMMA 3.3. *Let $N \neq \{0\}$ be a type* I *von Neumann algebra with centre Z, and $Z \subset M \subset N$ a von Neumann subalgebra. Then there exists a set $\mathcal{P}$ of mutually orthogonal, non-zero projections in $M' \cap N$ such that $\sum_{p\in\mathcal{P}} p = 1_N$ and, for every $p \in \mathcal{P}$,*

$$pNe = \overline{Me}^s$$

for some abelian projection $e \in N$ satisfying $e \leqslant p, \mathrm{z}_N(e) = \mathrm{z}_N(p)$.

Proof. Let $\mathcal{P}$ be a maximal set of mutually orthogonal, non-zero projections in $M' \cap N$ such that, for every $p \in \mathcal{P}$,

$$pNe_p = \overline{Me_p}^s$$

for some abelian projection $e_p \in N$ satisfying $e_p \leqslant p, \mathrm{z}_N(e_p) = \mathrm{z}_N(p)$. Such family $\mathcal{P}$ exists by Lemma 1.4 and by Zorn's Lemma. We will show that $\sum_{p\in\mathcal{P}} p = 1_N$.

Suppose the contrary, that is $1_N - \sum_{p\in\mathcal{P}} p \neq 0$. By Lemma 1.1 there exists an abelian projection $e \in N$ such that $e \leqslant 1_N - \sum_{p\in\mathcal{P}} p, \mathrm{z}_N(e) = \mathrm{z}_N\Big(1_N - \sum_{p\in\mathcal{P}} p\Big)$. In particular, $e \neq 0$. Further, by Lemma 1.4 $\overline{Me}^s = p_0Ne$ for some projection $p_0 \in M' \cap N$ with $e \leqslant p_0$.

Let $y \in N$ be arbitrary. Since $p_0ye \in p_0Ne = \overline{Me}^s$, there is a net $\{x_\lambda\}_\lambda$ in M such that $x_\lambda e \xrightarrow{s} p_0ye$. Since $\mathcal{P} \subset M' \cap N$, it follows that

$$x_\lambda e = x_\lambda\Big(1_N - \sum_{p\in\mathcal{P}} p\Big)e = \Big(1_N - \sum_{p\in\mathcal{P}} p\Big)x_\lambda e \xrightarrow{s} \Big(1_N - \sum_{p\in\mathcal{P}} p\Big)p_0ye.$$

Consequently $p_0ye = \Big(1_N - \sum_{p\in\mathcal{P}} p\Big)p_0ye$, i.e. $\sum_{p\in\mathcal{P}} pp_0ye = 0$.

We conclude that $\sum_{p\in\mathcal{P}} pp_0Ne = \{0\}$ and so, since $\mathrm{z}_N(e)$ is the orthogonal projection onto the closed linear span of $Ne\mathcal{H}$, $\sum_{p\in\mathcal{P}} pp_0\,\mathrm{z}_N(e) = 0$. Thus

$$M' \cap N \ni p_0\,\mathrm{z}_N(e) = \Big(1_N - \sum_{p\in\mathcal{P}} p\Big)p_0\,\mathrm{z}_N(e) \leqslant 1_N - \sum_{p\in\mathcal{P}} p.$$

Furthermore, $\mathrm{z}_N(e) \geqslant p_0\,\mathrm{z}_N(e)p_0 \geqslant p_0ep_0 = e \neq 0$ implies that $p_0\,\mathrm{z}_N(e) \neq 0$ and $\mathrm{z}_N(p_0\,\mathrm{z}_N(e)) = \mathrm{z}_N(e)$.

Thus $p_0\,\mathrm{z}_N(e)$ is a non-zero projection in $M' \cap N$ such that $p_0\,\mathrm{z}_N(e)Ne = p_0Ne = \overline{Me}^s$ with e an abelian projection in N satisfying $e \leqslant p_0\,\mathrm{z}_N(e)$ and $\mathrm{z}_N(e) = \mathrm{z}_N(p_0\,\mathrm{z}_N(e))$. But, since $p_0\,\mathrm{z}_N(e) \leqslant 1_N - \sum_{p\in\mathcal{P}} p$, this contradicts the maximality of $\mathcal{P}$. ∎

THEOREM 3.4. *Let C be a unital, abelian C^*-algebra with Gelfand spectrum Ω and let $(A_1, \iota_1), (A_2, \iota_2)$ be C^*-algebras over C. Let further $\pi_j : A_j \to \mathcal{B}(\mathcal{H})$, $j = 1,2,$*

be non-degenerate $*$-representations, such that

$$M(\pi_1)\circ\iota_1 = M(\pi_2)\circ\iota_2 \quad \textit{and} \quad \pi_1(A_1)\subset N, \pi_2(A_2)\subset N'$$

for a type I *von Neumann algebra $N\subset\mathcal{B}(\mathcal{H})$ with centre $Z=(M(\pi_j)\circ\iota_j)(C)''$, and $\pi : A_1\otimes A_2\to\mathcal{B}(\mathcal{H})$ the $*$-homomorphism defined by*

$$\pi(a_1\otimes a_2)=\pi_1(a_1)\pi_2(a_2), \quad a_1\in A_1, a_2\in A_2.$$

Then

$$\|\pi(a)\|\leqslant \sup_{t\in\Omega}\|(\pi_{\iota_1,t}\otimes\pi_{\iota_2,t})(a)\|_{\min}=\|a\|_{C,\min}, \quad a\in A_1\otimes A_2 \tag{3.7}$$

and thus there is a unique $*$-representation $\widetilde{\pi} : A_1\otimes_{C,\min}A_2\to\mathcal{B}(\mathcal{H})$ such that

$$\pi(a)=\widetilde{\pi}(a/\mathcal{J}_C), \quad a\in A_1\otimes A_2,$$

where $a/\mathcal{J}_C$ denotes the natural image of $a\in A_1\otimes A_2$ in the quotient $*$-algebra $(A_1\otimes A_2)/\mathcal{J}_C\subset A_1\otimes_{C,\min}A_2$.

Proof. If $\mathcal{H}=\{0\}$, then (3.7) holds trivially. It remains to prove it in the case $\mathcal{H}\neq\{0\}$.

By Lemma 3.3 there exists a set $\mathcal{P}\subset\pi_1(A_1)'\cap N$ of mutually orthogonal, non-zero projections such that $\sum\limits_{p\in\mathcal{P}}p=1_{\mathcal{H}}$ and, for every $p\in\mathcal{P}$,

$$pNe_p=\overline{\pi_1(A_1)''e_p}^{s}$$

for some abelian projection $e_p\in N$ satisfying $e_p\leqslant p, \mathrm{z}_N(e_p)=\mathrm{z}_N(p)$.

Similarly, there exists a set $\mathcal{Q}\subset\pi_2(A_2)'\cap N'$ of mutually orthogonal, non-zero projections such that $\sum\limits_{q\in\mathcal{Q}}q=1_{\mathcal{H}}$ and, for every $q\in\mathcal{Q}$,

$$qN'f_q=\overline{\pi_2(A_2)''f_q}^{s}$$

for some abelian projection $f_q\in N'$ satisfying $f_q\leqslant q$, $\mathrm{z}_{N'}(f_q)=\mathrm{z}_{N'}(q)$.

Let $a\in A_1\otimes A_2$ be arbitrary. By Lemma 3.2 we have $\|\pi(a)pq\|\leqslant\|a\|_{C,\min}$ for every $p\in\mathcal{P}, q\in\mathcal{Q}$. Since $\sum\limits_{p\in\mathcal{P}}p=\sum\limits_{q\in\mathcal{Q}}q=1_{\mathcal{H}}$ and $\mathcal{P}\cup\mathcal{Q}\subset\pi_1(A_1)'\cap\pi_2(A_2)'\subset\pi(A_1\otimes A_2)'$, we have $\pi(a^*a)=\sum\limits_{p,q}\pi(a^*a)pq$, where the operators $\pi(a^*a)pq$ are positive and mutually orthogonal. Consequently:

$$\|\pi(a)\|^2=\|\pi(a^*a)\|=\sup_{p,q}\|\pi(a^*a)pq\|=\sup_{p,q}\|\pi(a)pq\|^2\leqslant\|a\|^2_{C,\min}. \quad \blacksquare$$

We will denote $\widetilde{\pi}$ in Theorem 3.4 by $\pi_1\otimes_{C,\min}\pi_2$ and call it the *tensor product of π_1 and π_2 over C*. We notice that the $*$-representation $\pi_1\otimes_{C,\min}\pi_2$ maps $A_1\otimes_{C,\min}A_2$ onto the C^*-subalgebra $\overline{\mathrm{lin}(\pi_1(A_1)\pi_2(A_2))}\subset\mathcal{B}(\mathcal{H})$ and it is non-degenerate. Indeed, if $\{u_\lambda\}_\lambda$ is an increasing approximate unit for A_1 and $\{v_\mu\}_\mu$ is an increasing approximate unit for A_2, then we have

$$\pi_1(u_\lambda)\xrightarrow{\text{so}}1_{\mathcal{H}} \quad \text{and} \quad \pi_2(v_\mu)\xrightarrow{\text{so}}1_{\mathcal{H}}$$

(see e.g. Lemma 3/4.1 of [21]), so

$$(\pi_1 \otimes_{C,\min} \pi_2)((u_\lambda \otimes v_\mu)/\mathcal{J}_C) = \pi_1(u_\lambda)\pi_2(v_\mu) \xrightarrow{\text{so}} 1_{\mathcal{H}}.$$

Therefore $M(\overline{\text{lin}(\pi_1(A_1)\pi_2(A_2))})$ can be identified with

$$\{T \in \mathcal{B}(\mathcal{H}) : \pi_1(A_1)\pi_2(A_2)T \cup T\pi_1(A_1)\pi_2(A_2) \subset \overline{\text{lin}(\pi_1(A_1)\pi_2(A_2))}\}.$$

