J. OPERATOR THEORY
58:2(2007), 387–414

DICHOTOMY AND FREDHOLM PROPERTIES OF EVOLUTION EQUATIONS

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Communicated by Nikolai K. Nikolski

ABSTRACT. Under minimal assumptions, we characterize the Fredholm property and compute the Fredholm index of abstract differential operators $-\frac{\mathrm{d}}{\mathrm{d}t}$ $+A(\cdot)$ acting on spaces of functions $f : \mathbb{R} \to X$. Here $A(t)$ are (in general) unbounded operators on the Banach space X and our results are formulated in terms of exponential dichotomies on two halflines for the propagator solving the evolution equation $\dot{u}(t) = A(t)u(t)$ in a mild sense.

KEYWORDS: *Fredholm operator and index, exponential dichotomy, node operator, evolution family, evolution equation, weighted shift operator, input-output method.*

MSC (2000): Primary 47A53, 47D06; Secondary 34G10, 35P05.

1. INTRODUCTION

In this paper we obtain the final version of the infinite dimensional Dichotomy Theorem for well-posed differential equations

$$(Gu)(t) := -u'(t) + A(t)u(t) = f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

on a Banach space X. Our main Dichotomy Theorem 1.1 characterizes the Fredholm property of the (closure of the) operator G on, say, $L^p(\mathbb{R}, X)$ and determines its Fredholm index in terms of the exponential dichotomies on half lines of the propagator solving (1.1). The linear operators $A(t)$, $t \in \mathbb{R}$, on X are unbounded, in general, and we only require that the corresponding initial value problem (1.3) below is well-posed in a mild sense. We reduce the problem to the study of a weighted shift operator on X-valued sequence spaces, and give a purely operator theoretical proof of our Theorem 1.1 based on the discrete version of the "input-output" method from the theory of differential equations.

The Dichotomy Theorem is related to problems arising from finite dimensional dynamics, Morse theory, and the theory of traveling waves. For a detailed discussion concerning these connections, we refer to Section 7 in [13]. This theorem can further be viewed as an extension of a simple form of the celebrated Atiyah-Patodi-Singer Index Theorem, cf. [22].

For finite dimensional $X = \mathbb{C}^d$, versions of the Dichotomy Theorem were established in the papers [6], [17], [18], and [23]. Here $A(t)$ are matrices and $G = -\frac{\mathrm{d}}{\mathrm{d}t} + A(\cdot)$ is defined on the Sobolev space $W^{1,p}(\mathbb{R}, \mathbb{C}^d)$, for instance. In this case G is Fredholm if and only if the propagator (or evolution family) $\{U(t,\tau)\}_{t\geqslant\tau}$ solving (1.1) has exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$. However, applications to partial differential equations require an infinite dimensional version of the Dichotomy Theorem for unbounded $A(t)$. Progresses in this direction have been made in [2], [3], [4], [5], [9], [10], [12], [13], [19], [20], [21], [24], and the references therein. We stress that the proofs of the finite and infinite dimensional versions of the Dichotomy Theorem are quite different due to many new difficulties arising in the infinite dimensional setting, as described in Sections 1 and 7 of [13].

Recently, several authors discussed the Fredholm property of the operator G and related questions (such as perturbation results) in specific infinite dimensional settings. In [20] and [21] a differential equation of the form (1.1) on a Banach space X having the UMD property was studied, where the constant domain of the operators $A(t)$ is compactly embedded in X and $A(t) \to A_\pm$ as $t \to \pm\infty$. Assuming that the spectra of $A_\pm$ do not intersect $\mathrm{i}\mathbb{R}$, it was proved that G is Fredholm on $L^p(\mathbb{R}, X)$ for $p \in (1, \infty)$, and its index was computed in terms of the spectral flow of $A(\cdot)$. (Here the Cauchy problem (1.3) could be ill-posed.) In [9] and [10] theorems of this type are established for general (well-posed) parabolic problems. The latter approach is based on a detailed study of the maximal regularity property of the solutions to the (inhomogeneous) differential equation. The case of bounded operators $A(t)$ was considered in [1] in connection with applications to infinite dimensional Morse theory. In [19] and [24] necessary and sufficient conditions for the Fredholm property of G were given for a special class of infinite dimensional differential equations having a backward uniqueness property, cf. (BU) below. This work is related to a detailed study of traveling waves for elliptic problems on cylinders. All these papers dealt with the asymptotically autonomous case (except for [19]) and imposed restrictive regularity hypotheses ensuring the closedness of $G = -\frac{\mathrm{d}}{\mathrm{d}t} + A(\cdot)$ defined on $\mathrm{dom}(\frac{\mathrm{d}}{\mathrm{d}t}) \cap \mathrm{dom}(A(\cdot))$. See [9], [10], [13] for more details.

In a different line of research, one starts with a strongly continuous evolution family $\{U(t,\tau)\}_{t\geqslant\tau}$, and constructs an operator $\mathbf{G}$ on, say, $L^p(\mathbb{R}, X)$ as described below. There are no additional restrictions on the regularity or the asymptotic behaviour of $A(\cdot)$. If (1.3) is well-posed in a classical sense, then $\mathbf{G}$ is the closure of $G = -\frac{\mathrm{d}}{\mathrm{d}t} + A(\cdot)$. In [5] (see also [2], [3], [4]) it was further assumed *a priori* that $\{U(t,\tau)\}_{t\geqslant\tau}$ has exponential dichotomies on semi-lines. Then a "node operator" was introduced, and it was proved that G and the node operator are Fredholm at the same time with equal indices. On the other hand, the authors in [13] required X to be reflexive and imposed a condition of backward uniqueness on the evolution family. Under these hypotheses, they could characterize the Fredholm property of $\mathbf{G}$ as we do below. In the current paper we discard any additional assumption and establish the following theorem (the relevant definitions are given in Section 2).

THEOREM 1.1. *Assume that $\mathcal{U} = \{U(t,\tau) : t \geqslant \tau; t,\tau \in \mathbb{R}\}$ is a strongly continuous, exponentially bounded evolution family on a Banach space X, and let $\mathbf{G}$ be the generator of the associated evolution semigroup defined on $\mathcal{E}(\mathbb{R}) = L^p(\mathbb{R}, X)$, $p \in [1,\infty)$, or on $\mathcal{E}(\mathbb{R}) = \mathcal{C}_0(\mathbb{R}, X)$. Then the operator $\mathbf{G}$ is Fredholm if and only if there exist real numbers $a \leqslant b$ such that the following two conditions hold:*

(i) *The evolution family $\mathcal{U}$ has exponential dichotomies with the family of projections $\{P_t^-\}_{t\leqslant a}$ and $\{P_t^+\}_{t\geqslant b}$ on $(-\infty, a]$ and $[b,\infty)$, respectively.*

(ii) *The node operator $N(b,a)$, acting from* $\ker P_a^-$ *to* $\ker P_b^+$ *and defined by the rule $N(b,a) = (I - P_b^+)U(b,a)|_{\ker P_a^-}$, is Fredholm.*

Moreover, if $\mathbf{G}$ is Fredholm, then we have the equalities $\dim\ker \mathbf{G} = \dim\ker N(b,a)$, $\operatorname{codim}\operatorname{im}\mathbf{G} = \operatorname{codim}\operatorname{im} N(b,a)$, *and* $\operatorname{ind}\mathbf{G} = \operatorname{ind} N(b,a)$. *In particular, the Fredholm properties of $\mathbf{G}$ are independent of the choice of the function space $\mathcal{E}(\mathbb{R})$.*

In Proposition 6.1 we further give a description of the range of $\mathbf{G}$ in the spirit of the classical Fredholm alternative using the adjoint evolution family.

The evolution semigroup $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ mentioned in Theorem 1.1 is defined on $L^p(\mathbb{R}, X)$, $p \in [1,\infty)$, or on $\mathcal{C}_0(\mathbb{R}, X)$ by the formula $(T(t)f)(\tau) = U(\tau, \tau - t)f(\tau - t)$, $\tau \in \mathbb{R}$, $t \geqslant 0$, see [2], [7], [25]. This is a strongly continuous semigroup, and we denote its generator by $\mathbf{G}$. The operator $\mathbf{G}$ can be described in terms of mild solutions to an inhomogenous evolution equation, as shown by the following lemma, see Proposition 4.32 of [7].

LEMMA 1.2. *A function u belongs to the domain* $\operatorname{dom}\mathbf{G}$ *of the operator* $\mathbf{G}$ *on $L^p(\mathbb{R}, X)$, $p \in [1,\infty)$, respectively, on $\mathcal{C}_0(\mathbb{R}, X)$, if and only if $u \in L^p(\mathbb{R}, X) \cap \mathcal{C}_0(\mathbb{R}, X)$, respectively, $u \in \mathcal{C}_0(\mathbb{R}, X)$, and there exists an $f \in L^p(\mathbb{R}, X)$, respectively, $f \in \mathcal{C}_0(\mathbb{R}, X)$, with*

$$u(t) = U(t,\tau)u(\tau) - \int_\tau^t U(t,\sigma)f(\sigma)\mathrm{d}\sigma \quad \textit{for all } t \geqslant \tau \textit{ in } \mathbb{R}. \tag{1.2}$$

If (1.2) *holds, then* $\mathbf{G}u = f$.

Suppose for a moment that the differential equation

$$u'(t) = A(t)u(t), \quad t \geqslant \tau, \quad u(\tau) = x \in \operatorname{dom}(A(\tau)), \tag{1.3}$$

is well-posed in a classical sense, i.e., the operators $A(t)$ are all densely defined and there is an evolution family $\mathcal{U}$ such that $U(t,\tau)\operatorname{dom}(A(\tau)) \subseteq \operatorname{dom}(A(t))$ for $t \geqslant \tau$ and $u(t) = U(t,\tau)x$ is the unique C^1-solution of (1.3). Then $\mathbf{G}$ is the closure of the operator $G = -\frac{\mathrm{d}}{\mathrm{d}t} + A(\cdot)$ on $L_p(\mathbb{R}, X)$, $p \in [1,\infty)$, respectively, on $\mathcal{C}_0(\mathbb{R}, X)$, with the domain $\operatorname{dom} G = \{u \in W^{1,p}(\mathbb{R}, X) : u(t) \in \operatorname{dom} A(t)$ a.e., $A(\cdot)u(\cdot) \in L^p(\mathbb{R}, X)\}$, respectively $\{u \in \mathcal{C}_0(\mathbb{R}, X) : u(t) \in \operatorname{dom} A(t)$ for $t \in \mathbb{R}$, $u'(\cdot), A(\cdot)u(\cdot) \in \mathcal{C}_0(\mathbb{R}, X)\}$, where $W^{1,p}(\mathbb{R}, X)$, $p \in [1, \infty)$, is the usual Sobolev space, cf. Theorem 3.12 of [7]. However, one knows only rather restrictive assumptions on the operators $A(t)$ implying well-posedness in the above sense, and almost no necessary conditions, see the survey given in [25]. Thus we only assume that the evolution family $\mathcal{U}$ exists, without any reference to operators $A(t)$.

Our Theorem 1.1 was shown in Theorem 1.1 of [13] assuming in addition that X is reflexive and $\mathcal{U}$ has the following backward uniqueness property (BU):

(BU.1): If $u \in \mathcal{C}_0(\mathbb{R}, X)$, $u(t) = U(t, \tau)u(\tau)$ for all $t \geqslant \tau$ in $\mathbb{R}$, and $u(\tau) = 0$ for some $\tau \in \mathbb{R}$, then $u = 0$.

(BU.2): If $v \in \mathcal{C}_{\mathrm{b}}^{w,*}(\mathbb{R}, X^*)$, $v(\tau) = U(t, \tau)^* v(t)$ for all $t \geqslant \tau$ in $\mathbb{R}$, and $v(\tau) = 0$ for some $\tau \in \mathbb{R}$, then $v = 0$.

(See also Remark 7.4 saying that for our purposes $C_{\mathrm{b}}^{w,*}(\mathbb{R}, X^*)$ can be replaced by $C_0^{w,*}(\mathbb{R}, X^*)$ in (BU.2)). We point out that these properties do not hold for certain evolution families solving parabolic partial differential equations. Some sufficient conditions for (BU) are known for specific classes of partial differential equations. However, in general it is rather difficult to verify (BU), cf. [9] and references therein. In Section 7 we present two examples, where $\mathbf{G}$ is Fredholm but (BU) fails.

Our proof also shows that if $\mathcal{U}$ does satisfy the backward uniqueness property (BU), then we can take $a = b = 0$ in our Theorem 1.1, see Proposition 7.1. Using a different method, this result was proved in Theorem 1.2 of [13] for reflexive X. As shown in Example 7.3, the conclusion of Theorem 1.1 with $a = b = 0$ is false in general if (BU) is violated.

The proof of the (simpler) "if" part of Theorem 1.1 given in [13] or [5] works without the reflexivity assumption and without the backward uniqueness property. The main objective of the current paper is to remove these additional conditions in the proof of the "only if" part. Without these hypotheses the problem at hand becomes significantly more involved, and thus the methods used in the current paper are quite different from those in [13]. We use an approach going back to Daletskii and Krein, [8], and Levitan and Zhikov, [14], which is sometimes called the "input-output method."

In [8] this technique was used to characterize the exponential stability of an evolution family $\mathcal{U}$. The basic idea is to solve the equation $\mathbf{G}u = f$ on $\mathbb{R}_+$ for functions of the form $f(t) = \varphi'(t)U(t, s)x$ (where φ is a suitable scalar function). For such f it can be seen that $u(t) = -\varphi(t)U(t, s)x$ using a version of Lemma 1.2. If $\mathbf{G}$ is invertible on $\mathbb{R}_+$, one can then deduce the required exponential estimate by means of the boundedness of $\mathbf{G}^{-1}$. A variant of this argument shows that the stable and unstable subspaces of $\mathcal{U}$ yield a time depending decomposition of X if $\mathbf{G}$ is invertible on $\mathbb{R}$, leading to a characterization of exponential dichotomy on $\mathbb{R}$ given in [14]. In the more recent contributions [15] and [16], this approach was employed to characterize exponential dichotomy on $\mathbb{R}_+$. Here additional difficulties appear at the initial time $t = 0$ which correspond to the fact that the dichotomy projections are not unique in the half line case, in general. We point out that the input-output method is quite different from the approach used in [2], [3], [4], and [7] (and its modifications in [5] and [13]), where the main tool for the construction of the exponential dichotomy, say, on $\mathbb{R}$ was the Riesz projection of the semigroup generated by $\mathbf{G}$.

In the present paper we deal with operators $\mathbf{G}$ being Fredholm. This fact forces us to "delete" the kernel and co-kernel of $\mathbf{G}$. Moreover, we can only expect to obtain exponential dichotomies of $\mathcal{U}$ on (possibly disjoint) semi-lines $(-\infty, a]$ and $[b, \infty)$, see Example 7.3. Thus we must control the behaviour of $U(t,s)$ at a, b, and in between. In order to achieve this, we first discretize the problem (see Section 2). In Section 3, we then treat the stable subspaces on $\mathbb{Z}_+$ and the unstable subspaces on $\mathbb{Z}_-$. These spaces are somewhat easier to handle since they are given explicitly in terms of $\mathcal{U}$, see (3.1) and (3.2). The main difficulty is the construction of the correct complements of these spaces. Here we need several decompositions of X given in Lemma 3.6. In Sections 4 and 5 we construct the dichotomies on $[b, \infty)$ and $(-\infty, a]$ by propagating the "traces" of the kernel and co-kernel of $\mathbf{G}$ at the points b and a (Lemmas 4.2 and 5.2). In Section 6 we deal with the node operator to show condition (ii) in Theorem 1.1, and the formulas for the defect numbers. In Section 7 we describe the backward uniqueness properties in terms of the traces of the kernel and co-kernel of $\mathbf{G}$, and show that one can take $a = b = 0$ in Theorem 1.1 when the backward uniqueness properties hold, see Proposition 7.1.

