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C^*-ALGEBRAS OF LABELLED GRAPHS

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ABSTRACT. We describe a class of C^*-algebras which simultaneously generalise the ultragraph algebras of Tomforde and the shift space C^*-algebras of Matsumoto. In doing so we shed some new light on the different C^*-algebras that may be associated to a shift space. Finally, we show how to associate a simple C^*-algebra to an irreducible sofic shift.

KEYWORDS: *C^*-algebras, labelled graph, ultragraph, shift space, Matsumoto algebra.*

MSC (2000): 46L05, 37B10.

1. INTRODUCTION

The purpose of this paper is to introduce a class of C^*-algebras associated to labelled graphs. Our motivation is to provide a common framework for working with the ultragraph algebras of Tomforde (see [26], [27]) and the C^*-algebras associated to shift spaces studied by Matsumoto and Carlsen (see [14], [16], [6], [8] amongst others). Here a labelled graph $(E, \mathcal{L})$ over an alphabet $\mathcal{A}$ is a directed graph E, together with a map $\mathcal{L} : E^1 \to \mathcal{A}$. An ultragraph $\mathcal{G}$ is a particular example of a labelled graph (see Example 3.3 (ii)), and a shift space Λ has many presentations as a labelled graph (see Example 3.3 (iii) of [13]). Hence it is natural to give our common framework in terms of labelled graphs.

To a two-sided shift space Λ over a finite alphabet, Matsumoto associates two C^*-algebras $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda^*}$ generated by partial isometries (see [8]). Although $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda^*}$ are generated by elements satisfying the same relations, it turns out that they are not isomorphic in general (see Theorem 4.1 of [8]). This fact manifests itself in our realisation in Section 6.2 of $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda^*}$ as the C^*-algebras of the labelled graphs $(E_\Lambda, \mathcal{L}_\Lambda)$ and $(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*})$ respectively, which are not necessarily isomorphic as labelled graphs. Moreover, in Corollary 6.9 we show that using labelled graphs gives us the facility to canonically associate a simple C^*-algebra to an irreducible sofic shift (cf. [8], [6], [7]).

In fact we can associate a number of (possibly different) C^*-algebras to a labelled graph. This leads us to the notion of a labelled space, which we describe in Section 3. Briefly, a labelled space $(E, \mathcal{L}, \mathcal{B})$ consists of a labelled graph $(E, \mathcal{L})$ together with a collection $\mathcal{B} \subseteq 2^{E^0}$ which plays the same role as $\mathcal{G}^0$ in [26] and is related to the abelian AF-subalgebra A_Λ (respectively A_{Λ^*}) in $\mathcal{O}_\Lambda$ (respectively $\mathcal{O}_{\Lambda^*}$) generated by the source projections.

In Section 4 we define a representation of a labelled space in terms of partial isometries $\{s_a : a \in \mathcal{A}\}$ and projections $\{p_A : A \in \mathcal{B}\}$ subject to certain relations. Our relations generalise those found in [26], [14]. In order to build a nondegenerate C^*-algebra from a representation of $(E, \mathcal{L}, \mathcal{B})$ it is necessary for $\mathcal{B}$ to be weakly left-resolving: a condition which is a generalisation of the left-resolving property for labelled graphs. Hence we may define $C^*(E, \mathcal{L}, \mathcal{B})$ to be the C^*-algebra which is universal for representations of the weakly left-resolving labelled space $(E, \mathcal{L}, \mathcal{B})$. Since any ultragraph has a natural realisation as a left-resolving labelled graph, the class of C^*-algebras of labelled spaces contains the ultragraph algebras (and hence, graph algebras and Exel-Laca algebras).

In Section 5 we give a version of the gauge-invariant uniqueness theorem for $C^*(E, \mathcal{L}, \mathcal{B})$ which will ultimately allow us to make the connection with the Matsumoto algebras.

In Section 6 we give three applications of our uniqueness theorem: In Section 6.1 we show how to construct a dual labelled space, which is the analogue of the higher block presentation of a shift space (cf. [13]). We give an isomorphism theorem for dual labelled spaces which is a generalisation of Corollary 2.5 in [4] and forms a starting point for future work (see [3]). In Section 6.2 we show that if $\mathcal{O}_\Lambda$ (respectively $\mathcal{O}_{\Lambda^*}$) has a gauge action, then it is isomorphic to the C^*-algebra of a certain labelled space. Then in Section 6.3 we give necessary conditions for the C^*-algebra of a labelled space to be isomorphic to the C^*-algebra of the underlying directed graph. We then show how to associate a simple C^*-algebra to an irreducible shift space. By example, we show that in general the C^*-algebra of a labelled space will not be isomorphic to the C^*-algebra of any directed graph; hence labelled graph C^*-algebras form a strictly larger class of C^*-algebras than graph algebras.

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Since we seek to generalise them, we begin by giving a brief description of ultragraph algebras and Matsumoto algebras.

2. ULTRAGRAPH ALGEBRAS AND MATSUMOTO ALGEBRAS

2.1. ULTRAGRAPH ALGEBRAS. Following [26], an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consists of a countable set of vertices G^0, a countable set of edges $\mathcal{G}^1$, and functions $s : \mathcal{G}^1 \to G^0$ and $r : \mathcal{G}^1 \to 2^{G^0}$. Let $\mathcal{G}^0$ be the smallest collection of 2^{G^0}

which contains $s(e)$ and $r(e)$ for all $e \in \mathcal{G}^1$ and is closed under finite intersections and unions. The ultragraph algebra $C^*(\mathcal{G})$ is the universal C^*-algebra for Cuntz-Krieger $\mathcal{G}$-families: collections of partial isometries $\{s_e : e \in \mathcal{G}^1\}$ with mutually orthogonal ranges, and projections $\{p_A : A \in \mathcal{G}^0\}$ satisfying the relations:

(1.) $p_\emptyset = 0$, $p_A p_B = p_{A\cap B}$ and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ for all $A, B \in \mathcal{G}^0$;
(2.) $s_e^* s_e = p_{r(e)}$ and $s_e s_e^* \leqslant p_{s(e)}$ for all $e \in \mathcal{G}^1$;
(3.) $p_v = \sum\limits_{s(e)=v} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$;

(see Definition 2.7 in [26]). Recall that $v \in G^0$ is an infinite emitter if $|s^{-1}(v)| = \infty$.

If $\mathcal{G}$ has no infinite emitters, then the underlying graph (see Examples 3.3 (ii)) can still fail to be row-finite. With this in mind we make the following definition (cf. Remark 2.6 in [26]):

DEFINITION 2.1. The ultragraph $\mathcal{G}$ is *row-finite* if there are no infinite emitters and $r(e)$ is finite for all $e \in \mathcal{G}^1$.

Ultragraph algebras simultaneously generalise graph C^*-algebras and Exel-Laca algebras (see Sections 3 and 4 of [26]). By Corollary 5.5 in [27] there is a non row-finite ultragraph whose C^*-algebra is not isomorphic to a graph algebra or an Exel-Laca algebra.

2.2. MATSUMOTO ALGEBRAS. For an introduction to shift spaces we refer the reader to the excellent treatment in [13]. Let Λ be a two-sided shift space over a finite alphabet $\mathcal{A}$. Let

$$X_\Lambda = \{(x_i)_{i\geqslant 1} : (x_i)_{i\in\mathbb{Z}} \in \Lambda\} \tag{2.1}$$

denote the set of all right-infinite sequences in Λ.

For each $k \geqslant 1$, let Λ^k be the set of all words with length k appearing in some $x \in \Lambda$. We set $\Lambda_\ell = \bigcup\limits_{k=0}^{\ell} \Lambda^k$ and $\Lambda^* = \bigcup\limits_{k=0}^{\infty} \Lambda^k$ where Λ^0 denotes the empty word $\emptyset$.

Following [8] there are two C^*-algebras associated to Λ. Each C^*-algebra is generated by partial isometries $\{t_a : a \in \mathcal{A}\}$ subject to

$$\sum_{a\in\mathcal{A}} t_a t_a^* = 1, \quad \text{and} \quad t_\alpha^* t_\alpha t_\beta = t_\beta t_{\alpha\beta}^* t_{\alpha\beta}, \quad \text{where } \alpha, \beta, \alpha\beta \in \Lambda^*. \tag{2.2}$$

As in [8] we denote by $\mathcal{O}_\Lambda$ the C^*-algebra defined directly on Hilbert space in [18], [20] and by $\mathcal{O}_{\Lambda^*}$ the C^*-algebra defined using the Fock space construction in [14], [16], [17], [15], [19]. Because of the different ways in which the relations (2.2) are realised it turns out that $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda^*}$ are not isomorphic in general (see Section 6 in [8]).

There is a uniqueness theorem for $\mathcal{O}_\Lambda$ (respectively $\mathcal{O}_{\Lambda^*}$) when Λ satisfies Condition (I) (respectively Condition (I*)) given in Section 4 of [8] (respectively Section 3 of [8]).

CONDITION I. For $x \in X_\Lambda$ and $l \in \mathbb{N}$ put $\Lambda_l(x) = \{\mu \in \Lambda_l : \mu x \in X_\Lambda\}$. Two infinite paths $x, y \in X_\Lambda$ are *l-past equivalent* (written $x \sim_l y$) if $\Lambda_l(x) = \Lambda_l(y)$. The shift space X_Λ satisfies Condition (I) if for any $l \in \mathbb{N}$ and $x \in X_\Lambda$ there exists $y \in X_\Lambda$ such that $y \neq x$, $y \sim_l x$.

CONDITION I*. For $\omega \in \Lambda^*$ and $l \in \mathbb{N}$ we set $\Lambda_l(\omega) = \{\mu : |\mu| \leqslant l,\ \mu\omega \in \Lambda^*\}$. Two words $\mu, \nu \in \Lambda^*$ are said to be *l-past equivalent* (written $\mu \sim_l \nu$) if $\Lambda_l(\mu) = \Lambda_l(\nu)$. The subset $\Lambda_l^* \subseteq \Lambda^*$ is defined by

$$\Lambda_l^* := \{\omega \in \Lambda^* : |\{\mu \in \Lambda^* : \mu \sim_l \omega\}| < \infty\}.$$

The shift space Λ satisfies Condition (I*) if for every $l \in \mathbb{N}$ and $\mu \in \Lambda_l^*$ there exist distinct words $\xi_1, \xi_2 \in \Lambda^*$ with $|\xi_1| = |\xi_2| = m$ such that

$$\mu \sim_l \xi_1\gamma_1 \quad \text{and} \quad \mu \sim_l \xi_2\gamma_2$$

for some $\gamma_1, \gamma_2 \in \Lambda_{l+m}^*$.