It is easy to see that, with the above identification,

$$\pi_1(A_1) \cup \pi_2(A_2) \subset M(\overline{\text{lin}(\pi_1(A_1)\pi_2(A_2))}) \text{ and}$$

$$\pi_1(a_1)\pi_2(v_\mu) \xrightarrow{\text{strictly}} \pi_1(a_1), \quad a_1 \in A_1, \tag{3.8}$$

$$\pi_1(u_\lambda)\pi_2(a_2) \xrightarrow{\text{strictly}} \pi_2(a_2), \quad a_2 \in A_2.$$

We notice that it can happen that, for given non-zero C^*-algebras (A_1, ι_1), (A_2, ι_2) over C, only the $*$-representations $\pi_1 : A_1 \to \{0\}$ and $\pi_2 : A_2 \to \{0\}$ satisfy the assumptions in Theorem 3.4. Let, for example, (A_1, ι_2), (A_2, ι_2) be the C^*-algebras over $C([0,1])$ defined in [1] before Proposition 3.3, for which $A_1 \otimes_{C([0,1]),\min} A_2 = \{0\}$. Then, if $\pi_j : A_j \to \mathcal{B}(\mathcal{H})$, $j = 1,2$, are any non-degenerate $*$-representations satisfying the conditions in Theorem 3.4, then the $*$-representation $\pi_1 \otimes_{C,\min} \pi_2$ can be non-degenerate only if $\mathcal{H} = \{0\}$. Nevertheless, this pathology is possible only in the case of non-unital A_1 and A_2 (cf. Corollary 5.8).

Criteria for the faithfulness of $\pi_1 \otimes_{C,\min} \pi_2$ will be proved in Section 5.

4. DESCRIPTION OF THE GLIMM IDEALS IN SPATIALLY REPRESENTED C^*-ALGEBRAS

If A is a unital C^*-algebra and $1_A \in C \subset Z(A)$ is a C^*-subalgebra with Gelfand spectrum Ω, then we shall denote by $I_{C\subset A}(t)$ the ideal $I_\iota(t)$, where ι is the inclusion map of C in $Z(A)$. In other words,

$$I_{C\subset A}(t) = \overline{\{c \in C : c(t) = 0\}A}, \quad t \in \Omega. \tag{4.1}$$

Proposition 2.1 (ii) implies the following dependence of $I_{C\subset A}(t)$ on A: If M is a unital C^*-algebra and $1_M \in C \subset A \subset M$ are C^*-subalgebras such that $C \subset Z(M)$, then

$$I_{C\subset A}(t) = A \cap I_{C\subset M}(t), \quad t \in \Omega. \tag{4.2}$$

The dependence of $I_{C\subset A}(t)$ on C is described in the following lemma:

LEMMA 4.1. *Let M be a unital C^*-algebra, $1_M \in Z \subset Z(M)$ a C^*-subalgebra with Gelfand spectrum $\widetilde{\Omega}$, and $1_M \in C \subset Z$ a C^*-subalgebra with Gelfand spectrum Ω. Then*

$$I_{C\subset M}(t) = \bigcap\{I_{Z\subset M}(\chi) : \chi \in \widetilde{\Omega}, \chi(c) = c(t) \text{ for all } c \in C\}, \quad t \in \Omega.$$

Proof. Let $t \in \Omega$ be arbitrary and let us denote

$$\widetilde{\Omega}_t = \{\chi \in \widetilde{\Omega} : \chi(c) = c(t) \text{ for all } c \in C\} = \{\chi \in \widetilde{\Omega} : \chi|_{I_{C\subset Z}(t)} = 0\}.$$

The inclusion $I_{C\subset M}(t) \subset \bigcap\limits_{\chi \in \widetilde{\Omega}_t} I_{Z\subset M}(\chi)$ follows at once from definition (4.1): if $c \in C, c(t) = 0$ and $\chi \in \widetilde{\Omega}_t$, then $\chi(c) = c(t) = 0$, so $cM \subset I_{Z\subset M}(\chi)$. Thus it remains to show the converse inclusion.

According to (4.2) $I_{C\subset Z}(t) = Z \cap I_{C\subset M}(t)$, so $Z_t = Z/I_{C\subset Z}(t) \ni z/I_{C\subset Z}(t) \longmapsto z/I_{C\subset M}(t) \in M/I_{C\subset M}(t) = M_t$ is an injective $*$-homomorphism, through which we can identify Z_t with a C^*-subalgebra of M_t. On the other hand, the map which associates to $\chi \in \widetilde{\Omega}_t$ the character $\chi_t : Z_t \ni z/I_{C\subset Z}(t) \mapsto \chi(z)$, is a homeomorphism of $\widetilde{\Omega}_t$ onto the Gelfand spectrum of Z_t. Thus

$$\bigcap_{\chi \in \widetilde{\Omega}_t} I_{Z_t\subset M_t}(\chi_t) = \{0\}.$$

Now let $x \bigcap\limits_{\chi \in \widetilde{\Omega}_t} I_{Z\subset M}(\chi)$ be arbitrary. For every $\chi \in \widetilde{\Omega}_t$, the quotient map $M \to M_t$ maps $I_{Z\subset M}(\chi)$ into $I_{Z_t\subset M_t}(\chi_t)$: if $z \in Z, \chi(z) = 0$ and $y \in M$, then we have $(zy)/I_{C\subset M}(t) = (z/I_{C\subset Z}(t))(y/I_{C\subset M}(t))$ with $\chi_t(z/I_{C\subset Z}(t)) = \chi(z) = 0$, hence $(zy)/I_{C\subset M}(t) \in I_{Z_t\subset M_t}(\chi_t)$. Consequently,

$$x/I_{C\subset M}(t) \in \bigcap_{\chi \in \widetilde{\Omega}_t} I_{Z_t\subset M_t}(\chi_t) = \{0\},$$

that is $x \in I_{C\subset M}(t)$. ∎

The next simple result should be known, but we have no reference for it:

LEMMA 4.2. *Let N be a type* I *von Neumann algebra with centre Z, $e_0 \in N$ an abelian projection of central support 1_N, and $b \in N$. Then there exists an abelian projection $e \in N$ of central support 1_N such that*

$$\Phi_{e_0}(b^*xb) = \Phi_{e_0}(b^*b)\Phi_e(x), \quad x \in N. \tag{4.3}$$

Proof. Let $be_0 = w|be_0|$ be the polar decomposition of be_0 and let p denote the central support of b^*b. Then $|be_0| = (e_0b^*be_0)^{1/2} = ze_0$ with $0 \leqslant z \in Zp$ and $w^*w = \mathrm{s}_N(e_0b^*be_0) \leqslant e_0$, so that $w^*w = \mathrm{z}_N(w^*w)e_0 = pe_0$.

Since pe_0 is an abelian, hence finite projection in N, there is a unitary $\widetilde{w} \in N$ such that $w = \widetilde{w}pe_0$ (see e.g. E.4.9 of [19] or 6.9.7 of [10]). Then $e = \widetilde{w}e_0\widetilde{w}^*$ is an abelian projection of central support 1_N in N. For every $x \in N$, since $exe = \widetilde{w}(e_0\widetilde{w}^*x\widetilde{w}e_0)\widetilde{w}^* = \Phi_{e_0}(\widetilde{w}^*x\widetilde{w})\widetilde{w}e_0\widetilde{w}^* = \Phi_{e_0}(\widetilde{w}^*x\widetilde{w})e$, we have

$$\Phi_{e_0}(\widetilde{w}^*x\widetilde{w}) = \Phi_e(x), \tag{4.4}$$

hence

$$\begin{aligned}\Phi_{e_0}(b^*xb) &= \Phi_{e_0}((be_0)^*xbe_0) = \Phi_{e_0}(e_0zw^*xwze_0) = z^2\Phi_{e_0}(w^*xw)\\ &= z^2\Phi_{e_0}(e_0p\widetilde{w}^*x\widetilde{w}pe_0) \stackrel{(4.4)}{=} z^2p\Phi_e(x) = z^2\Phi_e(x).\end{aligned}$$

In particular, for $x = 1_N$, $\Phi_{e_0}(b^*b) = z^2\Phi_e(1_N) = z^2$ and so (4.3) holds. ∎

The following result is essentially Lemma 5.13 of [24].

LEMMA 4.3. *Let N be a type* I *von Neumann algebra with centre Z, $\widetilde{\Omega}$ the Gelfand spectrum of Z, e_0 an abelian projection of central support 1_N in N, and $\chi \in \widetilde{\Omega}$. Then*

$$\begin{aligned}I_{Z\subset N}(\chi) &= \{x \in N : \chi(\Phi_{e_0}(b^*xb)) = 0 \textit{ for every } b \in N\}\\ &= \{x \in N : \chi(\Phi_e(x)) = 0 \textit{ for every}\\ &\qquad\qquad \textit{abelian projection } e \in N \textit{ with } \mathrm{z}_N(e) = 1_N\}.\end{aligned}$$

Proof. Clearly, $\{x \in N : \chi(\Phi_{e_0}(b^*xb)) = 0$ for every $b \in N\}$ is a norm-closed two-sided ideal $\mathcal{J}$ of N, which contains $I_{Z\subset N}(\chi)$. Let us assume that this inclusion is strict. Then there exists a positive element in $\mathcal{J}\setminus I_{Z\subset N}(\chi)$ and an appropriate spectral projection f of it will still belong to $\mathcal{J}\setminus I_{Z\subset N}(\chi)$. Since $\mathrm{z}_N(f)e_0 \prec f$, there exists $u \in N$ such that $u^*u = \mathrm{z}_N(f)e_0$ and $uu^* \leqslant f$. Thus $\mathrm{z}_N(f)e_0 = u^*fu \in \mathcal{J}$ and it follows that $\chi(\mathrm{z}_N(f)) = \Phi_{e_0}(\mathrm{z}_N(f)e_0) = 0$. But then, by definition (4.1), $f = \mathrm{z}_N(f)f \in I_{Z\subset N}(\chi)$, in contradiction with the assumption $f \in \mathcal{J}\setminus I_{Z\subset N}(\chi)$.