2. NOTATION, DEFINITIONS, AND PRELIMINARY RESULTS

We set $\mathbb{R}_+ = \{t \in \mathbb{R} : t \geqslant 0\}$, $\mathbb{R}_- = \{t \in \mathbb{R} : t \leqslant 0\}$, $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geqslant 0\}$, $\mathbb{Z}_- = \{n \in \mathbb{Z} : n \leqslant 0\}$, and we use t, τ, σ to denote real numbers and n, m, k, j to denote integers. We write c for a generic (positive) constant. A^*, $\operatorname{dom}(A)$, $\ker A$, $\operatorname{im} A$ are the adjoint, the domain, the kernel and the range of an operator A on a Banach space X with dual space X^*, and $A_{|Y}$ is the restriction of A on the subspace Y of X. The set of all bounded linear operators from a Banach space X to a Banach space Y is designated by $\mathcal{B}(X,Y)$, and $\mathcal{B}(X,X) =: \mathcal{B}(X)$. For a subspace $Y_* \subseteq X^*$, we use the (non-standard !) notation $Y_*^{\perp} = \{x \in X : \langle x, \xi \rangle = 0 \text{ for all } \xi \in Y_*\}$ for the preannihilator, where $\langle \cdot, \cdot \rangle$ is the (X, X^*)-pairing. If P and Q are two projections on X, then $X = \operatorname{im} P \oplus \ker P = \operatorname{im} Q \oplus \ker Q$, where throughout "$\oplus$" denotes a decomposition of a Banach space into closed subspaces with trivial intersection. With respect to these decompositions, each $A \in \mathcal{B}(X)$ can be written as the 2×2 operator matrix

$$A = \begin{bmatrix} PAQ & PA(I-Q) \\ (I-P)AQ & (I-P)A(I-Q) \end{bmatrix}.$$

$\mathcal{C}_0(\mathbb{R}, X)$ is the space of continuous functions $f : \mathbb{R} \to X$ vanishing at $\pm\infty$; $\mathcal{C}_b^{w,*}(\mathbb{R}, X^*)$ is the space of bounded, weak star continuous functions $f : \mathbb{R} \to X^*$; $L_p(\mathbb{R}, X)$ is the space of (equivalence classes of) Bochner p-integrable functions $f : \mathbb{R} \to X$, where $p \in [1, \infty)$. We denote by χ_M the characteristic function of a set M. If $(\varphi_k)_{k\in\mathbb{Z}}$ is a numerical sequence and $x \in X$, then $\varphi \otimes x$ denotes the X-valued sequence $(\varphi_k x)_{k\in\mathbb{Z}}$.

An *evolution family* $\mathcal{U} = \{U(t,\tau)\}_{t\geqslant\tau}$ on a set $J \subseteq \mathbb{R}$ is a family of operators $U(t,\tau) \in \mathcal{B}(X)$, $t \geqslant \tau$, $t, \tau \in J$, satisfying

$$U(t,t) = I \quad \text{(the identity operator on} X);$$
$$U(t,\tau)U(\tau,\sigma) = U(t,\sigma) \quad \text{for all } t \geqslant \tau \geqslant \sigma \text{ with } t,\tau,\sigma \in J.$$

It is called *strongly continuous* if the map $(t,\tau) \mapsto U(t,\tau)x$ is continuous for all $x \in X$ and $t \geqslant \tau$ in J. If $\|U(t,\tau)\| \leqslant M e^{\omega(t-\tau)}$ for some constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ and all $t \geqslant \tau$ in J, then $\mathcal{U}$ is *exponentially bounded*.

DEFINITION ED. An evolution family $\mathcal{U}$ has an *exponential dichotomy* on $J \subseteq \mathbb{R}$ if there exist closed subspaces $\{X_s(t)\}_{t\in J}$ and $\{X_u(t)\}_{t\in J}$ of X such that:

($\mathrm{i_J}$) $X = X_s(t) \oplus X_u(t)$ for all $t \in J$ and $U(t,\tau)X_s(\tau) \subseteq X_s(t)$, $U(t,\tau)X_u(\tau) \subseteq X_u(t)$ for all $t \geqslant \tau$ in J;

($\mathrm{ii_J}$) $U(t,\tau)_{|X_u(\tau)}$ is an invertible from $X_u(\tau)$ to $X_u(t)$ for all $t \geqslant \tau$ in J;

($\mathrm{iii_J}$) there are constants $N, \nu > 0$ such that

$$\|U(t,\tau)_{|X_s(\tau)}\| \leqslant N e^{-\nu(t-\tau)}, \quad \|(U(t,\tau)_{|X_u(\tau)})^{-1}\| \leqslant N e^{-\nu(t-\tau)} \quad \text{for all } t \geqslant \tau \text{ in } J.$$

We denote by P_t the projection onto $X_s(t)$ parallel to $X_u(t)$. If $J = [b, \infty)$ or $J = \mathbb{Z} \cap [b, \infty)$ we write $X_{s,u}^+(t)$ and P_t^+ for the respective dichotomy subspaces and the dichotomy projections, and if $J = (-\infty, a]$ or $J = \mathbb{Z} \cap (-\infty, a]$ we write $X_{s,u}^-(t)$ and P_t^- for the respective dichotomy subspaces and the dichotomy projections. If $\mathcal{U}$ is strongly continuous and exponentially bounded on an unbounded interval J and ($\mathrm{i_J}$) $-$ ($\mathrm{iii_J}$) hold, then the function $t \mapsto P_t$ is strongly continuous and uniformly bounded on J, see Lemma IV.1.1, IV.3.2 of [8] or Lemma 4.2 of [16].

In order to prove Theorem 1.1, we pass from continuous time to discrete time; i.e., we replace the operator $\mathbf{G}$ in the statement of Theorem 1.1 by the difference operator D defined by the formula

$$D(x_n)_{n\in\mathbb{Z}} = (x_n - U(n, n-1)x_{n-1})_{n\in\mathbb{Z}}, \tag{2.1}$$

cf. [3], [7], [11]. The operator D is acting on the sequence space $\mathcal{E}(\mathbb{Z})$, where $\mathcal{E}(\mathbb{Z}) = \ell^p(\mathbb{Z}, X)$ if $\mathcal{E}(\mathbb{R}) = L^p(\mathbb{R}, X)$, $p \in [1, \infty)$ and $\mathcal{E}(\mathbb{Z}) = c_0(\mathbb{Z}, X)$ if $\mathcal{E}(\mathbb{R}) = \mathcal{C}_0(\mathbb{R}, X)$. This replacement is possible due to Theorem 1.4 and Lemma 1.5 of [13] (cf. also Theorem 7.6.5 of [11], Theorem 1 of [3], and Theorem 2 of [4]). These results say that $\mathcal{U}$ has an exponential dichotomy on $\mathbb{R}_\pm$ if it has an exponential dichotomy on $\mathbb{Z}_\pm$ and that $\operatorname{im} \mathbf{G}$ is closed if and only if $\operatorname{im} D$ is closed, $\dim\ker \mathbf{G} = \dim\ker D$, and $\operatorname{codim}\operatorname{im} \mathbf{G} = \operatorname{codim}\operatorname{im} D$. In particular, the operator $\mathbf{G}$ is Fredholm if and only if D is Fredholm, and $\operatorname{ind} \mathbf{G} = \operatorname{ind} D$. Since we focus our attention on the proof of the "only if" part of Theorem 1.1, throughout Sections 2–5 we will assume that D is a Fredholm operator.

In the following we collect some basic properties of the spaces

$$X_n = \{x \in X : \exists\ (x_k)_{k\in\mathbb{Z}} \in \ker D \text{ so that } x = x_n\} \quad \text{and} \tag{2.2}$$

$$X_{n,*} = \{\xi \in X^* : \exists\ (\xi_k)_{k\in\mathbb{Z}} \in \ker D^* \text{ so that } \xi = \xi_n\}, \tag{2.3}$$

where $n \in \mathbb{Z}$. Simple computations show that

$$D^*(\xi_n)_{n\in\mathbb{Z}} = (\xi_n - U(n+1,n)^*\xi_{n+1})_{n\in\mathbb{Z}},$$

$$\ker D = \{(x_n)_{n\in\mathbb{Z}} \in \mathcal{E}(\mathbb{Z}) : x_n = U(n,m)x_m \text{ for all } n \geqslant m\} \tag{2.4}$$

$$\ker D^* = \{(\xi_n)_{n\in\mathbb{Z}} \in \mathcal{E}(\mathbb{Z})^* : \xi_m = U(n,m)^*\xi_n \text{ for all } n \geqslant m\}. \tag{2.5}$$

These formulas imply that $U(n,m)X_m = X_n$ and $U(n,m)^*X_{n,*} = X_{m,*}$ for all $n \geqslant m$. Because of these identities and the Fredholm property of D, we obtain $0 \leqslant \dim X_{n+1} \leqslant \dim X_n \leqslant \dim\ker D < \infty$ and $0 \leqslant \dim X_{n,*} \leqslant \dim X_{n+1,*} \leqslant \dim\ker D^* < \infty$ for all $n \in \mathbb{Z}$. Hence, there are $a, b \in \mathbb{Z}$ with $a \leqslant b$ such that $\dim X_n$ and $\dim X_{n,*}$ are constant for $n \leqslant a$ and $n \geqslant b$.

Without loss of generality, we may assume that $a = 0$ and $b \geqslant 1$ due to the following translation argument: For $a \in \mathbb{Z}$, consider the strongly continuous evolution family $\mathcal{U}_a$ defined by $U_a(t,\tau) = U(t+a, \tau+a)$ for $t \geqslant \tau$ in $\mathbb{R}$, and the shift operator S_a on $\mathcal{E}(\mathbb{Z})$ acting by $S_a(x_n)_{n\in\mathbb{Z}} = (x_{n+a})_{n\in\mathbb{Z}}$. If D_a is the difference operator associated to $\mathcal{U}_a$ as in (2.1), then $D_a = S_a D S_a^{-1}$, and thus D_a and D have the same Fredholm properties. So, choosing an appropriate a, we have that $\dim X_n(\mathcal{U}_a)$ and $\dim X_{n,*}(\mathcal{U}_a)$ are constant for $n \leqslant 0$. To sum things up, we impose the following assumption, without loss of generality.

HYPOTHESIS 1. $\mathcal{U}$ is a discrete, exponentially bounded evolution family on $\mathbb{Z}$, D is a Fredholm operator, and $\dim X_n$ and $\dim X_{n,*}$ are constant for $n \geqslant b$ and $n \leqslant 0$, for some $1 \leqslant b \in \mathbb{Z}$.

LEMMA 2.1. *Let Hypothesis* 1 *be satisfied. Then* $\dim X_n \leqslant \dim\ker D < \infty$, $\dim X_{n,*} \leqslant \dim\ker D^* < \infty$ *for* $n \in \mathbb{Z}$, *and the following assertions hold:*

(i) $U(n,m)X_m = X_n$ *for all* $n \geqslant m$.

(ii) $U(n,m)^*X_{n,*} = X_{m,*}$ *for all* $n \geqslant m$.

(iii) $U(n,m)_{|X_m} : X_m \to X_n$ *is invertible if* $m \leqslant n \leqslant 0$ *or* $n \geqslant m \geqslant b$.

(iv) $U(n,m)^*_{|X_{n,*}} : X_{n,*} \to X_{m,*}$ *is invertible if* $m \leqslant n \leqslant 0$ *or* $n \geqslant m \geqslant b$.

(v) $X_n \subseteq X_{n,*}^{\perp}$ *for all* $n \in \mathbb{Z}$.

(vi) $x \in X_{m,*}^{\perp}$ *if and only if* $U(n,m)x \in X_{n,*}^{\perp}$, *where* $n \geqslant m$ *in* $\mathbb{Z}$.

Proof. We already observed after (2.4) and (2.5) that the first assertion and statements (i) and (ii) hold. Assertions (iii) and (iv) follow from these assertions and Hypothesis 1. In order to show (v), take $\mathbf{x} = (x_k)_{k\in\mathbb{Z}} \in \ker D$, $\boldsymbol{\xi} = (\xi_k)_{k\in\mathbb{Z}} \in \ker D^*$, and $n \in \mathbb{Z}$. Then (2.5) and (2.4) imply that

$$\langle x_n, \xi_n\rangle = \langle x_n, U(k,n)^*\xi_k\rangle = \langle U(k,n)x_n, \xi_k\rangle = \langle x_k, \xi_k\rangle$$

for all $k \geqslant n$. Letting $k \to \infty$, we deduce $\langle x_n, \xi_n \rangle = 0$ since $x \in c_0(\mathbb{Z}, X)$ and $\boldsymbol{\xi}$ is bounded. Thus assertion (v) holds. The last assertion follows from the identities

$$\langle x, \xi_m \rangle = \langle x, U(n,m)^* \xi_n \rangle = \langle U(n,m)x, \xi_n \rangle$$

for all $n \geqslant m$ and all $\boldsymbol{\xi} = (\xi_n)_{n\in\mathbb{Z}} \in \ker D^*$. ∎

Since $X_0 \subseteq X_{0,*}^{\perp}$ and $\dim X_0 < \infty$, we can choose a closed subspace X_0' of X with

$$X_{0,*}^{\perp} = X_0 \oplus X_0'. \tag{2.6}$$

Moreover, we define the following closed subspaces of $\mathcal{E}(\mathbb{Z})$ and $\mathcal{E}(\mathbb{Z})^*$:

$$\mathcal{F} = \{\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in \mathcal{E}(\mathbb{Z}) : x_n \in X_{n,*}^{\perp} \text{ for all } n \in \mathbb{Z}\}, \tag{2.7}$$

$$\mathcal{F}_0 = \{\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in \mathcal{F} : x_0 \in X_0'\}, \tag{2.8}$$

$$\mathcal{F}_{b,*} = \{\boldsymbol{\xi} = (\xi_n)_{n\in\mathbb{Z}} \in \mathcal{E}(\mathbb{Z})^* : \xi_n \in X_{n,*} \text{ for all } n \in \mathbb{Z},\ \xi_b = 0\}. \tag{2.9}$$

On these spaces the operators $D_0 := D_{|\mathcal{F}_0}$ and $D_{b,*} := D^*_{|\mathcal{F}_{b,*}}$ have better properties than D and D^*, respectively, as stated in the next lemma.