PROPOSITION 2.2. *Let Λ be a two-sided shift space over a finite alphabet which satisfies Condition* (I). *Then there is a strongly continuous action β of $\mathbb{T}$ on $\mathcal{O}_\Lambda$ such that $\beta_z(t_a) = zt_a$ for all $a \in \mathcal{A}$ and $z \in \mathbb{T}$.*

Proof. That each β_z is an automorphism of $\mathcal{O}_\Lambda$ for each $z \in \mathbb{T}$ follows from Proposition 4.2 in [8]. A standard $\epsilon/3$ argument shows that β is strongly continuous. ∎

From p. 363 in [14] there is always a gauge action on $\mathcal{O}_{\Lambda^*}$. In [19] Matsumoto defines λ-graph systems $\mathcal{L}_\Lambda$ and $\mathcal{L}_{\Lambda^*}$ associated to a two-sided shift space Λ together with corresponding C^*-algebras $\mathcal{O}_{\mathcal{L}_\Lambda}$ and $\mathcal{O}_{\mathcal{L}_{\Lambda^*}}$. By Theorem 5.6 in [8] we see that if Λ satisfies Condition (I) then $\mathcal{O}_\Lambda \cong \mathcal{O}_{\mathcal{L}_\Lambda}$ and if Λ satisfies Condition (I*) then $\mathcal{O}_{\Lambda^*} \cong \mathcal{O}_{\mathcal{L}_{\Lambda^*}}$. Hence, for our purposes, it suffices to work with $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda^*}$.

3. LABELLED SPACES

A directed graph E consists of a quadruple (E^0, E^1, r, s) where E^0 and E^1 are countable sets of vertices and edges respectively and $r, s : E^1 \to E^0$ are maps giving the direction of each edge. A path $\lambda = e_1 \cdots e_n$ is a sequence of edges $e_i \in E^1$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n-1$. The collection of paths of length n in E is denoted E^n and the collection of all finite paths in E by E^*, so that $E^* = \bigcup_{n \geqslant 0} E^n$. The edge shift (X_E, σ_E) associated to a directed graph E with no sinks or sources is defined by:

$$\mathsf{X}_E = \{x \in (E^1)^{\mathbb{Z}} : s(x_{i+1}) = r(x_i) \text{ for all } i \in \mathbb{Z}\} \quad \text{and} \quad (\sigma_E x)_i = x_{i+1} \quad \text{for } i \in \mathbb{Z}.$$

The following definition is adapted from Definition 3.1.1 in [13]:

DEFINITION 3.1. A *labelled graph* $(E, \mathcal{L})$ over an alphabet $\mathcal{A}$ consists of a directed graph E together with a labelling map $\mathcal{L} : E^1 \to \mathcal{A}$.

Without loss of generality we may assume that the map $\mathcal{L}$ is onto. We say that the labelled graph $(E, \mathcal{L})$ is *row-finite* if the underlying graph E is row-finite.

Given a labelled graph $(E, \mathcal{L})$ such that every vertex in E emits and receives an edge, we may define a subshift $(\mathsf{X}_{(E,\mathcal{L})}, \sigma)$ of $\mathcal{A}^{\mathbb{Z}}$ by

$$\mathsf{X}_{(E,\mathcal{L})} = \{y \in \mathcal{A}^{\mathbb{Z}} : \text{ there exists } x \in \mathsf{X}_E \text{ such that } y_i = \mathcal{L}(x_i) \text{ for all } i \in \mathbb{Z}\},$$

where σ is the shift map. The labelled graph $(E, \mathcal{L})$ is said to be a *presentation* of the shift space $X = \mathsf{X}_{(E,\mathcal{L})}$. As shown in Section 3.1 of [13] a shift space may have many different presentations (see Examples 3.3 (ii), (vi), (vii)).

Let $\mathcal{A}^*$ be the collection of all *words* in the symbols of $\mathcal{A}$ (see Section 0.2 of [25]). The map $\mathcal{L}$ extends naturally to a map $\mathcal{L} : E^n \to \mathcal{A}^*$, where $n \geqslant 1$: for $\lambda = e_1 \cdots e_n \in E^n$ put $\mathcal{L}(\lambda) = \mathcal{L}(e_1) \cdots \mathcal{L}(e_n)$; in this case the path $\lambda \in E^n$ is said to be a *representative* of the *labelled path* $\mathcal{L}(e_1) \cdots \mathcal{L}(e_n)$. Let $\mathcal{L}(E^n)$ denote the collection of all labelled paths in $(E, \mathcal{L})$ of length n, then $\mathcal{L}^*(E) = \bigcup\limits_{n \geqslant 1} \mathcal{L}(E^n)$ denotes the collection of all words in the alphabet $\mathcal{A}$ which may be represented by paths in the labelled graph $(E, \mathcal{L})$. In this way $\mathcal{L}$ induces a map from the language $\bigcup\limits_{n \geqslant 1} E^n$ of the subshift of finite type X_E associated to E into $\mathcal{L}^*(E)$, the language of the shift space $\mathsf{X}_{(E,\mathcal{L})}$ presented by $(E, \mathcal{L})$ (see Section 3 of [13]). The usual length function $|\cdot| : E^* \to \mathbb{N}$ transfers naturally over to $\mathcal{L}^*(E)$.

For α in $\mathcal{L}^*(E)$ we put

$$s_{\mathcal{L}}(\alpha) = \{s(\lambda) \in E^0 : \mathcal{L}(\lambda) = \alpha\} \quad \text{and} \quad r_{\mathcal{L}}(\alpha) = \{r(\lambda) \in E^0 : \mathcal{L}(\lambda) = \alpha\},$$

so that $r_{\mathcal{L}}, s_{\mathcal{L}} : \mathcal{L}^*(E) \to 2^{E^0}$. We shall drop the subscript on $r_{\mathcal{L}}$ and $s_{\mathcal{L}}$ if the context in which it is being used is clear. For $\alpha, \beta \in \mathcal{L}^*(E)$ we have $\alpha\beta \in \mathcal{L}^*(E)$ if and only if $r(\alpha) \cap s(\beta) \neq \emptyset$.

Where possible we shall denote the elements of $\mathcal{A} = \mathcal{L}(E^1)$ as a, b, etc., elements of $\mathcal{L}^*(E)$ as α, β, etc., leaving e, f for elements of E^1 and λ, μ for elements of E^*.

Let $(E, \mathcal{L})$ and $(F, \mathcal{L}')$ be graphs labelled by the same alphabet. A graph isomorphism $\phi : E \to F$ is a *labelled graph isomorphism* if $\mathcal{L}'(\phi(e)) = \mathcal{L}(e)$ for all $e \in E^1$ and we write $\phi : (E, \mathcal{L}) \to (F, \mathcal{L}')$.

DEFINITION 3.2. The labelled graph $(E, \mathcal{L})$ is *left-resolving* if for all $v \in E^0$ the map $\mathcal{L} : r^{-1}(v) \to \mathcal{A}$ is injective.

The left-resolving condition ensures that for all $v \in E^0$ the labels $\{\mathcal{L}(e) : r(e) = v\}$ of all incoming edges to v are all different. In particular if $\lambda, \mu \in \bigcup\limits_{n \geqslant 1} E^n$ satisfy $\mathcal{L}(\lambda) = \mathcal{L}(\mu)$ and $r(\lambda) = r(\mu)$ then $\lambda = \mu$.

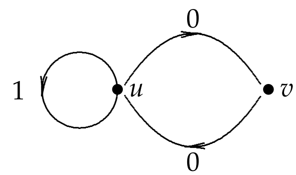
EXAMPLES 3.3. (i) Let E be a directed graph. Put $\mathcal{A} = E^1$ and let $\mathcal{L} : E^1 \to E^1$ be the identity map (the *trivial labelling*); then $(E, \mathcal{L})$ is a left-resolving labelled graph.

(ii) Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph. Define $E = E_{\mathcal{G}}$ by putting $E^0 = G^0$, $E^1 = \{(e,w) : e \in \mathcal{G}^1, w \in r(e)\}$ and defining $r', s' : E^1 \to E^0$ by $s'(e,w) = s(e)$, $r'(e,w) = w$. Set $\mathcal{A} = \mathcal{G}^1$ and define $\mathcal{L}_{\mathcal{G}} : E^1 \to \mathcal{A}$ by $\mathcal{L}_{\mathcal{G}}(e,w) = e$. The resulting labelled graph $(E_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}})$ is left-resolving since the source map is single-valued. If $\mathcal{G}$ is row-finite in the sense of Definition 2.1 then $E_{\mathcal{G}}$ is row-finite.

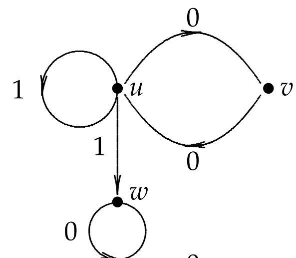
On the other hand, given a left-resolving labelled graph $(E, \mathcal{L})$ over an alphabet $\mathcal{A}$ where $s_{\mathcal{L}} : \mathcal{L}^*(E) \to 2^{E^0}$ is single-valued, we can form a ultragraph $\mathcal{G}_{(E,\mathcal{L})} = (E^0, \mathcal{A}, r', s')$ with $s' = s_{\mathcal{L}}$ and $r' = r_{\mathcal{L}}$. If $(E,\mathcal{L})$ is row-finite then the ultragraph $\mathcal{G}_{(E,\mathcal{L})}$ is row-finite.

(iii) Following Section 3 of [13] the labelled graphs

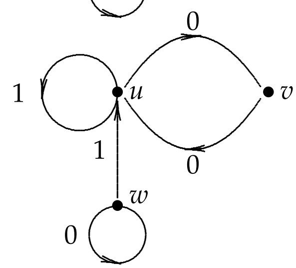
$(E_1, \mathcal{L}_1) :=$



$(E_2, \mathcal{L}_2) :=$



$(E_3, \mathcal{L}_3) :=$



have the same language as the even shift Y since between any two 1′s there must be an even number of 0′s. Hence $\mathsf{X}_{(E_i,\mathcal{L}_i)} = Y$ for $i = 1,2,3$ by Proposition 1.3.4 (3) of [13]. Only graphs $(E_1, \mathcal{L}_1)$ and $(E_2, \mathcal{L}_2)$ are left-resolving.

