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NEW C^*-ALGEBRAS FROM SUBSTITUTION TILINGS

DANIEL GONÇALVES

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ABSTRACT. Given a tiling with finite local complexity and a finite number of patterns up to translation, we associate a C^*-algebra to it. We show that this C^*-algebra is a recursive subhomogeneous algebra and characterize its ideals. In the case of a substitution tiling, that also has primitivity and recognizability, we use the construction mentioned above, on each of the inflated tilings, to obtain a inductive limit C^*-algebra that encodes the dynamics of the inflation map. We show that this C^*-algebra is simple.

KEYWORDS: *C^*-algebras, tilings, substitution tilings, operator algebras.*

MSC (2000): 47L40.

1. INTRODUCTION AND NOTATION

C^*-algebras from tilings are closely related with the physics of quasicrystals. In [4], Kellendonk and Putnam consider the motion of a particle moving through a solid which is modeled by an aperiodic tiling and describe how to replace momentum operators with "translation" (actually partial translation) operators. The C^*-algebra generated by these operators is the standard C^*-algebra associated to a tiling, introduced by Bellissard, see [1], [4]. In the case of an aperiodic substitution tiling, the C^*-algebra above coincides with the C^*-algebra arising from the unstable equivalence relation in a Smale space. Under some hypothesis, $(\Omega, \mathrm{d}, \omega)$, where Ω is the hull of a tiling and ω is the inflation map, has the structure of a Smale space (roughly this means that (Ω, d) is a compact space and ω is a homeomorphism of Ω possessing canonical coordinates of contracting and expanding directions, see [8] for details) and this is the space we want to consider. Now two tilings in Ω, say T and T′, are unstably equivalent if the distance between $\omega^{-n}(\mathrm{T})$ and $\omega^{-n}(\mathrm{T}')$ converges to zero as n goes to infinity (which means they are translations of each other) and similarly T and T′ are stably equivalent if the distance between $\omega^{n}(\mathrm{T})$ and $\omega^{n}(\mathrm{T}')$ converges to zero as n goes to infinity (which means that there exists $n \in \mathbb{N}$ such that $\omega^{n}(\mathrm{T})$ and $\omega^{n}(\mathrm{T}')$ agree on a small ball around the origin). The C^*-algebra of Bellissard is the groupoid C^*-algebra associated with the unstable equivalence relation and can be seen as the cross product $C(\Omega) \times \mathbb{R}^d$ (see [1], [10]). The groupoid C^*-algebra of the stable equivalence relation in Ω is the focus of this paper. This C^*-algebra is strongly Morita equivalent to the groupoid C^*-algebra of an equivalence relation, which we will define in Section 5, on the usual d-dimensional Euclidean space, $\mathbb{R}^d$.

In order to obtain the C^*-algebra mentioned above, i.e., the C^*-algebra arising from the stable equivalence relation, we first need to construct a C^*-algebra associated to any tiling with finite local complexity and a finite number of tiles up to translation. This is done in Section 2. In Section 3, under the additional assumption that the tilings have a cellular structure, we give an explicit description for the ideals of the C^*-algebras introduced in Section 2. We also show, in Section 4, that these C^*-algebras can be obtained by starting with $C(X, K(l_2[\mathbb{N}]))$, for some compact Hausdorff space X, and then applying a finite number of pullbacks. This is a natural generalization of the notion of a recursive subhomogeneous algebra, introduced by N.C. Philips in [7], and so we say that the C^*-algebras of Section 2 are recursive subhomogeneous algebras. In the case of a substitution tiling, (ω, T) (where ω is the inflation map), that also has primitivity and recognizability, we use the construction of Section 2 on the tilings $\omega^k(T)$, $k = 1, 2, \ldots$, to obtain a inductive limit C^*-algebra, which is the C^*-algebra associated to the stable equivalence relation in (Ω, d, ω). This is done is Section 5. Finally, in Section 6 we show that this C^*-algebra is simple. Before we proceed we need to introduce some notation.

A tiling, T, of $\mathbb{R}^d$ is a collection of subsets $\{t_1, t_2, \ldots\}$, called tiles, such that their union is $\mathbb{R}^d$ and their interiors are pairwise disjoint. We will also assume that each tile is homeomorphic to the closed unit ball, $\overline{B(0,1)}$. More generally, we can also have a set, called labels. A labeled tile is then a tile with a label. (The idea being that we now have a way of distinguishing two tiles that may be exactly the same geometric object). It is clear how to extend the definitions below to the situation of labeled tiles and all the results of this paper apply equally well to this situation.

We may think of a tiling, T, as a multi-valued function, i.e., for $u \in \mathbb{R}^d$ we set $T(u) = \{t \in T : u \in t\}$ and for $U \subseteq \mathbb{R}^d$ we set $T(U) = \bigcup_{v \in U} T(v)$. To define a substitution tiling, we start with a finite set of tiles, $p_1, p_2, \ldots, p_n$, which we call prototiles and let $\widehat{\Omega}$ be the collection of all patches made up by translations of these prototiles. By the translation of a tile t by a vector $v \in \mathbb{R}^d$ we mean the set $t + v = \{x + v : x \in t\}$. Suppose there is an inflation constant $\lambda > 1$ and a substitution rule that associates to each prototile p_i a patch P_i with support p_i and such that $\lambda P_i \in \widehat{\Omega}$. Let $\widehat{\omega}(p_i) := \lambda P_i$. For any tile $t_i = p_j + u$ in T, we define $\widehat{\omega}(p_j + u) := \widehat{\omega}(p_j) + \lambda u$ and for any patch $P \in \widehat{\Omega}$ we have that $\widehat{\omega}(P) := \bigcup_{t \in P} \widehat{\omega}(t)$. Finally we restrict our attention to the set Ω of all tilings, T, in $\widehat{\Omega}$ such that for any patch $P \subseteq \mathrm{T}$, with bounded support, we have $P \subseteq \widehat{\omega}^k(p_i + u)$, for some $k \geqslant 1$, $1 \leqslant i \leqslant n$ and $u \in \mathbb{R}^d$. The restriction of $\widehat{\omega}$ to Ω is denoted by ω. A substitution tiling is a tiling that is contained in Ω as above. It is shown in [1] that Ω is non-empty. The following properties are standard conditions on a substitution:

(i) ω is *primitive* that is, there exists an integer $N_0 > 0$ such that for all pairs of prototiles p_i and p_j, the partial tiling $\omega^{N_0}(p_i)$ contains a translation of p_j.

(ii) Ω satisfies the *finite local complexity* condition that is, for all $r > 0$, there are only finitely many partial tilings, up to translation, that are subsets of tilings in Ω and whose support has diameter less than r. Also, tilings in Ω have only a finite number of tiles, up to translation.

(iii) $\omega : \Omega \to \Omega$ is one to one. This condition is often referred as *recognizability* of ω and implies that all tilings in Ω are aperiodic (see [1]).

With all three assumptions above it follows that ω is actually a bijection. The following lemma will be needed later:

LEMMA 1.1. *Let $p_1, p_2, \ldots, p_n$ be a finite set of tiles and Δ be a collection of tilings such that every tiling in Δ contain only translations of the tiles $p_1, p_2, \ldots, p_n$. Then there exists $\delta > 0$ such that, if $\mathrm{T} \in \Delta$, t is a tile in T and u is a vector with $0 < |u| < \delta$, then $t + u$ is not a tile in T.*

2. CONSTRUCTION OF THE STEP C^*-ALGEBRAS

In this section, we introduce a new C^*-algebra from a tiling that satisfies the finite local complexity property. At this point we do not need a substitution rule, but in the case of a substitution tiling we will use these C^*-algebras as building blocks for a C^*-algebra encoding the dynamics of the inflation map.