To complete the proof, we have to prove that

$$\begin{aligned}\mathcal{J} = \{x \in N : \chi(\Phi_e(x)) = 0 &\text{ for every}\\ &\text{abelian projection } e \in N \text{ with } \mathrm{z}_N(e) = 1_N\}.\end{aligned}$$

If $x \in \mathcal{J}$ and $e \in N$ is an abelian projection, then there exists $v \in N$ with $v^*v \leqslant e_0$, $vv^* = e$ and, taking into account that $v^*v = \mathrm{z}_N(v^*v)e_0$ and $\Phi_e(x) \in Z\,\mathrm{z}_N(e) = Z\,\mathrm{z}_N(v^*v)$, we obtain successively

$$\begin{aligned}&v^*xv = v^*(exe)v \stackrel{(1.2)}{=} v^*(\Phi_e(x)e)v = \Phi_e(x)v^*v = \Phi_e(x)\,\mathrm{z}_N(v^*v)e_0 = \Phi_e(x)e_0,\\ &\chi(\Phi_e(x)) = \chi(\Phi_{e_0}(v^*xv)) = 0.\end{aligned}$$

This proves the inclusion $\subset$.

For the converse inclusion, let $x \in N$ be such that $\chi(\Phi_e(x)) = 0$ for every abelian projection $e \in N$ of central support 1_N. For every $b \in N$, according to Lemma 4.2, there exists an abelian projection $e \in N$ with central support 1_N such that $\Phi_{e_0}(b^*xb) = \Phi_{e_0}(b^*b)\Phi_e(x)$. Then

$$\chi(\Phi_{e_0}(b^*xb)) = \chi(\Phi_{e_0}(b^*b))\chi(\Phi_e(x)) = 0. \quad ∎$$

Lemmas 4.1 and 4.2 enable us to prove the following extension of Theorem 4.2 in [18] (see also Theorem 4.17 of [24]) in the case of type I von Neumann algebras:

THEOREM 4.4. *Let N be a type* I *von Neumann algebra with centre Z, $\widetilde{\Omega}$ the Gelfand spectrum of Z, $1_N \in C \subset Z$ a C^*-subalgebra with Gelfand spectrum Ω, and $C \subset A \subset N$ an intermediate C^*-algebra. Then*

$$\begin{aligned} I_{C\subset A}(t) = \{a \in A : \chi(\Phi_e(a)) = 0 \textit{ for every} \\ \textit{abelian projection } e \in N \textit{ with } \mathrm{z}_N(e) = 1_N \\ \textit{and } \chi \in \widetilde{\Omega} \textit{ with } \chi(c) = c(t) = 0, c \in C\}, \quad t \in \Omega. \end{aligned}$$

Proof. Let $t \in \Omega$ be arbitrary.

By Lemmas 4.1 and 4.3 we have

$$\begin{aligned} I_{C\subset N}(t) &= \bigcap \{I_{Z\subset N}(\chi) : \chi \in \widetilde{\Omega}, \chi(c) = c(t) \text{ for all } c \in C\} \\ &= \{x \in N : \chi(\Phi_e(x)) = 0 \text{ for every} \\ &\qquad \text{abelian projection } e \in N \text{ with } \mathrm{z}_N(e) = 1_N \\ &\qquad \text{and } \chi \in \widetilde{\Omega} \text{ with } \chi(c) = c(t) = 0, c \in C\} \end{aligned}$$

and, using (4.2), we conclude that

$$\begin{aligned} I_{C\subset A}(t) &= A \cap I_{C\subset N}(t) \\ &= \{a \in A : \chi(\Phi_e(a)) = 0 \text{ for every} \\ &\qquad \text{abelian projection } e \in N \text{ with } \mathrm{z}_N(e) = 1_N \\ &\qquad \text{and } \chi \in \widetilde{\Omega} \text{ with } \chi(c) = c(t) = 0, c \in C\}. \quad \blacksquare \end{aligned}$$

COROLLARY 4.5. *Let N be a type* I *von Neumann algebra with centre Z, $\widetilde{\Omega}$ the Gelfand spectrum of Z, $1_N \in C \subset Z$ a C^*-subalgebra with Gelfand spectrum Ω, $C \subset A \subset N$ an intermediate C^*-algebra and $t \in \Omega$. Then every pure state φ on A with $\varphi(c) = c(t), c \in C$, belongs to the weak* closure of*

$$\begin{aligned} \{\chi \circ \Phi_e : e \in N \textit{ abelian projection with } \mathrm{z}_N(e) = 1_N \\ \chi \in \widetilde{\Omega} \textit{ with } \chi(c) = c(t) = 0 \textit{ for all } c \in C\}. \end{aligned}$$

Proof. For every abelian projection $e \in N$ with $\mathrm{z}_N(e) = 1_N$ and every $\chi \in \widetilde{\Omega}$ with $\chi(c) = c(t) = 0$, $c \in C$, let $\pi_{e,\chi} : A \to \mathcal{B}(\mathcal{H}_{e,\chi})$ be the GNS representation associated to the restriction of $\chi \circ \Phi_e$ to A and let $\xi_{e,\chi}$ denote its canonical cyclic vector. By Theorem 4.4 and Proposition 2.4 we have $\bigcap\limits_{e,\chi} \ker(\pi_{e,\chi}) = I_{C\subset A}(t) \subset \ker(\varphi)$, so we can apply Proposition 3.4.2 of [4] or Theorem 5.1.15 of [14], deducing that φ belongs to the weak* closure of the states

$$\bigcup_{e,\chi} \{A \ni a \longmapsto (\pi_{e,\chi}(a)\xi|\xi) : \xi \in \mathcal{H}_{e,\chi}, \|\xi\| = 1\}.$$

Since every $\xi \in \mathcal{H}_{e,\chi}$ with $\|\xi\| = 1$ is norm-limit in $\mathcal{H}_{e,\chi}$ of unit vectors of the form $\pi_{e,\chi}(b)\xi_{e,\chi}$ and then $\chi(\Phi_e(b^*b)) = (\pi_{e,\chi}(b^*b)\xi_{e,\chi}|\xi_{e,\chi}) = 1$, it follows that φ is in the weak* closure of the linear functionals

$$A \ni a \longmapsto (\pi_{e,\chi}(a)\pi_{e,\chi}(b)\xi_{e,\chi}|\pi_{e,\chi}(b)\xi_{e,\chi}) = \chi(\Phi_e(b^*ab))$$

with $\chi(\Phi_e(b^*b)) = 1$.

But, according to Lemma 4.2, for every abelian projection $e \in N$ of central support 1_N and every $b \in N$, there exists an abelian projection $e(b) \in N$ of central support 1_N such that $\Phi_e(b^*xb) = \Phi_e(b^*b)\Phi_{e(b)}(x)$, $x \in N$. Therefore every linear functional $A \ni a \longmapsto \chi(\Phi_e(b^*ab))$ with $\chi(\Phi_e(b^*b)) = 1$ is of the form $A \ni a \longmapsto \chi(\Phi_{e(b)}(a)) = (\chi \circ \Phi_{e(b)})(a)$. ∎

Corollary 4.5 implies the following description of $\mathcal{J}_C$ in terms of an appropriate spatial representation:

COROLLARY 4.6. *Let $(A_1, \iota_1), (A_2, \iota_2)$ be C^*-algebras over a unital, abelian C^*-algebra C, and $\pi_j : A_j \longrightarrow \mathcal{B}(\mathcal{H})$, $j = 1, 2$, two faithful non-degenerate $*$-representations such that*

$$M(\pi_1) \circ \iota_1 = M(\pi_2) \circ \iota_2 \quad \text{and} \quad \pi_1(A_1) \subset N, \pi_2(A_2) \subset N'$$

for a type I *von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $Z = (M(\pi_j) \circ \iota_j)(C)''$. Let $\widetilde{\Omega}$ denote the Gelfand spectrum of Z. Then $a \in A_1 \otimes A_2$ belongs to $\mathcal{J}_C$ if and only if*

$$((\chi_1 \circ \Phi_e \circ \pi_1) \otimes (\chi_2 \circ \Phi_f \circ \pi_2))(a) = 0$$

for all

$$\text{abelian projections } e \in N, f \in N' \text{ with } \mathrm{z}_N(e) = \mathrm{z}_{N'}(f) = 1_{\mathcal{H}},$$
$$\chi_1, \chi_2 \in \widetilde{\Omega} \text{ with } \chi_1 \circ M(\pi_1) \circ \iota_1 = \chi_2 \circ M(\pi_2) \circ \iota_2.$$

Proof. According to Corollary 2.3, we can assume without loss of generality that A_1 and A_2 are unital. Let Ω denote the Gelfand spectrum of C.