(iv) Let E be a directed graph and Γ a group which acts on (the right of) E. Define $\mathcal{L}_q : E^1 \to E^1/\Gamma$ by $\mathcal{L}_q(e) = q(e)$ where $q : E^1 \to E^1/\Gamma$ is the quotient map. If the action of Γ is free on E^1, then the resulting labelled graph $(E, \mathcal{L}_q)$ is left-resolving. More generally, if $p : F \to E$ is a graph morphism then there is a labelling $\mathcal{L}_p : F^1 \to E^1$ given by $\mathcal{L}_p(f) = p(f)$ for all $f \in E^1$. If p is a covering map then $\mathcal{L}_p$ is left-resolving.

(v) Recall from Section 3 of [2], that an out-splitting of a directed graph E is formed by a partition $\mathcal{P}$ of $s^{-1}(v)$ into $m(v) \geqslant 1$ non-empty subsets for each

$v \in E^0$ (if $s^{-1}(v) = \emptyset$ then $m(v) = 0$). Given such a partition $\mathcal{P}$ one may construct a directed graph $E_s(\mathcal{P})$ where $E_s(\mathcal{P}^1) = \{e^j : e \in E^1, 1 \leqslant j \leqslant m(r(e))\} \cup \{e : m(r(e)) = 0\}$. Define $\mathcal{L} : E_s(\mathcal{P})^1 \to E^1$ by $\mathcal{L}(e^j) = e$ for $1 \leqslant j \leqslant m(r(e))$ and $\mathcal{L}(e) = e$ if $m(r(e)) = 0$. For an in-splitting (see Section 5 in [2]) of E using a partition $\mathcal{P}$, a similar construction also yields a labelled graph. However the resulting labelling $\mathcal{L}$ of the in-split graph $E_r(\mathcal{P})$ will not be left-resolving in general.

(vi) Let Λ be a two-sided shift space over a finite alphabet $\mathcal{A}$ with X_Λ defined as in (2.1). Let $X_\Lambda^- = \{(x_i)_{i\leqslant 0} : (x_i)_{i\in\mathbb{Z}} \in \Lambda\}$ so that any element $x \in \Lambda$ may be written as $x = x^-x^+$. For arbitrary $x^+ \in X_\Lambda$ and $x^- \in X_\Lambda^-$ the bi-infinite sequence $y = x^-x^+$ may not belong to Λ. Define the past set of $t \in X_\Lambda$ as

$$P_\infty(t) = \{x^- \in X_\Lambda^- \ : \ x^-t \in \Lambda\}.$$

A shift is *sofic* if and only if the number of past sets is finite [11], [13].

For $s, t \in X_\Lambda$, we say that s is *past equivalent* to t (denoted $s \sim_\infty t$) if $P_\infty(s) = P_\infty(t)$. Define a labelled graph $(E_\Lambda, \mathcal{L}_\Lambda)$ as follows: let $E_\Lambda^0 = \{[v] : v \in X_\Lambda/\sim_\infty\}$, $E_\Lambda^1 = \{([v], a, [w]) : a \in \mathcal{A}, aw \sim_\infty v\}$ with $s([v], a, [w]) = [v]$ and $r([v], a, [w]) = [w]$. If $([v], a, [w]) \in E_\Lambda^1$ we put $\mathcal{L}_\Lambda([v], a, [w]) = a$. The resulting left-resolving labelled graph is usually referred to as the *left-Krieger cover* of Λ and the construction is evidently independent of the choice of representatives (see [11]).

If Y is the even shift then $(E_Y, \mathcal{L}_Y)$ is labelled graph isomorphic to $(E_2, \mathcal{L}_2)$ in (iii) above. Let Z be shift over the alphabet $\{1, 2, 3, 4\}$ in which the words

$$\{12^k1, 32^k12, 32^k13, 42^k14 : k \geqslant 0\}$$

do not occur (see Section 4 of [8]) then $(E_Z, \mathcal{L}_Z)$ has six vertices.

(vii) Let Λ be a two-sided shift over a finite alphabet $\mathcal{A}$. We construct a variant of the predecessor graph $(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*})$ in the following way. For $\mu \in \Lambda^*$ we define

$$P(\mu) := \{\lambda : \lambda\mu \in \Lambda^*\}$$

and define an equivalence relation by $\mu \sim \nu$ if $P(\mu) = P(\nu)$. A shift is sofic if and only if the number of predecessor sets is finite [13].

Let Λ_∞^* denote those $\mu \in \Lambda^*$ which have an infinite equivalence class. Since $\mathcal{A}$ is finite $\Lambda_\infty^*/\sim$ can be identified with $\Omega_{\Lambda^*} = \varprojlim \Omega_l^*$ as described in Section 2 of [16]. We set $E_{\Lambda^*}^0 = \Lambda_\infty^*/\sim$, $E_{\Lambda^*}^1 = \{([\mu], a, [\nu]) : a \in \mathcal{A}, [\mu] = [a\nu]\}$, $r([\mu], a, [\nu]) = [\nu]$ and $s([\mu], a, [\nu]) = [\mu]$. The labelling map is defined by $\mathcal{L}_{\Lambda^*}([\mu], a, [\nu]) = a$. The resulting labelled graph is evidently left-resolving.

If Y is the even shift then $(E_{Y^*}, \mathcal{L}_{Y^*})$ is labelled graph isomorphic to $(E_2, \mathcal{L}_2)$ in (iii) above (cf. [6], [16]). If Z is the sofic shift described in Example 3.3 (vi) then $(E_{Z^*}, \mathcal{L}_{Z^*})$ has seven vertices and contains $(E_Z, \mathcal{L}_Z)$ as a subgraph.

DEFINITION 3.4. Let $(E, \mathcal{L})$ be a labelled graph. For $A \subseteq E^0$ and $\alpha \in \mathcal{L}^*(E)$ the *relative range of α with respect to A* is defined to be

$$r(A, \alpha) = \{r(\lambda) : \lambda \in E^*, \mathcal{L}(\lambda) = \alpha, s(\lambda) \in A\}.$$

REMARK 3.5. For any $A, B \subseteq E^0$ and $\alpha \in \mathcal{L}^*(E)$ we have

$$r(A \cap B, \alpha) \subseteq r(A, \alpha) \cap r(B, \alpha) \quad \text{and} \quad r(A \cup B, \alpha) = r(A, \alpha) \cup r(B, \alpha).$$

For all $A \subseteq E^0$ and $\alpha \in \mathcal{L}^*(E)$ we have $r(A, \alpha) = r(A \cap s(\alpha), \alpha)$.

A collection $\mathcal{B} \subseteq 2^{E^0}$ of subsets of E^0 is said to be *closed under relative ranges for* $(E, \mathcal{L})$ if for all $A \in \mathcal{B}$ and $\alpha \in \mathcal{L}^*(E)$ we have $r(A, \alpha) \in \mathcal{B}$. If $\mathcal{B}$ is closed under relative ranges for $(E, \mathcal{L})$, contains $r(\alpha)$ for all $\alpha \in \mathcal{L}^*(E)$ and is also closed under finite intersections and unions, then we say that $\mathcal{B}$ is *accommodating* for $(E, \mathcal{L})$.

DEFINITION 3.6. A *labelled space* consists of a triple $(E, \mathcal{L}, \mathcal{B})$, where $(E, \mathcal{L})$ is a labelled graph and $\mathcal{B}$ is accommodating for $(E, \mathcal{L})$.

DEFINITION 3.7. A labelled space $(E, \mathcal{L}, \mathcal{B})$ is *weakly left-resolving* if for every $A, B \in \mathcal{B}$ and every $\alpha \in \mathcal{L}^*(E)$ we have $r(A, \alpha) \cap r(B, \alpha) = r(A \cap B, \alpha)$.

In particular, the labelled space $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving if no pair of disjoint sets $A, B \in \mathcal{B}$ can emit paths λ, μ respectively with $\mathcal{L}(\lambda) = \mathcal{L}(\mu)$ and $r(\lambda) = r(\mu)$. If $(E, \mathcal{L})$ is left-resolving then $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving for any $\mathcal{B}$. Evidently if $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving, then $(E, \mathcal{L}, \mathcal{B}')$ is weakly left-resolving for any $\mathcal{B}' \subseteq \mathcal{B}$.

Consider the following subsets of 2^{E^0}:

$$\mathcal{E} = \{\{v\} : v \in E^0 \text{ is a source or a sink }\} \cup \{r(\alpha) : \alpha \in \mathcal{L}^*(E)\} \cup \{s(\alpha) : \alpha \in \mathcal{L}^*(E)\},$$

$$\mathcal{E}^- = \{\{v\} : v \in E^0 \text{ is a sink }\} \cup \{r(\alpha) : \alpha \in \mathcal{L}^*(E)\}.$$

The following definition is analogous to the definition of $\mathcal{G}^0$ in [26].

DEFINITION 3.8. Let $\mathcal{E}^0$ (respectively $\mathcal{E}^{0,-}$) denote the smallest subset of 2^{E^0} containing $\mathcal{E}$ (respectively $\mathcal{E}^-$) which is accommodating for $(E, \mathcal{L})$.

REMARK 3.9. If $\alpha, \beta \in \mathcal{L}^*(E)$ are such that $\alpha\beta \in \mathcal{L}^*(E)$ then

$$r(s(\alpha), \alpha\beta) = r(\alpha\beta) \quad \text{and} \quad r(r(\alpha), \beta) = r(\alpha\beta).$$

For $\alpha, \beta \in \mathcal{L}^*(E)$ with $\alpha\beta \in \mathcal{L}^*(E)$ and $A \subseteq E^0$ we have $r(r(A, \alpha), \beta) = r(A, \alpha\beta)$.