Throughout this section, T is a tiling on $\mathbb{R}^d$ with finite local complexity and only a finite number of tiles up to translation. Let $\mathrm{T}(x) = \{t \in \mathrm{T} : x \in t\}$. We define an equivalence relation in $\mathbb{R}^d$ by

$$\mathrm{G}(\mathrm{T}) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : \mathrm{T}(x) - x = T(y) - y\} \tag{2.1}$$

and we give G(T) the usual topology of $\mathbb{R}^d \times \mathbb{R}^d$. Where no confusion arises we denote G(T) by G. We intend to consider the reduced groupoid C^*-algebra of G, $\mathrm{C}^*_{\mathrm{r}}(\mathrm{G})$, as defined in [10], but for this we need to show that G is etale, which in the language of [10] means that G is an r-discrete groupoid with counting measure as a Haar system. In our context, this means we have to show that G is σ-compact, $\Delta = \{(x, x) \in \mathrm{G} : x \in \mathbb{R}^d\}$ is an open subset of G and the source and range maps defined by $s, r : \mathrm{G} \to \mathbb{R}^d$, $r(x, y) = x$, $s(x, y) = y$ are local homeomorphisms, i.e., for all $(x, y) \in \mathrm{G}$ there exists a neighborhood U of (x, y) in G, such that r restricted to U and s restricted to U are homeomorphisms from U to $r(\mathrm{U})$ and $s(\mathrm{U})$ respectively, and $r(\mathrm{U})$ and $s(\mathrm{U})$ are open subsets of $\mathbb{R}^d$. These properties follow straightforwardly once we have the two lemmas below.

LEMMA 2.1. *Given $x \in \mathbb{R}^d$, there exists $\delta > 0$ such that if y is in the ball of center x and radius δ, $B(x, \delta)$, then $T(y) \subseteq T(x)$.*

LEMMA 2.2. *Let $(x, y) \in G$. Then there exists $\delta_0 > 0$ such that*

$$B((x,y),\delta) \cap G = \left\{(x+v, y+v) : v \in B(0, \frac{\delta}{\sqrt{2}})\right\}$$

for all $\delta \leqslant \delta_0$.

Proof. Let δ_0 be small enough so that both Lemmas 2.1 and 1.1 hold. Let $\delta < \frac{\delta_0}{2}$.

Step 1. First we show that if $v \in B(0, \frac{\delta}{\sqrt{2}})$ then $(x+v, y+v)$ is contained in $G \cap B((x,y),\delta)$. It is straightforward to show that $(x+v, y+v) \subseteq B((x,y),\delta)$. We now show that $(x+v, y+v) \in G$, i.e., that $T(x+v) - (x+v) = T(y+v) - (y+v)$. By Lemma 2.1 and the choice of δ, we have that $T(x+v) + y - x \subseteq T(x) + y - x = T(y)$ and hence $T(x+v) + y - x$ is a patch in T. Since $y + v = (x+v) + y - x$, we have that $T(y+v) = T(x+v) + y - x$ as desired.

Step 2. To get the other inclusion, suppose $(x', y') \in B((x,y),\delta) \cap G$. This implies that $x' = x + a$ and $y' = y + b$, where $a, b \in B(0,\delta)$. Since $(x', y') \in G$ we have that $T(y') = T(x') + y' - x'$, i.e., $T(y+b) = T(x+a) + y + b - x - a$, and by the choice of δ and the fact that $(x, y) \in G$, we have that $T(y+b) \subseteq T(x) + y - x + b - a = T(y) + b - a$. Also by the choice of δ we have $T(y+b) \subseteq T(y)$. We now can use Lemma 1.1 to prove that $a = b$. Fix a tile $p_i + w$ in $T(y+b)$. From what is done above we have that $p_i + w \in T(y)$ and $p_i + w \in T(y) + b - a$. Let $u = w - b + a$. Then $p_i + u$ and $p_i + u + (b-a)$ belong to $T(y)$, and since $|b - a| \leqslant |b| + |a| \leqslant \delta_0$, by Lemma 1.1, we must have $a = b$. We proved that $(x', y') = (x+a, y+a)$ and since $(x', y') \in B((x,y),\delta)$ we have that $\sqrt{a^2 + a^2} < \delta$ and hence $|a| < \frac{\delta}{\sqrt{2}}$ as desired. ∎

PROPOSITION 2.3. G *is an etale equivalence relation.*

Now, to obtain the reduced C^*-algebra of G, we consider the complex linear space of compactly supported, continuous and complex valued functions, $C_c(G)$, equipped with multiplication and involution defined by $(f * g)(x,y) = \sum_{z \in [x]} f(x,z) g(z,y)$, $f^*(x,y) = \overline{f(y,x)}$, where $(x,y) \in G$ and $f, g \in C_c(G)$, and complete it with respect to the reduced norm, see [10]. In this paper we will restrict our attention to the reduced C^*-algebra, since the C^*-algebras considered are amenable. We refer the reader to [10], [6] or [13] for a detailed description of how to construct the full and reduced C^*-algebra from a groupoid. Below we give an example of a C^*-algebra constructed through the approach above. This C^*-algebra will play a crucial role in understanding the C^*-algebras from tilings.

EXAMPLE 2.4. *Let X be a locally compact, σ-compact space and Y a countable set with the discrete topology. Let $G = \Delta_X \times (Y \times Y)$ on $X \times Y$, with the product topology* *(here $\Delta_x = \{(x, x) : x \in X\}$, the diagonal of X). So a point (x_1, y_1) is equivalent to (x_2, y_2) if and only if $x_1 = x_2$. Then G is etale and*

$$C_r^*(G) \cong C_0(X, K(l^2(Y))),$$

where $C_0(X, K(l^2(Y)))$ denotes the continuous functions that vanish at infinity with values in the compact operators, $K(l^2(Y))$, in the Hilbert space $l^2(Y)$.

Proof. For simplicity we assume $Y = \{1, 2, 3, \ldots\}$. We define an isometric $*$-homomorphism ψ from $C_c(G)$ into $C_0(X, K(l^2(Y)))$ by

$$\psi(f)(t)(\xi)(j) = \sum_{z=1,2,\ldots} f((t, j), (t, z))\xi(z) = \sum_{(t,z)\in[(t,j)]} f((t, j), (t, z))\xi(z)$$

for $t \in X$ and $\xi \in l_2(Y)$ and $j \in Y$.

First we need to show that $\psi(f)$ is well defined, i.e., we need to show that $\psi(f)(t)$ is compact for all $t \in X$ and that $\psi(f)$ is continuous and has limit 0 at infinity. Then we need to show that ψ is a isometric $*$-homomorphism with dense range and hence we can extend it to $C_r^*(G)$. The details follow below.

Step 1. $\psi(f)(t) \in K(l_2[N])$ for all $t \in X$.

Notice that for any $i, j \in Y$ the set $X \times \{i\} \times \{j\} = \{((x, i), (x, j)) : x \in X\}$ is open and hence the collection of sets $\{X \times \{i\} \times \{j\}\}_{i,j\in Y}$ covers the support of f. Since the support of f is compact, there exists a finite subcover and so there exists $N \in \mathbb{N}$ such that if either $i > N$ or $j > N$ then $f((t, i)(t, j)) = 0$ for all $t \in X$. But this implies that for any $j > N$, $\psi(f)(t)(\xi)(j) = 0$ for any $\xi \in l_2(Y)$ and hence $\psi(f)(t)$ is a finite rank operator for all $t \in X$.

Step 2. $\psi(f)$ is continuous.