LEMMA 2.2. *Let Hypothesis* 1 *be satisfied. Then the following assertions hold:*

(i) *$\mathcal{F}$ is D-invariant and $D_{|\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ is surjective.*

(ii) *The operator $D_0 = D_{|\mathcal{F}_0} : \mathcal{F}_0 \to \mathcal{F}$ is invertible;*

(iii) *$D_{b,*} = D^*_{|\mathcal{F}_{b,*}}$ is uniformly injective, that is, $\|D_{b,*}\boldsymbol{\xi}\|_{(\mathcal{E}(\mathbb{Z}))^*} \geqslant c\|\xi\|_{(\mathcal{E}(\mathbb{Z}))^*}$ for all $\boldsymbol{\xi} \in \mathcal{F}_{b,*}$ and a constant $c > 0$.*

Proof. Assertions (i) and (ii) can be shown exactly as Lemma 2.2 of [13] and Lemma 2.3 of [13], respectively. To prove (iii), we have to verify that $D_{b,*} : \mathcal{F}_{b,*} \to \mathcal{E}(\mathbb{Z})^*$ is injective and has closed range. If $\boldsymbol{\xi} = (\xi_n)_{n\in\mathbb{Z}} \in \ker D_{b,*}$ then $\xi_n = U(b,n)^*\xi_b = 0$ for $n \leqslant b$ and $U(n,b)^*\xi_n = \xi_b = 0$ for $n \geqslant b$ by (2.5). Furthermore, Lemma 2.1(iv) implies that $\xi_n = 0$ for $n \geqslant b$, proving that $D_{b,*}$ is injective. Next, take $\boldsymbol{\eta} = \lim_{n\to\infty} D_{b,*}\boldsymbol{\xi}_n$ with $\boldsymbol{\xi}_n \in \mathcal{F}_{b,*}$. Since D^* is Fredholm, $\operatorname{im} D^*$ is closed and thus there is $\boldsymbol{\zeta} \in \mathcal{E}(\mathbb{Z})^*$ with $\boldsymbol{\eta} = D^*\boldsymbol{\zeta}$. Moreover, there exist an operator $D^\dagger \in \mathcal{B}(\mathcal{E}(\mathbb{Z})^*)$ and a finite rank operator R such that $D^\dagger D^* = I + R$ and $\operatorname{im} R \subseteq \ker D^*$. Observe that $D^*(\boldsymbol{\zeta} - \boldsymbol{\xi}_n) \to 0$ as $n \to \infty$. Then it follows that $\boldsymbol{\zeta} - \boldsymbol{\xi}_n + \mathbf{w}_n \to 0$ as $n \to \infty$ for $\mathbf{w}_n = R(\boldsymbol{\zeta} - \boldsymbol{\xi}_n) \in \ker D^*$. Passing to the elements of the sequences, we deduce that $\zeta_k = \lim_{n\to\infty} (\xi_{k,n} - w_{k,n}) \in X_{k,*}$ for each $k \in \mathbb{Z}$, where $\boldsymbol{\zeta} = (\zeta_k)_{k\in\mathbb{Z}}$, $\boldsymbol{\xi}_n = (\xi_{k,n})_{k\in\mathbb{Z}}$ and $\mathbf{w}_n = (w_{k,n})_{k\in\mathbb{Z}}$. There is a vector $\boldsymbol{\theta} = (\theta_k)_{k\in\mathbb{Z}} \in \ker D^*$ with $\zeta_b = \theta_b$ by (2.3). Hence, $\boldsymbol{\zeta} - \boldsymbol{\theta} \in \mathcal{F}_{b,*}$ by (2.9) and $\boldsymbol{\eta} = D^*(\boldsymbol{\zeta} - \boldsymbol{\theta}) = D_{b,*}(\boldsymbol{\zeta} - \boldsymbol{\theta})$. So the range of $D_{b,*}$ is closed. ∎

We will need the next elementary lemma which is probably well-known.

LEMMA 2.3. *Let V be a subspace of X, $\{\xi_1, \dots, \xi_d\}$ be a set of linearly independent vectors in X^*, and $Y_* = \operatorname{Span}\{\xi_1, \dots, \xi_d\}$. Then the following assertions hold:*

(i) *There are $x_1, \dots, x_d \in X$ such that $\langle x_i, \xi_j \rangle = \delta_{ij}$ for all $i, j \in \{1, \dots, d\}$, where δ_{ij} is the Kronecker Delta.*

(ii) *Let* $v_1, \dots, v_d \in V$ *satisfy* $\langle v_i, \xi_j \rangle = \delta_{ij}$ *for all* $i, j \in \{1, \dots, d\}$ *and set* $W = \operatorname{Span}\{v_1, \dots, v_d\}$. *Then* $V = (V \cap Y_*^{\perp}) \oplus W$.

(iii) $\operatorname{codim} Y_*^{\perp} = d < \infty$.

Proof. (i) It is clear that assertion (i) holds if $d = 1$. Assume that it is true for some $d \in \mathbb{N}$ and let $\{\xi_1, \dots, \xi_d, \xi_{d+1}\}$ be a system of linearly independent vectors. We want to prove by contradiction that

$$\bigcap_{i=1}^{d} \ker \xi_i \not\subseteq \ker \xi_{d+1}. \tag{2.10}$$

Take $x \in X$ and let $\{x_1, \dots, x_d\}$ satisfy the induction hypothesis. If (2.10) were false, then we would obtain

$$x - \sum_{j=1}^{d} \langle x, \xi_j \rangle x_j \in \bigcap_{i=1}^{d} \ker \xi_i \subseteq \ker \xi_{d+1}, \quad \text{i.e.,} \ \xi_{d+1} = \sum_{j=1}^{d} \langle x_j, \xi_{d+1} \rangle \xi_j.$$

This is a contradiction, and so (2.10) is true. Thus there exists $x_{d+1} \in \bigcap_{i=1}^{d} \ker \xi_i$ with $\langle x_{d+1}, \xi_{d+1} \rangle = 1$, concluding the proof of (i).

(ii) Let $x \in V$ and set $y = x - \sum_{j=1}^{d} \langle x, \xi_j \rangle v_j \in V$. Then

$$\langle y, \xi_i \rangle = \langle x, \xi_i \rangle - \sum_{j=1}^{d} \langle x, \xi_j \rangle \delta_{ji} = 0$$

for all $i \in \{1, \dots, d\}$. As a consequence, $y \in V \cap Y_*^{\perp}$ and so $x \in (V \cap Y_*^{\perp}) + W$. We have shown that $V \subseteq (V \cap Y_*^{\perp}) + W$. The converse inclusion follows directly from $W \subseteq V$. If $x \in (V \cap Y_*^{\perp}) \cap W$, then there are $\lambda_1, \dots, \lambda_d \in \mathbb{C}$ such that $x = \sum_{j=1}^{d} \lambda_j v_j$. Therefore

$$\lambda_i = \sum_{j=1}^{d} \lambda_j \delta_{ji} = \sum_{j=1}^{d} \langle \lambda_j v_j, \xi_i \rangle = \langle x, \xi_i \rangle = 0$$

for all $i \in \{1, \dots, d\}$, and hence $(V \cap Y_*^{\perp}) \cap W = \{0\}$. Thus (ii) holds.

(iii) The third assertion follows from (i) and (ii). ∎

LEMMA 2.4. *Let* $(a_n)_{n \in \mathbb{Z}_+}$ *be a sequence of positive numbers and* $(b_n)_{n \in \mathbb{Z}_+} \in c_0(\mathbb{Z}_+, \mathbb{R}_+)$ *such that* $a_{m+n} \leqslant b_n a_m$, *for all* $n, m \in \mathbb{Z}_+$. *Then there are* $N, \nu > 0$, *depending only on* $(b_n)_{n \in \mathbb{Z}_+}$ *such that* $a_{n+m} \leqslant N e^{-\nu n} a_m$ *for all* $n, m \in \mathbb{Z}_+$.

Proof. Take $n_0 \in \mathbb{Z}_+$ such that $b_{n_0} < e^{-1}$. We set $N = e(\max\{b_0, \dots, b_{n_0}\} + 1)$, $\nu = \frac{1}{n_0}$, and $p = [\frac{n}{n_0}]$ for $n, m \in \mathbb{Z}_+$. Then we obtain

$$a_{n+m} \leqslant b_{n - pn_0} a_{pn_0 + m} \leqslant \frac{N}{e} a_{pn_0 + m} \leqslant \frac{N}{e} (b_{n_0})^p a_m$$

$$\leqslant Ne^{-p-1}a_m \leqslant Ne^{-n/n_0}a_m = Ne^{-\nu n}a_m . \qquad \blacksquare$$

3. THE STABLE SUBSPACES ON $\mathbb{Z}_+$ AND THE UNSTABLE SUBSPACES ON $\mathbb{Z}_-$

In this section we will use the notations $\mathcal{E}(\mathbb{Z}_\pm) = \ell^p(\mathbb{Z}_\pm, X)$ if $\mathcal{E}(\mathbb{Z}) = \ell^p(\mathbb{Z}, X)$, $p \in [1,\infty)$, and $\mathcal{E}(\mathbb{Z}_\pm) = c_0(\mathbb{Z}_\pm, X)$ if $\mathcal{E}(\mathbb{Z}) = c_0(\mathbb{Z}, X)$. We introduce the stable and unstable subspaces of $\mathcal{U}$ on $\mathbb{Z}_+$ and $\mathbb{Z}_-$, respectively, by

$$X_s^+(k) = \{x \in X : (U(n+k,k)x)_{n\in\mathbb{Z}_+} \in \mathcal{E}(\mathbb{Z}_+)\}, \quad k \geqslant 0, \tag{3.1}$$

$$X_u^-(k) = \{x \in X : \exists\ (x_n)_{n\in\mathbb{Z}_-} \in \mathcal{E}(\mathbb{Z}_-) \text{ with } x_n = U(n,m)x_m \text{ for } m \leqslant n \leqslant 0 \text{ and } x_k = x\}, \quad k \leqslant 0. \tag{3.2}$$

We observe that

$$U(n,m)X_s^+(m) \subseteq X_s^+(n) \quad \text{for all } n \geqslant m \geqslant 0, \tag{3.3}$$

$$U(n,m)X_u^-(m) = X_u^-(n) \quad \text{for all } m \leqslant n \leqslant 0. \tag{3.4}$$

Let $U_s^+(n,m) : X_s^+(m) \to X_s^+(n)$ and $U_u^-(n,m) : X_u^-(m) \to X_u^-(n)$ be the linear operators defined by $U_s^+(n,m)x = U(n,m)x$ for $n \geqslant m \geqslant 0$ and $x \in X_s^+(m)$ and by $U_u^-(n,m)x = U(n,m)x$ for $m \leqslant n \leqslant 0$ and $x \in X_u^-(m)$. The following lemma shows in particular that the above subspaces do not match at $n = 0$, in general.

LEMMA 3.1. *Let Hypothesis* 1 *be satisfied. Then the following assertions hold:*
(i) $X_s^+(0) + X_u^-(0) = X_{0,*}^\perp$.
(ii) $X_s^+(0) \cap X_u^-(0) = X_0$.

Proof. (i) Let $\boldsymbol{\xi} = (\xi_n)_{n\in\mathbb{Z}} \in \ker D^*$. Then $\boldsymbol{\xi}$ is bounded, and $U(k,0)^*\xi_k = \xi_0$ by (2.5). For $x \in X_s^+(0)$, (3.1) yields $U(k,0)x \to 0$ as $k \to \infty$. We compute

$$\langle x, \xi_0\rangle = \langle x, U(k,0)^*\xi_k\rangle = \langle U(k,0)x, \xi_k\rangle$$

for all $k \geqslant 0$. Letting $k \to \infty$, we deduce $\langle x, \xi_0\rangle = 0$ and thus $x \in X_{0,*}^\perp$. For $x \in X_u^-(0)$, there is $(x_k)_{k\in\mathbb{Z}_-} \in \mathcal{E}(\mathbb{Z}_-)$ such that $x_n = U(n,m)x_m$ for all $m \leqslant n \leqslant 0$ and $x_0 = x$ due to (3.2). In this case we have $x_k \to 0$ as $k \to -\infty$ and

$$\langle x, \xi_0\rangle = \langle x_0, \xi_0\rangle = \langle U(0,k)x_k, \xi_0\rangle = \langle x_k, U(0,k)^*\xi_0\rangle = \langle x_k, \xi_k\rangle$$

for all $k \leqslant 0$. Letting $k \to -\infty$, we infer $x \in X_{0,*}^\perp$. Hence, $X_s^+ + X_u^- \subseteq X_{0,*}^\perp$.

Assume that $x \in X_{0,*}^\perp$. Then the sequence $\mathbf{y} = -\chi_{\{1\}} \otimes U(1,0)x$ belongs to $\mathcal{F}$ due to (2.7) and Lemma 2.1(vi). Lemma 2.2(i) gives a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in \mathcal{F}$ with $D\mathbf{x} = \mathbf{y}$. This equation implies that $x_1 - U(1,0)x_0 = y_1 = -U(1,0)x$ and $x_n - U(n,1)x_1 = y_n = 0$ for $n \geqslant 2$. We conclude that $U(n,0)(x - x_0) = -x_n$ for all $n \geqslant 1$, and thus $x - x_0 \in X_s^+(0)$ by (3.1). Using $D\mathbf{x} = \mathbf{y}$ again, we obtain $x_n - U(n,m)x_m = y_n = 0$ for all $m \leqslant n \leqslant 0$, so that $x_0 \in X_u^-(0)$ by (3.2). Therefore, $x = x - x_0 + x_0 \in X_s^+(0) + X_u^-(0)$, proving (i).

(ii) Let $x \in X_s^+(0) \cap X_u^-(0)$. Then $x_n = U(n,0)x$ defines a sequence $(x_n)_{n\in\mathbb{Z}_+} \in \mathcal{E}(\mathbb{Z}_+)$ by (3.1), and there is a sequence $(x_n)_{n\in\mathbb{Z}_-} \in \mathcal{E}(\mathbb{Z}_-)$ so that $x = x_0$

and $x_n = U(n,m)x_m$ for all $m \leqslant n \leqslant 0$ due to (3.2). It is easy to check that $x_n = U(n,m)x_m$ for all $n \geqslant m$ in $\mathbb{Z}$, and thus $x \in X_0$ by (2.2) and (2.4). Hence, $X_s^+(0) \cap X_u^-(0) \subseteq X_0$. The converse inclusion follows directly from the definitions of X_0, $X_s^+(0)$, and $X_u^-(0)$ in (2.2), (3.1), and (3.2). ∎

REMARK 3.2. Using the same arguments as in the proof of Lemma 3.1(i), one can establish that $X_s^+(k) \subseteq X_{k,*}^{\perp}$ for all $k \geqslant 0$ and $X_u^-(k) \subseteq X_{k,*}^{\perp}$ for all $k \leqslant 0$.

In the derivation of the dichotomy estimates we make use of the following sequences, where $n \in \mathbb{Z}_+$ and $p \in [1,\infty)$:

$$\alpha_n = \begin{cases} (n+1)^{1-(1/p)}, & \text{if } \mathcal{E}(\mathbb{Z}) = \ell^p(\mathbb{Z},X), \\ (n+1), & \text{if } \mathcal{E}(\mathbb{Z}) = c_0(\mathbb{Z},X), \end{cases} \qquad \beta_n = \begin{cases} (n+1)^{1/p}, & \text{if } \mathcal{E}(\mathbb{Z}) = \ell^p(\mathbb{Z},X), \\ 1, & \text{if } \mathcal{E}(\mathbb{Z}) = c_0(\mathbb{Z},X). \end{cases}$$

REMARK 3.3. We note some obvious properties of the above sequences:

(i) $\alpha_n \beta_n = n+1$ for all $n \geqslant 0$.

(ii) $\sum_{k=m}^{m+n} \|x_k\| \leqslant \alpha_n \|\mathbf{x}\|_{\mathcal{E}(\mathbb{Z})}$ for all $m \in \mathbb{Z}$, $n \geqslant 0$, $\mathbf{x} = (x_k)_{k\in\mathbb{Z}} \in \mathcal{E}(\mathbb{Z})$.

(iii) $\|\chi_{\{m,\dots,m+n\}} \otimes x\|_{\mathcal{E}(\mathbb{Z})} = \beta_n \|x\|$ for all $x \in X$, $m \in \mathbb{Z}$, $n \geqslant 0$.

We can now establish the dichotomy estimates of $U_s^+(n,m)$ for $n \geqslant m \geqslant 0$, as well as the invertibility of $U_u^-(n,m)$ and the dichotomy estimates of $U_u^-(n,m)^{-1}$ for $m \leqslant n \leqslant 0$.