For labelled spaces $(E, \mathcal{L}, \mathcal{E}^0)$ which are weakly left-resolving Remark 3.5 and Remark 3.9 show that to form $\mathcal{E}^0$ it suffices to form

$$\mathcal{E} \cup \{r(A, \alpha) : A \in \mathcal{E}, \alpha \in \mathcal{L}^*(E)\}$$

and then close under finite intersections and unions. To form $\mathcal{E}^{0,-}$, by Remark 3.5 it suffices to close $\mathcal{E}^-$ under finite intersections and unions. Evidently, $\mathcal{E}^{0,-} \subseteq \mathcal{E}^0$; the containment can be strict, for instance this occurs when E has sources. One can show that $\mathcal{E}^0 = \mathcal{E}^{0,-}$ if and only if for every $\alpha \in \mathcal{L}^*(E)$, $s(\alpha)$ can be written as a finite union of sets of the form $\bigcap_{i=1}^{n} r(\beta_i)$. Since E^0, $\mathcal{L}^*(E)$ and $\mathcal{E}$ are countable it follows that $\mathcal{E}^0$ and $\mathcal{E}^{0,-}$ are countable.

For $A \in 2^{E^0}$ and $n \geqslant 1$ let

$$L_A^n = \{\alpha \in \mathcal{L}(E^n) : A \cap s(\alpha) \neq \emptyset\}$$

denote those labelled paths of length n whose source intersects A nontrivially.

4. C^*-ALGEBRAS OF LABELLED SPACES

DEFINITION 4.1. Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space. A representation of $(E, \mathcal{L}, \mathcal{B})$ consists of projections $\{p_A : A \in \mathcal{B}\}$ and partial isometries $\{s_a : a \in \mathcal{L}(E^1)\}$ with the properties that:

(i) If $A, B \in \mathcal{B}$ then $p_A p_B = p_{A\cap B}$ and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$, where $p_\emptyset = 0$.

(ii) If $a \in \mathcal{L}(E^1)$ and $A \in \mathcal{B}$ then $p_A s_a = s_a p_{r(A,a)}$.

(iii) If $a, b \in \mathcal{L}(E^1)$ then $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$.

(iv) For $A \in \mathcal{B}$, if L_A^1 is finite and non-empty we have

$$p_A = \sum_{a \in L_A^1} s_a p_{r(A,a)} s_a^*. \tag{4.1}$$

If $a, b \in \mathcal{L}(E^1)$ are such that $ab \in \mathcal{L}^*(E)$ then we have

$$(s_a^* s_a)(s_b s_b^*) = p_{r(a)} s_b s_b^* = s_b p_{r(r(a),b)} s_b^* = s_b s_b^* p_{r(a)} = (s_b s_b^*)(s_a^* s_a).$$

Hence $s_a s_b$ is a partial isometry which is nonzero if and only if s_a and s_b are. Therefore we may define $s_{ab} = s_a s_b$ and similarly define s_α for all $\alpha \in \mathcal{L}^*(E)$. One checks that Definition 4.1 (ii) holds for $\alpha \in \mathcal{L}^*(E)$, Definition 4.1 (iii) holds for $\alpha, \beta \in \mathcal{L}(E^n)$ for $n \geqslant 1$ and Definition 4.1 (iv) holds for $A \in \mathcal{B}$ with finite and nonempty L_A^n for $n \geqslant 1$. Then (cf. (2.2)) we have

$$s_\alpha^* s_\alpha s_\beta = p_{r(\alpha)} s_\beta = s_\beta p_{r(r(\alpha),\beta))} = s_\beta p_{r(\alpha\beta)} = s_\beta s_{\alpha\beta}^* s_{\alpha\beta}.$$

To justify the requirement that $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving in Definitions 4.1, consider the following: Let $\{p_A, s_a\}$ be a representation of $(E, \mathcal{L}, \mathcal{B})$ in which $p_A \neq 0$ for all $A \in \mathcal{B}$. By Definition 4.1 (i) we have $(p_A - p_{A\cap B})(p_B - p_{A\cap B}) = 0$ for all $A, B \in \mathcal{B}$. Suppose, for contradiction, that there is $\alpha \in \mathcal{L}^*(E)$ such that $r(A, \alpha) \cap r(B, \alpha) \neq r(A \cap B, \alpha)$. From Definition 4.1 (iv) we have

$$p_A - p_{A\cap B} \geqslant s_\alpha (p_{r(A,\alpha)} - p_{r(A\cap B,\alpha)}) s_\alpha^* \quad \text{and} \quad p_B - p_{A\cap B} \geqslant s_\alpha (p_{r(B,\alpha)} - p_{r(A\cap B,\alpha)}) s_\alpha^*$$

so $(p_A - p_{A\cap B})(p_B - p_{A\cap B}) \neq 0$, a contradiction. Thus a representation of $(E, \mathcal{L}, \mathcal{B})$ will be degenerate if $(E, \mathcal{L}, \mathcal{B})$ is not weakly left-resolving.

Relation (iv) in Definition 4.1 can make sense even if $A \in \mathcal{B}$ emits infinitely many edges in E: If there are only finitely many different labels attached to the edges which A emits then L_A^1 is finite. For directed graphs the analogue of equation (4.1) holds when a vertex has finite valency; when this is true at every vertex, the graph is called row-finite. With this in mind, we make the following definition:

DEFINITION 4.2. Let $(E, \mathcal{L}, \mathcal{B})$ be a labelled space. We say that $A \in \mathcal{B}$ is *singular* if L_A^1 is infinite. If no set $A \in \mathcal{B}$ is singular we say that $(E, \mathcal{L}, \mathcal{B})$ is *set-finite*.

If $(E, \mathcal{L}, \mathcal{B})$ is set-finite, then L_A^n is finite for all $A \in \mathcal{B}$ and all $n \geqslant 1$. In the examples below, the resulting labelled space will be set-finite whenever the original graph is row-finite.

EXAMPLES 4.3. (i) Let E be a directed graph with the trivial labelling $\mathcal{L}$. Then $\mathcal{E}^0$ consists of all the finite subsets of E^0. If E is row-finite then $(E, \mathcal{L}, \mathcal{E}^0)$ and $(E, \mathcal{L}, \mathcal{E}^{0,-})$ are set-finite. One may show that a representation of $(E, \mathcal{L}, \mathcal{E}^0)$ is a Cuntz-Krieger E-family and conversely (see [1], [4] for instance). If all sources in E have finite valency, then the $*$-algebra generated by a representation of $(E, \mathcal{L}, \mathcal{E}^{0,-})$ contains a representation of $(E, \mathcal{L}, \mathcal{E}^0)$. If there is a source $v \in E^0$ with infinite valency then there is no representative of p_v in the $*$-algebra generated by a representation of $(E, \mathcal{L}, \mathcal{E}^{0,-})$.

(ii) Under the identification of an ultragraph $\mathcal{G}$ with a labelled graph $(E_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}})$ we have $\mathcal{E}^0_{\mathcal{G}} = \mathcal{G}^0$. Since $\mathcal{A} = \mathcal{G}^1$ a representation of $(E_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}}, \mathcal{E}^0_{\mathcal{G}})$ is a Cuntz-Krieger $\mathcal{G}$-family (see Definition 2.7 of [26]). If $\mathcal{G}$ has sources which are singular then we get similar behaviour to that described in (i) above.

(iii) In Examples 3.3 (iii) we have $\mathcal{E}_i^0 = 2^{E_i^0}$ for $i = 1, 2, 3$. Though $\mathcal{E}_1^{0,-} = 2^{E_1^0}$, we find that $\mathcal{E}_2^{0,-} = \{\{w\}, \{u, w\}, \{v, w\}, \{u, v, w\}\}$ and $\mathcal{E}_3^{0,-} = \{\emptyset, \{u\}, \{v\}, \{u, v\}, \{u, v, w\}\}$. A representation of $(E_2, \mathcal{L}_2, \mathcal{E}_2^{0,-})$ is generated by partial isometries s_0, s_1 satisfying the relations in Proposition 8.3 of [14] and Section 2 of [6] for $\mathcal{O}_Y$, where Y is the even shift.

(iv) A covering $p : F \to E$ of directed graphs yields a labelling $\mathcal{L}_p : F^1 \to E^1$. We may identify $\mathcal{F}^0$ with the collection of inverse images of the finite subsets of E^0. A representation of $(F, \mathcal{L}_p, \mathcal{F}^0)$ is a Cuntz-Krieger E-family. If F has sources with infinite valency, then we get similar behaviour to that described in (i) above.

(v) An outsplitting $E_s(\mathcal{P})$ of E gives rise to a labelling $\mathcal{L} : E_s(\mathcal{P})^1 \to E^1$. If $\mathcal{P}$ is proper then we may identify $\mathcal{E}_S(\mathcal{P})^0$ with the collection of finite subsets of E^0, and a representation of $(E_s(\mathcal{P}), \mathcal{L}, \mathcal{E}_S(\mathcal{P})^0)$ is a Cuntz-Krieger E-family. If E has sources with infinite valency then, we get similar behaviour to that described in (i) above, even when the outsplitting is proper.

(vi) An arbitrary shift $\Lambda \subseteq \mathcal{A}^{\mathbb{Z}}$ gives rise to a left-resolving labelled graph $(E_\Lambda, \mathcal{L}_\Lambda)$ with no sources or sinks, called the Left Krieger cover. If $\mathcal{A}$ is finite then the generators of $\mathcal{O}_\Lambda$ form a representation of $(E_\Lambda, \mathcal{L}_\Lambda, \mathcal{E}_\Lambda^{0,-})$ (cf. [8], [14]).

(vii) An arbitrary shift $\Lambda \subseteq \mathcal{A}^{\mathbb{Z}}$ gives rise to a left-resolving labelled graph $(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*})$ with no sources or sinks, called the predecessor graph. If $\mathcal{A}$ is finite then the generators of $\mathcal{O}_{\Lambda^*}$ form a representation of the $(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*}, \mathcal{E}_{\Lambda^*}^{0,-})$ (cf. [8], [14]).

Examples 4.3 (i)–(v) show that it is possible for $\mathcal{E}^0$ and $\mathcal{E}^{0,-}$ to be different, but for the $*$-algebras generated by representations of $(E,\mathcal{L},\mathcal{E}^0)$ and $(E,\mathcal{L},\mathcal{E}^{0,-})$ to be the same.