Let $t_0 \in X$ and give $\varepsilon > 0$. Let N as above. Choose an open neighborhood V_{t_0} such that if $t \in V_{t_0}$ then

$$|f((t, j)(t, z)) - f((t_0, j)(t_0, z))| < \frac{\sqrt{\varepsilon}}{\sqrt{N}}$$

for all $j = 1, 2, \ldots, N$ and for all $z = 1, 2, \ldots, N$. So, if $\xi \in l_2(Y)$ with $\|\xi\| = 1$ and $t \in V_{t_0}$ then

$$\begin{aligned}
\|(\psi(f)(t) - \psi(f)(t_0))\xi\|^2 &= \sum_{j=1,2,3,\ldots} \|(\psi(f)(t) - \psi(f)(t_0))\xi(j)\|^2 \\
&= \sum_{j=1,\ldots,N} \|(\psi(f)(t) - \psi(f)(t_0))\xi(j)\|^2 \\
&= \sum_{j=1,\ldots,N} \Big| \sum_{z=1,2,\ldots} f((t, j)(t, z))\xi(z) - f((t_0, j)(t_0, z))\xi(z) \Big|^2 \\
&= \sum_{j=1,\ldots,N} \Big| \sum_{z=1,\ldots,N} (f((t, j)(t, z)) - f((t_0, j)(t_0, z)))\xi(z) \Big|^2 \\
&\leqslant \sum_{j=1,\ldots,N} \sum_{z=1,\ldots,N} |f((t, j)(t, z)) - f((t_0, j)(t_0, z))|^2 |\xi(z)|^2
\end{aligned}$$

$$\leqslant \sum_{j=1,\ldots,N}\sum_{z=1,\ldots,N}\frac{\varepsilon}{N}|\xi(z)|^2 \leqslant \frac{\varepsilon}{N}\sum_{j=1,\ldots,N}\sum_{z=1,2,\ldots}|\xi(z)|^2$$
$$= \frac{\varepsilon}{N}\sum_{j=1,\ldots,N}\|\xi\|^2 = \frac{\varepsilon}{N}\sum_{j=1,\ldots,N}1 = \varepsilon$$

and this proves that ψ is continuous at any $t_0 \in X$.

Step 3. $\lim_{t\to\infty}\psi(f)(t) = 0$.

Give $\varepsilon > 0$. Denote by π_x the projection of $X \times Y$ onto X, i.e., $\pi_x(t,y) = t$. Let $K = \pi_x(r(\mathrm{supp} f))$ where r is the range map. Notice that K is compact, since it is the image by a continuous function of a compact set. Now if $t \notin K$ then, for any y,z in Y, we have that $((t,y),(t,z)) \notin \mathrm{supp} f$ and hence $\psi(f)(t) = 0$.

Step 4. ψ is a isometric $*$-homomorphism with dense range.

We only need to show that ψ has dense range as the other properties follow without further difficulties. First observe that for a fixed t in X, the set $\{\psi(f)(t) : f \in C_c(G)\}$ is dense in $K(l_2(Y))$. Let $f \in C_c(X, K(l_2(Y)))$ and $\varepsilon > 0$. Cover the support of f by open sets where f is uniformly compact, that is, for every t in $\mathrm{supp}(f)$ let V_t be an open set such that $\|f(z) - f(z')\| < \varepsilon$ for all $z, z' \in V_t$ (such sets do exist from continuity of f). Since the support of f is compact, there exists a finite cover, say $V_{t_1}, \ldots, V_{t_M}$. For each t_i, $1 \leqslant i \leqslant M$, from the observation above, there exists $f_i \in C_c(G)$ such that $\|f(t_i) - \psi(f_i)(t_i)\| < \varepsilon$. For simplicity we put $k_i := \psi(f_i)(t_i)$. Let α_i be a partition of unity with respect to the open sets V_i, $1 \leqslant i \leqslant M$, see [12]. Then for all $1 \leqslant i \leqslant M$ we have that $\alpha_i \in C_c(X)$, $\mathrm{supp}(\alpha_i) \subset V_i$, $0 \leqslant \alpha_i \leqslant 1$ and $\sum_{i=1,\ldots,M}\alpha_i(t) = 1$ for any t in $\mathrm{supp}(f)$. Now for any $t \in X$ we define

$$g(t) = \sum_{i=1,\ldots,M}\alpha_i(t)k_i.$$

This g is a good approximation for f. To see this let $t \in X$. Notice that if $\alpha_i(t) \neq 0$, $1 \leqslant i \leqslant M$, then $t \in \mathrm{supp}(\alpha_i) \subset V_i$ and hence $\|f(t_i) - f(t)\| < \varepsilon$. With this in mind we get that, for any t in X,

$$\|g(t) - f(t)\| = \left\|\sum_{i=1,\ldots,M}\alpha_i(t)k_i - \sum_{i=1,\ldots,M}\alpha_i(t)f(t)\right\| \leqslant \sum_{i=1,\ldots,M}\|\alpha_i(t)[k_i - f(t)]\|$$
$$\leqslant \sum_{i=1,\ldots,M}\alpha_i(t)(\|k_i - f(t_i)\| + \|f(t_i) - f(t)\|) \leqslant \sum_{i=1,\ldots,M}\alpha_i(t)(\varepsilon+\varepsilon) \leqslant 2\varepsilon.$$

So $\|f - g\| = \sup_{t\in X}\|f(t) - g(t)\| \leqslant 2\varepsilon$. Finally we have that g belongs to $C_c(X, BL(l_2(N)))$, since sums of block operators and multiplication by scalar yields another block operator. Since $\psi(C_c(G)) = C_c(X, BL(l_2(N)))$, there exists $h \in C_c(G)$ such that $\psi(h) = g$ and hence ψ has dense range as desired. ∎

3. IDEALS OF $C_r^*(G)$

The C^*-algebras defined in the previous section have an ideal structure that we present below. This is particular interesting in the case of a substitution tiling, where the inductive limit C^*-algebra we will introduce later is simple, but the step C^*-algebras are not.

By a G-invariant set we mean a set Z such that whenever $(x,y) \in G$ and $x \in Z$, then $y \in Z$. For each G-invariant set of $\mathbb{R}^d$ we associate an ideal, namely $I_Z = \overline{\{f \in C_c(G) : f|_{Z\times Z} = 0\}}$. Ideals arising in this way are particularly important, because we can give a good description of the quotient on $C_r^*(G)$ by them. Most of the ideal structure of $C_r^*(G)$, that we will explore, depends on the proposition below.

PROPOSITION 3.1. *Let* I *be an ideal of* $C_r^*(G)$. *Then*

$$I = \overline{\{f \in C_c(G) : f|_Q = 0\}}$$

where Q *is a closed subset of* G *such that* $G \circ Q \circ G \subseteq Q$, *i.e., if* (x,y) *and* $(z,w) \in G$ *and* $(y,z) \in Q$ *then* $(x,w) = (x,y)(y,z)(z,w) \in Q$. *If* Z *is a closed,* G*-invariant subset of* $\mathbb{R}^d$ *then* $G \cap Z \times Z$ *is etale with unit space* Z. *Moreover, the map that restricts a function in* $C_c(G)$ *to* $G|_{Z\times Z}$ *is a* $*$*-homomorphism, which extends to a surjection from* $C_r^*(G)$ *onto* $C_r^*(G|_{Z\times Z})$ *with kernel equal to* I_Z.

Proof. The proof of this proposition can be found in [5] and [10]. ∎

We now focus on the tiling situation. From now on, we assume every tiling has a cellular structure; for example, if $d = 2$ we assume each tiling has vertices, edges and faces. With this assumption we can consider the ideal of all functions that vanish at the edges and the ideal of all functions that vanish at the vertices. More precisely, for $0 \leqslant i < d$, let X_i be the set of all $x \in \mathbb{R}^d$ such that if $x \in t$, for some tile $t \in T$, then x is in the i-skeleton of t. So X_0 is the set of vertices in T, X_1 is the set of all points of $\mathbb{R}^d$ that are contained in an edge of a tile in T and so on. Before considering the ideals induced by this sets we need to show G-invariance.