Assume first that $a \in \mathcal{J}_C$ and let $e \in N, f \in N'$ be abelian projections with $\mathrm{z}_N(e) = \mathrm{z}_{N'}(f) = 1_{\mathcal{H}}$, while $\chi_1, \chi_2 \in \widetilde{\Omega}$ with $\chi_1 \circ M(\pi_1) \circ \iota_1 = \chi_2 \circ M(\pi_2) \circ \iota_2$. Then $\chi_j \circ M(\pi_j) \circ \iota_j$ is $C \ni c \longmapsto c(t)$ for some $t \in \Omega$. Since $(\chi_1 \circ \Phi_e \circ \pi_1)(\iota_1(c)a) = \chi_1((\pi_1 \circ \iota_1)(c)\Phi_e(\pi_1(a))) = c(t)(\chi_1 \circ \Phi_e \circ \pi_1)(a)$ for all $a \in A_1$ and $c \in C$, Proposition 2.4 yields $\chi_1 \circ \Phi_e \circ \pi_1|_{I_{\iota_1}(t)} = 0$. Similarly, $\chi_2 \circ \Phi_f \circ \pi_2|_{I_{\iota_2}(t)} = 0$. Thus $\chi_1 \circ \Phi_e \circ \pi_1 = \theta_1 \circ \pi_{\iota_1,t}$ for some state θ_1 on $A_1/I_{\iota_1}(t)$ and $\chi_2 \circ \Phi_f \circ \pi_2 = \theta_2 \circ \pi_{\iota_2,t}$ for some state θ_2 on $A_2/I_{\iota_2}(t)$. Consequently

$$|((\chi_1 \circ \Phi_e \circ \pi_1) \otimes (\chi_2 \circ \Phi_f \circ \pi_2))(a)| \leqslant \|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|_{\min} \leqslant \|a\|_{C,\min} = 0.$$

Now let us assume that $a \in A_1 \otimes A_2$ is such that

$$((\chi_1 \circ \Phi_e \circ \pi_1) \otimes (\chi_2 \circ \Phi_f \circ \pi_2))(a) = 0$$

for all abelian projections $e \in N, f \in N'$ with $\mathrm{z}_N(e) = \mathrm{z}_{N'}(f) = 1_{\mathcal{H}}$ and all χ_1, $\chi_2 \in \widetilde{\Omega}$ with $\chi_1 \circ \pi_1 \circ \iota_1 = \chi_2 \circ \pi_2 \circ \iota_2$. Taking into account that π_1, π_2 are injective and using Corollary 4.5, we obtain that $(\varphi_1 \otimes \varphi_2)(a) = 0$ for all $\varphi_1 \in P(A_1), \varphi_2 \in P(A_2)$ with $\varphi_1 \circ \iota_1 = \varphi_2 \circ \iota_2$. In other words,

$$(\psi_1 \otimes \psi_2)((\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)) = 0, \quad \psi_j \in P(A_j/I_{\iota_j}(t)), j = 1, 2, \quad t \in \Omega.$$

It follows that $(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a) = 0$ for every $t \in \Omega$, that is $a \in \mathcal{J}_C$. ∎

5. FAITHFUL TENSOR PRODUCTS OF $*$-REPRESENTATIONS OVER ABELIAN C^*-ALGEBRAS

Let C be a unital, abelian C^*-algebra, (A_1, ι_1) and (A_2, ι_2) C^*-algebras over C, and $\pi_j : A_j \to \mathcal{B}(\mathcal{H})$, $j = 1, 2$ non-degenerate $*$-representations such that

$$M(\pi_1) \circ \iota_1 = M(\pi_2) \circ \iota_2 \quad \text{and} \quad \pi_1(A_1) \subset N, \pi_2(A_2) \subset N'$$

for some type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $(\pi_j \circ \iota_j)(C)''$. In this section we prove criteria for the faithfulness if $\pi_1 \otimes_{C,\min} \pi_2$.

We notice that $\pi_1 \otimes_{C,\min} \pi_2$ can be faithful without π_1, π_2 being faithful. Indeed, in [1], before Proposition 3.3, an example of non-zero A_1, A_2 is given such that $\mathcal{J}_C = A_1 \otimes A_2$, that is $A_1 \otimes_{C,\min} A_2 = \{0\}$. Then, choosing for π_1 and π_2 the zero $*$-representation, $\pi_1 \otimes_{C,\min} \pi_2$ is faithful, while π_1 and π_2 are not. Nevertheless:

PROPOSITION 5.1. *Let C be a unital, abelian C^*-algebra with Gelfand spectrum Ω, $(A_1, \iota_1), (A_2, \iota_2)$ C^*-algebras over C, and $\pi_j : A_j \longrightarrow \mathcal{B}(\mathcal{H})$, $j = 1, 2$, non-degenerate $*$-representations such that*

$$M(\pi_1) \circ \iota_1 = M(\pi_2) \circ \iota_2 \quad \textit{and} \quad \pi_1(A_1) \subset N, \pi_2(A_2) \subset N'$$

for a type I *von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $Z = (M(\pi_j) \circ \iota_j)(C)''$. If $\pi_1 \otimes_{C,\min} \pi_2$ is faithful and $I_{\iota_2}(t) \neq A_2$ for all $t \in \Omega$, then π_1 is faithful. In particular, if $M(\pi_1) \otimes_{C,\min} M(\pi_2)$ is faithful and $A_2 \neq \{0\}$, then π_1 is faithful.*

Proof. Let us assume that $\pi_1 \otimes_{C,\min} \pi_2$ is faithful, $I_{\iota_2}(t) \neq A_2$ for every $t \in \Omega$, and $a_1 \in A_1, \pi_1(a_1) = 0$.

Let $a_2 \in A_2$ be arbitrary. The injectivity of $\pi_1 \otimes_{C,\min} \pi_2$ and

$$(\pi_1 \otimes_{C,\min} \pi_2)((a_1 \otimes a_2)/\mathcal{J}_C) = \pi_1(a_1)\pi_2(a_2) = 0$$

imply that $a_1 \otimes a_2 \in \mathcal{J}_C$, that is $\pi_{\iota_1,t}(a_1) \otimes \pi_{\iota_2,t}(a_2) = 0$, $t \in \Omega$. Since, for any $t \in \Omega$, $\pi_{\iota_2,t}(a_2) \neq 0$ for some $a_2 \in A_2$, it follows that $\pi_{\iota_1,t}(a_1) = 0$, $t \in \Omega$. Consequently, $\|a_1\| = \sup\limits_{t \in \Omega} \|\pi_{\iota_1,t}(a_1)\| = 0$, that is $a_1 = 0$.

Now, if $A_2 \neq \{0\}$, then $1_{M(A_2)} \notin \widetilde{I}_{\iota_2}(t)$, so $\widetilde{I}_{\iota_2}(t) \neq M(A_2)$ for all $t \in \Omega$. Therefore, by the above part of the proof,

$$M(\pi_1) \otimes_{C,\min} M(\pi_2) \text{ faithful} \implies M(\pi_1) \text{ faithful.} \quad \blacksquare$$

According to Proposition 5.1, by looking for the faithfulness of $\pi_1 \otimes_{C,\min} \pi_2$ it is natural to assume the faithfulness of π_1 and π_2. However, the faithfulness of π_1 and π_2 alone does not imply the faithfulness of $\pi_1 \otimes_{C,\min} \pi_2$, as the next proposition will show.

We shall denote by $l^\infty(\mathbb{N})$ the C^*-algebra of all bounded complex sequences, by $c(\mathbb{N})$ the C^*-subalgebra of $l^\infty(\mathbb{N})$ consisting of all convergent sequences, and by $l^2(\mathbb{N})$ the Hilbert space of all square-summable complex sequences.

PROPOSITION 5.2. *Let us consider the unital, abelian C^*-algebras $C = c(\mathbb{N})$, $A_1 = A_2 = l^\infty(\mathbb{N})$ and the inclusion maps $\iota_j : C \to A_j, j = 1, 2$. Let further π_j*

denote the faithful, unital $*$-homomorphism $A_j \to \mathcal{B}(l^2(\mathbb{N}))$ which associates to every $a \in l^\infty(\mathbb{N})$ the multiplication operator with a on $l^2(\mathbb{N})$. Then $\pi_1 \otimes_{C,\min} \pi_2$ is not faithful.

Proof. We notice that the Gelfand spectrum of $c(\mathbb{N})$ can be identified with the one-point compactification $\widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ of $\mathbb{N}$.

Let $\chi_{\text{odds}} \in l^\infty(\mathbb{N})$ denote the characteristic function of all odd natural numbers, and χ_{evens} the characteristic function of all even natural numbers. Then

$$(\pi_1 \otimes_{C,\min} \pi_2)((\chi_{\text{odds}} \otimes \chi_{\text{evens}})/\mathcal{J}_C) = \pi_1(\chi_{\text{odds}})\pi_2(\chi_{\text{evens}}) = 0.$$

We shall show that $\|\chi_{\text{odds}} \otimes \chi_{\text{evens}}\|_{C,\min} = 1$, hence $(\chi_{\text{odds}} \otimes \chi_{\text{evens}})/\mathcal{J}_C \neq 0$, which completes the proof of the non-injectivity of $\pi_1 \otimes_{C,\min} \pi_2$.

Let ev_n denote the evaluation map $l^\infty(\mathbb{N}) \ni a \longmapsto a(n)$. Then every ev_n is a state on $l^\infty(\mathbb{N})$. Let φ_1 be a weak*limit point of $\{\text{ev}_n\}_{n\,\text{odd}}$, and φ_2 a weak*limit point of $\{\text{ev}_n\}_{n\,\text{even}}$. Clearly, $\varphi_1(\chi_{\text{odds}}) = 1$ and φ_1 carries $c \in C$ in $c(\infty)$, so by Proposition 2.4 we have $\varphi_1|_{I_{\iota_1}(\infty)} = 0$. Therefore $\varphi_1 = \psi_1 \circ \pi_{\iota_1,\infty}$ for some state ψ_1 on $A_1/I_{\iota_1}(\infty)$. Similarly, $\varphi_2(\chi_{\text{evens}}) = 1$ and $\varphi_2 = \psi_2 \circ \pi_{\iota_2,\infty}$ for some state ψ_2 on $A_2/I_{\iota_2}(\infty)$. Since

$$\begin{aligned} 1 &= (\varphi_1 \otimes \varphi_2)(\chi_{\text{odds}} \otimes \chi_{\text{evens}}) = (\psi_1 \otimes \psi_2)((\pi_{\iota_1,\infty} \otimes \pi_{\iota_2,\infty})(\chi_{\text{odds}} \otimes \chi_{\text{evens}})) \\ &\leqslant \|(\pi_{\iota_1,\infty} \otimes \pi_{\iota_2,\infty})(\chi_{\text{odds}} \otimes \chi_{\text{evens}})\|_{\min} \leqslant \|\chi_{\text{odds}} \otimes \chi_{\text{evens}}\|_{C,\min} \leqslant 1, \end{aligned}$$

we conclude that $\|\chi_{\text{odds}} \otimes \chi_{\text{evens}}\|_{C,\min} = 1$. ∎

In the sequel we shall prove criteria in order that the tensor product of two faithful $*$-representations over a unital, abelian C^*-algebra be still faithful.