Let $(E,\mathcal{L},\mathcal{B})$ be a labelled space. Let $\mathcal{B}^* = \mathcal{L}^*(E) \cup \mathcal{B}$ and extend r, s to $\mathcal{B}^*$ by $r(A) = A$, $s(A) = A$ for all $A \in \mathcal{B}$. For $A \in \mathcal{B}$, put $s_A = p_A$, so s_β is defined for all $\beta \in \mathcal{B}^*$.

LEMMA 4.4. *Let $(E,\mathcal{L},\mathcal{B})$ be a weakly left-resolving labelled space and $\{s_a, p_A\}$ be a representation of $(E,\mathcal{L},\mathcal{B})$. Then any nonzero product of s_a, p_A and s_b^* can be written as a finite combination of elements of the form $s_\alpha p_A s_\beta^*$ for some $A \in \mathcal{B}$, and $\alpha, \beta \in \mathcal{B}^*$ satisfying $A \subseteq r(\alpha) \cap r(\beta) \neq \emptyset$.*

Proof. Since $s_\alpha p_A s_\beta^* = s_\alpha p_{r(\alpha)\cap A \cap r(\beta)} s_\beta^*$ it follows that $s_\alpha p_A s_\beta^*$ is zero unless $A \cap r(\alpha) \cap r(\beta) \neq \emptyset$ and without loss of generality we may assume that $A \subseteq r(\alpha) \cap r(\beta)$. For $\alpha, \beta, \gamma, \delta \in \mathcal{L}^*(E)$ and $A, B \in \mathcal{B}$ we have

$$(s_\alpha p_A s_\beta^*)(s_\gamma p_B s_\delta^*) = \begin{cases} s_{\alpha\gamma'} p_{r(A,\gamma')\cap B} s_\delta^* & \text{if } \gamma = \beta\gamma', \\ s_\alpha p_{A \cap r(B,\beta')} s_{\delta\beta'}^* & \text{if } \beta = \gamma\beta', \\ s_\alpha p_{A\cap B} s_\delta^* & \text{if } \beta = \gamma, \\ 0 & \text{otherwise .} \end{cases} \tag{4.2}$$

To see this, suppose $\gamma = \beta\gamma'$ then as $A \subseteq r(\beta) \cap r(\alpha)$

$$\begin{aligned} s_\alpha p_A s_\beta^* s_\gamma p_B s_\delta^* &= s_\alpha p_A s_\beta^* s_\beta s_{\gamma'} p_B s_\delta^* = s_\alpha p_A p_{r(\beta)} s_{\gamma'} p_B s_\delta^* \\ &= s_\alpha p_A s_{\gamma'} p_B s_\delta^* = s_{\alpha\gamma'} p_{r(A,\gamma')\cap B} s_\delta^* . \end{aligned}$$

A similar calculation gives the desired formulas in the cases $\beta = \gamma\beta'$ and $\beta = \gamma$. If β and γ have no common initial segment, then without loss of generality, assume that $\beta \in \mathcal{L}(E^n)$ and $\gamma \in \mathcal{L}(E^m)$ with $n > m$. Write $\beta = \beta'\beta''$ where $\beta' \in \mathcal{L}(E^m)$, and then by Definition 4.1(iv) we have $s_\beta^* s_\gamma = s_{\beta''}^* s_{\beta'}^* s_\gamma = 0$ since $\beta' \neq \gamma$ and so $s_\alpha p_A s_\beta^* s_\gamma p_A s_\delta^* = 0$. By Definition 4.1 (i) and (ii) we may extend (4.2) to the case when $\alpha, \beta, \gamma, \delta \in \mathcal{B}^*$. ∎

THEOREM 4.5. *Let $(E,\mathcal{L},\mathcal{B})$ be a weakly left-resolving labelled space. There exists a C^*-algebra B generated by a universal representation of $\{s_a, p_A\}$ of $(E,\mathcal{L},\mathcal{B})$. Furthermore the s_a's are nonzero and every p_A with $A \neq \emptyset$ is nonzero.*

Proof. Let $S_{(E,\mathcal{L},\mathcal{B})} := \{(\alpha, A, \beta) : \alpha, \beta \in \mathcal{B}^*, A \in \mathcal{B}, A \subseteq r(\alpha) \cap r(\beta)\}$ and let $k_{(E,\mathcal{L},\mathcal{B})}$ be the space of functions of finite support on $S_{(E,\mathcal{L},\mathcal{B})}$. The set of point masses $\{e_\tau : \tau \in S_{(E,\mathcal{L},\mathcal{B})}\}$ forms a basis for $k_{(E,\mathcal{L},\mathcal{B})}$. Set $(\alpha, A, \beta)^* := (\beta, A, \alpha)$; then thinking of $e_{(\alpha,A,\beta)}$ as $s_\alpha p_A s_\beta^*$ and using (4.2) we can define a multiplication with respect to which $k_{(E,\mathcal{L},\mathcal{B})}$ is a $*$-algebra.

As a $*$-algebra $k_{(E,\mathcal{L},\mathcal{B})}$ is generated by the elements $q_A := e_{(A,A,A)}$ for $A \in \mathcal{B}$ and $t_a := e_{(a,r(a),r(a))}$ for $a \in \mathcal{L}(E^1)$. Our definition of multiplication ensures that properties (ii) and (iii) of Definition 4.1 hold; moreover $q_A q_B = q_{A\cap B}$. We mod out by the ideal J generated by the elements $q_{A\cup B} - q_A - q_B + q_{A\cap B}$ for $A, B \in \mathcal{B}$, and

$q_A - \sum_{a \in L_A^1} s_a p_{r(A,a)} s_a^*$ for $A \in \mathcal{B}$ with L_A^1 nonempty and finite. Then the images r_A of q_A and u_a of t_a in $k_{(E,\mathcal{L},\mathcal{B})}/J$ form a representation of $(E, \mathcal{L}, \mathcal{B})$ that generates $k_{(E,\mathcal{L},\mathcal{B})}/J$. The triple $(k_{(E,\mathcal{L},\mathcal{B})}/J, r_A, u_a)$ has the required universal property, but is not a C^*-algebra. Using a standard argument we can convert this triple to a C^*-algebra B satisfying the required properties (see Theorem 2.1 of [10] for instance).

Now for each $a \in \mathcal{L}(E^1)$ and $e \in \mathcal{L}^{-1}(a)$, let $\mathcal{H}_{(a,e)}$ be an infinite-dimensional Hilbert space. Also for each $v \in s(a)$ we define $\mathcal{H}_{(a,v)} := \bigoplus_{\{e : s(e)=v, \mathcal{L}(e)=a\}} \mathcal{H}_{(a,e)}$. If v is a sink let $\mathcal{H}_v$ be an infinite-dimensional Hilbert space. For $A \in \mathcal{B}$ we define $\mathcal{H}_A := \bigoplus_{b \in L_A^1} \bigoplus_{v \in s(b) \cap A} \mathcal{H}_{(b,v)}$ and then note that each Hilbert space we have defined is a subspace of

$$\mathcal{H} := \Big(\bigoplus_{a \in \mathcal{L}(E^1)} \bigoplus_{v \in s(a)} \mathcal{H}_{(a,v)} \Big) \bigoplus_{\{v : s^{-1}(v) = \emptyset\}} \mathcal{H}_v.$$

For each $a \in \mathcal{L}(E^1)$, let S_a be a partial isometry with initial space $\mathcal{H}_{r(a)}$ and final space $\bigoplus_{v \in s(a)} \mathcal{H}_{(a,v)} \subseteq \mathcal{H}_{s(a)}$. For $A \in \mathcal{B}$, define P_A to be the projection of $\mathcal{H}$ onto $\mathcal{H}_A$, where this is interpreted as the zero projection when $A = \emptyset$.

It is easy to verify that since $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving, the operators $\{S_a, P_A\}$ form a representation of $(E, \mathcal{L}, \mathcal{B})$ in which S_a, P_A are nonzero. By the universal property there exists a homomorphism $\pi_{S,P} : B \to C^*(\{S_a, P_A\})$. Since the S_a's and P_A's are nonzero, it follows that the s_a's and p_A's are also nonzero. ∎

DEFINITION 4.6. Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space, then $C^*(E, \mathcal{L}, \mathcal{B})$ is the universal C^*-algebra generated by a representation of $(E, \mathcal{L}, \mathcal{B})$.

Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space and $\{s_a, p_A\}$ be the universal representation of $(E, \mathcal{L}, \mathcal{B})$, then by Lemma 4.4

$$\operatorname{span}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{L}^*(E), A \in \mathcal{B}, A \subseteq r(\alpha) \cap r(\beta)\}$$

is a dense $*$-subalgebra of $C^*(E, \mathcal{L}, \mathcal{B})$. The following result may be proved along the same lines as Lemma 3.2 of [26].

LEMMA 4.7. *Let $\mathcal{A}$ be finite, E have no sinks, and $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space. Then $C^*(E, \mathcal{L}, \mathcal{B})$ is unital.*

Proof. Observe that $\sum_{a \in \mathcal{A}} s_a s_a^*$ is a unit for $C^*(E, \mathcal{L}, \mathcal{B})$. ∎

LEMMA 4.8. *If $\phi : (E, \mathcal{L}) \to (F, \mathcal{L}')$ is a labelled graph isomorphism, then for all $\mathcal{B}$ which are accommodating for $(E, \mathcal{L})$ we have $C^*(E, \mathcal{L}, \mathcal{B}) \cong C^*(F, \mathcal{L}', \phi(\mathcal{B}))$.*

Proof. The map ϕ induces a bijection between the generators of $C^*(E, \mathcal{L}, \mathcal{B})$ and $C^*(F, \mathcal{L}', \phi(\mathcal{B}))$ and so by the universal property there are homomorphisms from one C^*-algebra to the other which are also inverses of each other. ∎

5. GAUGE INVARIANT UNIQUENESS THEOREM

Let $\{s_a, p_A\}$ be the universal representation of $(E, \mathcal{L}, \mathcal{B})$ which generates $C^*(E, \mathcal{L}, \mathcal{B})$. For $z \in \mathbb{T}$, $a \in \mathcal{L}(E^1)$ and $A \in \mathcal{B}$ let

$$t_a := \gamma_z s_a = z s_a \quad \text{and} \quad q_A := \gamma_z p_A = p_A;$$

then the family $\{t_a, q_A\} \in C^*(E, \mathcal{L}, \mathcal{B})$ is also a representation of $(E, \mathcal{L}, \mathcal{B})$. By universality of $C^*(E, \mathcal{L}, \mathcal{B})$ and a routine $\epsilon/3$ argument we see that γ extends to a strongly continuous action

$$\gamma : \mathbb{T} \to \mathrm{Aut}\ C^*(E, \mathcal{L}, \mathcal{B})$$

which we call the *gauge action*.