PROPOSITION 3.2. *For* $0 \leqslant i < d$, X_i *is closed and* G*-invariant.*

The proposition above allows us to consider the ideals

$$I_{X_i} = \overline{\{f \in C_c(G) : f|_{X_i\times X_i} = 0\}}$$

for $0 \leqslant i < d$. Notice that $X_{d-1} \supseteq \cdots \supseteq X_1 \supseteq X_0$ and hence $I_{X_{d-1}} \subseteq \cdots \subseteq I_{X_1} \subseteq I_{X_0}$.

For simplicity we restrict our attention to $d = 2$. Proposition 3.1 gives us the following exact sequence:

$$0 \to I_{X_1} \to C_r^*(G) \to C_r^*(G|_{X_1}) \to 0 \tag{3.1}$$

where $G|_{X_1}$ denotes $G \cap (X_1 \times X_1)$. We can describe I_{X_1} nicely, but we are unable to do the same for $C_r^*(G|_{X_1})$. But we can apply Proposition 3.1 for $X_0 \times X_0 \subseteq G|_{X_1}$

and get the following exact sequence:

$$0 \to C_r^*(G|_{X_1-X_0}) \to C_r^*(G|_{X_1}) \to C_r^*(G|_{X_0}) \to 0 \tag{3.2}$$

where $C_r^*(G|_{X_1-X_0}) = \overline{\{f \in C_c(G|_{X_1}) : f|_{X_0\times X_0} = 0\}}$ and we can realize both end terms of this exact sequence. In order to do so we need a few lemmas:

LEMMA 3.3. *Let* T *be a tiling with finite local complexity and a cell structure (i.e. vertices, edges and faces). Then the equivalence class of any* $x \in \mathbb{R}^d$ *is countable. In particular if* T *is a substitution tiling with finite local complexity, recognizability and primitivity then the equivalence class of any* $x \in \mathbb{R}^d$ *is infinitely countable.*

LEMMA 3.4. *There are only a finite number of equivalence classes of vertices in* G*, i.e., there are, up to translation, only a finite number of patterns of vertices.*

Proof. Follows from finite local complexity. ∎

NOTATION 3.5. We denote the set of all vertex equivalence classes by $\mathcal{V}$. So from the lemma above we can write the set of vertices, X_0, as a finite disjoint union $\dot{\bigcup}_{[v]\in\mathcal{V}} [v]$.

We have a similar result for the edge patterns. We say two edges are equivalent, if the pattern defined by the interior points of the edges are the same up to translation. So there is only a finite number of edge equivalence classes in G, i.e., there is only a finite number of edge patterns, defined by the interior points of the edges, up to translation.

NOTATION 3.6. We denote the set of all edge equivalence classes by $\mathcal{E}$. Observe that we can write the set of edges as a finite disjoint union $\dot{\bigcup}_{[e]\in\mathcal{E}} [e]$.

With the setting above we can describe many of the terms in the exact sequences (3.1) and (3.2).

PROPOSITION 3.7. $C_r^*(G|_{X_0}) \cong \bigoplus_{[v]\in\mathcal{V}} K(l^2([v]))$ *where the* $[v]$*'s are as in Notation* 3.5.

Proof. Since we can write the set of vertices, X_0, as a finite disjoint union $\dot{\bigcup}_{[v]\in\mathcal{V}} [v]$, where each v is a representative of a vertex equivalence class, we have that $G|_{X_0} = \dot{\bigcup}_{[v]\in\mathcal{V}} \{[v]\} \times \{[v]\}$ and from a simplified version of example 2.4 we have the desired result. ∎

PROPOSITION 3.8. $C_r^*(G|_{X_1-X_0}) \cong \bigoplus_{[e]\in\mathcal{E}} C_0(e, K(l^2([e])))$ *where the* $[e]$*'s are as in Notation* 3.6.

Proof. Follows from Example 2.4. ∎

Finally, remember that we only have a finite number of tiles up to translation. This means we can also write the set of tiles as a finite disjoint union $\dot{\bigcup\limits_{[p]\in\mathcal{P}}} [p]$, where $[p]$ denote the equivalence class of p, i.e., the set of all tiles that are translations of p and $\mathcal{P}$ denote the set of all tile equivalence classes. We can now state the last proposition of this section.

PROPOSITION 3.9. $\mathrm{I}_{\mathrm{X}_1} \cong \bigoplus\limits_{p\in\mathcal{P}} \mathrm{C}_0(p, \mathrm{K}(l^2([p])))$ *where the* $[p]$*'s are as described above.*

4. $C^*_{\mathrm{r}}(\mathrm{G})$ AS A RECURSIVE SUBHOMOGENEOUS C^*-ALGEBRA

The notion of a recursive subhomogeneous C^*-algebra was introduced by N.C. Philips in [7]. In this section, we will describe $C^*_{\mathrm{r}}(\mathrm{G})$ as a recursive subhomogeneous algebra. But we will need a slight generalization of the notion introduced in [7]. Basically we will have to replace continuous functions taking values in the matrices, by continuous functions taking values in the compact operators in some Hilbert space. This is a rather natural generalization, but in doing this we get C^*-algebras that are non unital. This means that we will not be able to apply directly the results of [7] for $C^*_{\mathrm{r}}(\mathrm{G})$. But it seems reasonable to expect that they can be generalized. Below, we introduce the definition of a recursive subhomogeneous C^*-algebra, RSA, already modified to fit our needs.

DEFINITION 4.1. Let X be a compact Hausdorff space. Then:
(i) $\mathrm{C}(\mathrm{X}, \mathrm{K}(l_2[\mathbb{N}]))$ is an RSA.
(ii) If A is RSA, $\mathrm{X}^{(0)} \subseteq \mathrm{X}$, $\mathrm{X}^{(0)}$ is closed; $\varphi : \mathrm{A} \to \mathrm{C}(\mathrm{X}^{(0)}, \mathrm{K}(l_2[\mathbb{N}]))$ is any homomorphism, and $\rho : \mathrm{C}(\mathrm{X}, \mathrm{K}(l_2[\mathbb{N}])) \to \mathrm{C}(\mathrm{X}^{(0)}, \mathrm{K}(l_2[\mathbb{N}]))$ is the restriction homomorphism, then the pull back

$$\{(a, f) \in \mathrm{A} \oplus \mathrm{C}(\mathrm{X}, \mathrm{K}(l_2[\mathbb{N}])) : \varphi(a) = \rho(f)\}$$

denoted by $\mathrm{A} \oplus_{\mathrm{C}(\mathrm{X}^{(0)}, \mathrm{K}(l_2[\mathbb{N}]))} \mathrm{C}(\mathrm{X}, \mathrm{K}(l_2[\mathbb{N}]))$ is an RSA.

EXAMPLE 4.2. Any direct sum of RSA's is an RSA.

We will show that $C^*_{\mathrm{r}}(\mathrm{G})$ is an RSA. The first step is to show that $C^*_{\mathrm{r}}(\mathrm{G}|_{\mathrm{X}_1})$ is isomorphic to an RSA which we construct below.

From Notition 3.6 we know that there is only a finite number of edge equivalence classes. So we can write the set of edges, E, as

$$E = \{e_j + u^j_l : j = 1, \ldots, k; l = 0, 1, 2, \ldots\}$$

where each u^j_l is a vector in $\mathbb{R}^n$, $u^j_0 = 0$ for every j and each e_j is a representative of one edge equivalence class.