Let $\mathcal{H}$ be a Hilbert space, $A, B \subset \mathcal{B}(\mathcal{H})$ C^*-subalgebras with B containing $1_{\mathcal{H}}$, and $\varphi \in S(A)$. If $C^*(A \cup B)$ denotes the C^*-algebra generated by $A \cup B$, then

$$\{\theta \in S(C^*(A \cup B)) : \theta|_A = \varphi\}$$

is a weak*closed, convex subset of $S(C^*(A \cup B))$, so the subset

$$K(A, B; \varphi) = \{\theta|_B : \theta \in S(C^*(A \cup B)), \theta|_A = \varphi\} \subset S(B)$$

is convex and weak*closed.

Let X be a non-empty convex set in some vector space. We recall that $x \in X$ is an extreme point of X if and only if $x = \frac{1}{2}(x_1 + x_2), x_1, x_2 \in X$, is possible only for $x_1 = x_2$ (cf. Theorem 5.2 of [24]). We denote the set of all extreme points of X (the extreme boundary of X) by $\partial_e X$.

LEMMA 5.3. *Let $\mathcal{H}$ be a Hilbert space, $A, B \subset \mathcal{B}(\mathcal{H})$ C^*-subalgebras with B containing $1_{\mathcal{H}}$, and $\varphi \in P(A)$. Then*

$$\partial_e K(A, B; \varphi) \subset \{\theta|_B : \theta \in P(C^*(A \cup B)), \theta|_A = \varphi\}.$$

If additionally $B \subset A'$, then

$$\{\theta|_B : \theta \in P(C^*(A \cup B)), \theta|_A = \varphi\} \subset P(B),$$

hence also the converse inclusion holds.

Proof. Let $\psi \in \partial_e K(A, B; \varphi)$ be arbitrary. Then

$$K_\psi = \{\theta \in S(C^*(A \cup B)) : \theta|_A = \varphi, \theta|_B = \psi\}$$

is a non-empty weak*compact, convex set, so by the Krein-Milman Theorem it has an extreme point θ_0. We claim that $\theta_0 \in P(C^*(A \cup B))$.

For let us assume that $\theta_0 = \frac{1}{2}(\theta_1 + \theta_2)$ with $\theta_1, \theta_2 \in S(C^*(A \cup B))$. Since $\varphi \in P(A) = \partial_e S(A)$ and $\varphi = \theta_0|_A = \frac{1}{2}(\theta_1|_A + \theta_2|_A)$, we have $\theta_1|_A = \theta_2|_A = \varphi$. Therefore $\theta_1|_B$ and $\theta_2|_B$ belong to $K(A, B; \varphi)$. But $\psi = \theta_0|_B = \frac{1}{2}(\theta_1|_B + \theta_2|_B)$, so, using that $\psi \in \partial_e K(A, B; \varphi)$, we obtain $\theta_1|_B = \theta_2|_B = \psi$. Consequently $\theta_1, \theta_2 \in K_\psi$ and the extremality of θ_0 in K_ψ yields $\theta_1 = \theta_2 = \theta_0$.

Now let us assume that $B \subset A'$ and $\psi = \theta|_B$ for some $\theta \in P(C^*(A \cup B))$ with $\theta|_A = \varphi$. Let $\pi_\theta : C^*(A \cup B) \longrightarrow \mathcal{B}(\mathcal{H}_\theta)$ be the GNS representation associated to θ, and ξ_θ its canonical cyclic vector. Since θ is a pure state, π_θ is irreducible.

Let p_0 denote the unit of the weak operator closed $*$-subalgebra $\overline{\pi_\theta(A)}^{\text{wo}}$ of $\mathcal{B}(\mathcal{H}_\theta)$. Then $p_0 \in \pi_\theta(A)' \cap \pi_\theta(B)' = \pi_\theta(C^*(A \cup B))' = \mathbb{C}1_{\mathcal{H}_\theta}$. Moreover, since $\theta|_A = \varphi \neq 0$, p_0 is non-zero. Consequently $p_0 = 1_{\mathcal{H}_\theta}$, and so $\overline{\pi_\theta(A)}^{\text{wo}}$ is a von Neumann algebra. In particular, ξ_θ belongs to $\mathcal{H}_{\theta,\varphi} = \overline{\pi_\theta(A)\xi_\theta} \subset \mathcal{H}_\theta$.

The orthogonal projection P' onto $\mathcal{H}_{\theta,\varphi}$ clearly belongs to the commutant $\pi_\theta(A)'$ of $\overline{\pi_\theta(A)}^{\text{wo}}$. The central support of P' is the orthogonal projection on $\overline{\text{lin}(\pi_\theta(A)'P'\mathcal{H}_\theta)} \supset \overline{\text{lin}(\pi_\theta(B)\pi_\theta(A)\xi_\theta)} = \overline{\text{lin}(\pi_\theta(C^*(A \cup B))\xi_\theta)} = \mathcal{H}_\theta$, so $z_{\pi_\theta(A)'}(P') = 1_{\mathcal{H}_\theta}$. Therefore the induction $*$-homomorphism

$$\rho_{\theta,\varphi} : \overline{\pi_\theta(A)}^{\text{wo}} \ni T \longmapsto T|_{\mathcal{H}_{\theta,\varphi}} \in \mathcal{B}(\mathcal{H}_{\theta,\varphi})$$

is injective. But the $*$-representation $\pi_{\theta,\varphi} : A \ni a \longmapsto \pi_\theta(a)|_{\mathcal{H}_{\theta,\varphi}} \in \mathcal{B}(\mathcal{H}_{\theta,\varphi})$ is unitarily equivalent to the GNS representation $\pi_\varphi : A \longrightarrow \mathcal{B}(\mathcal{H}_\varphi)$ of φ and $\varphi \in P(A)$, so $\pi_{\theta,\varphi}$ is irreducible and consequently the range of $\rho_{\theta,\varphi}$ is equal to $\overline{\pi_{\theta,\varphi}(A)}^{\text{wo}} = \mathcal{B}(\mathcal{H}_{\theta,\varphi})$. Therefore $N = \overline{\pi_\theta(A)}^{\text{wo}} = \rho_{\theta,\varphi}^{-1}(\mathcal{B}(\mathcal{H}_{\theta,\varphi}))$ is a type I factor.

Now, $\pi_\theta(B) \subset N'$ and the relative commutant of $\pi_\theta(B)$ in N' is $\pi_\theta(B)' \cap N' = \pi_\theta(B)' \cap \pi_\theta(A)' = \pi_\theta(C^*(A \cup B))' = \mathbb{C}1_{\mathcal{H}_\theta}$. Since the bicommutant theorem holds in type I factors, we get $\overline{\pi_\theta(B)}^{\text{wo}} = N'$. We claim that P' is a minimal projection of N'.

For let $T' \in N', 0 \leqslant T' \leqslant 1_{\mathcal{H}_\theta}$, be arbitrary. Since

$$(\pi_\theta(a)T'\xi_\theta|\xi_\theta) \leqslant (\pi_\theta(a)\xi_\theta|\xi_\theta) = \varphi(a), \quad a \in A^+$$

and $\varphi \in P(A)$, there exists $0 \leqslant \lambda \leqslant 1$ such that $(\pi_\theta(a)T'\xi_\theta|\xi_\theta) = \lambda\varphi(a)$ for all $a \in A$ (see e.g. 4.7 of [21]). Consequently

$$((T' - \lambda 1_{\mathcal{H}_\theta})\pi_\theta(a_1)\xi_\theta|\pi_\theta(a_2)\xi_\theta) = (\pi_\theta(a_2^*a_1)T'\xi_\theta|\xi_\theta) - \lambda\varphi(a_2^*a_1) = 0$$

for all $a_1, a_2 \in A$ and it follows that $P'(T' - \lambda 1_{\mathcal{H}_\theta})P' = 0$, i.e. $P'T'P' = \lambda P'$.

By the minimality of P' in N', for every $b \in B$ there exists $\lambda_b \in \mathbb{C}$ such that $P'\pi_\theta(b)P' = \lambda_b P'$. Since $\lambda_b = (\lambda_b P'\xi_\theta|\xi_\theta) = (P'\pi_\theta(b)P'\xi_\theta|\xi_\theta) = \theta(b) = \psi(b)$, we have $P'\pi_\theta(b)P' = \psi(b)P'$.

