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FINITE RANK PERTURBATIONS OF LOCALLY DEFINITIZABLE SELF-ADJOINT OPERATORS IN KREIN SPACES

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ABSTRACT. In this paper the well-known result that a definitizable operator in a Krein space remains definitizable after a finite dimensional perturbation is generalized to a class of self-adjoint operators in Krein spaces which locally have the same spectral properties as definitizable operators. As an application the spectral properties of direct sums of indefinite Sturm-Liouville operators are studied.

KEYWORDS: *Self-adjoint operators in Krein spaces, finite rank perturbations, (locally) definitizable operators, spectral points of positive and negative type, self-adjoint extensions of symmetric operators, indefinite Sturm-Liouville operators.*

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INTRODUCTION

A self-adjoint operator A in a Krein space $(\mathcal{K}, [\cdot,\cdot])$ is called *definitizable* if the resolvent set $\rho(A)$ is nonempty and there exists a polynomial p such that $[p(A)x, x] \geqslant 0$ for all $x \in \operatorname{dom}(p(A))$. Spectral and perturbation theory of definitizable operators is well developed and of great importance in many applications (see e.g. [6], [7], [8], [14], [26], [30], [31], [32], [35]). It was shown by H. Langer in [31], [32] (see also [19]) that a definitizable operator A has a spectral function and with the help of this spectral function the real points of the spectrum $\sigma(A)$ can be classified in points of positive and negative type and a finite set of so-called critical points. A fundamental paper on perturbations of definitizable operators is [26] by P. Jonas and H. Langer where it is proved that a definitizable operator remains definitizable after a finite dimensional perturbation in resolvent sense if the perturbed operator is self-adjoint and has a nonempty resolvent set.

The aim of this paper is to generalize this perturbation result to a class of self-adjoint operators in Krein spaces which locally have the same spectral properties as definitizable operators. More precisely, let Ω be some domain in $\overline{\mathbb{C}}$ symmetric with respect to the real line such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections of Ω with the upper and lower open half-planes are simply connected. A self-adjoint operator A is said to be *definitizable over* Ω if every point $\mu \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighbourhood I_μ in $\overline{\mathbb{R}}$ such that the spectral points in each component of $I_\mu \setminus \{\mu\}$ are all of the same sign type, the nonreal spectrum of A in $\Omega \setminus \overline{\mathbb{R}}$ does not accumulate to $\Omega \cap \overline{\mathbb{R}}$, consists of isolated points which are poles of the resolvent of A and the resolvent is of finite order growth near to $\Omega \cap \overline{\mathbb{R}}$ (cf. [24]). Perturbations of locally definitizable operators and stability properties of spectral points of positive and negative type and so-called spectral points or intervals of type π_+ and type π_- were investigated in e.g. [2], [4], [20], [21], [22], [34].

The main result of this note is Theorem 2.2 where we prove that a self-adjoint operator, or more generally a self-adjoint relation, which is locally definitizable over Ω remains locally definitizable over Ω after a finite dimensional perturbation in resolvent sense if the perturbed operator or relation is self-adjoint and the unperturbed and perturbed operator or relation have a common point in their resolvent sets belonging to Ω. For the special case of definitizable operators this result coincides with Theorem 1 in [26] mentioned above. The methods used in the proof of Theorem 2.2 differ from those applied in [26]. Our proof is based on a variant of Theorem 2.4 in [4] (see Theorem 2.1) on the stability of intervals of type π_+ and type π_- under compact perturbations and a recent result from [1] on the spectral properties of the inverses of certain matrix-valued functions associated to locally definitizable operators and relations.

We briefly describe the contents of this paper. In Section 1 we introduce the spectral points of positive and negative type with the help of approximative eigensequences and we recall the definitions and connections between locally definitizable self-adjoint relations and locally definitizable functions from [24] and [25]. In particular, Theorem 1.8 on the representation of a locally definitizable function with the help of the resolvent of a locally definitizable self-adjoint relation is an important tool in the proof of our main result (Theorem 2.2) which is the focus of Section 2. In Section 3.1 we apply our perturbation result to the self-adjoint extensions of symmetric operators or relations of finite defect. We use the concept of boundary value spaces and associated Weyl functions for the parametrization of the closed extensions of a symmetric relation and the description of their spectral properties (see e.g. [10], [11], [12], [13]). As an example we consider in Section 3.2 the direct sum of a regular and a singular Sturm-Liouville differential operator with the indefinite weight $\operatorname{sgn} x$ and we show in Section 3.3 that in such a general setting self-adjoint differential operators with an empty resolvent set can appear.