Consider the set $[0,1] \times E$ and identify each point $(t, e_j + u_l^j)$ in this set with the point $r_{e_j}(t) + u_l^j$, where r_{e_j} is a parametrization of the edge e_j. On $[0,1] \times E$ introduce the equivalence relation $\Delta\times \sim$, where $\sim$ is the equivalence relation on the edges (so $e_j + u_l^j$ is equivalent to any other translation $e_j + u_h^j$) and Δ is the diagonal of $[0,1]$. Observe that $(0, e_j + u_l^j)$ and $(1, e_j + u_l^j)$ are the endpoints of the edge $e_j + u_l^j$ and we also consider the set $\{0,1\} \times E$ with the equivalence relation $\Delta\times \sim$. Also notice that $\mathrm{C}_\mathrm{r}^*([0,1] \times E, \Delta\times \sim)$ is isomorphic to $\bigoplus_{j=1,\ldots,k} \mathrm{C}([0,1], \mathrm{K}(l_2([e_j])))$ and $\mathrm{C}_\mathrm{r}^*(\{0,1\} \times E, \Delta\times \sim)$ is isomorphic to $\bigoplus_{j=1,\ldots,k} \mathrm{C}(\{0,1\}, \mathrm{K}(l_2([e_j])))$ and hence the restriction map ρ from $\mathrm{C}_\mathrm{r}^*([0,1] \times E, \Delta\times \sim)$ onto $\mathrm{C}_\mathrm{r}^*(\{0,1\} \times E, \Delta\times \sim)$ is defined as required by definition 4.1.

Now observe that $\mathrm{C}_\mathrm{r}^*(\mathrm{G}|_{\mathrm{X}_0})$ is an RSA, since from Proposition 3.7 we have that $\mathrm{C}_\mathrm{r}^*(\mathrm{G}|_{\mathrm{X}_0}) \cong \bigoplus_{v \in \mathcal{V}} \mathrm{K}(l^2([v]))$. Let $\varphi : \mathrm{C}_\mathrm{c}(\mathrm{G}|_{\mathrm{X}_0}) \to \mathrm{C}_\mathrm{c}(\{0,1\} \times E, \Delta\times \sim)$ be the $*$-homomorphism defined by

$$\varphi(f)((i, e_j + u_l^j), (i, e_j + u_h^j)) = f(r_{e_j}(i) + u_l^j, r_{e_j}(i) + u_h^j)$$

if $(r_{e_j}(i) + u_l^j, r_{e_j}(i) + u_h^j) \in \mathrm{G}$, $i \in \{0,1\}$ and to be equal to 0 otherwise, for $f \in \mathrm{C}_\mathrm{c}(\mathrm{G}|_{\mathrm{X}_0})$.

The $*$-homomorphism φ above can be extended to a $*$-homomorphism from $\mathrm{C}_\mathrm{r}^*(\mathrm{G}|_{\mathrm{X}_0})$ onto $\mathrm{C}_\mathrm{r}^*(\{0,1\} \times E, \Delta\times \sim)$. We will still denote the extension by φ as this should not bring any confusion. With this setting we can consider the pull back C^*-algebra,

$$\mathrm{C}_\mathrm{r}^*(\mathrm{G}|_{\mathrm{X}_0}) \oplus_{\mathrm{C}_\mathrm{r}^*(\{0,1\} \times E, \Delta\times\sim)} \mathrm{C}_\mathrm{r}^*([0,1] \times E, \Delta\times \sim)$$

which by Definition 4.1 is an RSA. We will show that the pull back C^*-algebra above is isomorphic to $\mathrm{C}_\mathrm{r}^*(\mathrm{G}|_{\mathrm{X}_1})$. For this we need a lemma:

LEMMA 4.3. $\ker\rho \cap \mathrm{C}_\mathrm{c}([0,1] \times E, \Delta\times \sim)$ *is dense in* $\ker\rho$.

Proof. From Proposition 3.1, we know that there exists a closed set Q contained in $([0,1] \times E, \Delta\times \sim)$ such that

$$\ker\rho = \overline{\{f \in \mathrm{C}_\mathrm{c}([0,1] \times E, \Delta\times \sim) : f|_\mathrm{Q} = 0\}}.$$

It follows that $\ker\rho \cap \mathrm{C}_\mathrm{c}([0,1] \times E, \Delta\times \sim)$ contains

$$\{f \in \mathrm{C}_\mathrm{c}([0,1] \times E, \Delta\times \sim) : f|_\mathrm{Q} = 0\}$$

and hence the lemma follows. ∎

PROPOSITION 4.4. $\mathrm{C}_\mathrm{r}^*(\mathrm{G}|_{\mathrm{X}_0}) \oplus_{\mathrm{C}_\mathrm{r}^*(\{0,1\} \times E, \Delta\times\sim)} \mathrm{C}_\mathrm{r}^*([0,1] \times E, \Delta\times \sim)$ *is isomorphic to* $\mathrm{C}_\mathrm{r}^*(\mathrm{G}|_{\mathrm{X}_1})$*. In particular,* $\mathrm{C}_\mathrm{r}^*(\mathrm{G}|_{\mathrm{X}_1})$ *is an* RSA.

Proof. Let α be the homomorphism defined by

$$\begin{array}{rcl}\alpha : C_c(G|_{X_1}) & \rightarrow & C^*_r(G|_{X_0}) \oplus_{C^*_r(\{0,1\}\times E,\Delta\times\sim)} C^*_r([0,1]\times E,\Delta\times\sim),\\ f & \mapsto & (\alpha_1(f),\alpha_2(f))\end{array}$$

where $\alpha_1(f)$ is the restriction to the vertices map ($\alpha_1(f) = f|_{X_0\times X_0}$)) and

$$\alpha_2(f)((t,e_j+u^j_l),(t,e_j+u^j_h)) = f(r_{e_j}(t)+u^j_l, r_{e_j}(t)+u^j_h)$$

if $(r_{e_j}(t)+u^j_l, r_{e_j}(t)+u^j_h) \in G$ and to be equal to 0 otherwise.

It follows from the definitions above that $\varphi(\alpha_1(f)) = \rho(\alpha_2(f))$ and hence $\alpha(f)$ is well defined. One can also check that α is a $*$-homomorphism. Below we show that α is isometric and has dense range.

Step 1. α is isometric.

Since $\|\alpha(f)\| = \max\{\|\alpha_1(f)\|,\|\alpha_2(f)\|\}$ it is enough to show that $\|f\| = \|\alpha_2(f)\|$. Notice that

$$\|\alpha_2(f)\| = \sup_{(t,e_j+u^j_l)\in[0,1]\times E} \|\lambda_{(t,e_j+u^j_l)}(\alpha_2(f))\|$$

where $\lambda_{(t,e_j+u^j_l)}$ is as in the definition of the reduced norm. So if $t\in(0,1)$ then $l_2[(t,e_j+u^j_l)] = l_2([r_{e_j}(t)+u^j_l])$ and hence

$$\lambda_{(t,e_j+u^j_l)}(\alpha_2(f)) = \lambda_{r_{e_j}(t)+u^j_l}(f).$$

Also if $t\in\{0,1\}$ then $[(t,e_j+u^j_l)]_{\Delta\times\sim}$ is equal to the disjoint union $\bigcup_{h=1}^{h_0}[(t,e_j+u^j_h)]_0$, where each $r_{e_j}(t)+u^j_h$ is a representative of a different vertex equivalence class in G and $[(t,e_j+u^j_h)]_0$ is the set of all points (t,e) in $[0,1]\times E$ such that $r_{e_j}(t)+u^j_h$ is equivalent to $r_e(t)$ in G. This implies that

$$\lambda_{(t,e_j+u^j_l)}(\alpha_2(f)) = \bigoplus_{h=1}^{h_0} \lambda_{r_{e_j}(t)+u^j_h}(f)$$

and hence $\|\lambda_{(t,e_j+u^j_l)}(\lambda_2(f))\| = \max_{h=1,\dots,h_0} \|\lambda_{r_{e_j}(t)+u^j_h}(f)\|$, which implies that α is isometric.

Step 2. α has dense range.