Let π be a $*$-isomorphism of the type I factor N' onto some $\mathcal{B}(\mathcal{K})$. Then $\pi(P')$ is an one-dimensional projection and, choosing a vector $\eta \in \pi(P')\mathcal{K}$, $\|\eta\| = 1$, we have $\psi(b) = ((\pi \circ \pi_\theta)(b)\eta|\eta)$, $b \in B$. Since $(\pi \circ \pi_\theta)(B)$ is weak operator dense in $\mathcal{B}(\mathcal{K})$, we conclude that ψ is a pure state. ∎

Now we study the extreme points of the intersection of $K(A_1, B; \varphi_1)$ and $K(A_2, B; \varphi_2)$:

LEMMA 5.4. *Let $\mathcal{H}$ be a Hilbert space, $A_1, A_2, B \subset \mathcal{B}(\mathcal{H})$ C^*-subalgebras with B abelian and $1_\mathcal{H} \in B \subset {A_1}' \cap {A_2}'$, and $\varphi_1 \in P(A_1)$, $\varphi_2 \in P(A_2)$. If*

$$\psi \in \partial_e(K(A_1, B; \varphi_1) \cap K(A_2, B; \varphi_2))$$

then, for $j = 1, 2$, there exists $\tau_j \in P(C^*(A_j \cup B))$ such that

$$\tau_j|_{A_j} = \varphi_j, \tau_j|_B = \psi \quad \text{and} \quad \tau_j(ab) = \tau_j(a)\tau_j(b), \quad a \in C^*(A_j \cup B), b \in B.$$

In particular,

$$\partial_e(K(A_1, B; \varphi_1) \cap K(A_2, B; \varphi_2)) = \partial_e K(A_1, B; \varphi_1) \cap \partial_e K(A_2, B; \varphi_2).$$

Proof. Let us denote, for convenience, $K_1 = K(A_1, B; \varphi_1), K_2 = K(A_2, B; \varphi_2)$ and set

$$\begin{aligned} K_\psi = \{(\theta_1, \theta_2) \in S(C^*(A_1 \cup B)) \times S(C^*(A_2 \cup B)) : \theta_j|_{A_j} = \varphi_j, \theta_j|_B = \psi \\ \text{for } j = 1, 2\}, \\ K = \{(\theta_1, \theta_2) \in S(C^*(A_1 \cup B)) \times S(C^*(A_2 \cup B)) : \theta_1|_B = \theta_2|_B\}. \end{aligned}$$

Since $K_\psi \neq \emptyset$ is convex and compact with respect to the product of the weak* topologies, by the Krein-Milman Theorem it has an extreme point (τ_1, τ_2).

First we show that $(\tau_1, \tau_2) \in \partial_e K$. For let $({\theta_1}', {\theta_2}'), ({\theta_1}'', {\theta_2}'') \in K$ be such that

$$(\tau_1, \tau_2) = \frac{1}{2}(({\theta_1}', {\theta_2}') + ({\theta_1}'', {\theta_2}'')). \tag{5.1}$$

Then, for $j = 1, 2$, we have $\varphi_j = \tau_j|_{A_j} = \frac{1}{2}({\theta_j}'|_{A_j} + {\theta_j}''|_{A_j})$ and, since $\varphi_j \in P(A_j)$, it follows that ${\theta_j}'|_{A_j} = {\theta_j}''|_{A_j} = \varphi_j$, hence ${\theta_j}'|_B, {\theta_j}''|_B \in K_j$. But ${\theta_1}'|_B = {\theta_2}'|_B$ and ${\theta_1}''|_B = {\theta_2}''|_B$, so actually ${\theta_1}'|_B = {\theta_2}'|_B \in K_1 \cap K_2$ and ${\theta_1}''|_B = {\theta_2}''|_B \in K_1 \cap K_2$. Now $\psi = \tau_1|_B \overset{(5.1)}{=} \frac{1}{2}({\theta_1}'|_B + {\theta_1}''|_B)$ and $\psi \in \partial_e(K_1 \cap K_2)$, yields ${\theta_j}'|_B = {\theta_j}''|_B = \psi, j = 1, 2$, and therefore $({\theta_1}', {\theta_2}'), ({\theta_1}'', {\theta_2}'') \in K_\psi$. So, by the extremality of (τ_1, τ_2) in K_ψ, we conclude that

$$({\theta_1}', {\theta_2}') = ({\theta_1}'', {\theta_2}'') = (\tau_1, \tau_2).$$

Next we prove

$$\tau_j(ab) = \tau_j(a)\tau_j(b) = \varphi_j(a)\psi(b), a \in C^*(A_j \cup B), b \in B, \quad j = 1, 2. \tag{5.2}$$

Clearly, it is enough to prove (5.2) in the case that $\varepsilon 1_{\mathcal{H}} \leqslant b \leqslant (1-\varepsilon)1_{\mathcal{H}}$ for some $\varepsilon > 0$. Set for $j = 1, 2$:

$$\theta_j{}' = \frac{1}{\psi(b)}\tau_j(\cdot b), \theta_j{}'' = \frac{1}{\psi(1_{\mathcal{H}} - b)}\tau_j(\cdot(1_{\mathcal{H}} - b)) \in S(C^*(A_j \cup B)).$$

Since $\tau_1|_B = \psi = \tau_2|_B$, both pairs $(\theta_1{}', \theta_2{}')$ and $(\theta_1{}'', \theta_2{}'')$ belong to K. Thus

$$(\tau_1, \tau_2) = \psi(b)(\theta_1{}', \theta_2{}') + \psi(1_{\mathcal{H}} - b)(\theta_1{}'', \theta_2{}'') \quad \text{and} \quad (\tau_1, \tau_2) \in \partial_e K$$

imply that $(\theta_1{}', \theta_2{}') = (\tau_1, \tau_2)$, i.e. (5.2).

Finally we prove that $\tau_j \in P(C^*(A_j \cup B))$, $j = 1, 2$. Then, by Lemma 5.3, we have also $\psi \in \partial_e K(A_1, B; \varphi_1) \cap \partial_e K(A_2, B; \varphi_2)$.

For $\tau_1 \in P(C^*(A_1 \cup B))$, let us assume that

$$\tau_1 = \frac{1}{2}(\theta' + \theta'') \quad \text{for some } \theta', \theta'' \in S(C^*(A_1 \cup B)).$$

By (5.2) τ_1 is multiplicative on B, so $\tau_1|_B$ is a pure state on B. Therefore the above relation implies $\theta'|_B = \theta''|_B = \tau_1|_B = \psi = \tau_2|_B$ and it follows that

$$(\tau_1, \tau_2) = \frac{1}{2}((\theta', \tau_2) + (\theta'', \tau_2)), \quad \text{where } (\theta', \tau_2), (\theta'', \tau_2) \in K.$$

Using $(\tau_1, \tau_2) \in \partial_e K$, we get $(\theta', \tau_2) = (\theta'', \tau_2) = (\tau_1, \tau_2)$, hence $\theta' = \theta'' = \tau_1$.

The proof of $\tau_2 \in P(C^*(A_2 \cup B))$ is completely similar. ∎

The main result of this section is the next theorem, which yields faithfulness criteria for $\pi_1 \otimes_{C,\min} \pi_2$:

THEOREM 5.5. *Let C be a unital, abelian C^*-algebra with Gelfand spectrum Ω and let $(A_1, \iota_1), (A_2, \iota_2)$ be C^*-algebras over C. Let further $\pi_j : A_j \to \mathcal{B}(\mathcal{H})$, $j = 1, 2$, be faithful, non-degenerate $*$-representations, such that*

$$M(\pi_1) \circ \iota_1 = M(\pi_2) \circ \iota_2 \quad \textit{and} \quad \pi_1(A_1) \subset N, \pi_2(A_2) \subset N'$$

for a type I *von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $Z = (M(\pi_j) \circ \iota_j)(C)''$, $\widetilde{\Omega}$ the Gelfand spectrum of Z, and $\pi : A_1 \otimes A_2 \to \mathcal{B}(\mathcal{H})$ the $*$-homomorphism defined by*

$$\pi(a_1 \otimes a_2) = \pi_1(a_1)\pi_2(a_2), \quad a_1 \in A_1, a_2 \in A_2.$$

Then the following statements are equivalent:

(i) *$\pi_1 \otimes_{C,\min} \pi_2$ is faithful;*

(ii) *the kernel of π is equal to $\mathcal{J}_C$;*

(iii) *if $T_{j,k} \in \pi_j(A_j)$, $j = 1, 2$, $1 \leqslant k \leqslant n$, and $\sum\limits_{1 \leqslant k \leqslant n} T_{1,k}T_{2,k} = 0$, then*

$$\sum_{1 \leqslant k \leqslant n} (\chi_1 \circ \Phi_e)(T_{1,k})(\chi_2 \circ \Phi_f)(T_{2,k}) = 0$$

for all abelian projections $e \in N, f \in N'$ with $\mathrm{z}_N(e) = \mathrm{z}_{N'}(f) = 1_{\mathcal{H}}$ and all $\chi_1, \chi_2 \in \widetilde{\Omega}$ with $\chi_1 \circ M(\pi_1) \circ \iota_1 = \chi_2 \circ M(\pi_2) \circ \iota_2$;

(iv) *for any* $\varphi_1 \in P(A_1)$ *and* $\varphi_2 \in P(A_2)$ *with* $\varphi_1 \circ \iota_1 = \varphi_2 \circ \iota_2$ *we have*

$$K(\pi_1(A_1), Z; \varphi_1 \circ \pi_1^{-1}) \cap K(\pi_2(A_2), Z; \varphi_2 \circ \pi_2^{-1}) \neq \emptyset.$$

Proof. By the definition of $\pi_1 \otimes_{C,\min} \pi_2$, (ii) is equivalent to the injectivity of the restriction of $\pi_1 \otimes_{C,\min} \pi_2$ to $(A_1 \otimes A_2)/\mathcal{J}_C$, so (i) implies (ii). Conversely, if (ii) is satisfied, then the C^*-seminorm $A_1 \otimes A_2 \ni a \longmapsto \|\pi(a)\|$ vanishes exactly on $\mathcal{J}_C$, so Proposition 2.6 entails that $\|\pi(a)\| \geqslant \|a\|_{C,\min}$ for all $a \in A_1 \otimes A_2$. Taking into account (3.7), it follows that $\pi_1 \otimes_{C,\min} \pi_2$ is isometric on $(A_1 \otimes A_2)/\mathcal{J}_C$, hence on the whole $A_1 \otimes_{C,\min} A_2$.