1. LOCALLY DEFINITIZABLE SELF-ADJOINT RELATIONS AND LOCALLY DEFINITIZABLE FUNCTIONS

1.1. PRELIMINARIES. The linear space of bounded linear operators defined on a Krein space $\mathcal{K}_1$ with values in a Krein space $\mathcal{K}_2$ is denoted by $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. If $\mathcal{K} := \mathcal{K}_1 = \mathcal{K}_2$ we simply write $\mathcal{L}(\mathcal{K})$. We study linear relations from $\mathcal{K}_1$ to $\mathcal{K}_2$, that is, linear subspaces of $\mathcal{K}_1 \times \mathcal{K}_2$. The set of all closed linear relations from $\mathcal{K}_1$ to $\mathcal{K}_2$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{K}_1, \mathcal{K}_2)$. If $\mathcal{K} = \mathcal{K}_1 = \mathcal{K}_2$ we write $\widetilde{\mathcal{C}}(\mathcal{K})$. Linear operators from $\mathcal{K}_1$ into $\mathcal{K}_2$ are viewed as linear relations via their graphs. For the usual definitions of the linear operations with relations, the inverse etc., we refer to [15]. The sum and the direct sum of subspaces in $\mathcal{K}_1 \times \mathcal{K}_2$ will be denoted by $\hat{+}$ and $\dot{\hat{+}}$.

In the following let $(\mathcal{K}, [\cdot, \cdot])$ be a separable Krein space and let S be a closed linear relation in $\mathcal{K}$. The *resolvent set* $\rho(S)$ of S is the set of all $\lambda \in \mathbb{C}$ such that $(S - \lambda)^{-1} \in \mathcal{L}(\mathcal{K})$, the *spectrum* $\sigma(S)$ of S is the complement of $\rho(S)$ in $\mathbb{C}$. The *extended spectrum* $\widetilde{\sigma}(S)$ of S is defined by $\widetilde{\sigma}(S) = \sigma(S)$ if $S \in \mathcal{L}(\mathcal{K})$ and $\widetilde{\sigma}(S) = \sigma(S) \cup \{\infty\}$ otherwise. We say that $\lambda \in \mathbb{C}$ is a *point of regular type* of S, $\lambda \in r(S)$, if $(S - \lambda)^{-1}$ is a bounded operator. A point $\lambda \in \mathbb{C}$ is an *eigenvalue* of the relation S if $\ker(S - \lambda) \neq \{0\}$; we write $\lambda \in \sigma_{\mathrm{p}}(S)$. We say that $\lambda \in \mathbb{C}$ belongs to the *continuous spectrum* $\sigma_{\mathrm{c}}(S)$ (the *residual spectrum* $\sigma_{\mathrm{r}}(S)$) of S if $\ker(S - \lambda) = \{0\}$ and $\operatorname{ran}(S - \lambda)$ is dense in $\mathcal{K}$ (respectively if $\ker(S - \lambda) = \{0\}$ and $\operatorname{ran}(S - \lambda)$ is not dense in $\mathcal{K}$). An eigenvalue $\lambda \in \mathbb{C}$ of a closed linear relation S is called *normal* if the root manifold $\mathcal{L}_\lambda$ corresponding to λ is finite-dimensional and there is a projection P with $P\mathcal{K} = \mathcal{L}_\lambda$ such that

$$S = S \cap (P\mathcal{K})^2 \dot{\hat{+}} S \cap ((I - P)\mathcal{K})^2$$

and $\lambda \in \rho(S \cap ((1 - P)\mathcal{K})^2)$. The set of normal eigenvalues of S will be denoted by $\sigma_{\mathrm{p,norm}}(S)$.