Let $(a_2,a_1) \in C^*_r(G|_{X_0}) \oplus_{C^*_r(\{0,1\}\times E,\Delta\times\sim)} C^*_r([0,1]\times E,\Delta\times\sim)$. Notice that $\varphi(a_2) = \rho(a_1)$.

Let $\varepsilon>0$. Using the continuity of φ find $a'_2 \in C_c(G|_{X_0})$ such that $\|a'_2 - a_2\| < \varepsilon$ and $\|\varphi(a'_2)-\varphi(a_2)\| < \varepsilon$. Since ρ is onto, there exists

$$a'_1 \in C_c([0,1]\times E,\Delta\times\sim)$$

such that $\rho(a'_1) = \varphi(a'_2)$ (This is actually the Tietze extension theorem, see [12], i.e., since $\varphi(a'_2)$ belongs to $C_c(\{0,1\}\times E,\Delta\times\sim)$, we can extend it to a continuous function with compact support in $([0,1] \times E, \Delta\times \sim)$. It follows that $\|\rho(a_1' - a_1)\| = \|\varphi(a_2' - a_2)\| < \varepsilon$. Next observe that $\|\rho(a_1' - a_1)\|$ is a quotient norm, that is, since ρ is onto and $\frac{C_r^*([0,1]\times E,\Delta\times\sim)}{\ker\rho} \cong \mathrm{Im}\rho$ via $f + \ker\rho \to \rho(f)$, we have that $\|\rho(a_1' - a_1)\| = \|(a_1' - a_1) + \ker\rho\| = \inf\{\|a_1' - a_1 + c\| : c \in \ker\rho\}$. Hence there exists $c \in \ker\rho$ such that $\|a_1' + c - a_1\| < 2\varepsilon$. Furthermore from Lemma 4.3 there exists $c' \in \ker\rho$, $c' \in C_c([0,1] \times E, \Delta\times \sim)$, such that $\|a_1' + c' - a_1\| < 3\varepsilon$.

Now notice that $\rho(a_1' + c') = \rho(a_1') = \varphi(a_2')$ and hence $(a_2', a_1' + c')$ belongs to the pull back $C_r^*(G|_{X_0}) \oplus_{C_r^*(\{0,1\}\times E,\Delta\times\sim)} C_r^*([0,1] \times E, \Delta\times \sim)$. Also $\|(a_2, a_1) - (a_2', a_1' + c')\| < 3\varepsilon$. Finally since $(a_2', a_1' + c')$ belongs to the pull back and both a_2' and $a_1' + c'$ have compact support, we can find $f \in C_c(G|_{X_1})$ such that $\alpha(f) = (a_2', a_1' + c')$, which implies that α has dense range. ∎

The proposition above also allow us to state the theorem below:

THEOREM 4.5. *$C_r^*(G) \cong C_r^*(G|_{X_1}) \oplus_{C_r^*(S\times\tau,\Delta\times\sim)} C_r^*(D \times \tau, \Delta\times \sim)$, where D is the closed unit disk and S is the circle. In particular $C_r^*(G)$ is an* RSA.

Proof. The first part of the the theorem follows analogously to the proposition above and since $C_r^*(G|_{X_1})$ is an RSA, the second part follows. ∎

5. THE C^*-ALGEBRA INDUCED BY THE INFLATION MAP

The C^*-algebras from tilings defined in the previous section could be constructed not only for substitution tilings, but for any tiling with finite local complexity and a finite number of tiles up to translation. This is not the case for the C^*-algebras we are about to define. For them, we need a substitution tiling with primitivity, finite local complexity and recognizability. Examples of such tilings include the Penrose, Octagonal, Table and many others. We refer the reader to [3], [4], [1] for examples of aperiodic tilings. We want a way to capture the dynamics of the inflation rule on the equivalence relation induced by the tilings $\omega^k(T)$, $k = 1, 2, 3, \ldots$. For this, we define a equivalence relation on each tiling $\omega^k(T)$ and show that this relations are comparable by inclusion. This gives rise to a inductive limit of C^*-algebras.

From now on, call G_0 what we were calling G and define:

$$G_k(T) = \{(\lambda^{-k}x, \lambda^{-k}y) \in \mathbb{R}^2 \times \mathbb{R}^2 : (x, y) \in G(\omega^k(T))\}$$

for $k = 1, 2, 3, \ldots$.

PROPOSITION 5.1. *Each G_k is contained in G_{k+1}, i.e., $G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$.*

Proof. We will show that $G_0 \subseteq G_1$. The others inclusions are proved analogously. Let $(x, y) \in G_0$. We need to show that $(x, y) \in G_1$, i.e.,

$$\omega(T)(\lambda x) - \lambda x = \omega(T)(\lambda y) - \lambda y.$$

From the definition of ω, $\omega(\mathrm{T})(\lambda x) \subseteq \omega(\mathrm{T}x)$. Also $(x,y) \in \mathrm{G}_0$ implies that $\omega(\mathrm{T}y) = \omega(\mathrm{T}x) + \lambda(y-x)$. So we have

$$\omega(\mathrm{T})(\lambda x) + \lambda(y-x) \subseteq \omega(\mathrm{T}x) + \lambda(y-x) = \omega(\mathrm{T}y)$$

and hence $\omega(\mathrm{T})(\lambda x) + \lambda(y-x)$ is a patch in $\omega(\mathrm{T})$. Moreover, since $\lambda y = \lambda x + \lambda(y-x)$ this patch is exactly $\omega(\mathrm{T})(\lambda y)$ and it follows that $(x,y) \in \mathrm{G}_1$. ∎

The next step is to show that each G_k is open in the next. In order to do so we need two lemmas that are generalizations of Lemmas 2.1 and 2.2.

LEMMA 5.2. *Fix* $\mathrm{N} \in \mathbb{N}$. *Given* $x \in \mathbb{R}^d$, *there exists* $\delta > 0$ *such that if* y *belongs to the ball of center* x *and radius* δ, $\mathrm{B}(x,\delta)$, *then* $\mathrm{T}_k(\lambda^k(y)) \subseteq \mathrm{T}_k(x)(\lambda^k(x))$ *for all* $0 \leqslant k \leqslant \mathrm{N}$.

Proof. For any $0 \leqslant k \leqslant \mathrm{N}$ the set $\mathrm{U} = \frac{1}{\lambda^k}(\mathrm{int}(\mathrm{T}_k(\lambda^k x)))$ is open and $x \in \mathrm{U}$. So there exists δ_k such that $\mathrm{B}(x,\delta_k) \subseteq \mathrm{U}$. Now if $y \in \mathrm{B}(x,\delta_k)$ then $\lambda^k y \in \mathrm{int}(T_k(\lambda^k x))$ and hence $\mathrm{T}_k(\lambda^k y) \subseteq \mathrm{T}_k(\lambda^k x)$. Finally choose δ as the minimum of all δ_k. ∎

The next lemma characterizes neighborhoods of a point $(x,y) \in \mathrm{G}_k$.

LEMMA 5.3. *Let* $(x,y) \in \mathrm{G}_k$ *with* $k \in \mathbb{Z}$, $k \geqslant 0$. *Then there exists* $\delta_k > 0$ *such that*

$$\mathrm{B}((x,y),\delta) \cap \mathrm{G}_k = \left\{(x+v, y+v) : v \in \mathrm{B}(0, \frac{\delta}{\sqrt{2}})\right\} \quad \textit{for all } \delta \leqslant \delta_k.$$

Proof. Let δ be small enough so that both Lemmas 5.2, applied for x, and 1.1 hold. Let $\delta_k < \frac{\delta}{2\lambda^k}$. The rest of the proof is a straightforward adaptation of the proof of Lemma 2.2, just replacing T with T_k and multiplying by λ^k where necessary. ∎

OBSERVATION 5.4. It is clear from the lemma above that each G_k is etale.