By the above we have (i)$\Leftrightarrow$(ii). Next we prove that (i)$\Rightarrow$(iii)$\Rightarrow$(ii).

Let us assume that (i) is satisfied and $T_{j,k} \in \pi_j(A_j), j = 1,2, 1 \leqslant k \leqslant n$ are such that $\sum\limits_{1 \leqslant k \leqslant n} T_{1,k}T_{2,k} = 0$. Then $T_{j,k} = \pi_j(a_{j,k})$ for some $a_{j,k} \in A_j$ and, setting $a = \sum\limits_{1 \leqslant k \leqslant n} a_{1,k} \otimes a_{2,k} \in A_1 \otimes A_2$, we have $(\pi_1 \otimes_{C,\min} \pi_2)(a/\mathcal{J}_C) = \pi(a) = \sum\limits_{1 \leqslant k \leqslant n} T_{1,k}T_{2,k} = 0$, and by (i) it follows that $a \in \mathcal{J}_C$. Using Corollary 4.6, we conclude that, for any abelian projections $e \in N, f \in N'$ with $\mathrm{z}_N(e) = \mathrm{z}_{N'}(f) = 1_{\mathcal{H}}$, and any $\chi_1, \chi_2 \in \widetilde{\Omega}$ satisfying $\chi_1 \circ M(\pi_1) \circ \iota_1 = \chi_2 \circ M(\pi_2) \circ \iota_2$,

$$\begin{aligned}\sum_{1 \leqslant k \leqslant n} (\chi_1 \circ \Phi_e)(T_{1,k})(\chi_2 \circ \Phi_f)(T_{2,k}) &= \sum_{1 \leqslant k \leqslant n} (\chi_1 \circ \Phi_e \circ \pi_1)(a_{1,k})(\chi_2 \circ \Phi_f \circ \pi_2)(a_{2,k}) \\ &= ((\chi_1 \circ \Phi_e \circ \pi_1) \otimes (\chi_2 \circ \Phi_f \circ \pi_2))(a) = 0.\end{aligned}$$

Now we assume that (iii) is satisfied and $a \in A_1 \otimes A_2$ is such that $\pi(a) = 0$. Then $a = \sum\limits_{1 \leqslant k \leqslant n} a_{1,k} \otimes a_{2,k}$ with $a_{j,k} \in A_j$, so $\sum\limits_{1 \leqslant k \leqslant n} \pi_1(a_{1,k})\pi_2(a_{2,k}) = \pi(a) = 0$. By (iii) it follows that

$$\begin{aligned}&((\chi_1 \circ \Phi_e \circ \pi_1) \otimes (\chi_2 \circ \Phi_f \circ \pi_2))(a) \\ &\qquad = \sum_{1 \leqslant k \leqslant n} (\chi_1 \circ \Phi_e)(\pi_1(a_{1,k}))(\chi_2 \circ \Phi_f)(\pi_2(a_{2,k})) = 0\end{aligned}$$

for all abelian projections $e \in N, f \in N'$ with $\mathrm{z}_N(e) = \mathrm{z}_{N'}(f) = 1_{\mathcal{H}}$ and all $\chi_1, \chi_2 \in \widetilde{\Omega}$ satisfying $\chi_1 \circ M(\pi_1) \circ \iota_1 = \chi_2 \circ M(\pi_2) \circ \iota_2$. By Corollary 4.6 it follows that $a \in \mathcal{J}_C$.

Finally we prove that (i)$\Rightarrow$(iv)$\Rightarrow$(ii).

Let us assume that (i) holds and let $\varphi_1 \in P(A_1)$ and $\varphi_2 \in P(A_2)$ be such that $\varphi_1 \circ \iota_1 = \varphi_2 \circ \iota_2$. Then there is $t \in \Omega$ such that $\varphi_1(\iota_1(c)) = \varphi_2(\iota_2(c)) = c(t)$ for all $c \in C$ and by Proposition 2.4 it follows that $\varphi_1|_{I_{\iota_1}(t)} = 0$, $\varphi_2|_{I_{\iota_2}(t)} = 0$. Therefore $|(\varphi_1 \otimes \varphi_2)(a)| \leqslant \|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|_{\min} \leqslant \|a\|_{C,\min}, a \in A_1 \otimes A_2$ and so there exists a state $\widetilde{\varphi}$ on $A_1 \otimes_{C,\min} A_2$ such that $(\varphi_1 \otimes \varphi_2)(a) = \widetilde{\varphi}(a/\mathcal{J}_C), a \in A_1 \otimes A_2$. Then $\tau = \widetilde{\varphi} \circ (\pi_1 \otimes_{C,\min} \pi_2)^{-1}$ is a state on $\overline{\mathrm{lin}\pi_1(A_1)\pi_2(A_2)}$, which can be extended by strict continuity to a state on $M(\overline{\mathrm{lin}\pi_1(A_1)\pi_2(A_2)})$, still denoted by τ. We notice that, by (3.8), $C^*(\pi_1(A_1) \cup \pi_2(A_2)) \subset M(\overline{\mathrm{lin}\pi_1(A_1)\pi_2(A_2)})$. Since $\tau(\pi(a)) = \tau((\pi_1 \otimes_{C,\min} \pi_2)(a/\mathcal{J}_C)) = \widetilde{\varphi}(a/\mathcal{J}_C) = (\varphi_1 \otimes \varphi_2)(a)$ for all

$a \in A_1 \otimes A_2$, choosing some increasing approximate units $\{u_\lambda\}_\lambda, \{v_\mu\}_\mu$ for A_1 respectively A_2 and using (3.8), we obtain

$$\tau(\pi_1(a_1)) = \lim_\mu \tau(\pi_1(a_1)\pi_2(v_\mu)) = \lim_\mu \varphi_1(a_1)\varphi_2(v_\mu) = \varphi_1(a_1), \quad a_1 \in A_1,$$
$$\tau(\pi_2(a_2)) = \lim_\mu \tau(\pi_1(u_\lambda)\pi_2(a_2)) = \lim_\mu \varphi_1(u_\lambda)\varphi_2(a_2) = \varphi_2(a_2), \quad a_2 \in A_2,$$

(for $\varphi_2(v_\mu) \longrightarrow \|\varphi_2\| = 1$ and $\varphi_1(u_\lambda) \longrightarrow \|\varphi_1\| = 1$; see, for example Theorem 4.5(i) of [21]). Consequently, if θ is an extension of $\tau|_{C^*(\pi_1(A_1)\cup\pi_2(A_2))}$ to a state on $C^*(\pi_1(A_1) \cup Z \cup \pi_2(A_2))$, then $\theta|_{\pi_j(A_j)} = \varphi_j \circ \pi_j^{-1}, j = 1, 2$, and so

$$\theta|_Z \in K(\pi_1(A_1), Z; \varphi_1 \circ \pi_1^{-1}) \cap K(\pi_2(A_2), Z; \varphi_2 \circ \pi_2^{-1}).$$

Now let us assume that (iv) holds and let $a \in A_1 \otimes A_2$ with $\pi(a) = 0$ and $\varphi_1 \in P(A_1)$, $\varphi_2 \in P(A_2)$ with $\varphi_1 \circ \iota_1 = \varphi_2 \circ \iota_2$ be arbitrary.

By (iv) the weak*compact, convex set $K(\pi_1(A_1), Z; \varphi_1 \circ \pi_1^{-1}) \cap K(\pi_2(A_2), Z; \varphi_2 \circ \pi_2^{-1})$ is not empty, so by the Krein-Milman Theorem it has some extreme point ψ. Now, by Lemma 5.4, there exist $\theta_j \in P(C^*(\pi_j(A_j) \cup Z))$, $j = 1, 2$, such that

$$\begin{aligned} &\theta_j|_{\pi_j(A_j)} = \varphi_j \circ \pi_j^{-1}, \quad \theta_j|_Z = \psi, \\ &\theta_j(Tz) = \theta_j(T)\theta_j(z), \quad T \in C^*(\pi_j(A_j) \cup Z), z \in Z. \end{aligned} \tag{5.3}$$

On the other hand, if $a = \sum_{1\leqslant k\leqslant n} a_{1,k} \otimes a_{2,k}$ with $a_{1,k} \in A_1$, $a_{2,k} \in A_2$, then $\sum_{1\leqslant k\leqslant n} \pi_1(a_{1,k})\pi_2(a_{2,k}) = \pi(a) = 0$ and $\pi_1(a_{1,k}) \in N, \pi_2(a_{2,k}) \in N'$. By a classical result of Murray, von Neumann and Kadison (see e.g. Theorem 1.20.5 of [16] or Theorem 5.5.4 of [10], or Proposition 7.20 of [21]) it follows that there are $z_{j,k} \in Z, 1 \leqslant j, k \leqslant n$, such that $\sum_{1\leqslant j\leqslant n} \pi_1(a_{1,j})z_{jk} = 0$ for every $1 \leqslant k \leqslant n$, and $\sum_{1\leqslant k\leqslant n} z_{j,k}\pi_2(a_{2,k}) = \pi_2(a_{2,j})$ for every $1 \leqslant j \leqslant n$. Using (5.3) and the above equalities, we deduce that

$$\begin{aligned} \sum_{1\leqslant j\leqslant n} \varphi_1(a_{1,j})\psi(z_{j,k}) &= \sum_{1\leqslant j\leqslant n} \theta_1(\pi_1(a_{1,j}))\theta_1(z_{j,k}) = \theta_1\Big(\sum_{1\leqslant j\leqslant n} \pi_1(a_{1,j})z_{j,k}\Big) \\ &= 0 \quad \text{for every } 1 \leqslant k \leqslant n, \end{aligned}$$

$$\begin{aligned} \sum_{1\leqslant k\leqslant n} \psi(z_{j,k})\varphi_2(a_{2,k}) &= \sum_{1\leqslant k\leqslant n} \theta_2(z_{j,k})\theta_2(\pi_2(a_{2,k})) = \theta_2\Big(\sum_{1\leqslant k\leqslant n} z_{j,k}\pi_2(a_{2,k})\Big) \\ &= \theta_2(\pi_2(a_{2,j})) = \varphi_2(a_{2,j}) \quad \text{for every } 1 \leqslant j \leqslant n. \end{aligned}$$