We say that $\lambda \in \mathbb{C}$ belongs to the *approximate point spectrum of* S, denoted by $\sigma_{\mathrm{ap}}(S)$, if there exists a sequence $\binom{x_n}{y_n} \in S$, $n = 1, 2, \dots$, such that $\|x_n\| = 1$ and $\lim\limits_{n\to\infty} \|y_n - \lambda x_n\| = 0$. The *extended approximate point spectrum* $\widetilde{\sigma}_{\mathrm{ap}}(S)$ *of* S is defined by

$$\widetilde{\sigma}_{\mathrm{ap}}(S) := \begin{cases} \sigma_{\mathrm{ap}}(S) \cup \{\infty\} & \text{if} \quad 0 \in \sigma_{\mathrm{ap}}(S^{-1}), \\ \sigma_{\mathrm{ap}}(S) & \text{if} \quad 0 \notin \sigma_{\mathrm{ap}}(S^{-1}). \end{cases}$$

Next we recall the definitions of the spectra of positive and negative type of a closed linear relation (see e.g. [24], [34]).

DEFINITION 1.1. Let S be a closed linear relation in $\mathcal{K}$. A point $\lambda \in \sigma_{\mathrm{ap}}(S)$ is said to be of *positive type* (*negative type*) with respect to S, if for every sequence $\binom{x_n}{y_n} \in S$, $n = 1, 2, \dots$, with $\|x_n\| = 1$, $\lim\limits_{n\to\infty} \|y_n - \lambda x_n\| = 0$ we have

$$\liminf_{n\to\infty} \, [x_n, x_n] > 0 \quad \text{(respectively } \limsup_{n\to\infty} \, [x_n, x_n] < 0\text{).}$$

If $\infty \in \widetilde{\sigma}_{\rm ap}(S)$, ∞ is said to be of *positive type* (*negative type*) with respect to S if for every sequence $\binom{x_n}{y_n} \in S$, $n = 1, 2, \ldots$, with $\lim_{n\to\infty} \|x_n\| = 0$, $\|y_n\| = 1$ we have

$$\liminf_{n\to\infty}\, [y_n, y_n] > 0 \quad \text{(respectively } \limsup_{n\to\infty}\, [y_n, y_n] < 0\text{)}.$$

The set of all points of positive type (negative type) with respect to S will be denoted by $\sigma_{++}(S)$ (respectively $\sigma_{--}(S)$).

Note, that $\infty \in \widetilde{\sigma}_{\rm ap}(S)$ is of positive (negative) type with respect to S if and only if 0 is of positive (respectively negative) type with respect to S^{-1}.

Let S be a linear relation in $\mathcal{K}$. The *adjoint relation* $S^+ \in \widetilde{\mathcal{C}}(\mathcal{K})$ is defined by

$$S^+ := \left\{ \begin{pmatrix} h \\ h' \end{pmatrix} : [h, f'] = [h', f] \ \text{ for all } \begin{pmatrix} f \\ f' \end{pmatrix} \in S \right\}.$$

A linear relation A in $\mathcal{K}$ is said to be *symmetric* (*self-adjoint*) if $A \subset A^+$ (respectively $A = A^+$). We remark that for a self-adjoint relation A the points of positive and negative type introduced in Definition 1.1 belong to $\overline{\mathbb{R}}$. An open subset Δ of $\overline{\mathbb{R}}$ is said to be of *positive type* (*negative type*) with respect to the self-adjoint relation A if

$$\Delta \cap \widetilde{\sigma}(A) \subset \sigma_{++}(A) \quad \text{(respectively } \Delta \cap \widetilde{\sigma}(A) \subset \sigma_{--}(A)\text{)}.$$

An open subset Δ of $\overline{\mathbb{R}}$ is called of *definite type* with respect to A if Δ is of positive or negative type with respect to A.

1.2. LOCALLY DEFINITIZABLE SELF-ADJOINT RELATIONS. Let Ω be some domain in $\overline{\mathbb{C}}$ symmetric with respect to the real axis such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections of Ω with the upper and lower open half-planes are simply connected.