PROPOSITION 5.1. (i) *Let E be a directed graph with the trivial labelling $\mathcal{L}$. Then $C^*(E, \mathcal{L}, \mathcal{E}^0) \cong C^*(E)$.*

(ii) *Let $\mathcal{G}$ be an ultragraph. Then $C^*(E_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}}, \mathcal{E}^0_{\mathcal{G}}) \cong C^*(\mathcal{G})$, where $(E_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}})$ is the labelled graph associated to $\mathcal{G}$.*

(iii) *Let $p : F \to E$ be a covering map with induced labelling $\mathcal{L}_p : F^1 \to E^1$. Then $C^*(F, \mathcal{L}_p, \mathcal{F}^0) \cong C^*(E)$.*

(iv) *Let E be a directed graph and let $E_s(\mathcal{P})$ be an outsplitting. Let $\mathcal{L}$ be the labelling of $E_s(\mathcal{P})$ induced by the outsplitting. If $\mathcal{P}$ is a proper partition then $C^*(E_s(\mathcal{P}), \mathcal{L}, \mathcal{E}_s(\mathcal{P})^0) \cong C^*(E)$.*

Proof. In each case the left hand side contains a generating set for the C^*-algebra on the right as shown in Examples 4.3. We apply the appropriate gauge-invariant uniqueness theorem for the algebra on the right hand side to obtain the isomorphism. ∎

To establish connections with the Matsumoto algebras we need a version of the gauge-invariant uniqueness theorem for labelled graph algebras.

LEMMA 5.2. *Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space, $\{s_a, p_A\}$ a representation of $(E, \mathcal{L}, \mathcal{B})$, and $Y = \{s_{\alpha_i} p_{A_i} s^*_{\beta_i} : i = 1, \dots, N\}$ be a set of partial isometries in $C^*(E, \mathcal{L}, \mathcal{B})$ which is closed under multiplication and taking adjoints. If q is a minimal projection in $C^*(Y)$ then either*

(i) *$q = s_{\alpha_i} p_{A_i} s^*_{\alpha_i}$ for some $1 \leqslant i \leqslant N$; or*

(ii) *$q = s_{\alpha_i} p_{A_i} s^*_{\alpha_i} - q'$ where $q' = \sum_{l=1}^{m} s_{\alpha_{k(l)}} p_{A_{k(l)}} s^*_{\alpha_{k(l)}}$ and $1 \leqslant i \leqslant N$; moreover there is a nonzero $r = s_{\alpha_i \beta} p_{r(A_i, \beta)} s^*_{\alpha_i \beta} \in C^*(E, \mathcal{L}, \mathcal{B})$ such that $q'r = 0$ and $q \geqslant r$.*

Proof. By 4.2 any projection in $C^*(Y)$ may be written as

$$\sum_{j=1}^{n} s_{\alpha_{i(j)}} p_{A_{i(j)}} s^*_{\alpha_{i(j)}} - \sum_{l=1}^{m} s_{\alpha_{k(l)}} p_{A_{k(l)}} s^*_{\alpha_{k(l)}}$$

where the projections in each sum are mutually orthogonal and for each l there is a unique j such that $s_{\alpha_{i(j)}} p_{A_{i(j)}} s^*_{\alpha_{i(j)}} \geqslant s_{\alpha_{k(l)}} p_{A_{k(l)}} s^*_{\alpha_{k(l)}}$.

If $q = \sum_{j=1}^{n} s_{\alpha_{i(j)}} p_{A_{i(j)}} s^*_{\alpha_{i(j)}} - \sum_{l=1}^{m} s_{\alpha_{k(l)}} p_{A_{k(l)}} s^*_{\alpha_{k(l)}}$ is a minimal projection in $C^*(Y)$ then we must have $n = 1$. If $m = 0$ then $q = s_{\alpha_i} p_{A_i} s^*_{\alpha_i}$ for some $1 \leqslant i \leqslant N$. If $m \neq 0$ then $q = s_{\alpha_i} p_{A_i} s^*_{\alpha_i} - q'$ where $q' = \sum_{l=1}^{m} s_{\alpha_{k(l)}} p_{A_{k(l)}} s^*_{\alpha_{k(l)}}$ and $1 \leqslant k \leqslant N$. Since q' is the sum of finitely many projections and $q \neq 0$ it follows by repeated use of Definition 4.1 (iv) that there is a nonzero $r = s_{\alpha_i\beta} p_{r(A_i,\beta)} s^*_{\alpha_i\beta}$ in $C^*(E, \mathcal{L}, \mathcal{B})$ such that $rq' = 0$ and $q \geqslant r$. ∎

THEOREM 5.3. *Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space and let $\{S_a, P_A\}$ be a representation of $(E, \mathcal{L}, \mathcal{B})$ on Hilbert space. Take $\pi_{S,P}$ to be the representation of $C^*(E, \mathcal{L}, \mathcal{B})$ satisfying $\pi_{S,P}(s_a) = S_a$ and $\pi_{S,P}(p_A) = P_A$. Suppose that each P_A is non-zero whenever $A \neq \emptyset$, and that there is a strongly continuous action β of $\mathbb{T}$ on $C^*(S_\alpha, P_A)$ such that for all $z \in \mathbb{T}$, $\beta_z \circ \pi_{S,P} = \pi_{S,P} \circ \gamma_z$. Then $\pi_{S,P}$ is faithful.*

Proof. A straightforward argument along the lines of Lemma 2.2.3 of [22] shows that

$$C^*(E, \mathcal{L}, \mathcal{B})^\gamma = \overline{\text{span}}\{s_\alpha p_A s^*_\beta : \alpha, \beta \in \mathcal{L}(E^n) \text{ for some } n \text{ and } A \subseteq r(\alpha) \cap r(\beta)\}$$

where $C^*(E, \mathcal{L}, \mathcal{B})^\gamma$ is the fixed point algebra of $C^*(E, \mathcal{L}, \mathcal{B})$ under the gauge action γ. We claim that $C^*(E, \mathcal{L}, \mathcal{B})^\gamma$ is AF. Let Y be a finite subset of $C^*(E, \mathcal{L}, \mathcal{B})^\gamma$. Since $y \in Y$ may be approximated by a finite linear combination of elements of the form $s_\alpha p_A s^*_\beta$ where $|\alpha| = |\beta|$ we may assume that $Y = \{s_{\alpha_i} p_{A_i} s^*_{\beta_i} : |\alpha_i| = |\beta_i|, i = 1, \ldots, N\}$.

Let M be the length of the longest word in $\{\alpha_1, \ldots, \alpha_N\}$. Let W denote the collection of all words in $\mathcal{L}^*(E)$ of length at most M that can be formed from composing subwords of $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N$. Let $\mathcal{C}$ be the collection all finite intersections of $\{A_i\}_{i=1}^n$ and $\{r(A_i, \gamma) : 1 \leqslant i \leqslant N, \gamma \in W\}$. By (4.2) a nonzero product of elements of Y is of the form $s_\gamma p_A s^*_\delta$ where $\gamma, \delta \in W$ and $A \in \mathcal{C}$. Since W and $\mathcal{C}$ are finite it follows that $Y' = \{s_\gamma p_A s^*_\delta : \gamma, \delta \in W, A \in \mathcal{C}\}$ is finite, closed under adjoints and $C^*(Y) = C^*(Y')$. Hence we may assume that Y is closed under multiplication and taking adjoints. Thus $C^*(Y) = \overline{\text{span}}(Y)$ is finite dimensional and so $C^*(E, \mathcal{L}, \mathcal{B})^\gamma$ is AF by Theorem 2.2 of [5], establishing our claim.

To show that the canonical map $\pi_{S,P} : C^*(E, \mathcal{L}, \mathcal{B}) \to C^*(S_a, P_A)$ is injective on $C^*(E, \mathcal{L}, \mathcal{B})^\gamma$ we write $C^*(E, \mathcal{L}, \mathcal{B})^\gamma$ as $\overline{\bigcup C^*(Y_n)}$ where $\{Y_n : n \geqslant 1\}$ is an increasing family of finite sets which are closed under multiplication and taking adjoints. Suppose, for contradiction, that $\pi_{S,P}$ is not faithful on $C^*(Y_n)$ for some n. Then its kernel is an ideal and so must contain a nonzero minimal projection q. If $Y_n = \{s_{\alpha_i} p_{A_i} s^*_{\beta_i} : i = 1, \ldots, N(n)\}$ then by Lemma 5.2 either $q = s_{\alpha_i} p_{A_i} s^*_{\alpha_i}$

for some $1 \leqslant i \leqslant N(n)$ or $q = s_{\alpha_i} p_{A_i} s_{\alpha_i}^* - q'$ where $q' = \sum_{k=1}^{m} s_{\alpha_{i(k)}} p_{A_{i(k)}} s_{\alpha_{i(k)}}^*$ and $1 \leqslant i \leqslant N(n)$. In the first case $\pi_{S,P}(s_{\alpha_i} p_{A_i}) = S_{\alpha_i} P_{A_i}$ is a partial isometry with initial projection P_{A_i} and final projection $S_{\alpha_i} P_{A_i} S_{\alpha_i}^*$. But $P_{A_i} = \pi_{S,P}(p_{A_i}) \neq 0$ by hypothesis and so $\pi_{S,P}(q) = \pi_{S,P}(s_{\alpha_i} p_{A_i} s_{\alpha_i}^*) = S_{\alpha_i} P_{A_i} S_{\alpha_i}^* \neq 0$ which is a contradiction. In the second case by Lemma 5.2 (ii) there is $r = s_{\alpha_i\beta} p_{r(A_i,\beta)} s_{\alpha_i\beta}^*$ such that $q \geqslant r$ and $q'r = 0$. We may apply the above argument to show that $\pi_{S,P}(r) \neq 0$ and hence $\pi_{S,P}(q) \geqslant \pi_{S,P}(r) \neq 0$ which is also a contradiction. Hence $\pi_{S,P}$ is injective on $C^*(Y_n)$ and the result follows by arguments similar to those in Theorem 2.1 of [4]. ∎