LEMMA 3.4. *Let Hypothesis* 1 *be satisfied. Then the following assertions hold:*

(i) *There are constants $N, \nu > 0$ such that*

$$\|U_s^+(n,m)\| \leqslant N e^{-\nu(n-m)} \quad \textit{for all } \; n \geqslant m \geqslant 0.$$

(ii) *$X_s^+(m)$ is a closed subspace of X for all $m \geqslant 0$.*

Proof. Let $m \geqslant 0$, $x \in X_s^+(m)$, and $(\varphi_k)_{k\in\mathbb{Z}}$ be a finitely supported numerical sequence. We define the sequences $\mathbf{x} = (x_k)_{k\in\mathbb{Z}}$ and $\mathbf{y} = (y_k)_{k\in\mathbb{Z}}$ by

$$x_k = \begin{cases} 0, & k \leqslant m, \\ \Big(\sum_{j=m+1}^{k} \varphi_j\Big) U(k,m)x, & k > m, \end{cases} \qquad y_k = \begin{cases} 0, & k \leqslant m, \\ \varphi_k U(k,m)x, & k > m. \end{cases} \tag{3.5}$$

Remark 3.2 and (3.3) imply that $\mathbf{x} \in \mathcal{F}_0$, see (2.8). It is straightforward to check that $\mathbf{y} = D\mathbf{x} = D_0\mathbf{x}$. We first take $(\varphi_k)_{k\in\mathbb{Z}} = \chi_{\{m+1\}}$. Lemma 2.2(ii) and the exponential boundedness of the evolution family $\mathcal{U}$ yield

$$\begin{aligned} \|U(n,m)x\| &= \Big\| \sum_{j=m+1}^{n} \chi_{\{m+1\}}(j)\, U(n,m)x \Big\| \leqslant \|\mathbf{x}\|_{\mathcal{E}(\mathbb{Z})} \\ &\leqslant c\|D_0\mathbf{x}\|_{\mathcal{E}(\mathbb{Z})} = c\|\mathbf{y}\|_{\mathcal{E}(\mathbb{Z})} = c\|U(m+1,m)x\| \leqslant cMe^{\omega}\|x\| \end{aligned}$$

for all $n \geqslant m+1$. It follows that

$$\|U_s^+(k,j)\| \leqslant c \quad \text{for all } \; k \geqslant j \geqslant 0. \tag{3.6}$$

Second, we take $n > l > m$ and set $(\varphi_k)_{k\in\mathbb{Z}} = \chi_{\{l,\dots,n\}}$. For $\mathbf{x}$ and $\mathbf{y}$ defined in (3.5), estimate (3.6), Remark 3.3, and Lemma 2.2(ii) imply that

$$\begin{aligned}
&\frac{1}{2}(n-l+2)(n-l+1)\,\|U_s^+(n,m)x\| \\
&\quad= \sum_{k=l}^{n}(k-l+1)\|U_s^+(n,k)U_s^+(k,m)x\| \leqslant c\sum_{k=l}^{n}\sum_{j=m+1}^{k}\varphi_j\|U(k,m)x\| \\
&\quad= c\sum_{k=l}^{n}\|x_k\| \leqslant c\alpha_{n-l}\|\mathbf{x}\|_{\mathcal{E}(\mathbb{Z})} \leqslant c\alpha_{n-l}\|\mathbf{y}\|_{\mathcal{E}(\mathbb{Z})} \leqslant c\alpha_{n-l}\|\chi_{\{l,\dots,n\}}\otimes U_s^+(l,m)x\|_{\mathcal{E}(\mathbb{Z})} \\
&\quad= c\alpha_{n-l}\beta_{n-l}\|U_s^+(l,m)x\| = c(n-l+1)\|U_s^+(l,m)x\|.
\end{aligned}$$

So we have shown that $\|U(n,m)x\| \leqslant b_{n-l}\|U(l,m)x\|$ for all $n \geqslant l \geqslant m \geqslant 0$ and all $x \in X_s^+(m)$, where $b_0 = 1$ and $b_j = c(j+2)^{-1}$ for $j \geqslant 1$. By Lemma 2.4, there are $N, \nu > 0$ such that $\|U(n,m)x\| \leqslant N\mathrm{e}^{-\nu(n-l)}\|U(l,m)x\|$ for all $n \geqslant l \geqslant m$ and all $x \in X_s^+(m)$, which proves (i). Assertion (ii) follows easily from (i) and (3.1). ∎

LEMMA 3.5. *Let Hypothesis* 1 *be satisfied. Then the following assertions hold:*

(i) $U_u^-(n,m) : X_u^-(m) \to X_u^-(n)$ *is bijective for* $m \leqslant n \leqslant 0$.

(ii) *There are constants* $N, \nu > 0$ *such that*

$$\|(U_u^-(n,m))^{-1}\| \leqslant N\mathrm{e}^{-\nu(n-m)} \quad \text{for all } m \leqslant n \leqslant 0.$$

(iii) $X_u^-(k)$ *is a closed subspace of* X *for* $k \leqslant 0$.

Proof. (i) Fix $m \leqslant n \leqslant 0$. The surjectivity of $U_u^-(n,m)$ was already stated in (3.4). Take $x \in X_u^-(m)$ with $0 = U_u^-(n,m)x = U(n,m)x$. By (3.2) there is a sequence $\mathbf{x} = (x_k)_{k\in\mathbb{Z}_-} \in \mathcal{E}(\mathbb{Z}_-)$ such that $x_k = U(k,j)x_j$ for all $j \leqslant k \leqslant 0$ and $x = x_m$. We extend $\mathbf{x}$ to a sequence from $\mathbf{x} \in \mathcal{E}(\mathbb{Z})$ by setting $x_k = 0$ for $k > 0$. Since $x_0 = U(0,n)U(n,m)x = 0$, the sequence $\mathbf{x}$ belongs to $\ker D$. Hence, $x \in X_m$ by (2.2). Lemma 2.1(iii) now yields $x = 0$, and so (i) is established.

(ii) Take $\mathbf{w} = (w_k)_{k\in\mathbb{Z}_-} \in \mathcal{E}(\mathbb{Z}_-)$ with $w_k = U(k,j)w_j$ for all $j \leqslant k \leqslant 0$. Let $(\varphi_k)_{k\in\mathbb{Z}} \subseteq \mathbb{C}$ have finite support. We define $\mathbf{x} = (x_k)_{k\in\mathbb{Z}}$ and $\mathbf{y} = (y_k)_{k\in\mathbb{Z}}$ by

$$x_k = \begin{cases} 0, & k \geqslant 0, \\ \Big(\sum\limits_{j=k+1}^{0}\varphi_j\Big)w_k, & k \leqslant -1, \end{cases} \qquad y_k = \begin{cases} 0, & k \geqslant 1, \\ -\varphi_k w_k, & k \leqslant 0. \end{cases} \tag{3.7}$$

Observe that $\mathbf{x} \in \mathcal{F}_0$ since $w_k \in X_u^-(k) \subseteq X_{k,*}^\perp$ for all $k \in \mathbb{Z}_-$ (see (2.8), (3.2), and Remark 3.2). Moreover, $\mathbf{y} = D\mathbf{x} = D_0\mathbf{x}$. Let $m \leqslant n-1 < 0$ and choose first $(\varphi_k)_{k\in\mathbb{Z}} = \chi_{\{n\}}$. Lemma 2.2(ii) implies that

$$\|w_m\| = \|x_m\| \leqslant \|\mathbf{x}\|_{\mathcal{E}(\mathbb{Z})} \leqslant c\|\mathbf{y}\|_{\mathcal{E}(\mathbb{Z})} = c\|w_n\|. \tag{3.8}$$

Second, take $(\varphi_k)_{k\in\mathbb{Z}} = \chi_{\{m+1,\dots,n\}}$. Estimate (3.8), Lemma 2.2(ii), and Remark 3.3 now yield

$$\frac{1}{2}(n-m)(n-m+1)\|w_m\|$$

$$=\sum_{k=m}^{n-1}(n-k)\|w_m\|\leqslant c\sum_{k=m}^{n-1}\sum_{j=k+1}^{n}\varphi_j\|w_k\|=c\sum_{k=m}^{n-1}\|x_k\|\leqslant c\alpha_{n-m-1}\|\mathbf{x}\|_{\mathcal{E}(\mathbb{Z})}\leqslant c\alpha_{n-m-1}\|\mathbf{y}\|_{\mathcal{E}(\mathbb{Z})}$$

$$\leqslant c\alpha_{n-m-1}\|\chi_{\{m+1,\dots,n\}}\otimes w_n\|_{\mathcal{E}(\mathbb{Z})}=c\alpha_{n-m-1}\beta_{n-m-1}\|w_n\|=c(n-m)\|w_n\|,$$

which implies that $\|w_m\| \leqslant \frac{c}{n-m+1}\|w_n\|$ for all $m \leqslant n-1 < 0$. Applying Lemma 2.4 to the sequences $a_n = \|w_{-n}\|$ and $b_n = c(n+1)^{-1}$, we obtain constants $N, \nu > 0$ (independent of the choice of $\mathbf{w} = (w_k)_{k\in\mathbb{Z}}$) such that $\|w_m\| \leqslant N\mathrm{e}^{-\nu(n-m)}\|w_n\|$ for all $m \leqslant n \leqslant 0$. We can now deduce (ii) from the definition of $\mathbf{w} = (w_k)_{k\in\mathbb{Z}}$ and (i).

(iii) It suffices to consider $k = 0$ due to (i) and (ii). Take $x \in X$ and $x^{(n)} \in X_u^+(0)$, $n \in \mathbb{Z}_+$, with $x^{(n)} \to x$ as $n \to \infty$. Let $\mathbf{y}^{(n)} = (y_k^{(n)})_{k\in\mathbb{Z}_-}$ be a sequence in $\mathcal{E}(\mathbb{Z}_-)$ such that $y_k^{(n)} = U(k,j)y_j^{(n)}$ for all $j \leqslant k \leqslant 0$ and $y_0^{(n)} = x^{(n)}$ for all $n \geqslant 0$. Assertion (ii) yields

$$\|y_k^{(n)} - y_k^{(m)}\| = \|(U_u^-(0,k))^{-1}(x^{(n)} - x^{(m)})\| \leqslant N\mathrm{e}^{\nu k}\|x^{(n)} - x^{(m)}\|$$

for all $n, m \geqslant 0$ and all $k \leqslant 0$, and thus

$$\|\mathbf{y}^{(n)} - \mathbf{y}^{(m)}\|_{\mathcal{E}(\mathbb{Z})} \leqslant c\|x^{(n)} - x^{(m)}\| \quad \text{for all } n, m \geqslant 0.$$

As a result, there exists $\mathbf{y} = (y_k)_{k\in\mathbb{Z}_-} \in \mathcal{E}(\mathbb{Z}_-)$ with $\mathbf{y}^{(n)} \to \mathbf{y}$ in $\mathcal{E}(\mathbb{Z}_-)$ as $n \to \infty$. It follows that $y_k = U(k,j)y_j$ for all $j \leqslant k \leqslant 0$ and $y_0 = x$; i.e., $x \in X_u^-(0)$. ∎

As a preparation for the following two sections, we construct several splittings of X. Recall from Lemma 2.1 that $X_{0,*}$ is finite dimensional, and let $\{\xi_0^{(1)}, \dots, \xi_0^{(d_0)}\}$ be a basis of $X_{0,*}$. By Lemma 2.3 there exist vectors $\{x_0^{(1)}, \dots, x_0^{(d_0)}\} \subseteq X$ such that $\langle x_0^{(i)}, \xi_0^{(j)}\rangle = \delta_{ij}$ for all $i, j \in \{1, \dots, d_0\}$ and

$$X = X_{0,*}^{\perp} \oplus Y, \quad \text{where } Y := \operatorname{Span}\{x_0^{(1)}, \dots, x_0^{(d_0)}\}. \tag{3.9}$$

Recall from (2.6) that we have $X_{0,*}^{\perp} = X_0 \oplus X_0'$ where X_0 is given by (2.2). In order to relate these spaces with $X_s^+(0)$ and $X_u^-(0)$, we further introduce the subspaces

$$Z_1 = X_0' \cap X_s^+(0) \quad \text{and} \quad Z_2 = X_0' \cap X_u^-(0). \tag{3.10}$$

LEMMA 3.6. *Let Hypothesis* 1 *be satisfied. Then the following assertions hold:*

(i) $X_s^+(0) = Z_1 \oplus X_0$ *and* $X_u^-(0) = Z_2 \oplus X_0$.

(ii) $X_0' = Z_1 \oplus Z_2$.

(iii) $X = X_s^+(0) \oplus (Z_2 \oplus Y) = X_u^-(0) \oplus (Z_1 \oplus Y)$.

Proof. (i) We have seen in Lemma 3.4(ii) and Lemma 3.5(iii) that $X_s^+(0)$ and $X_u^-(0)$ are closed subspaces of X. Since X_0' is also a closed subspace of X, the spaces Z_1 and Z_2 are closed in X. We have $Z_1 \cap X_0 = \{0\}$ and $Z_1 \subseteq X_s^+(0)$ by (3.10) and (2.6). Lemma 3.1(ii) yields $X_0 \subseteq X_s^+(0)$, so that $X_0 + Z_1 \subseteq X_s^+(0)$. Let $x \in X_s^+(0)$. Then $x \in X_{0,*}^{\perp} = X_0 \oplus X_0'$ by Lemma 3.1(ii) and (2.6). So we can write $x = x_0 + x_0'$ for some $x_0 \in X_0$ and $x_0' \in X_0'$, implying $x_0' = x - x_0 \in X_s^+(0)$. Hence,

$x_0' \in Z_1$ by (3.10). Thus the first equation in (i) is verified. The second one can be established in the same way.

(ii) The identities (3.10), Lemma 3.1(ii), and (2.6) yield $Z_1 \subseteq X_0'$, $Z_2 \subseteq X_0'$, and

$$Z_1 \cap Z_2 = X_0' \cap X_s^+(0) \cap X_u^-(0) = X_0' \cap X_0 = \{0\}.$$

Let $x \in X_0'$. Then we deduce from (2.6) and Lemma 3.1(i) that $x \in X_{0,*}^{\perp} = X_s^+(0) + X_u^-(0)$. So assertion (i) provides us with $z_1 \in Z_1, z_2 \in Z_2$, and $v_1, v_2 \in X_0$ such that $x = z_1 + z_2 + v_1 + v_2$. Using again $Z_j \subseteq X_0'$, we obtain that $v_1 + v_2 = x - z_1 - z_2 \in X_0'$. Hence, $v_1 + v_2 \in X_0' \cap X_0 = \{0\}$. So we have shown that $X_0' \subseteq Z_1 + Z_2$, and the desired decomposition holds.

(iii) The spaces $Z_1 \oplus Y$ and $Z_2 \oplus Y$ are closed subspaces of X since Z_1 and Z_2 are closed in X by (i) and $\dim Y < \infty$ by (3.9). We then derive the splitting $X = X_0 \oplus Z_1 \oplus Z_2 \oplus Y$ from (3.9), (2.6), and (ii). Hence, (iii) follows from (i). ∎

4. EXPONENTIAL DICHOTOMY ON $\mathbb{Z}_+ \cap [b, \infty)$

The main difficulty in establishing the dichotomy on $\mathbb{Z}_+ \cap [b, \infty)$ is the construction of the correct complement of the stable subspace $X_s^+(k)$. To that purpose, we first deal with the "good part" of $X_u^+(k)$ by propagating the space Z_2 from (3.10); i.e., we set

$$Z_2(k) = U(k, 0)Z_2 \quad \text{for } k \in \mathbb{Z}_+. \tag{4.1}$$

Observe that, due to (3.10), a vector $x \in Z_2$ can be propagated backwards to an element $(x_n)_{n\in\mathbb{Z}_-}$ of $\mathcal{E}(\mathbb{Z}_-)$ with $x = U(0, n)x_n$, but this sequence cannot be extended to a non-zero element of $\ker D$. These facts are crucial for the next result.