PROPOSITION 5.5. *Each* G_k *is open in* G_{k+1}; $k = 0, 1, 2, \ldots$.

Proof. We will show that G_0 is open on G_1. The proof for k is the same, one just need to change indices.

Let $(x,y) \in \mathrm{G}_0$. Notice that (x,y) is also in G_1. Let $\delta = \min\{\delta_0, \delta_1\}$, where δ_0 and δ_1 are obtained from Lemma 5.3 applied for G_0 and G_1, respectively. Then

$$B((x,y),\delta) \cap \mathrm{G}_1 = \left\{(x+v, y+v) : v \in B(0, \frac{\delta}{\sqrt{2}})\right\} = B((x,y),\delta) \cap \mathrm{G}_0$$

and hence $B((x,y),\delta) \cap \mathrm{G}_1$ is a neighborhood of (x,y) in G_1 completely contained in G_0. ∎

With the two propositions above we are now able to show that $\mathrm{C}^*_\mathrm{r}(\mathrm{G}_k)$ is a sub-algebra of $\mathrm{C}^*_\mathrm{r}(\mathrm{G}_{k+1})$.

PROPOSITION 5.6. $\mathrm{C}^*_\mathrm{r}(\mathrm{G}_k)$ *is a subalgebra of* $\mathrm{C}^*_\mathrm{r}(\mathrm{G}_{k+1})$, *for all* $k = 0, 1, 2, \ldots$.

Proof. It is enough to show the proposition for $k = 0$. The idea is to include a function $f \in C_c(G_0)$ into $C_c(G_1)$ by extending it to 0 on $G_1 - G_0$. More explicitly, we define $\iota : C_c(G_0) \to C_c(G_1)$ by

$$\iota(f)(x,y) = \widetilde{f}(x,y) = \begin{cases} f(x,y) & (x,y) \in G_0, \\ 0 & (x,y) \in G_1 - G_0, \end{cases}$$

ι is a $*$-homomorphism. We show that ι is isometric and hence it can be extended to an isometric $*$-homomorphism from $C_r^*(G_0)$ into $C_r^*(G_1)$.

Remember that the reduced norm of a function $f \in C_c(G_1)$ is given by $\|\widetilde{f}\| = \sup_{x\in X} \|\lambda_x(\widetilde{f})\|$, where

$$\lambda_x(\widetilde{f})(\xi)(y) = \sum_{z\in[x]_1} \widetilde{f}(y,z)\xi(z)$$

for $\xi \in l_2([x]_1)$. We have that $[x]_1$ is equal to the disjoint union $[x_0]_0 \cup [x_1]_0 \cup [x_2]_0 \cup \cdots$ and hence $l_2([x]_1) = l_2([x_0]_0) \oplus l_2([x_1]_0) \oplus l_2([x_2]_0) \oplus \cdots$. Let $\lambda_x^1(\widetilde{f}) := \lambda_x(\widetilde{f})$ and $\lambda_{x_i}^0(f) = \lambda_{x_i}(f)$. Observe that $\lambda_x^1(\widetilde{f})$ acts on $l^2([x]_1)$ and $\lambda_{x_i}^0(f)$ acts on $l_2([x_i]_0)$. Once we show that $\lambda_x^1(\widetilde{f}) = \bigoplus \lambda_{x_i}^0(f)$, we have that

$$\|\widetilde{f}\| = \sup_{x\in X} \|\lambda_x^1(\widetilde{f})\| = \sup_{x\in X} \Big\| \bigoplus \lambda_{x_i}^0(f) \Big\|.$$

It is a fact from functional analysis that if a Hilbert space $H = \bigoplus H_i$, $T = \bigoplus T_i$ is a bounded operator in H and $\|T_i\|$ is uniformly bounded then $\|T\| = \sup \|T_i\|$. Using this fact we have that

$$\sup_{x\in X} \Big\| \bigoplus \lambda_{x_i}^0(f) \Big\| = \sup_{x\in X} \Big\{ \sup_{x_i : [x]_1 = \cup [x_i]_0} \|\lambda_{x_i}^0(f)\| \Big\} = \sup_{x\in X} \|\lambda_x^0(f)\| = \|f\|$$

and hence ι is isometric.

We still need to prove that $\lambda_x^1(\widetilde{f}) = \bigoplus \lambda_{x_i}^0(f)$. Notice that $l_2([x_i]_0) \subseteq l_2([x]_1)$ is invariant under $\lambda_x^1(\widetilde{f})$ for any $i \in \mathbb{N}$. To see this let let $i \in \mathbb{N}$, and $\xi \in l_2([x]_1)$ be supported in $l_2([x_i]_0)$. Now suppose $y \notin [x_i]_0$. We want to show that $\lambda_x^1(\widetilde{f})(\xi)(y) = \sum_{z\in[x]_1} f(y,z)\xi(z) = 0$. But this follows promptly once we notice that if $(y,z) \notin G_0$ then $\widetilde{f}(y,z) = 0$ and if $(y,z) \in G_0$ then $z \notin [x_i]_0$ and hence $\xi(z) = 0$.

Next we show that $\lambda_x^1(\widetilde{f})|_{l_2([x_i]_0)} = \lambda_{x_i}^0(f)$ and hence the desired results follows, since $l_2([x]_1) = l_2([x_0]_0) \oplus l_2([x_1]_0) \oplus l_2([x_2]_0) \oplus \cdots$. So take ξ supported on $l_2([x_i]_0)$ and let $y \in [x_i]_0$. Let $Z_2 = \{z \in [x]_1 : (y,z) \notin G_0\}$ and $Z_1 = \{z \in [x]_1 : (y,z) \in G_0 \to z \in [x_i]_0\}$. Observe that $\widetilde{f}(y,z) = 0$ in Z_2. Then we have as desired:

$$\begin{aligned}\lambda_x^1(\widetilde{f})(\xi)(y) &= \sum_{z\in[x]_1} \widetilde{f}(y,z)\xi(z) = \sum_{z\in Z_1} \widetilde{f}(y,z)\xi(z) + \sum_{z\in Z_2} \widetilde{f}(y,z)\xi(z) \\ &= \sum_{z\in[x_i]_0} f(y,z)\xi(z) = \lambda_{x_i}^0(f)(\xi|_{[x_i]_0})(y). \quad \blacksquare\end{aligned}$$

It is natural for us now to consider the inductive limit of the C^*-algebras $\mathrm{C}^*_{\mathrm{r}}(\mathrm{G}_k)$, with inclusion as connecting map. This is the C^*-algebra encoding the inflation map that we were looking for. The next proposition characterizes this inductive limit.

PROPOSITION 5.7. *The inductive limit C^*-algebra $\mathrm{C}^*_{\mathrm{r}}(\mathrm{G}_k)_{\rightarrow}$ is isomorphic to $\mathrm{C}^*_{\mathrm{r}}(\bigcup \mathrm{G}_k)$ where a basis for the topology of $\bigcup \mathrm{G}_k$ is given by the sets of the form $\mathrm{U}_k \cap (\bigcup \mathrm{G}_k)$, where U_k is any open set of G_k.*

Proof. In order to prove this proposition we first need to define a family of inclusions of $\mathrm{C}^*_{\mathrm{r}}(\mathrm{G}_k)$ into $\mathrm{C}^*_{\mathrm{r}}(\bigcup \mathrm{G}_k)$. We define these inclusions in the same way we included $\mathrm{C}^*_{\mathrm{r}}(\mathrm{G}_0)$ into $\mathrm{C}^*_{\mathrm{r}}(\mathrm{G}_1)$ in Proposition 5.6. So for $n = 0, 1, 2, \ldots$ we define $\lambda_n : \mathrm{C_c}(\mathrm{G}_n) \rightarrow \mathrm{C_c}(\bigcup \mathrm{G}_k)$ by

$$\lambda_n(f)(x,y) = \begin{cases} f(x,y) & (x,y) \in \mathrm{G}_n, \\ 0 & (x,y) \in (\bigcup \mathrm{G}_k) - \mathrm{G}_n. \end{cases}$$