6. APPLICATIONS

6.1. DUAL LABELLED GRAPHS. Let E have no sinks and $(E, \mathcal{L})$ be a labelled graph over alphabet $\mathcal{A}$. From this data we may form the *dual labelled graph* $(\widehat{E}, \widehat{\mathcal{L}})$ over alphabet $\widehat{\mathcal{A}} := \mathcal{L}(E^2)$ as follows: Let $\widehat{E}^0 = E^1$, $\widehat{E}^1 = E^2$ and the maps $r', s' : \widehat{E}^1 \to \widehat{E}^0$ be given by $r'(ef) = f$ and $s'(ef) = e$. The labelling $\widehat{\mathcal{L}} : \widehat{E}^1 \to \widehat{\mathcal{A}}$ is induced by the original labelling, so that $\widehat{\mathcal{L}}(ef) = \mathcal{L}(e)\mathcal{L}(f)$. For $ab \in \widehat{\mathcal{L}}(\widehat{E}^1) = \mathcal{L}(E^2)$ we have

$$r_{\widehat{\mathcal{L}}}(ab) = \{f : \widehat{\mathcal{L}}(ef) = ab\}, \quad \text{and} \quad s_{\widehat{\mathcal{L}}}(ab) = \{e : \widehat{\mathcal{L}}(ef) = ab\}$$

and for $B \in 2^{E^1}$

$$r_{\widehat{\mathcal{L}}}(B, ab) = \{f : \widehat{\mathcal{L}}(ef) = ab, e \in B\}.$$

These maps extend naturally to $\widehat{\mathcal{L}}^*(\widehat{E}) = \bigcup_{n \geqslant 1} \widehat{\mathcal{L}}(\widehat{E}^n)$ where for $n \geqslant 1$, $\widehat{\mathcal{L}}(\widehat{E}^n)$ is identified with $\mathcal{L}(E^{n+1})$. Consider the following subsets of 2^{E^1}

$$\widehat{\mathcal{E}} = \{\{e\} : s(e) \text{ is a source}\} \cup \{r_{\widehat{\mathcal{L}}}(\alpha) : \alpha \in \widehat{\mathcal{L}}^*(\widehat{E})\} \cup \{s_{\widehat{\mathcal{L}}}(\alpha) : \alpha \in \widehat{\mathcal{L}}^*(\widehat{E})\},$$
$$\widehat{\mathcal{E}}^- = \{r_{\widehat{\mathcal{L}}}(\alpha) : \alpha \in \widehat{\mathcal{L}}^*(\widehat{E})\}.$$

Let $\widehat{\mathcal{E}}^0$ (respectively $\widehat{\mathcal{E}}^{0,-}$) be the smallest collection of subsets of 2^{E^1} containing $\widehat{\mathcal{E}}$ (respectively $\widehat{\mathcal{E}}^-$) which is accommodating for $(\widehat{E}, \widehat{\mathcal{L}})$. One checks easily that if $(E, \mathcal{L}, \mathcal{B})$ is left-resolving, then $(\widehat{E}, \widehat{\mathcal{L}}, \widehat{\mathcal{B}})$ is weakly left-resolving for $\mathcal{B} = \mathcal{E}^0, \mathcal{E}^{0,-}$.

For $B \in \widehat{\mathcal{E}}^0$ (respectively $B \in \widehat{\mathcal{E}}^{0,-}$) we set

$$\widehat{L}_B^1 = \{ab \in \widehat{\mathcal{L}}(\widehat{E}^1) : s_{\widehat{\mathcal{L}}}(ab) \cap B \neq \emptyset\}.$$

If E has no sources and sinks, the shift $\mathsf{X}_{(\widehat{E},\widehat{\mathcal{L}})}$ determined by the dual labelled graph $(\widehat{E}, \widehat{\mathcal{L}})$ of $(E, \mathcal{L})$ is the second higher block shift $\mathsf{X}_{(E,\mathcal{L})}^{[2]}$ formed from $\mathsf{X}_{(E,\mathcal{L})}$ (cf. Section 1.4 of [13]).

REMARK 6.1. Suppose that $ab \in \mathcal{L}(E^2)$ then $c \in L^1_{r(ab)}$ if and only if $bc \in \widehat{L}^1_{r_{\widehat{\mathcal{L}}}(ab)}$; moreover $r(r(ab),c) = r(s(r_{\widehat{\mathcal{L}}}(ab)),bc)$. Suppose that $A \in \mathcal{E}^0$ (respectively $A \in \mathcal{E}^{0,-}$) then $a \in L^1_A$ and $ab \in \mathcal{L}(E^2)$ if and only if $ab \in \widehat{L}^1_{s^{-1}(A)}$.

THEOREM 6.2. *Let $(E,\mathcal{L})$ be a set-finite, left-resolving labelled graph with no sinks; then $C^*(E,\mathcal{L},\mathcal{E}^0) \cong C^*(\widehat{E},\widehat{\mathcal{L}},\widehat{\mathcal{E}}^0)$, moreover $C^*(E,\mathcal{L},\mathcal{E}^{0,-}) \cong C^*(\widehat{E},\widehat{\mathcal{L}},\widehat{\mathcal{E}}^{0,-})$.*

Proof. Let $\{s_a, p_A\}$ be a representation of $(E,\mathcal{L},\mathcal{E}^0)$ and $\{t_{ab}, q_B\}$ be a representation of $(\widehat{E},\widehat{\mathcal{L}},\widehat{\mathcal{E}}^0)$. For $ab \in \widehat{\mathcal{L}}(\widehat{E}^1)$ and $B \in \widehat{\mathcal{E}}^0$ let $T_{ab} = s_a s_b s_b^*$ and

$$Q_B := \sum_{ab \in \widehat{L}^1_B} s_{ab} p_{r(s(B),ab)} s_{ab}^*.$$

Since $(E,\mathcal{L},\mathcal{E}^0)$ is set-finite $(\widehat{E},\widehat{\mathcal{L}},\widehat{\mathcal{E}}^0)$ is set-finite by Remark 6.1 and so the above sum is finite. One checks that $\{T_{ab}, Q_B\}$ is a representation of $(\widehat{E},\widehat{\mathcal{L}},\widehat{\mathcal{E}}^0)$.

By the universal property there is a homomorphism $\pi_{T,Q} : C^*(\widehat{E},\widehat{\mathcal{L}},\widehat{\mathcal{E}}^0) \to C^*(E,\mathcal{L},\mathcal{E}^0)$ with $\pi_{T,Q}(t_{ab}) = T_{ab}$ and $\pi_{T,Q}(q_B) = Q_B$. Since $\pi_{T,Q}$ intertwines the respective gauge actions and $Q_B \neq 0$ it follows from Theorem 5.3 that $\pi_{T,Q}$ is faithful. We claim that $\pi_{T,Q}$ is surjective. For $a \in \mathcal{L}(E^1)$ we have

$$\begin{aligned} s_a = s_a p_{r(a)} &= s_a \sum_{b \in L^1_{r(a)}} s_b p_{r(r(a),b)} s_b^* = \sum_{b \in L^1_{r(a)}} s_a s_b s_b^* s_b p_{r(ab)} s_b^* \\ &= \sum_{b \in L^1_{r(a)}} s_a s_b s_b^* \sum_{c \in L^1_{r(ab)}} s_{bc} p_{r(r(ab),c)} s_{bc}^* \\ &= \sum_{b \in L^1_{r(a)}} T_{ab} \sum_{bc \in \widehat{L}^1_{r_{\widehat{\mathcal{L}}}(ab)}} s_{bc} p_{r(s(r_{\widehat{\mathcal{L}}}(ab)),bc)} s_{bc}^* \text{ (by Remark 6.1) } = \sum_{b \in L^1_{r(a)}} T_{ab} Q_{r_{\widehat{\mathcal{L}}}(ab)} \end{aligned}$$

and so $s_a \in C^*(T_{ab}, Q_B)$. For $A \in \mathcal{E}^0$, by Remark 6.1 we have

$$p_A = \sum_{a \in L^1_A} s_a p_{r(A,a)} s_a^* = \sum_{a \in L^1_A} s_a \sum_{b \in L^1_{r(A,a)}} s_b p_{r(r(A,a),b)} s_b^* s_a^* = \sum_{ab \in \widehat{L}^1_{s^{-1}(A)}} s_{ab} p_{r(A,ab)} s_{ab}^* = Q_{s^{-1}(A)}$$

which establishes our claim. The second isomorphism is proved along similar lines. ∎

6.2. MATSUMOTO ALGEBRAS.

THEOREM 6.3. *Let Λ be a shift space over a finite alphabet $\mathcal{A}$ which satisfies Condition* (I) *and has left-Krieger cover $(E_\Lambda, \mathcal{L}_\Lambda)$ then $\mathcal{O}_\Lambda \cong C^*(E_\Lambda, \mathcal{L}_\Lambda, \mathcal{E}^{0,-}_\Lambda)$. Moreover, if Λ has predecessor graph $(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*})$ then $\mathcal{O}_{\Lambda^*} \cong C^*(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*}, \mathcal{E}^{0,-}_{\Lambda^*})$.*

Proof. By definition every $A \in \mathcal{E}^{0,-}_\Lambda$ can be written as a union of sets of the form $A_j = \bigcap_{i=1}^{m(j)} r(\mu_i^j)$ for $j = 1, \ldots, n$. For $\mu \in \Lambda^*$ let $q_{r(\mu)} = t_\mu^* t_\mu$, then since the projections $\{t_\mu^* t_\mu : \mu \in \Lambda^*\}$ are mutually commutative (see p. 686 of [16]) we may define $q_{r(\mu)\cap r(\nu)} = q_{r(\mu)}q_{r(\nu)}$, and hence define q_{A_j} for $1 \leqslant j \leqslant n$. By the inclusion-exclusion principle one may further define

$$q_A = \sum_{j=1}^{n} q_{A_j} - \sum_{j\neq k} q_{A_j}q_{A_k} + \cdots + (-1)^{n+1} q_{A_1}\cdots q_{A_n}.$$

Using calculations along the lines of those in Section 3 of [14] one checks that $\{t_a, q_A\}$ is a representation of $(E_\Lambda, \mathcal{L}_\Lambda, \mathcal{E}^{0,-}_\Lambda)$. Let $\{s_a, p_A\}$ be a representation of $(E_\Lambda, \mathcal{L}_\Lambda, \mathcal{E}^{0,-}_\Lambda)$. By the universal property for $C^*(E_\Lambda, \mathcal{L}_\Lambda, \mathcal{E}^{0,-}_\Lambda)$ there is a map $\pi_{t,q}$: $C^*(E_\Lambda, \mathcal{L}_\Lambda, \mathcal{E}^{0,-}_\Lambda) \to \mathcal{O}_\Lambda$ such that $\pi_{t,q}(s_a) = t_a$ and $\pi_{t,q}(p_A) = q_A$, in particular $\pi_{t,q}$ is surjective. Since Λ satisfies Condition (I) it follows by Proposition 2.2 that $\mathcal{O}_\Lambda$ carries a strongly continuous action β of $\mathbb{T}$. Since $\beta_z \circ \pi_{t,q} = \pi_{t,q} \circ \gamma_z$ for all $z \in \mathbb{T}$ and $\pi_{t,q}(p_A) = q_A \neq 0$ it follows from Theorem 5.3 that $\pi_{t,q}$ is injective, which completes the proof of the first statement.