LEMMA 4.1. *Let Hypothesis* 1 *be satisfied. Then the following assertions hold:*
(i) $U(n, m)_{|Z_2(m)}$ *is bijective from* $Z_2(m)$ *to* $Z_2(n)$ *for all* $n \geqslant m \geqslant 0$.
(ii) *There are constants* $N, \nu > 0$ *such that*

$$\|(U(n, m)_{|Z_2(m)})^{-1}\| \leqslant N e^{-\nu(n-m)} \quad \textit{for all } n \geqslant m \geqslant 0.$$

(iii) $Z_2(k)$ *is a closed subspace of* X *for all* $k \geqslant 0$.

Proof. (i) The definition (4.1) implies that $U(n, m)Z_2(m) = Z_2(n)$ for all $n \geqslant m \geqslant 0$. Take $x \in Z_2(m)$ with $U(n, m)x = 0$. By (4.1), there exists a vector $z_2 \in Z_2$ such that $x = U(m, 0)z_2$. Since

$$U(j, 0)z_2 = U(j, n)U(n, m)U(m, 0)z_2 = U(j, n)U(n, m)x = 0$$

for all $j \geqslant n$, we obtain $z_2 \in X_s^+(0)$ (see (3.1)). Lemma 3.6(iii) then shows that $z_2 = 0$, and so $x = 0$. Thus $U(n, m) : Z_2(m) \to Z_2(n)$ is bijective.

(ii) Let $z_2 \in Z_2 \setminus \{0\}$. By (3.10) and (3.2) there is a sequence $\mathbf{w} = (w_k)_{k\in\mathbb{Z}_-} \in \mathcal{E}(\mathbb{Z}_-)$ such that $w_k = U(k, j)w_j$ for all $j \leqslant k \leqslant 0$ and $w_0 = z_2$. Let $(\varphi_k)_{k\in\mathbb{Z}}$ be a finitely supported numerical sequence. Define $\mathbf{x} = (x_k)_{k\in\mathbb{Z}}$ and $\mathbf{y} = (y_k)_{k\in\mathbb{Z}}$ by

$$x_k = \begin{cases} \sum\limits_{j=k+1}^{\infty} \varphi_j U(k,0)z_2, & k \geqslant 1, \\ \sum\limits_{j=1}^{\infty} \varphi_j w_k, & k \leqslant 0, \end{cases} \qquad y_k = \begin{cases} -\varphi_k U(k,0)z_2, & k \geqslant 1, \\ 0, & k \leqslant 0. \end{cases}$$

We have $w_k \in X_u^-(k) \subseteq X_{k,*}^{\perp}$ for all $k \leqslant 0$ due to (3.2) and Remark 3.2. Equations (3.10) and (2.6) and Lemma 2.1(vi) further imply that $U(k,0)z_2 \in X_{k,*}^{\perp}$ for $k \geqslant 0$. Since also $\mathbf{x} \in \mathcal{E}(\mathbb{Z})$ and $w_0 = z_2 \in X_0'$ by (3.10), the vector $\mathbf{x}$ belongs to $\mathcal{F}_0$ (see (2.8)). Moreover, $\mathbf{y} = D\mathbf{x} = D_0\mathbf{x}$. Let $n > m \geqslant 0$. Choose first $(\varphi_k)_{k\in\mathbb{Z}} = \chi_{\{n\}}$. Then Lemma 2.2(ii) yields

$$\|U(m,0)z_2\| = \|x_m\| \leqslant \|\mathbf{x}\|_{\mathcal{E}(\mathbb{Z})} \leqslant c\|\mathbf{y}\|_{\mathcal{E}(\mathbb{Z})} \leqslant c\|U(n,0)z_2\|. \tag{4.2}$$

Second, take $(\varphi_k)_{k\in\mathbb{Z}} = \chi_{\{m+1,\dots,n\}}$. In this case, estimate (4.2), Remark 3.3, and Lemma 2.2(ii) imply that

$$\begin{aligned} &\frac{1}{2}(n-m)(n-m+1)\|U(m,0)z_2\| \\ &\quad = \sum_{k=m}^{n-1}(n-k)\|U(m,0)z_2\| = \sum_{k=m}^{n-1}\sum_{j=k+1}^{\infty}\varphi_j\|U(m,0)z_2\| \leqslant c\sum_{k=m}^{n-1}\sum_{j=k+1}^{\infty}\varphi_j\|U(k,0)z_2\| \\ &\quad = c\sum_{k=m}^{n-1}\|x_k\| \leqslant c\alpha_{n-m-1}\|\mathbf{x}\|_{\mathcal{E}(\mathbb{Z})} \leqslant c\alpha_{n-m-1}\|\mathbf{y}\|_{\mathcal{E}(\mathbb{Z})} \leqslant c\alpha_{n-m-1}\|c\chi_{\{m+1,\dots,n\}}\otimes U(n,0)z_2\|_{\mathcal{E}(\mathbb{Z})} \\ &\quad \leqslant c\alpha_{n-m-1}\beta_{n-m-1}\|U(n,0)z_2\| = c(n-m)\|U(n,0)z_2\|. \end{aligned}$$

Therefore $\|U(m,0)z_2\| \leqslant \frac{c}{n-m+1}\|U(n,0)z_2\|$, and in particular $U(n,0)z_2 \neq 0$, for all $n \geqslant m \geqslant 0$. Applying Lemma 2.4 to the sequences $a_n = \|U(n,0)z_2\|^{-1}$ and $b_n = c(n+1)^{-1}$, we obtain constants $N, \nu > 0$ (independent of z_2) such that $\|U(m,0)z_2\| \leqslant N\mathrm{e}^{-\nu(n-m)}\|U(n,0)z_2\|$ for all $n \geqslant m \geqslant 0$. Using (i), we can now conclude that (ii) holds.

(iii) Since $U(k,0)_{|Z_2(0)} : Z_2(0) \to Z_2(k)$ is an isomorphism by (i) and (ii), assertion (iii) follows from (4.1) and the closedness of Z_2 proved in Lemma 3.6(i). ∎

We next introduce the remaining complement of the unstable subspace. Let $\{\xi_b^{(1)}, \dots, \xi_b^{(d_b)}\}$ be a basis of $X_{b,*}$ (cf. Lemma 2.1). By Lemma 2.3(i), there are vectors $x_b^{(1)}, \dots, x_b^{(d_b)}$ in X such that $\langle x_b^{(i)}, \xi_b^{(j)}\rangle = \delta_{ij}$ for all $i,j \in \{1,\dots,d_b\}$. Lemma 2.3(ii) shows that

$$X = X_{b,*}^{\perp} \oplus Y^+(b), \quad \text{where } Y^+(b) := \operatorname{Span}\{x_b^{(1)}, \dots, x_b^{(d_b)}\}. \tag{4.3}$$

We note that Z_2 is contained in $X_{0,*}^{\perp}$ due to (3.10) and (2.6). Lemma 2.1(vi) and equation (4.1) then imply that

$$Z_2(n) = U(n,0)Z_2 \subseteq X_{n,*}^{\perp} \quad \text{for all } n \in \mathbb{Z}_+. \tag{4.4}$$

Hence, $Z_2(b) \cap Y^+(b) = \{0\}$. Moreover, $Z_2(b)$ is closed by Lemma 4.1(iii). So we can define a closed subspace of X by

$$X_u^+(b) = Z_2(b) \oplus Y^+(b). \tag{4.5}$$

We see below that $X_u^+(b)$ is indeed the unstable subspace. We propagate these spaces by the evolution family; i.e., we set

$$X_u^+(k) = U(k,b)X_u^+(b) \quad \text{and} \quad Y^+(k) = U(k,b)Y^+(b) \quad \text{for all } k \geqslant b. \tag{4.6}$$

Finally, we let $U_u^+(n,m) = U(n,m)_{|X_u^+(m)}$ for $n \geqslant m \geqslant b$. Here we take $k \geqslant b$ in order to make sure that $\dim X_{k,*}$, and thus $\dim Y^+(k)$, is constant.

LEMMA 4.2. *Let Hypothesis* 1 *be satisfied. Then the following assertions hold:*
(i) $X_u^+(k)$ *is closed in* X *and* $X_u^+(k) = Z_2(k) \oplus Y^+(k)$ *for all* $k \geqslant b$.
(ii) $U_u^+(n,m)$ *is invertible from* $X_u^+(m)$ *to* $X_u^+(n)$ *and* $U(n,m)_{|Y^+(m)}$ *is invertible from* $Y^+(m)$ *to* $Y^+(n)$ *for all* $n \geqslant m \geqslant b$.
(iii) $X = Y^+(k) \oplus X_{k,*}^\perp$ *for all* $k \geqslant b$.

Proof. (i) Let $w \in Z_2(k) \cap Y^+(k)$ for some $k \geqslant b$. Then $w = U(k,b)x$ for a vector $x \in Z_2(b) \cap Y^+(b)$ by Lemma 4.1(i) and (4.6). Thus equation (4.5) yields $x = 0$, and so $w = 0$. Moreover, $Z_2(k) \oplus Y^+(k)$ is closed since $Z_2(k)$ is closed by Lemma 4.1(iii) and $Y^+(k)$ is finite dimensional by (4.3). Assertion (i) is now a consequence of (4.6), (4.5), and Lemma 4.1(i).

(ii) Let $n \geqslant m \geqslant b$. The surjectivity of $U_u(n,m) : X_u^+(m) \to X_u^+(n)$ and of $U_u(n,m) : Y^+(m) \to Y^+(n)$ follows from (4.6). Take $x \in X_u^+(m)$ with $U_u^+(n,m)x = 0$. By our definitions (4.6), (4.5), and (4.1), there are $z_2 \in Z_2$ and $y_b \in Y^+(b)$ such that $x = U(m,b)(U(b,0)z_2 + y_b)$. Therefore, $0 = U(n,m)x = U(n,0)z_2 + U(n,b)y_b$. On the other hand, $U(n,0)z_2 \in X_{n,*}^\perp$ by (4.4). For $\boldsymbol{\xi} = (\xi_k)_{k\in\mathbb{Z}} \in \ker D^*$ equation (2.5) thus yields

$$\langle y_b, \xi_b\rangle = \langle y_b, U(n,b)^*\xi_n\rangle = \langle U(n,b)y_b, \xi_n\rangle = -\langle U(n,0)z_2, \xi_n\rangle = 0.$$

We obtain $y_b \in X_{b,*}^\perp \cap Y^+(b) = \{0\}$ taking into account (4.3). As a result, $U(j,0)z_2 = U(j,n)U(n,0)z_2 = 0$ for all $j \geqslant n$, which means that $z_2 \in X_s^+(0) \cap Z_2$. Lemma 3.6(iii) now yields $z_2 = 0$. This fact leads to $x = 0$, and so $U_u^+(n,m) : X_u^+(m) \to X_u^+(n)$ is also injective. The assertions then follow from (i) and (4.6).

(iii) As we have seen before (4.3), there exist bases $\{\xi_b^{(1)}, \dots, \xi_b^{(d_b)}\}$ of $X_{b,*}$ and $\{x_b^{(1)}, \dots, x_b^{(d_b)}\}$ of $Y^+(b)$ such that $\langle x_b^{(i)}, \xi_b^{(j)}\rangle = \delta_{ij}$ for all $i,j \in \{1,\dots,d_b\}$. Lemma 2.1(iv) and part (ii) show that $\{(U(k,b)^*)^{-1}\xi_b^{(1)}, \dots, (U(k,b)^*)^{-1}\xi_b^{(d_b)}\}$ is a basis of $X_{k,*}$ and $\{U(k,b)x_b^{(1)}, \dots, U(k,b)x_b^{(d_b)}\}$ is a basis of $Y^+(k)$. Moreover $\langle U(k,b)x_b^{(i)}, (U(k,b)^*)^{-1}\xi_b^{(j)}\rangle = \delta_{ij}$ for all $i,j \in \{1,\dots,d_b\}$. Lemma 2.3(ii) thus yields the assertion. ∎

Let $n \in \mathbb{Z}_+$ and $p \in [1, \infty)$. The following sequences are used below when we estimate the inverses of $U_u^+(n,m)$:

$$\alpha_n^* = \begin{cases} (n+1)^{1/p} & \text{if } \mathcal{E}(\mathbb{Z}) = \ell^p(\mathbb{Z}, X), \\ (n+1) & \text{if } \mathcal{E}(\mathbb{Z}) = c_0(\mathbb{Z}, X), \end{cases} \quad \beta_n^* = \begin{cases} (n+1)^{1-1/p} & \text{if } \mathcal{E}(\mathbb{Z}) = \ell^p(\mathbb{Z}, X), \\ 1 & \text{if } \mathcal{E}(\mathbb{Z}) = c_0(\mathbb{Z}, X). \end{cases}$$

REMARK 4.3. We note some immediate properties of the above defined sequences:

(i) $\alpha_n^* \beta_n^* = n + 1$ for $n \geqslant 0$.

(ii) $\sum_{k=m}^{n+m} \|\xi_k\| \leqslant \alpha_n^* \|\boldsymbol{\xi}\|_{\mathcal{E}(\mathbb{Z})^*}$ for $m \in \mathbb{Z}, n \geqslant 0, \boldsymbol{\xi} = (\xi_k)_{k \in \mathbb{Z}} \in \mathcal{E}(\mathbb{Z})^*$;

(iii) $\|\chi_{\{m,\dots,m+n\}} \otimes \xi\|_{\mathcal{E}(\mathbb{Z})^*} = \beta_n^* \|\xi\|$ for $\xi \in X^*, m \in \mathbb{Z}, n \geqslant 0$.