Let A be a self-adjoint relation in the Krein space $\mathcal{K}$ such that $\sigma(A) \cap (\Omega \backslash \overline{\mathbb{R}})$ consists of isolated points which are poles of the resolvent of A, and no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of the nonreal spectrum of A in Ω. Let Δ be an open subset of $\Omega \cap \overline{\mathbb{R}}$. We say that A belongs to the class $S^\infty(\Delta)$, if for every finite union Δ' of open connected subsets, $\overline{\Delta'} \subset \Delta$, there exists $m \geqslant 1$, $M > 0$ and an open neighbourhood $\mathcal{U}$ of $\overline{\Delta'}$ in $\overline{\mathbb{C}}$ such that

$$\|(A - \lambda)^{-1}\| \leqslant M(1 + |\lambda|)^{2m-2}\, |\mathrm{Im}\, \lambda|^{-m} \tag{1.1}$$

holds for all $\lambda \in \mathcal{U} \backslash \overline{\mathbb{R}}$. The next definition can be found in e.g. [21].

DEFINITION 1.2. Let Ω be a domain as above and let A be a self-adjoint relation in the Krein space $\mathcal{K}$ such that $\sigma(A) \cap (\Omega \backslash \overline{\mathbb{R}})$ consists of isolated points which are poles of the resolvent of A and no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of the nonreal spectrum of A in Ω. The relation A is said to be *definitizable over Ω*, if $A \in S^\infty(\Omega \cap \overline{\mathbb{R}})$ and every point $\mu \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighbourhood I_μ in $\overline{\mathbb{R}}$ such that both components of $I_\mu \backslash \{\mu\}$ are of definite type with respect to A.

By Theorem 4.7 in [24] a self-adjoint relation A is definitizable over $\overline{\mathbb{C}}$ if and only if A is *definitizable*, that is, the resolvent set of A is nonempty and there exists a real polynomial p such that

$$[p(A)x, x] \geqslant 0$$

holds for all $x \in \operatorname{dom}(p(A))$. For a detailed study of definitizable self-adjoint operators and relations we refer to the fundamental paper [32] of H. Langer and to Sections 4 and 5 in [16]. The next theorem is a simple modification of Theorem 4.8 in [24].

THEOREM 1.3. *Let A be a self-adjoint relation in $\mathcal{K}$ and let Ω be a domain as above. A is definitizable over Ω if and only if for every domain Ω' with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, there exists a self-adjoint projection E in $\mathcal{K}$ such that A can be decomposed in*

$$A = (A \cap (E\mathcal{K})^2) \dotplus (A \cap ((1-E)\mathcal{K})^2)$$

and the following holds:

(i) *$A \cap (E\mathcal{K})^2$ is a definitizable relation in the Krein space $E\mathcal{K}$.*

(ii) *$\widetilde{\sigma}(A \cap ((1-E)\mathcal{K})^2) \cap \Omega' = \emptyset$.*

Let $A = A^+$ be definitizable over Ω, let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and let E be a self-adjoint projection with the properties as in Theorem 1.3. If $E'(\cdot)$ is the spectral function of the definitizable self-adjoint relation $A \cap (E\mathcal{K})^2$ in the Krein space $E\mathcal{K}$, then the mapping

$$\delta \mapsto E'(\delta)E =: E_A(\delta)$$

defined for all finite unions δ of connected subsets of $\Omega' \cap \overline{\mathbb{R}}$ the endpoints of which belong to $\Omega' \cap \overline{\mathbb{R}}$ and are of definite type with respect to $A \cap (E\mathcal{K})^2$, is the spectral function of A on $\Omega' \cap \overline{\mathbb{R}}$ (see Section 3.4 and Remark 4.9 in [24]). With the help of the local spectral function $E_A(\cdot)$ the open subsets of definite type in $\Omega' \cap \overline{\mathbb{R}}$ can be characterized in the following way. An open subset $\Delta \subset \Omega' \cap \overline{\mathbb{R}}$ is of positive type (negative type) with respect to A if and only if for every finite union δ of open connected subsets of Δ, $\overline{\delta} \subset \Delta$, such that the boundary points of δ in $\overline{\mathbb{R}}$ are of definite type with respect to A, the spectral subspace $(E_A(\delta)\mathcal{K}, [\cdot,\cdot])$ (respectively $(E_A(\delta)\mathcal{K}, -[\cdot,\cdot])$) is a Hilbert space.