Consequently

$$(\varphi_1 \otimes \varphi_2)(a) = \sum_{1 \leqslant j \leqslant n} \varphi_1(a_{1,j})\varphi_2(a_{2,j}) = \sum_{1 \leqslant j \leqslant n} \varphi_1(a_{1,j})\Big(\sum_{1 \leqslant k \leqslant n} \psi(z_{j,k})\varphi_2(a_{2,k})\Big)$$
$$= \sum_{1 \leqslant k \leqslant n} \Big(\sum_{1 \leqslant j \leqslant n} \varphi_1(a_{1,j})\psi(z_{j,k})\Big)\varphi_2(a_{2,k}) = 0.$$

But if a belongs to the kernel of π, then all b^*ab, $b \in A_1 \otimes A_2$, belong to the kernel of π, so by the above we have

$$(\varphi_1 \otimes \varphi_2)(b^*ab) = 0$$

for all $\varphi_1 \in P(A_1), \varphi_2 \in P(A_2)$ with $\varphi_1 \circ \iota_1 = \varphi_2 \circ \iota_2$ and all $b \in A_1 \otimes A_2$. By Corollary 2.5 it follows that $a/\mathcal{J}_C = 0$, that is $a \in \mathcal{J}_C$. ∎

A first application concerns the proper C^*-algebras over C:

COROLLARY 5.6. *Let C be a unital, abelian C^*-algebra and let (A_1, ι_1), (A_2, ι_2) be C^*-algebras over C. If $\pi_1 : A_1 \longrightarrow \mathcal{B}(\mathcal{H})$ and $\pi_2 : A_2 \longrightarrow \mathcal{B}(\mathcal{H})$ are faithful, non-degenerate $*$-representations and*

$$M(\pi_1) \circ \iota_1 = M(\pi_2) \circ \iota_2 \quad \textit{and} \quad \pi_1(A_1) \subset N, \pi_2(A_2) \subset N'$$

for a type I *von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $(M(\pi_j) \circ \iota_j)(C)$, then $\pi_1 \otimes_{C,\min} \pi_2$ is faithful.*

Proof. Since $M(\pi_j) \circ \iota_j$ is injective and $(M(\pi_j) \circ \iota_j)(C) = (M(\pi_j) \circ \iota_j)(C)''$, any characters χ_1, χ_2 on $(M(\pi_j) \circ \iota_j)(C)''$ with $\chi_1 \circ M(\pi_1) \circ \iota_1 = \chi_2 \circ M(\pi_2) \circ \iota_2$ are equal. Thus condition (iii) in Theorem 5.5 is trivially satisfied. ∎

The next application of Theorem 5.5 concerns unital $*$-representations, whose normal extension on a substantial part of the second dual is faithful:

COROLLARY 5.7. *Let C be a unital, abelian C^*-algebra and let (A_1, ι_1), (A_2, ι_2) be unital C^*-algebras over C. If $\pi_j : A_1 \to \mathcal{B}(\mathcal{H})$, $j = 1, 2$, are unital $*$-representations, such that the normal extension $\widetilde{\pi}_j : A_j^{**} \longrightarrow \mathcal{B}(\mathcal{H})$ of π_j is faithful on $C^*(A_j \cup \iota_j(C)^{**})$, and*

$$\pi_1 \circ \iota_1 = \pi_2 \circ \iota_2 \quad \textit{and} \quad \pi_1(A_1) \subset N, \pi_2(A_2) \subset N'$$

for a type I *von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $(\pi_j \circ \iota_j)(C)''$, then $\pi_1 \otimes_{C,\min} \pi_2$ is faithful.*

Proof. Let Ω denote the Gelfand spectrum of C and set $Z = (\pi_j \circ \iota_j)(C)''$. We shall verify that condition (iv) in Theorem 5.5 is satisfied.

For let $\varphi_1 \in P(A_1)$ and $\varphi_2 \in P(A_2)$ be such that $\varphi_1 \circ \iota_1 = \varphi_2 \circ \iota_2$. Then $C \ni c \mapsto (\varphi_j \circ \iota_j)(c)$ is a character of C, whose normal extension to C^{**} is equal to the composition $\varphi_j \circ \iota_j^{**}$ of the normal state φ_j on A_j^{**} with the second transposed map ι_j^{**}. Since $\widetilde{\pi}_j \circ \iota_j^{**} : C^{**} \to \mathcal{B}(\mathcal{H})$ is a faithful, normal $*$-representation with range Z, which does not depend on $j = 1, 2$, we can consider the character $\chi = (\varphi_j \circ \iota_j^{**}) \circ (\widetilde{\pi}_j \circ \iota_j^{**})^{-1}$ of Z.

Now let $j = 1, 2$ be arbitrary. Let θ_j denote the composition of the normal state φ_j of A_j^{**} with $(\widetilde{\pi}_j|_{C^*(A_j \cup \iota_j(C)^{**})})^{-1}$. Then θ_j is a state on

$$\widetilde{\pi}_j(C^*(A_j \cup \iota_j(C)^{**})) = C^*(\pi_j(A_j) \cup (\widetilde{\pi}_j \circ \iota_j^{**})(C^{**})),$$

whose restrictions to $\pi_j(A_j)$ and to $Z = (\widetilde{\pi}_j \circ \iota_j^{**})(C^{**})$ are $\varphi_j \circ \pi_j^{-1}$ and χ, respectively.

Consequently $K(\pi_1(A_1), Z; \varphi_1 \circ \pi_1^{-1}) \cap K(\pi_2(A_2), Z; \varphi_2 \circ \pi_2^{-1}) \ni \chi$. ∎

The situation in Corollary 5.7 can occur for any pair of unital C^*-algebras (A_1, ι_1), (A_2, ι_2) over C. Indeed, then $\iota_j^{**} : C^{**} \longrightarrow Z(A_j^{**})$, $j = 1, 2$, are injective unital, normal $*$-homomorphisms, so by Lemma 5.2 of [20] there exist injective unital, normal $*$-representations $\widetilde{\pi}_j : A_j^{**} \longrightarrow \mathcal{B}(\mathcal{H})$, $j = 1, 2$, such that $\widetilde{\pi}_1 \circ \iota_1^{**} = \widetilde{\pi}_2 \circ \iota_2^{**}$ and $\widetilde{\pi}_1(A_1^{**}) \subset N$, $\widetilde{\pi}_2(A_2^{**}) \subset N'$ for some type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre equal to $(\widetilde{\pi}_j \circ \iota_j^{**})(C^{**})$ and, denoting $\pi_j = \widetilde{\pi}_j|_{A_j}$, $j = 1, 2$, the normal extension $\widetilde{\pi}_j$ of π_j to A_j^{**} is faithful and

$$\pi_1 \circ \iota_1 = \pi_2 \circ \iota_2, \quad \pi_1(A_1) \subset N, \pi_2(A_2) \subset N', \quad Z(N) = (\pi_j \circ \iota_j)(C)''.$$

The above remarks and Corollary 5.7 imply immediately:

COROLLARY 5.8. *Let C be a unital, abelian C^*-algebra and let (A_1, ι_1), (A_2, ι_2) be C^*-algebras over C. Then there exist faithful, unital $*$-representations $\rho_j : M(A_j) \to \mathcal{B}(\mathcal{H})$, $j = 1, 2$, such that*

$$\rho_1 \circ \iota_1 = \rho_2 \circ \iota_2 \quad \textit{and} \quad \rho_1(M(A_1)) \subset N, \rho_2(M(A_2)) \subset N'$$

for some type I *von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $(\rho_j \circ \iota_j)(C)''$ and $\rho_1 \otimes_{C,\min} \rho_2$ is faithful.*

According to Corollary 2.3, if ρ_1, ρ_2 are as in Corollary 5.8, then $\rho_1 \otimes_{C,\min} \rho_2$ is faithful on $A_1 \otimes_{C,\min} A_2 \subset M(A_1) \otimes_{C,\min} M(A_2)$. However, in general we do not have $\rho_j = M(\pi_j)$, and so $(\rho_1 \otimes_{C,\min} \rho_2)|_{A_1 \otimes_{C,\min} A_2} = \pi_1 \otimes_{C,\min} \pi_2$, for appropriate non-degenerate $*$-representations $\pi_j : A_j \longrightarrow \mathcal{B}(\mathcal{H})$, because $(\rho_1 \otimes_{C,\min} \rho_2)|_{A_1 \otimes_{C,\min} A_2}$ is not always non-degenerate. Taking, for example, for A_1, A_2 the non-zero C^*-algebras over $C([0,1])$ with $A_1 \otimes_{C([0,1]),\min} A_2 = \{0\}$, given in [1] before Proposition 3.3, we will have $\rho_1 \neq 0$ and $\rho_2 \neq 0$, hence $(\rho_1 \otimes_{C([0,1]),\min} \rho_2) \neq 0$, while $(\rho_1 \otimes_{C([0,1]),\min} \rho_2)|_{A_1 \otimes_{C([0,1]),\min} A_2} = 0$.

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