LEMMA 4.4. *Let Hypothesis* 1 *be satisfied. Then the following assertions hold:*

(i) *There are constants* $N, \nu > 0$ *such that*

$$\|(U(n,m)^*_{|X_{n,*}})^{-1}\| \leqslant N \mathrm{e}^{-\nu(n-m)} \quad \textit{for } \; n \geqslant m \geqslant b;$$

(ii) *There are constants* $N, \nu > 0$ *such that*

$$\|(U_u^+(n,m))^{-1}\| \leqslant N \mathrm{e}^{-\nu(n-m)} \quad \textit{for } \; n \geqslant m \geqslant b.$$

Proof. (i) Let $\boldsymbol{\xi} = (\xi_k)_{k \in \mathbb{Z}} \in \ker D^*$ and $(\varphi_k)_{k \in \mathbb{Z}}$ be a finitely supported numerical sequence. We define the sequences $\boldsymbol{\eta} = (\eta_k)_{k \in \mathbb{Z}}$ and $\boldsymbol{\zeta} = (\zeta_k)_{k \in \mathbb{Z}}$ by

$$\eta_k = \begin{cases} 0, & k \leqslant b, \\ \Big(\sum_{j=b+1}^{k} \varphi_j \Big) \xi_k, & k \geqslant b+1, \end{cases} \quad \zeta_k = \begin{cases} 0, & k \leqslant b-1, \\ -\varphi_{k+1} \xi_k, & k \geqslant b. \end{cases}$$

We have $\boldsymbol{\eta} \in \mathcal{F}_{b,*}$ since $\boldsymbol{\xi} \in \ker D^*$ and $\eta_b = 0$ (see (2.3) and (2.9)). Moreover, $\boldsymbol{\zeta} = D^* \boldsymbol{\eta} = D_{b,*} \boldsymbol{\eta}$. Let $n \geqslant m+1 \geqslant b$. We first choose $(\varphi_k)_{k \in \mathbb{Z}} = \chi_{\{m+1\}}$. Then Lemma 2.2(iii) yields

$$\|\xi_n\| = \|\eta_n\| \leqslant \|\boldsymbol{\eta}\|_{\mathcal{E}(\mathbb{Z})^*} \leqslant c \|\boldsymbol{\zeta}\|_{\mathcal{E}(\mathbb{Z})^*} = c \|\xi_m\|. \tag{4.7}$$

Second, choose $(\varphi_k)_{k \in \mathbb{Z}} = \chi_{\{m+1,\dots,n\}}$. Making use of estimate (4.7), Remark 4.3, and Lemma 2.2(iii), we calculate

$$\begin{aligned} \frac{1}{2}(n-m)(n-m+1)\|\xi_n\| &= \sum_{k=m+1}^{n} (k-m)\|\xi_n\| \leqslant c \sum_{k=m+1}^{n} \sum_{j=b+1}^{k} \varphi_j \|\xi_j\| \\ &= c \sum_{k=m+1}^{n} \|\eta_k\| \leqslant c\alpha_{n-m-1}^* \|\boldsymbol{\eta}\|_{\mathcal{E}(\mathbb{Z})^*} \leqslant c\alpha_{n-m-1}^* \|\boldsymbol{\zeta}\|_{\mathcal{E}(\mathbb{Z})^*} \\ &= c\alpha_{n-m-1}^* \beta_{n-m-1}^* \|\xi_m\| = c(n-m)\|\xi_m\|. \end{aligned}$$

As a result, $\|\xi_n\| \leqslant \frac{c}{n-m+1} \|\xi_m\|$ for all $n \geqslant m \geqslant b$. Lemma 2.4 provides constants $N, \nu > 0$ (independent of $\boldsymbol{\xi}$) such that $\|\xi_n\| \leqslant N \mathrm{e}^{-\nu(n-m)} \|\xi_m\|$ for all $n \geqslant m \geqslant b$ and $\boldsymbol{\xi} = (\xi_k)_{k \in \mathbb{Z}} \in \ker D^*$, proving (i).

(ii) The decomposition $X = Y^+(k) \oplus X_{k,*}^{\perp}$ from Lemma 4.2(iii) implies that $Y^+(k)^* = X_{k,*}$ for all $k \geqslant b$ since $X_{k,*}$ is finite dimensional. Thus we have

$$((U(n,m)_{|Y^+(m)})^{-1})^* = ((U(n,m)_{|Y^+(m)})^*)^{-1} = (U(n,m)^*_{|X_{n,*}})^{-1}$$

for all $n \geqslant m \geqslant b$ by Lemmas 4.2(ii) and 2.1(iv). Assertion (i) now yields

$$\|(U(n,m)_{|Y^+(m)})^{-1}\| \leqslant N\mathrm{e}^{-\nu(n-m)} \quad \text{for all } n \geqslant m \geqslant b. \tag{4.8}$$

Lemmas 4.1 and 4.2 show that $U_u^+(n,m)^{-1}$ has the matrix representation

$$\begin{bmatrix} (U(n,m)_{|Z_2(m)})^{-1} & 0 \\ 0 & (U(n,m)_{|Y^+(m)})^{-1} \end{bmatrix} : Z_2(m) \oplus Y^+(m) \longrightarrow Z_2(n) \oplus Y^+(n)$$

for all $n \geqslant m \geqslant b$. So the assertion follows from Lemma 4.1(ii) and (4.8). ∎

THEOREM 4.5. *Let Hypothesis* 1 *hold. Then $\mathcal{U}$ has an exponential dichotomy on $\mathbb{Z}_+ \cap [b,\infty)$ with subspaces $X_s^+(k)$ and $X_u^+(k)$ given by* (3.1) *and* (4.6)*, respectively.*

Proof. The spaces $X_s^+(m)$ and $X_u^+(m)$, $m \geqslant b$, are closed and invariant under $U(n,m)$ due to Lemmas 3.4 and 4.2 and formula (3.3). We have shown the invertibility of $U_u^+(n,m) : X_u^+(m) \to X_u^+(n)$ in Lemma 4.2(ii), and the exponential estimates of $U_s^+(n,m)$ and $U_u^+(n,m)^{-1}$ in Lemmas 3.4 and 4.4. It remains to verify that $X_s^+(m) \oplus X_u^+(m) = X$ for $m \geqslant b$. In view of Lemma 4.2 this fact follows from the decomposition

$$X_{m,*}^{\perp} = X_s^+(m) \oplus Z_2(m) \quad \text{for all } m \geqslant 0. \tag{4.9}$$

We prove (4.9). Let $x \in X_s^+(m) \cap Z_2(m)$ for some $m \geqslant 0$. Then Lemma 4.1(ii) and Lemma 3.4(i) yield

$$\|x\| \leqslant N\mathrm{e}^{-\nu(n-m)}\|U(n,m)x\| \leqslant N^2\mathrm{e}^{-2\nu(n-m)}\|x\| \quad \text{for all } n \geqslant m,$$

which implies that $x = 0$. Take $x \in X_{m,*}^{\perp}$ for some $m \geqslant 0$. We define the sequence $\mathbf{y} = (-\chi_{\{m+1\}} \otimes U(m+1,m)x)_{m\in\mathbb{Z}}$ which belongs to $\mathcal{F}$ by Lemma 2.1(ii) and (2.7). Lemma 2.2(i) gives a sequence $\mathbf{x} = (x_k)_{k\in\mathbb{Z}} \in \mathcal{F}$ such that $D\mathbf{x} = \mathbf{y}$. It follows that

$$\begin{aligned} &x_k - U(k,k-1)x_{k-1} = y_k = 0 \quad \text{for all } k \in \mathbb{Z}\backslash\{m+1\}, \\ &x_{m+1} - U(m+1,m)x_m = -U(m+1,m)x. \end{aligned} \tag{4.10}$$

Therefore $x_k = U(k,j)x_j$ for all $j \leqslant k \leqslant 0$, and so $x_0 \in X_u^-(0) = Z_2 \oplus X_0$ by (3.2) and Lemma 3.6(i). Thus we can write $x_0 = z_2 + v_0$ with $z_2 \in Z_2$ and $\mathbf{v} = (v_k)_{k\in\mathbb{Z}} \in \ker D$ (see (2.2)). The equations (4.10) further yield $x_j = U(j,m)(x_m - x)$ for all $j \geqslant m+1$ and $x_m = U(m,0)x_0 = U(m,0)z_2 + v_j$, using also (2.4). We then deduce

$$U(j,m)(x - U(m,0)z_2) = -x_j + U(j,m)(x_m - U(m,0)z_2) = -x_j + v_j$$

for all $j \geqslant m+1$. The vector $x - U(m,0)z_2$ thus belongs to $X_s^+(m)$ since $\mathbf{x},\mathbf{v} \in \mathcal{E}(\mathbb{Z})$ (see (3.1)). We thus obtain $x = (x - U(m,0)z_2) + U(m,0)z_2 \in X_s^+(m) + Z_2(m)$ due to Lemma 4.1(i); i.e., $X_{m,*}^{\perp} \subseteq X_s^+(m) + Z_2(m)$. The converse inclusion follows from Remark 3.2 and (4.4). ∎

5. EXPONENTIAL DICHOTOMY ON $\mathbb{Z}_-$

The situation on $\mathbb{Z}_-$ is simpler than in the previous section since we have dealt with the unstable subspaces already in Lemma 3.5. We first define our candidates for the stable subspaces on $\mathbb{Z}_-$ by setting

$$X_s^-(0) = Z_1 \oplus Y \quad \text{and} \quad X_s^-(k) = \{x \in X : U(0,k)x \in X_s^-(0)\} \tag{5.1}$$

for all $k \in \mathbb{Z}_-$. Recall from (3.9) that Y is finite dimensional and from Lemma 3.6 that Z_1 is closed and $Z_1 \cap Y = \{0\}$. We further denote $U_s^-(n,m) = U(n,m)_{|X_s^-(m)}$ for $m \leqslant n \leqslant 0$, and we introduce the auxiliary spaces

$$Z_1(k) = \{x \in X : U(0,k)x \in Z_1\} \subseteq X_s^-(k) \quad \text{for all } k \in \mathbb{Z}_- . \tag{5.2}$$

REMARK 5.1. Since the subspaces $X_s^-(0)$ and Z_1 are closed, $X_s^-(m)$ and $Z_1(m)$ are closed subspaces of X for all $m \in \mathbb{Z}_-$. Moreover, $U(n,m)X_s^-(m) \subseteq X_s^-(n)$ and $U(n,m)Z_1(m) \subseteq Z_1(n)$ for all $m \leqslant n \leqslant 0$.

LEMMA 5.2. *Let Hypothesis* 1 *hold. Then the following assertions hold for* $k \leqslant 0$*:*
(i) $Z_1(k) = X_s^-(k) \cap X_{k,*}^{\perp}$.
(ii) $X = X_s^-(k) \oplus X_u^-(k)$.

Proof. (i) Since $Z_1 \subseteq X_{0,*}^{\perp}$ by (3.10) and (2.6), Lemma 2.1(vi) and (5.2) yield $Z_1(k) \subseteq X_s^-(k) \cap X_{k,*}^{\perp}$ for $k \in \mathbb{Z}_-$. Let $x \in X_{k,*}^{\perp} \cap X_s^-(k)$. Due to (5.1), there are $z_1 \in Z_1$ and $y \in Y$ such that $U(0,k)x = y + z_1$. We take $\boldsymbol{\xi} = (\xi_n)_{n\in\mathbb{Z}} \in \ker D^*$ and calculate

$$\langle y, \xi_0 \rangle = \langle U(0,k)x, \xi_0 \rangle - \langle z_1, \xi_0 \rangle = \langle x, U(0,k)^* \xi_0 \rangle = \langle x, \xi_k \rangle = 0$$

using (2.5) and $Z_1 \subseteq X_{0,*}^{\perp}$. So we obtain $y \in Y \cap X_{0,*}^{\perp} = \{0\}$ employing also (3.9). Hence, $U(0,k)x = z_1 \in Z_1$; i.e., $x \in Z_1(k)$.

(ii) Lemma 3.6(iii) and (5.1) show that $X = X_s^-(0) \oplus X_u^-(0)$. Hence, given $x \in X$, there exist $x_1^- \in X_s^-(0)$ and $x_2^- \in X_u^-(0)$ with $U(0,k)x = x_1^- + x_2^-$. By (3.2) there is a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{Z}_-} \in \mathcal{E}(\mathbb{Z}_-)$ such that $x_n = U(n,m)x_m$ for all $m \leqslant n \leqslant 0$ and $x_0 = x_2^-$. Observe that $x_k \in X_u^-(k)$ by (3.2). We further compute

$$U(0,k)(x - x_k) = U(0,k)x - x_2^- = x_1^- \in X_s^-(0),$$

so that $x - x_k \in X_s^-(k)$ by (5.1). As a result, $X = X_s^-(k) + X_u^-(k)$. Take $x \in X_s^-(k) \cap X_u^-(k)$. Then equation (3.4) yields $U(0,k)x \in X_u^-(0)$. As above we see that $U(k,0)x \in X_u^-(0)$. Hence, $U(0,k)x = 0$ and Lemma 3.5(i) implies $x = 0$. ∎

LEMMA 5.3. *Let Hypothesis* 1 *hold. Then there are constants* $N, \nu > 0$ *such that*

$$\|U(n,m)_{|Z_1(m)}\| \leqslant N e^{-\nu(n-m)} \quad \textit{for all } m \leqslant n \leqslant 0.$$

Proof. Let $m \leqslant -1$, $x \in Z_1(m)$, and $(\varphi_k)_{k\in\mathbb{Z}}$ be a finitely supported numerical sequence. We define the sequences $\mathbf{x} = (x_k)_{k\in\mathbb{Z}}$ and $\mathbf{y} = (y_k)_{k\in\mathbb{Z}}$ by

$$x_k = \begin{cases} 0, & k \leqslant m-1, \\ \sum\limits_{j=m}^{k} \varphi_j U(k,m)x, & m \leqslant k \leqslant -1, \\ \sum\limits_{j=m}^{-1} \varphi_j U(k,m)x, & k \geqslant 0, \end{cases} \qquad y_k = \begin{cases} 0, & k \leqslant m-1, \\ \varphi_k U(k,m)x, & m \leqslant k \leqslant -1, \\ 0, & k \geqslant 0. \end{cases}$$

We have $\mathbf{x} \in \mathcal{E}(\mathbb{Z})$ and $x_0 \in X_0'$ because of $U(0,m)Z_1(m) \subseteq Z_1 = X_s^+(0) \cap X_0'$ (see (5.2), (3.10), and (3.1)). Lemmas 5.2(i) and 2.1(vi) further yield $U(k,m)x \in X_{k,*}^{\perp}$ for $k \geqslant m$. Therefore $\mathbf{x} \in \mathcal{F}_0$ (see (2.8)). Moreover, $\mathbf{y} = D\mathbf{x} = D_0\mathbf{x}$. Let $m \leqslant n \leqslant -1$. Choose first $(\varphi_k)_{k\in\mathbb{Z}} = \chi_{\{m\}}$. Using Lemma 2.2(ii), we estimate

$$\|U(n,m)x\| = \sum_{j=m}^{n} \varphi_j \|U(n,m)x\| = \|x_n\| \leqslant \|\mathbf{x}\|_{\mathcal{E}(\mathbb{Z})} \leqslant c\|\mathbf{y}\|_{\mathcal{E}(\mathbb{Z})} = c\|x\|. \tag{5.3}$$

As a consequence of estimate (5.3), Remark 3.3, and Lemma 2.2(ii), for $(\varphi_k)_{k\in\mathbb{Z}} = \chi_{\{m,\dots,n\}}$, we obtain that

$$\begin{aligned} &\frac{1}{2}(n-m+1)(n-m+2)\|U(n,m)x\| \\ &= \sum_{k=m}^{n}(k-m+1)\|U(n,m)x\| = \sum_{k=m}^{n}\sum_{j=m}^{k}\varphi_j\|U(n,k)U(k,m)x\| \leqslant \sum_{k=m}^{n}\sum_{j=m}^{k}\varphi_j c\|U(k,m)x\| \\ &= c\sum_{k=m}^{n}\|x_k\| \leqslant c\alpha_{n-m}\|\mathbf{x}\|_{\mathcal{E}(\mathbb{Z})} \leqslant c\alpha_{n-m}\|\mathbf{y}\|_{\mathcal{E}(\mathbb{Z})} = c\alpha_{n-m}\beta_{n-m}\|x\| = c(n-m+1)\|x\|. \end{aligned}$$

It follows that $\|U(n,m)_{|Z_1(m)}\| \leqslant \frac{c}{n-m+2}$ for all $m \leqslant n \leqslant 0$. This implies the assertion by a standard argument, cf. Theorem III.6.1 of [8]. ∎

LEMMA 5.4. *Let Hypothesis* 1 *hold. Then there are constants* $N, \nu > 0$ *such that*

$$\|U(n,m)^*_{|X_{n,*}}\| \leqslant N e^{-\nu(n-m)} \quad \textit{for all } m \leqslant n \leqslant 0.$$

Proof. Let $\boldsymbol{\xi} = (\xi_n)_{n\in\mathbb{Z}} \in \ker D^*$ and $(\varphi_k)_{k\in\mathbb{Z}}$ be a finitely supported sequence. Define the sequences $\boldsymbol{\eta} = (\eta_k)_{k\in\mathbb{Z}}$ and $\boldsymbol{\zeta} = (\zeta_k)_{k\in\mathbb{Z}}$ by setting

$$\eta_k = \begin{cases} 0, & k \geqslant 0, \\ \sum\limits_{j=k}^{-1} \varphi_j \xi_k, & k \leqslant -1, \end{cases} \qquad \zeta_k = \begin{cases} 0, & k \geqslant 0, \\ \varphi_k \xi_k, & k \leqslant -1. \end{cases}$$