As a generalization of open subsets of positive and negative type we introduce open subsets of type π_+ and type π_- in the next definition, see e.g. [21].

DEFINITION 1.4. Let Ω be a domain as in the beginning of this section and let A be a self-adjoint relation in $\mathcal{K}$ which is definitizable over Ω. An open subset Δ of $\Omega \cap \overline{\mathbb{R}}$ is said to be of *type* π_+ (*type* π_-) with respect to A if for every finite union δ of open connected subsets of Δ, $\overline{\delta} \subset \Delta$, such that the boundary points of δ in $\overline{\mathbb{R}}$ are of definite type with respect to A the spectral subspace $(E_A(\delta)\mathcal{K}, [\cdot,\cdot])$ is a Pontryagin space with finite rank of negativity (respectively positivity). We shall say that A is of *type π_+ over Ω* (*type π_- over Ω*) if $\Omega\cap\overline{\mathbb{R}}$ is of type π_+ (respectively type π_-) with respect to A.

We remark that spectral points in open subsets of type π_+ and type π_- can also be characterized with the help of approximative eigensequences (see [2]).

1.3. MATRIX-VALUED LOCALLY DEFINITIZABLE FUNCTIONS. Let Ω be a domain as in the beginning of Section 1.2 and let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued piecewise meromorphic function in $\Omega\backslash\overline{\mathbb{R}}$ which is symmetric with respect to the real axis, that is $\tau(\overline{\lambda}) = \tau(\lambda)^*$ for all points λ of holomorphy of τ. If, in addition, no point of $\Omega\cap\overline{\mathbb{R}}$ is an accumulation point of nonreal poles of τ we write $\tau\in M^{n\times n}(\Omega)$. The set of the points of holomorphy of τ in $\Omega\backslash\overline{\mathbb{R}}$ and all points $\mu\in\Omega\cap\mathbb{R}$ such that τ can be analytically continued to μ and the continuations from $\Omega\cap\mathbb{C}^+$ and $\Omega\cap\mathbb{C}^-$ coincide, is denoted by $\mathfrak{h}(\tau)$.

The following definition of open sets of positive and negative type with respect to matrix functions and Definition 1.6 below of locally definitizable matrix functions can be found in [25].

DEFINITION 1.5. Let $\tau\in M^{n\times n}(\Omega)$. An open subset $\Delta\subset\Omega\cap\overline{\mathbb{R}}$ is said to be of *positive type* with respect to τ if for every $x\in\mathbb{C}^n$ and every sequence (μ_k) of points in $\Omega\cap\mathbb{C}^+\cap\mathfrak{h}(\tau)$ which converges in $\overline{\mathbb{C}}$ to a point of Δ we have

$$\liminf_{k\to\infty}\,\mathrm{Im}\,(\tau(\mu_k)x,x)\geqslant 0.$$

An open subset $\Delta\subset\Omega\cap\overline{\mathbb{R}}$ is said to be of *negative type* with respect to τ if Δ is of positive type with respect to $-\tau$. Δ is said to be of *definite type* with respect to τ if Δ is of positive or of negative type with respect to τ.

DEFINITION 1.6. A function $\tau\in M^{n\times n}(\Omega)$ is called *definitizable in Ω* if the following holds:

(i) Every point $\mu\in\Omega\cap\overline{\mathbb{R}}$ has an open connected neighbourhood I_μ in $\overline{\mathbb{R}}$ such that both components of $I_\mu\backslash\{\mu\}$ are of definite type with respect to τ.

(ii) For every finite union Δ of open connected subsets in $\overline{\mathbb{R}}$, $\overline{\Delta}\subset\Omega\cap\overline{\mathbb{R}}$, there exist $m\geqslant 1$, $M>0$ and an open neighbourhood $\mathcal{U}$ of $\overline{\Delta}$ in $\overline{\mathbb{C}}$ such that

$$\|\tau(\lambda)\|\leqslant M(1+|\lambda|)^{2m}\,|\mathrm{Im}\,\lambda|^{-m}$$

holds for all $\lambda\in\mathcal{U}\backslash\overline{\mathbb{R}}$.