The second statement is proved similarly. ∎

REMARKS 6.4. (i) In Section 5 of [8] a Condition $(*)$ is given under which for shift spaces Λ satisfying $(*)$ Conditions (I) and (I*) are equivalent and $\mathcal{O}_\Lambda \cong \mathcal{O}_{\Lambda^*}$. This suggests that if Λ satisfies $(*)$ then $(E_\Lambda, \mathcal{L}_\Lambda)$ is labelled graph isomorphic to $(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*})$ and the isomorphism of $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda^*}$ can be deduced from Theorem 4.8. However Theorem 6.1 of [8] shows that, in general, $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda^*}$ are not isomorphic. In particular, $(E_\Lambda, \mathcal{L}_\Lambda)$ and $(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*})$ are not labelled graph isomorphic in general.

(ii) The isomorphism of $C^*(E_\Lambda, \mathcal{L}_\Lambda, \mathcal{E}^{0,-}_\Lambda)$ and $\mathcal{O}_\Lambda$ identifies $C^*(p_A : A \in \mathcal{E}^{0,-}_\Lambda)$ with $A_\Lambda \subset \mathcal{O}_\Lambda$. Recall from Corollary 4.7 of [16] that $A_\Lambda \cong C(\Omega_\Lambda)$, hence we may think of the elements of $\mathcal{E}^{0,-}_\Lambda$ as indexing closed sets in Ω_Λ.

(iii) In [7] Carlsen constructs a C^*-algebra which has $\mathcal{O}_\Lambda$ as a quotient, that is isomorphic to $\mathcal{O}_\Lambda$ if Λ satisfies Condition (I), and always carries a gauge action. A proof along the lines of Theorem 6.3 shows that this new algebra is isomorphic to $C^*(E_\Lambda, \mathcal{L}_\Lambda, \mathcal{E}^{0,-}_\Lambda)$ for all Λ.

6.3. FINITENESS CONDITIONS.

DEFINITION 6.5. A labelled graph $(E, \mathcal{L})$ is *label-finite* if $|\mathcal{L}^{-1}(a)| < \infty$ for all $a \in \mathcal{L}(E^1)$.

If $(E, \mathcal{L})$ is label-finite then $\mathcal{L}^{-1}(\alpha)$ is finite for all $\alpha \in \mathcal{L}^*(E)$ and so all sets in $\mathcal{E}^0$ are finite (and conversely). If $(E, \mathcal{L})$ is label-finite then $(\widehat{E}, \widehat{\mathcal{L}})$ is label-finite. If E is row-finite and $(E, \mathcal{L})$ is label-finite then $(E, \mathcal{L}, \mathcal{E}^0)$ is set-finite.

The following result generalises Corollary 2.5 in [4] (see also Remark 3.3 (i) in [2]).

THEOREM 6.6. *Let $(E, \mathcal{L})$ be a row-finite left-resolving labelled graph which is label-finite and satisfies $\{v\} \in \mathcal{E}^0$ for all $v \in E^0$. Then $C^*(E, \mathcal{L}, \mathcal{E}^0) \cong C^*(E)$; moreover if $\{v\} \in \mathcal{E}^{0,-}$ for all $v \in E^0$ then $C^*(E, \mathcal{L}, \mathcal{E}^{0,-}) \cong C^*(E)$.*

Proof. Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger E-family and $\{t_a, q_A\}$ be the canonical generators of $C^*(E, \mathcal{L}, \mathcal{E}^0)$. For $a \in \mathcal{L}(E^1)$ and $A \in \mathcal{E}^0$ let

$$T_a = \sum_{e \in E^1 : \mathcal{L}(e) = a} s_e , \quad \text{and} \quad Q_A = \sum_{v \in A} p_v .$$

The above sums make sense since $(E, \mathcal{L})$ is label-finite. Since E is row-finite one may easily check that these operators define a representation of $(E, \mathcal{L}, \mathcal{E}^0)$. By the universal property of $C^*(E, \mathcal{L}, \mathcal{E}^0)$ there is a homomorphism $\pi_{T,Q} : C^*(E, \mathcal{L}, \mathcal{E}^0) \to C^*(E)$ given by $\pi_{T,Q}(t_a) = T_a$ and $\pi_{T,Q}(q_A) = Q_A$ for all $a \in \mathcal{L}(E^1)$ and $A \in \mathcal{E}^0$. Since $\{v\} \in \mathcal{E}^0$ for all $v \in E^0$, we have $p_v = Q_v \in C^*(T_a, Q_A)$ for all $v \in E^0$. Since our labelled graph is left-resolving we have $s_e = T_{\mathcal{L}(e)} Q_{r(e)} \in C^*(T_a, Q_A)$ for all $e \in E^1$, and so $\pi_{T,Q}$ is surjective. The canonical gauge actions on $C^*(E)$ and $C^*(E, \mathcal{L}, \mathcal{E}^0)$ satisfy the required properties and $\pi_{T,Q}(q_A) = Q_A \neq 0$ for all $A \in \mathcal{E}^0$, so $\pi_{T,Q}$ is an isomorphism by Theorem 5.3.

The proof of the second isomorphism is essentially the same. ∎

COROLLARY 6.7. *Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be a row-finite ultragraph; then $C^*(\mathcal{G}) \cong C^*(E_{\mathcal{G}})$ where $E_{\mathcal{G}}$ is the underlying directed graph of $\mathcal{G}$.*

Proof. From Examples 3.3 (ii) a row-finite ultragraph $\mathcal{G}$ may be realised as a row-finite left-resolving labelled graph $(E_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}})$. As $E_{\mathcal{G}}$ is row-finite it follows that $(E_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}})$ is label-finite. Since the source map is single-valued it follows that $v \in \mathcal{E}^0_{\mathcal{G}}$ for all $v \in G^0 = E^0_{\mathcal{G}}$ and hence the result follows from Theorem 6.6. ∎

The following result was first observed in Theorem 3.5 of [6] (see also Corollary 3.4.5 in [24]).

COROLLARY 6.8. *Let Λ be a sofic shift over a finite alphabet; then $\mathcal{O}_\Lambda \cong C^*(E_\Lambda)$ where $(E_\Lambda, \mathcal{L}_\Lambda)$ is the left-Krieger cover of Λ.*

Proof. As E^0_Λ is finite and each $v \in E^0_\Lambda$ has a different past there are the word $\alpha_v \in \mathcal{L}^*(E_\Lambda)$ with $r_{\mathcal{L}_\Lambda}(\alpha_v) = \{v\}$. Hence $\{v\} \in \mathcal{E}^{0,-}_\Lambda$ for all $v \in E^0_\Lambda$. The result follows by Theorem 6.6. ∎

From Theorem 3.3.18 in [13] any two minimal left-resolving representations $(E, \mathcal{L})$, $(F, \mathcal{L}')$ of an irreducible sofic shift are labelled graph isomorphic and so $C^*(E, \mathcal{L}, \mathcal{E}^0_-) \cong C^*(F, \mathcal{L}', \mathcal{F}^0_-)$ by Lemma 4.8. Moreover, one may use the minimality of the representation to show that the underlying graph E is irreducible (cf. Lemma 3.3.10 in [13]). Hence we have:

COROLLARY 6.9. *Let $(E, \mathcal{L})$ be a minimal left-resolving presentation of an irreducible sofic shift over a finite alphabet, then $C^*(E, \mathcal{L}, \mathcal{E}^{0,-}) \cong C^*(E, \mathcal{L}, \mathcal{E}^0)$ is simple.*

REMARK 6.10. Recall that the graph $(E_2, \mathcal{L}_2)$ in Examples 3.3 (ii) is the left-Krieger cover of the even shift Y. Although Y is irreducible, $(E_2, \mathcal{L}_2)$ is not a minimal left-resolving presentation of Y and $\mathcal{O}_Y \cong C^*(E_2)$ is not simple. However the graph $(E_1, \mathcal{L}_1)$ Examples 3.3 (ii) is a minimal left-resolving cover of Y

and so

$$C^*(E_1, \mathcal{L}_1, \mathcal{E}_1^{0,-}) \cong C^*(E_1, \mathcal{L}_1, \mathcal{E}_1^0) \cong C^*(E_1)$$

is simple. Similarly $C^*(E_Z, \mathcal{L}_Z, \mathcal{E}_Z^{0,-}) \cong C^*(E_Z)$ is simple where Z is the irreducible shift introduced in Examples 3.3 (vi).

Thus, if one wishes to associate a simple C^*-algebra to an irreducible sofic shift Λ, then one should use the minimal left-resolving presentation of Λ ([6], [7]).

For a general shift space Λ, either $(E_\Lambda, \mathcal{L}_\Lambda)$ will not be row-finite or there will be $v \in E_\Lambda^0$ with $v \notin \mathcal{E}_\Lambda^{0,-}$. This indicates that the C^*-algebras corresponding to presentations of such shift spaces will not be Morita equivalent to graph algebras. The shift associated to a certain Shannon graph (see Theorem 7.7 of [21]) provides such an example.

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