Since $\boldsymbol{\xi} \in \ker D^*$, we obtain that $\boldsymbol{\eta} \in \mathcal{F}_{b,*}$ (see (2.9) and (2.3)). Moreover, $\boldsymbol{\zeta} = D\boldsymbol{\eta} = D_{b,*}\boldsymbol{\eta}$ due to (2.5). Let $m \leqslant n \leqslant -1$. First choose $(\varphi_k)_{k\in\mathbb{Z}} = \chi_{\{n\}}$. Then Lemma 2.2(iii) yields

$$\|\xi_m\| = \|\eta_m\| \leqslant \|\boldsymbol{\eta}\|_{\mathcal{E}(\mathbb{Z})^*} \leqslant c\|\boldsymbol{\zeta}\|_{\mathcal{E}(\mathbb{Z})^*} = c\|\xi_n\|. \tag{5.4}$$

Second, choose $(\varphi_k)_{k\in\mathbb{Z}} = \chi_{\{m,\dots,n\}}$. Employing inequality (5.4), Remark 4.3, and Lemma 2.2(iii), we can estimate

$$\begin{aligned}
&\frac{1}{2}(n-m+1)(n-m+2)\|\zeta_m\| \\
&\quad = \sum_{k=m}^{n}(n-k+1)\|\zeta_m\| = \sum_{k=m}^{n}\sum_{j=k}^{-1}\varphi_j\|\zeta_m\| \leqslant c\sum_{k=m}^{n}\sum_{j=k}^{-1}\varphi_j\|\zeta_k\| = c\sum_{k=m}^{n}\|\eta_k\| \\
&\quad \leqslant c\alpha^*_{n-m}\|\boldsymbol{\eta}\|_{\mathcal{E}(\mathbb{Z})^*} \leqslant c\alpha^*_{n-m}\|\boldsymbol{\zeta}\|_{\mathcal{E}(\mathbb{Z})^*} \leqslant c\alpha^*_{n-m}\beta^*_{n-m}\|\zeta_n\| = c(n-m+1)\|\zeta_n\|.
\end{aligned}$$

Taking into account that $\|\zeta_{-1}\| \leqslant M\mathrm{e}^{\omega}\|\zeta_0\|$, we infer $\|\zeta_m\| \leqslant \frac{c}{n-m+2}\|\zeta_n\|$ for all $m \leqslant n \leqslant 0$. An application of Lemma 2.4 to the sequences $a_j = \|\zeta_{-j}\|$ and $b_j = c(j+2)^{-1}$ gives $N, \nu > 0$ such that $\|\zeta_m\| \leqslant N\mathrm{e}^{-\nu(n-m)}\|\zeta_n\|$ for all $m \leqslant n \leqslant 0$, proving the lemma. ∎

THEOREM 5.5. *Let Hypothesis* 1 *be satisfied. Then $\mathcal{U}$ has an exponential dichotomy on $\mathbb{Z}_-$ with subspaces $X_s^-(k)$ and $X_u^-(k)$ given by* (5.1) *and* (3.2)*, respectively.*

Proof. Property (i) in the definition of exponential dichotomy was established in Lemma 5.2(ii), Remark 5.1, and (3.4). Lemma 3.5 yields property (ii) and the second exponential estimate in (iii). In order to prove the remaining estimate for $U_s^-(n,m)$, we fix a basis $\{\zeta_0^{(1)},\dots,\zeta_0^{(d_0)}\}$ of the space $X_{0,*}$ (which is finite dimensional by Lemma 2.1). There exist sequences $\boldsymbol{\eta}_1 = (\eta_k^{(1)})_{k\in\mathbb{Z}},\dots,$ $\boldsymbol{\eta}_{d_0} = (\eta_k^{(d_0)})_{k\in\mathbb{Z}}$ belonging to $\ker D^*$ such that $\eta_0^{(j)} = \zeta_0^{(j)}$ for all $j \in \{1,\dots,d_0\}$, see (2.5). Lemma 2.1(iv) implies that $\{\eta_k^{(1)},\dots,\eta_k^{(d_0)}\}$ is a basis of $X_{k,*}$ for all $k \leqslant 0$. Using Remark 3.2, we obtain $X_u^-(k) \subseteq X_{k,*}^{\perp} = \bigcap_{j=1}^{d_0}\ker \eta_k^{(j)}$ for all $k \leqslant 0$. As a consequence of Lemmas 2.3(i) and 5.2(ii) we then find vectors $y_k^{(1)}$, …, $y_k^{(d_0)}$ contained in $X_s^-(k)$ such that $\langle y_k^{(i)}, \eta_k^{(j)}\rangle = \delta_{ij}$ for all $i,j \in \{1,\dots,d_0\}$ and $k \leqslant 0$. We now define $Y^-(k) = \operatorname{Span}\{y_k^{(1)},\dots,y_k^{(d_0)}\}$. From Lemmas 2.3(ii) and 5.2(i) we deduce

$$X_s^-(k) = (X_s^-(k)\cap X_{k,*}^{\perp}) \oplus Y^-(k) = Z_1(k) \oplus Y^-(k) \quad \text{for all } k \leqslant 0. \tag{5.5}$$

Let $m \leqslant n \leqslant 0$. We further introduce the space

$$\widetilde{Y}^-(n,m) = \operatorname{Span}\{U(n,m)y_m^{(1)},\dots,U(n,m)y_m^{(d_0)}\} = U(n,m)Y^-(m), \tag{5.6}$$

where $U(n,m)y_m^{(j)} \in X_s^-(n)$ for all $j \in \{1,\dots,d_0\}$ due to Remark 5.1. Moreover, $\langle U(n,m)y_m^{(i)}, \eta_n^{(j)}\rangle = \langle y_m^{(i)}, U(n,m)^*\eta_n^{(j)}\rangle = \langle y_m^{(i)}, \eta_m^{(j)}\rangle = \delta_{ij}$ for all $i,j \in \{1,\dots,d_0\}$ by (2.5). As in (5.5) we can conclude by Lemma 2.3(iii) that

$$X_s^-(n) = (X_s^-(n)\cap X_{n,*}^{\perp}) \oplus \widetilde{Y}^-(n,m) = Z_1(n) \oplus \widetilde{Y}^-(n,m). \tag{5.7}$$

Our construction implies that $\dim \widetilde{Y}^-(n,m)^* = \dim X_{n,*} < \infty$. Therefore (5.7) yields $X_{n,*} \subseteq \widetilde{Y}^-(n,m)^*$, and hence $X_{n,*} = \widetilde{Y}^-(n,m)^*$. Similarly, the equality

$Y^-(m)^* = X_{m,*}$ follows from (5.5). Using Lemma 5.4, we arrive at

$$\|U(n,m)_{|Y^-(m)}\| = \|(U(n,m)_{|Y^-(m)})^*\| = \|U(n,m)^*_{|X_{n,*}}\| \leqslant N\mathrm{e}^{-\nu(n-m)} \tag{5.8}$$

for $m \leqslant n \leqslant 0$ and some constants $N, \nu > 0$. In view of (5.5), (5.7), (5.6), and Remark 5.1, the operator $U_s^-(n,m)$ has the matrix representation

$$\begin{bmatrix} U(n,m)_{|Z_1(m)} & 0 \\ 0 & U(n,m)_{|Y^-(m)} \end{bmatrix} : Z_1(m) \oplus Y^-(m) \longrightarrow Z_1(n) \oplus \widetilde{Y}^-(n,m).$$

The exponential estimate for $U_s^-(n,m)$ follows from Lemma 5.3 and (5.8). ∎

6. PROOF OF THEOREM 1.1

Sufficiency. Assume that (i) and (ii) in Theorem 1.1 hold. Then the Fredholm property of $\mathbf{G}$ can be shown exactly as in Theorem 1.1 of [13]. (At this point of the argument as well as in the proof of Theorem 1.4 and Lemma 1.5 the conditions (BU.1) and (BU.2) have not been used in [13].)

Necessity. We proceed similarly to [13]. Assume that $\mathbf{G}$ is Fredholm. As observed in Section 2, Theorem 1.4 of [13] then implies Hypothesis 1 for $\mathcal{U}$, where we may assume that $a = 0$ without loss of generality. Then Theorems 4.5 and 5.5 show that $\mathcal{U}$ has exponential dichotomies on $[b,\infty) \cap \mathbb{Z}_+$ and $\mathbb{Z}_-$. Lemma 1.5 of [13] (combined with a translation argument) further implies that $\mathcal{U}$ has exponential dichotomies on $\mathbb{R}_-$ and $[b,\infty)$.

We further have to prove (ii), i.e., the Fredholm property of the node operator $N(b,0) = (I - P_b^+)U(b,a) : \ker P_0^- \to \ker P_b^+$. Lemma 3.6(i) and (4.5) yield

$$\ker P_0^- = X_u^-(0) = Z_2 \oplus X_0 \quad \text{and} \quad \ker P_b^+ = X_u^+(b) = Z_2(b) \oplus Y^+(b). \tag{6.1}$$

Recall from Lemma 2.1 and (4.3) that X_0 and $Y^+(b)$ are finite dimensional. Thus the Fredholm property of $N(b,0)$ follows from the equations

$$\ker N(b,0) = X_0 \quad \text{and} \quad \operatorname{im} N(b,0) = Z_2(b). \tag{6.2}$$

For $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in \ker D$ we obtain $N(b,0)x_0 = (I - P_b^+)x_b = 0$ using (2.4), so that $X_0 \subseteq \ker N(b,0)$. Conversely, let $x \in \ker N(b,0) \subseteq \ker P_0^-$. Due to (6.1) there are $z_2 \in Z_2$ and $x_0 \in X_0$ with $x = z_2 + x_0$. We can then infer $N(b,0)z_2 = N(b,0)x = 0$ because of $X_0 \subseteq \ker N(b,0)$. Since further $U(b,0)z_2 \in Z_2(b) \subseteq X_u^+(b) = \ker P_b^+$ by (4.1) and (4.5), we arrive at $0 = N(b,0)x = U(b,0)z_2$. Now Lemma 4.1(i) shows that $z_2 = 0$, and thus $x = x_0 \in X_0$. By the same arguments we deduce $\operatorname{im} N(b,0) = N(b,0)Z_2 = (I - P_b^+)Z_2(b) = Z_2(b)$.

Finally, we want to show the index and dimension formulas in Theorem 1.1 assuming that $\mathbf{G}$ is Fredholm. Define $R_0 : \ker D \to X_0$ and $R_{b,*} : \ker D^* \to X_{b,*}$ by $R_0(x_n)_{n\in\mathbb{Z}} = x_0$ and $R_{b,*}(\xi_n)_{n\in\mathbb{Z}} = \xi_b$, respectively. The maps R_0 and $R_{b,*}$ are surjective linear operators, by (2.4) and (2.5). Lemma 2.1(iii) and (iv) then show that R_0 and $R_{b,*}$ are bijective, so that $\dim\ker D = \dim X_0$ and $\dim\ker D^* = \dim X_{b,*}$. Using Theorem 1.4 of [13] and (6.2), we conclude

$$\dim\ker \mathbf{G} = \dim\ker D = \dim X_0 = \dim\ker N(b,0).$$

Employing in addition (4.3) and (6.1), we further deduce

$$\operatorname{codim}\operatorname{im}\mathbf{G} = \operatorname{codim}\operatorname{im} D = \dim\ker D^* = \dim X_{b,*} = \dim Y^+(b) = \operatorname{codim}\operatorname{im} N(b,0).$$

Theorem 1.1 has been established. ∎

The image of $\mathbf{G}$ admits the following description in terms of trajectories $v(\tau) = U(t,\tau)^* v(t)$, i.e., the "solutions of the adjoint problem"; cf. [10] or [17]. In the following proof it is again convenient to work with D instead of $\mathbf{G}$ since we know D^* explicitely.

PROPOSITION 6.1. *Let* $\mathbf{G}$ *be Fredholm on* $\mathcal{E}(\mathbb{R})$. *Then* $f \in \operatorname{im}\mathbf{G}$ *if and only if*

$$\int_{\mathbb{R}} \langle f(\sigma), v(\sigma)\rangle \mathrm{d}\sigma = 0 \quad \forall\, v \in \mathcal{E}_*(\mathbb{R}) \cap C_{\mathrm{b}}^{w,*}(\mathbb{R}, X^*)$$

with $v(\tau) = U(t,\tau)^* v(t) \quad \forall\ t \geqslant \tau$, *where* $\mathcal{E}_*(\mathbb{R}) = \{v : \mathbb{R} \to X^* : v$ *is weakly star measurable*, $\|v(\cdot)\| \in L^q(\mathbb{R})\}$, $q = 1$ *if* $\mathcal{E}(\mathbb{R}) = C_0(\mathbb{R}, X)$, *and* $\frac{1}{p} + \frac{1}{q} = 1$ *if* $\mathcal{E}(\mathbb{R}) = L^p(\mathbb{R}, X)$ *with* $p \in [1,\infty)$.

Proof. Assume that $f \in \operatorname{im}\mathbf{G}$ and $v \in \mathcal{E}_*(\mathbb{R}) \cap C_{\mathrm{b}}^{w,*}(\mathbb{R}, X^*)$ with $v(\tau) = U(t,\tau)^* v(t)$ for all $t \geqslant \tau$. Due to Lemma 1.2, there is a function $u \in \mathcal{E}(\mathbb{R}) \cap C_0(\mathbb{R}, X)$ satisfying (1.2). So we can compute

$$\begin{aligned}
\int_\tau^t \langle f(\sigma), v(\sigma)\rangle \mathrm{d}\sigma &= \int_\tau^t \langle f(\sigma), U(t,\sigma)^* v(t)\rangle \mathrm{d}\sigma = \int_\tau^t \langle U(t,\sigma) f(\sigma), v(t)\rangle \mathrm{d}\sigma \\
&= \Big\langle \int_\tau^t U(t,\sigma) f(\sigma)\mathrm{d}\sigma, v(t)\Big\rangle = \langle U(t,\tau)u(\tau), v(t)\rangle - \langle u(t), v(t)\rangle \\
&= \langle u(\tau), v(\tau)\rangle - \langle u(t), v(t)\rangle
\end{aligned}$$

for all $t \geqslant \tau$. Letting $\tau \to -\infty$ and $t \to \infty$, we deduce that $\int_{\mathbb{R}} \langle f(\sigma), v(\sigma)\rangle \mathrm{d}\sigma = 0$ by means of $u \in C_0(\mathbb{R}, X)$ and $v \in C_{\mathrm{b}}^{w,*}(\mathbb{R}, X^*)$.