If $\tau\in M^{n\times n}(\overline{\mathbb{C}})$ is definitizable in $\overline{\mathbb{C}}$ there exists a scalar rational function g symmetric with respect to the real line such that the poles of g belong to the set $\mathfrak{h}(\tau)\cup\{\infty\}$ and $g\tau$ is the sum of a Nevanlinna function and a meromorphic function in $\overline{\mathbb{C}}$ (cf. Theorem 4.7 in [24]). In this case we shall say that τ is a *definitizable function*. For a comprehensive study of definitizable functions we refer to [23]. A famous subclass of the definitizable functions are the *generalized Nevanlinna functions* introduced and studied by M.G. Krein and H. Langer (see e.g. [29]). Recall that a function $\tau \in M^{n\times n}(\overline{\mathbb{C}})$ belongs to the class N_κ, $\kappa = 0, 1, 2, \ldots$, if the kernel K_τ,

$$K_\tau(\lambda, \mu) := \frac{\tau(\lambda) - \tau(\overline{\mu})}{\lambda - \overline{\mu}},$$

has κ negative squares. Note that the class N_0 coincides with the class of Nevanlinna functions.

In [25] it is proved that a function $\tau \in M^{n\times n}(\Omega)$ is definitizable in Ω if and only if for every finite union Δ of open connected subsets of $\overline{\mathbb{R}}$ such that $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$, τ can be written as the sum $\tau = \tau_0 + \tau_{(0)}$ of an $\mathcal{L}(\mathbb{C}^n)$-valued definitizable function τ_0 and an $\mathcal{L}(\mathbb{C}^n)$-valued function $\tau_{(0)}$ which is locally holomorphic on $\overline{\Delta}$.

Let $\tau \in M^{n\times n}(\Omega)$. We shall say that an open subset $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ is of *type* π_+ with respect to τ if for every open set δ which is the union of a finite number of pairwise disjoint connected open subsets of Δ such that $\overline{\delta} \subset \Delta$, τ can be written as the sum $\tau = \tau_0 + \tau_{(0)}$ of an $\mathcal{L}(\mathbb{C}^n)$-valued generalized Nevanlinna function τ_0 and an $\mathcal{L}(\mathbb{C}^n)$-valued function $\tau_{(0)}$ which is locally holomorphic on $\overline{\delta}$. We shall say that an open subset $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ is of *type* π_- with respect to τ if Δ is of type π_+ with respect to $-\tau$.

The following theorem is a consequence of the considerations in Section 3.1 in [25] and Theorem 3.18 in [24]. It establishes a connection between self-adjoint relations which are locally definitizable and $\mathcal{L}(\mathbb{C}^n)$-valued locally definitizable functions. For the convenience of the reader we give a short proof.

THEOREM 1.7. *Let Ω be a domain as above and let A be a self-adjoint relation in the Krein space $\mathcal{K}$ which is definitizable over Ω. Let $\gamma \in \mathcal{L}(\mathbb{C}^n, \mathcal{K})$ and $S = S^* \in \mathcal{L}(\mathbb{C}^n)$, fix some point $\lambda_0 \in \rho(A) \cap \Omega$ and define*

$$\tau(\lambda) := S + \gamma^+((\lambda - \operatorname{Re}\lambda_0) + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A - \lambda)^{-1})\gamma$$

for all $\lambda \in \rho(A) \cap \Omega$. Then the following holds:

(i) *The function τ is definitizable in Ω.*

(ii) *An open subset Δ of $\Omega \cap \overline{\mathbb{R}}$ which is of positive type (negative type) with respect to A is of positive type (respectively negative type) with respect to τ.*

(iii) *An open subset Δ of $\Omega \cap \overline{\mathbb{R}}$ which is of type π_+ (type π_-) with respect to A is of type π_+ (respectively type π_-) with respect to τ.*

Proof. In order to show assertions (i) and (ii) let $\mu \in \Omega \cap \overline{\mathbb{R}}$ and choose an open connected neighbourhood I_μ of μ in $\overline{\mathbb{R}}$ such that both components of $I_\mu \backslash \{\mu\}$ are of definite type with respect to A. Assume e.g. that a component Δ_+ of $I_\mu \backslash \{\mu\}$ is of positive type with respect to A. Let $x \in \mathbb{C}^n$ and let (λ_k) be a sequence in $\Omega \cap \mathbb{C}^+ \cap \mathfrak{h}(\tau)$ which converges to some point in Δ_+. Making use of