Each λ_n is an isometric $*$-homomorphism and hence it can be extended to an isometric $*$-homomorphism from $\mathrm{C}^*_{\mathrm{r}}(\mathrm{G}_n)$ into $\mathrm{C}^*_{\mathrm{r}}(\bigcup \mathrm{G}_k)$ (the proof of this statement is analogous to Proposition 5.6). Now denote by ι_k the inclusion of $\mathrm{C}^*_{\mathrm{r}}(\mathrm{G}_k)$ into $\mathrm{C}^*_{\mathrm{r}}(\mathrm{G}_{k+1})$. Then $\lambda_n = \lambda_{n+1} \circ \iota_n$ for all $n = 0, 1, 2, \ldots$ and by Definition 6.2.2(ii) in [11] it follows that $\mathrm{C}^*_{\mathrm{r}}(\bigcup \mathrm{G}_k)$ is isomorphic to the inductive limit $\mathrm{C}^*_{\mathrm{r}}(\mathrm{G}_k)_{\rightarrow}$ as desired. ∎

6. $C^*_{\mathrm{r}}(\bigcup \mathrm{G}_k)$ IS SIMPLE

One of the very interesting properties of $\mathrm{C}^*_{\mathrm{r}}(\bigcup \mathrm{G}_k)$ is simplicity. We will discuss the K-theory of $\mathrm{C}^*_{\mathrm{r}}(\bigcup \mathrm{G}_k)$ in a follow-up paper.

We need two more lemmas. From now on we denote the union $\bigcup \mathrm{G}_k$ by G and the equivalence class of a point $x \in \mathbb{R}^d$ with respect to G by $[x]$.

LEMMA 6.1. *Let p_i be a prototile. Then there exists $M > 0$ such that $\omega^M(p_i)$ contains at least one translation of all prototiles, vertex patterns and edge patterns.*

Proof. Take N from primitivity. Then $\omega^{\mathrm{N}}(p_i)$ contain a translation of any prototile. Notice that if $k \in \mathbb{N}$, then $\omega^{k+\mathrm{N}}(p_i)$ also contains a translation of all prototiles, since $\omega^{k+\mathrm{N}}(p_i) = \omega^{\mathrm{N}}(\omega^k(p_i))$ (use the definition of primitivity once again).

Now let P be a translation of one of the prototiles, or a translation of a vertex pattern, or a translation of an edge pattern. Then there exists $n_P \in \mathbb{N}$ such that $\omega^{n_P}(p_i)$ contains a translation of P (to see this, notice that from the definition of a substitution tiling, there exists a prototile p_j and $m \in \mathbb{N}$ such that $\omega^m(p_j)$ contains a translation of P. So take $n_P = m + \mathrm{N}$). Let MX be the maximum of all n_P as above. Observe that this is really a maximum, since we only have a finite number of prototiles, vertex and edge patterns. We will show that $MX + \mathrm{N}$ satisfies the lemma. So let P as before, that is, P is a translation of one of the prototiles, or a translation of a vertex pattern, or a translation of an edge pattern. Then $\omega^l(p_i) \supseteq P$ for some $l \in \mathbb{N}$. Now observe that $\omega^{MX+\mathrm{N}}(p_i) = \omega^l(\omega^{(MX-l)+\mathrm{N}}(p_i))$, and from the first paragraph of the this proof we have that $\omega^{(MX-l)+\mathrm{N}}(p_i)$ contains a translation of p_i. We conclude that $\omega^{MX+\mathrm{N}}(p_i)$ contains a translation of P as desired. ∎

LEMMA 6.2. *For any $x \in \mathbb{R}^d$, $[x]$ is dense in $\mathbb{R}^d$.*

Proof. Given a ball $\mathrm{B}(y,\varepsilon)$ in $\mathbb{R}^d$, we want to find a point in the equivalence class of x within the ball. Since the ball has a fixed radius we can find a $n \in \mathbb{N}$ such that one tile of $\lambda^{-n}\omega^n(\mathrm{T})$ is completely contained in $\mathrm{B}(y,\varepsilon)$. Choose M as in Lemma 6.1. Let $\mathrm{N} = M + n$. Then a translation of any prototile, vertex and edge patterns of $\lambda^{-\mathrm{N}}\omega^{\mathrm{N}}(\mathrm{T})$ appears at least once inside the ball $\mathrm{B}(y,\varepsilon)$. In particular a translation of $\lambda^{-\mathrm{N}}(\omega^{\mathrm{N}}(\mathrm{T})(\lambda^{\mathrm{N}}x))$ is contained in $\mathrm{B}(y,\varepsilon)$ and hence there exists a point $y \in \mathrm{B}(y,\varepsilon)$ such that $(\lambda^{-\mathrm{N}}\omega^{\mathrm{N}}(\mathrm{T}))(y) = \lambda^{-\mathrm{N}}(\omega^{\mathrm{N}}(\mathrm{T})(\lambda^{\mathrm{N}}x)) + y - x$. This last equality implies that $\omega^{\mathrm{N}}(\mathrm{T})(\lambda^{\mathrm{N}}y) = \omega^{\mathrm{N}}(\mathrm{T})(\lambda^{\mathrm{N}}(x)) + \lambda^{\mathrm{N}}(y-x)$ and hence $(y,x) \in \mathrm{G}_{\mathrm{N}}$. So y is equivalent to x in G and the proof is complete. ∎

THEOREM 6.3. *$C^*_{\mathrm{r}}(\bigcup \mathrm{G}_k)$ is simple.*

Proof. Let I be an ideal of $C^*_{\mathrm{r}}(\mathrm{G}) = C^*_{\mathrm{r}}(\bigcup \mathrm{G}_k)$. By Proposition 3.1, we know that $\mathrm{I} = \mathrm{I}(\mathrm{Q})$, for some closed set Q satisfying all the conditions of Proposition 3.1. Let $\mathrm{Z} = \{z \in \mathbb{R}^d : (z,z) \in \mathrm{Q}\}$. Then Z is closed, G-invariant and $\mathrm{Q} = \mathrm{Z} \times \mathrm{Z} \cap \mathrm{G}$. In the language of Proposition 3.1 we have that $\mathrm{I} = \mathrm{I}_{\mathrm{Z}}$. We will show that either Z is empty, which implies that $\mathrm{I} = C^*_{\mathrm{r}}(\mathrm{G})$, or Z is dense in G, which implies that $\mathrm{I} = \{0\}$.

Suppose Z is not empty. Let $\mathrm{U} \neq \emptyset$ be an open subset of $\mathbb{R}^d$. Let $z \in \mathrm{Z}$. By Lemma 6.2, there exists $w \in [z]$ such that $w \in \mathrm{U}$. From G-invariance, $w \in \mathrm{Z}$ and hence Z is dense in $\mathbb{R}^d$ as desired. ∎

It is now natural to wonder about the K-theory of these C^*-algebras. We can use the ideal structure of $\mathrm{C}^*(\mathrm{G}_k)$, given in Section 3, to compute the K-theory groups of the step C^*-algebras $\mathrm{C}^*(\mathrm{G}_k)$ and we can use the inductive limit construction of $C^*_{\mathrm{r}}(\bigcup \mathrm{G}_k)$, given in Section 5, to compute its K-theory groups. A detailed approach to the K-theory of these C^*-algebras, including the computation of K-theory for a number of examples, will be done in a follow up paper, see [2].

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DANIEL GONÇALVES, DEPARTAMENTO DE MATEMATICA, UNIVERSIDADE FEDERAL DE SANTA CATARINA, FLORIAN OLIS, 88.040-900, BRASIL
E-mail address: daemig@gmail.com

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