Assume that $f \in \mathcal{E}(\mathbb{R})$ satisfies the condition in the proposition. We define the operator $R : \mathcal{E}(\mathbb{R}) \to \mathcal{E}(\mathbb{Z})$ by setting

$$(Rg)_n = -\int_{n-1}^{n} U(n,\tau) g(\tau)\mathrm{d}\tau \quad \text{for all } n \in \mathbb{Z}.$$

We claim that $Rf \in \operatorname{im} D$. Since $\mathbf{G}$ is a Fredholm operator, Theorem 1.4 in [13] shows that $\operatorname{im} D$ is closed, and thus $\operatorname{im} D = (\ker D^*)^\perp$. For $\boldsymbol{\xi} = (\xi_n)_{n\in\mathbb{Z}} \in \ker D^*$, we define $v : \mathbb{R} \to X^*$ by $v(\tau) = U(n,\tau)^* \xi_n$ for $\tau \in (n-1, n]$ and $n \in \mathbb{Z}$. Due to (2.5), we obtain $v \in \mathcal{E}_*(\mathbb{R}) \cap \mathcal{C}_{\mathrm{b}}^{w,*}(\mathbb{R}, X^*)$ and $v(\tau) = U(t,\tau)^* v(t)$ for all $t \geqslant \tau$. Furthermore,

$$\langle Rf, \boldsymbol{\xi} \rangle = -\sum_{n\in\mathbb{Z}} \Big\langle \int_{n-1}^{n} U(n,\tau) f(\tau) \mathrm{d}\tau, \xi_n \Big\rangle = -\sum_{n\in\mathbb{Z}} \int_{n-1}^{n} \langle f(\tau), U(n,\tau)^* \xi_n \rangle \mathrm{d}\tau$$

$$= -\sum_{n\in\mathbb{Z}} \int_{n-1}^{n} \langle f(\tau), v(\tau) \rangle \mathrm{d}\tau = -\int_{\mathbb{R}} \langle f(\tau), v(\tau) \rangle \mathrm{d}\tau = 0,$$

proving the claim. Using Lemma 6.1(iv) of [13], we conclude that $f \in \operatorname{im} \mathbf{G}$. ∎

7. BACKWARD UNIQUENESS PROPERTY

In the following proposition we describe the backward uniqueness property (BU) (see the introduction) in terms of the spaces X_n and $X_{n,*}$.

PROPOSITION 7.1. *Assume that the operator* $\mathbf{G}$ *is Fredholm on* $\mathcal{E}(\mathbb{R})$. *Then the following assertions hold:*

(i) (BU.1) *holds if and only if* $\dim X_n$ *is constant for* $n \in \mathbb{Z}$.

(ii) (BU.2) *holds if and only if* $\dim X_{n,*}$ *is constant for* $n \in \mathbb{Z}$.

(iii) *If* (BU.1) *and* (BU.2) *hold, then we can take* $a = b = 0$ *in Theorem* 1.1.

Proof. (i) Assume that (BU.1) holds. Take $x \in X_m$ with $U(n,m)x = 0$ for some $n \geqslant m$. Then there is a sequence $\mathbf{x} = (x_k)_{k\in\mathbb{Z}} \in \ker D$ such that $x_m = x$ by (2.2). We define the function $u : \mathbb{R} \to X$ by $u(t) = U(t,j)x_j$ for $t \in [j, j+1)$ and $j \in \mathbb{Z}$. It is easy to check that $u \in \mathcal{C}_0(\mathbb{R}, X)$ and $u(t) = U(t,\tau)u(\tau)$ for all $t \geqslant \tau$ using (2.4). Since $u(n) = U(n,m)x_m = U(n,m)x = 0$, (BU.1) shows that $u(m) = x = 0$. This means that the map $U(n,m) : X_m \to X_n$ is injective, and hence it is bijective by Lemma 2.1(i). As a result, $\dim X_m = \dim X_n$ for all $n \geqslant m$.

Assume that $\dim X_n$ is constant on $\mathbb{Z}$. Let $u \in \mathcal{C}_0(\mathbb{R}, X)$ be a function satisfying $u(t) = U(t,\tau)u(\tau)$ for all $t \geqslant \tau$ and $u(\tau_0) = 0$ for some $\tau_0 \in \mathbb{R}$. Obviously, $u(t) = 0$ for all $t \geqslant \tau_0$. By Theorem 1.1, $\mathcal{U}$ has an exponential dichotomy on $(-\infty, a]$ for some $a \in \mathbb{R}$. Thus, using that $\sup_\tau \|P_\tau^-\| < \infty$, we can estimate

$$\|P_t^- u(t)\| = \|U(t,\tau) P_\tau^- u(\tau)\| \leqslant N \mathrm{e}^{-\nu(t-\tau)} \|P_\tau^- u(\tau)\| \leqslant N' \mathrm{e}^{-\nu(t-\tau)} \|u\|_\infty$$

for all $\tau \leqslant t \leqslant a$. Letting $\tau \to -\infty$, we obtain that $P_t^- u(t) = 0$, i.e., $u(t) \in X_u^-(t)$, for all $t \leqslant a$. Then we derive the inequality

$$\|u(t)\| = \|U_u^-(a,t)^{-1} u(a)\| \leqslant N \mathrm{e}^{-\nu(a-t)} \|u(a)\|$$

for all $t \leqslant a$. As a result, $(u(n))_{n\in\mathbb{Z}} \in \ker D$ which leads to $u(n) \in X_n$ for all $n \in \mathbb{Z}$ (see (2.4) and (2.2)). The identity $\dim X_n = \dim X_m$ and Lemma 2.1 then yield the invertibility of $U(n,m) : X_m \to X_n$ for all $n \geqslant m$. Thus $u(n) = 0$ for all $n \in \mathbb{Z}$ since $u(n) = 0$ for large n.

(ii) Assume that (BU.2) holds. Take $\xi \in X_{n,*}$ with $U(n,m)^*\xi = 0$. Then there is a sequence $\boldsymbol{\xi} = (\xi_k)_{k\in\mathbb{Z}} \in \ker D^*$ such that $\xi_n = \xi$ by (2.3). We define the function $v : \mathbb{R} \to X^*$ by $v(t) = U(j,t)^*\xi_j$ for $t \in (j-1, j]$ and $j \in \mathbb{Z}$. It is straightforward to see that $v \in C_{\mathrm{b}}^{w,*}(\mathbb{R}, X^*)$ and $v(\tau) = U(t,\tau)^*v(t)$ for all $t \geqslant \tau$. Since $v(m) = U(n,m)^*\xi_n = U(n,m)^*\xi = 0$, (BU.2) yields $v = 0$, and thus $\xi = 0$. Now Lemma 2.1(ii) implies that $\dim X_{n,*} = \dim X_{m,*}$ for $n \geqslant m$.

Assume that $\dim X_{n,*}$ is constant on $\mathbb{Z}$. Let $v \in C_{\mathrm{b}}^{w,*}(\mathbb{R}, X)$ satisfy $v(\tau) = U(t,\tau)^*v(t)$ for all $t \geqslant \tau$ and $v(\tau_0) = 0$ for some $\tau_0 \in \mathbb{R}$. Hence, $v(\tau) = 0$ for all $\tau \leqslant \tau_0$. Theorem 1.1 shows that $\mathcal{U}$ has an exponential dichotomy on $[b, \infty)$. This fact, due to $\sup_\tau \|P_\tau^+\| < \infty$, leads to the estimate

$$|\langle P_\tau^+ x, v(\tau)\rangle| = |\langle P_\tau^+ x, U(t,\tau)^* v(t)\rangle| = |\langle U(t,\tau)P_\tau^+ x, v(t)\rangle| \leqslant N' \mathrm{e}^{-\nu(t-\tau)} \|v\|_\infty \|x\|$$

for all $t \geqslant \tau \geqslant b$ and $x \in X$. Letting $t \to \infty$, we obtain that $\langle P_\tau^+ x, v(\tau)\rangle = 0$ for all $\tau \geqslant b$ and all $x \in X$. We can now conclude that

$$\begin{aligned}
|\langle x, v(\tau)\rangle| &= |\langle U(\tau,b)U_u^+(\tau,b)^{-1}(I-P_\tau^+)x, v(\tau)\rangle| = |\langle U_u^+(\tau,b)^{-1}(I-P_\tau^+)x, v(b)\rangle| \\
&\leqslant N\mathrm{e}^{-\nu(\tau-b)}\|I - P_\tau^+\|\|x\|\|v(b)\|, \\
\|v(\tau)\| &\leqslant c\mathrm{e}^{-\nu(\tau-b)}\|v(b)\|,
\end{aligned}$$

for all $\tau \geqslant b$ and all $x \in X$. Consequently, $(v(n))_{n\in\mathbb{Z}} \in \ker D^*$ and $v(n) \in X_{n,*}$ (see (2.5) and (2.3)). Since $\dim X_{n,*} = \dim X_{m,*}$ for all $n \geqslant m$, Lemma 2.1 implies the invertibility of $U(n,m)^* : X_{n,*} \to X_{n,*}$ for all $n \geqslant m$. So we arrive at $v(n) = 0$ for all $n \in \mathbb{Z}$, and hence $v = 0$.

(iii) The last assertion follows from (i), (ii), and the definition of a and b given after (2.5). ∎

We present the examples mentioned in the introduction. Observe that here X is a Hilbert space and $\mathcal{U}$ is generated by piecewise constant operators $A(t) = A_+$ for $t \geqslant 0$ and $A(t) = A_-$ for $t \leqslant 0$.

EXAMPLE 7.2. Let $X = L^2(\mathbb{R}_+)$, $f_0 = \chi_{[0,1]}$, and $P_0 : X \to X$, $P_0 f = \langle f, f_0\rangle f_0$, be the orthogonal projection onto $\mathrm{Span}\{f_0\}$, and set $Q_0 = I - P_0$. Define $(S_1(t)f)(\tau) = \mathrm{e}^{-t} f(t+\tau)$ for $t, \tau \geqslant 0$ and $f \in X$, and $S_2(t)f = \mathrm{e}^t P_0 f + \mathrm{e}^{-t} Q_0 f$ for $t \geqslant 0$ and $f \in X$. Let $\mathcal{U} = \{U(t,\tau)\}_{t\geqslant\tau}$ be the strongly continuous, exponentially bounded evolution family on X given by

$$U(t,\tau) = \begin{cases} S_1(t-\tau), & t \geqslant \tau \geqslant 0, \\ S_1(t)S_2(-\tau), & t \geqslant 0 \geqslant \tau, \\ S_2(t-\tau), & 0 \geqslant t \geqslant \tau. \end{cases}$$

$\mathbf{G}$ denotes the generator of the associated evolution semigroup on $L^2(\mathbb{R}, X)$.

We claim that $\dim\ker\mathbf{G} = 1$ and that, more precisely, $\ker\mathbf{G}$ is the set of functions u given by $u(t) = S_1(t)u(0)$ for $t \geqslant 0$, $u(t) = S_2(t)u(0)$ for $t \leqslant 0$, and $u(0) \in \mathrm{Span}\{f_0\}$. Indeed, if $u \in \ker\mathbf{G}$, then Lemma 1.2 shows that $u(t) =$

$U(t,0)u(0) = S_1(t)u(0)$ for all $t \geqslant 0$ and $u(0) = U(0,t)u(t) = S_2(-t)u(t)$ for all $t \leqslant 0$. Since $u \in L^2(\mathbb{R}, X)$, we must have $Q_0u(0) = 0$. The proof of the converse inclusion is straightforward. The claim is proved.

Let $f \in L^2(\mathbb{R}_+)$ and define $u : \mathbb{R} \to L^2(\mathbb{R}_+)$ by

$$u(t) = \begin{cases} -\int\limits_{-\infty}^{t} \mathrm{e}^{\tau - t} Q_0 f(\tau) \mathrm{d}\tau + \int\limits_{t}^{0} \mathrm{e}^{t-\tau} P_0 f(\tau) \mathrm{d}\tau, & t < 0, \\ -\int\limits_{0}^{t} S_1(t-\tau) f(\tau) \mathrm{d}\tau - S_1(t) Q_0 \int\limits_{-\infty}^{0} \mathrm{e}^{\tau} f(\tau) \mathrm{d}\tau, & t \geqslant 0. \end{cases}$$

Using Lemma 1.2 we see that $u \in \operatorname{dom} \mathbf{G}$ and $\mathbf{G}u = f$. Therefore $\mathbf{G}$ is surjective and thus Fredholm.

Define $u_0 \in \ker \mathbf{G}$ by $u_0(t) = \mathrm{e}^t f_0$ for $t < 0$ and $u_0(t) = S_1(t) f_0$ for $t \geqslant 0$. Then $u_0(t) = U(t,\tau)u_0(\tau)$ for all $t \geqslant \tau$. However, $u_0(0) = f_0 \neq 0$ and $(u_0(2))(\tau) = \mathrm{e}^{-2} f_0(2+\tau) = 0$ for $\tau \geqslant 0$. As a result, (BU.1) fails for $u = u_0$.

Using the adjoint of the evolution family $\mathcal{U}$ in Example 7.2, one can construct an example of the evolution family $\widetilde{\mathcal{U}}$ such that the operator $\mathbf{G}$ is Fredholm but (BU.2) fails (and, of course, for the direct sum of $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ both (BU.1) and (BU.2) fail).

EXAMPLE 7.3. With the notations in Example 7.2, we define the strongly continuous evolution family $\mathcal{V} = \{V(t,\tau)\}_{t \geqslant \tau}$ by,

$$V(t,\tau) = \begin{cases} S_2(t-\tau), & t \geqslant \tau \geqslant 1, \\ S_2(t-1) S_1(1-\tau)^*, & t \geqslant 1 \geqslant \tau, \\ S_1(t-\tau)^*, & 1 \geqslant t \geqslant \tau. \end{cases}$$

Arguing as in Example 7.2, we can establish the Fredholm property of the generator of the evolution semigroup on $L^2(\mathbb{R}, X)$ associated with $\mathcal{V}$. It is clear that $\mathcal{V}$ has exponential dichotomies on $\mathbb{R}_-$ and $[1,\infty)$ with projections $P_t^- = I$ for $t \leqslant 0$ and $P_t^+ = Q_0$ for $t \geqslant 1$, respectively. Looking for a contradiction, we suppose that $\mathcal{V}$ has an exponential dichotomy on $\mathbb{R}_+$. The definition of the exponential dichotomy implies that $X_s^+(\tau) = \{f \in L^2(\mathbb{R}_+) : V(t,\tau)f \to 0 \text{ as } t \to \infty\}$. Hence, $X_s^+(1) = \ker P_0$, so that $X_u^+(1)$ must be a (one dimensional) complement of $\ker P_0$. On the other hand, $P_0 V(1,0) = P_0 S_1(1)^* = 0$ contradicting the required surjectivity of $V(1,0) : X_u^+(0) \to X_u^+(1)$.

REMARK 7.4. In Propositions 6.1 and 7.1 we can replace $\mathcal{C}_{\mathrm{b}}^{w,*}(\mathbb{R}, X^*)$ by the space $\mathcal{C}_0^{w,*}(\mathbb{R}, X^*)$ of continuous functions vanishing at $\pm\infty$ if $\mathcal{E}(\mathbb{R}) = \mathcal{C}_0(\mathbb{R}, X)$ or $\mathcal{E}(\mathbb{R}) = L^p(\mathbb{R}, X)$ for $p \in (1,\infty)$. This fact follows from the proofs of these results because for an orbit $v(\cdot)$ satisfying $v(\tau) = U(t,\tau)^* v(t)$ for all $t \geqslant \tau$ in $\mathbb{R}$, the conditions $v \in \mathcal{C}_{\mathrm{b}}^{w,*}(\mathbb{R}, X^*)$ and $v \in \mathcal{C}_0^{w,*}(\mathbb{R}, X^*)$ are equivalent provided $\mathcal{U}$ has exponential dichotomies on $(-\infty, a]$ and $[b, \infty)$.

Acknowledgements. The authors thank Yuri Tomilov for many useful discussions and Nguyen Thieu Huy for important comments on a preliminary version of the paper. Research of the first author was supported in part by the NSF grants 0338743 and 0354339, by the CRDF grant UP1-2567-OD-03, and by the Research Board and the Research Council of the University of Missouri; also, the support of the EU Marie Curie "Transfer of Knowledge" program is gratefully acknowledged. Research of the second author was supported in part by the NSF grant 0338743. The second author wishes to thank R. Schnaubelt and the Analysis Group of Martin-Luther University for kind hospitality during his visit in Halle. The third author is supported in part by the DAAD grant D/03/36798.

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Received September 1, 2005; revised November 10, 2006.