Theorem 3.18 in [24] we obtain

$$\begin{aligned}&\liminf_{k\to\infty} \operatorname{Im}\,(\tau(\lambda_k)x,x)\\&\qquad = \liminf_{k\to\infty}\, \operatorname{Im}\,[((\lambda_k-\operatorname{Re}\lambda_0)+(\lambda_k-\lambda_0)(\lambda_k-\overline{\lambda}_0)(A-\lambda_k)^{-1})\gamma x,\gamma x]\geqslant 0\end{aligned}$$

and this implies that Δ_+ is of positive type with respect to τ. A similar reasoning shows that a component Δ_- of $I_\mu\backslash\{\mu\}$ which is of negative type with respect to A is also of negative type with respect to τ. Therefore property (i) of Definition 1.6 is fulfilled and assertion (ii) is proved. The growth properties of the resolvent of A (see (1.1)) imply condition (ii) of Definition 1.6 and hence τ is locally definitizable in Ω.

It remains to prove assertion (iii). Let δ be a finite union of open connected subsets of Δ, $\overline{\delta}\subset\Delta$, and choose a finite union δ_1 of open connected subsets of Δ such that $\overline{\delta}\subset\delta_1$, $\overline{\delta}_1\subset\Delta$ and the boundary points of δ_1 in $\overline{\mathbb{R}}$ are of definite type with respect to A. As A is of type π_+ over Δ the spectral subspace $(E_A(\delta_1),[\cdot,\cdot])$ is a Pontryagin space with finite rank of negativity. Therefore

$$\tau_0(\lambda):=S+\gamma^+((\lambda-\operatorname{Re}\lambda_0)+(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(A-\lambda)^{-1})E_A(\delta_1)\gamma$$

is a generalized Nevanlinna function and from $\widetilde{\sigma}(A\cap((1-E_A(\delta_1))\mathcal{K}^2)\cap\overline{\delta}=\emptyset$ we obtain that

$$\tau_{(0)}(\lambda):=\gamma^+((\lambda-\operatorname{Re}\lambda_0)+(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(A-\lambda)^{-1})(1-E_A(\delta_1))\gamma$$

is holomorphic in a neighbourhood of $\overline{\delta}$. Hence Δ is of type π_+ with respect to τ. A similar argument shows that an open subset $\Delta\subset\Omega\cap\overline{\mathbb{R}}$ which is of type π_- with respect to A is also of type π_- with respect to τ. ∎

The next theorem states that a locally definitizable function can be represented with the help of the resolvent of a locally definitizable self-adjoint relation. A proof can be found in [25].

THEOREM 1.8. *Let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued function definitizable in Ω (an $\mathcal{L}(\mathbb{C}^n)$-valued local generalized Nevanlinna function in Ω) and let Ω' be a domain with the same properties as Ω such that $\overline{\Omega'}\subset\Omega$.*

Then there exists a Krein space $\mathcal{G}$, a self-adjoint relation T in $\mathcal{G}$ which is definitizable over Ω' (respectively of type π_+ over Ω') and a mapping $\gamma\in\mathcal{L}(\mathbb{C}^n,\mathcal{G})$ with the following properties:

(i) $\rho(T)\cap\Omega'=\mathfrak{h}(\tau)\cap\Omega'$.

(ii) *For a fixed $\lambda_0\in\rho(T)\cap\Omega'$ and all $\lambda\in\rho(T)\cap\Omega'$ we have*

$$\tau(\lambda)=\operatorname{Re}\tau(\lambda_0)+\gamma^+((\lambda-\operatorname{Re}\lambda_0)+(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(T-\lambda)^{-1})\gamma.$$

(iii) *For any finite union Δ of open connected subsets of $\overline{\mathbb{R}}$, $\overline{\Delta}\subset\Omega'\cap\overline{\mathbb{R}}$, such that the boundary points of Δ are of definite type with respect to τ the spectral projection $E_T(\Delta)$ is defined. If Ω'' is a domain with the same properties as Ω, $\overline{\Omega''}\subset\Omega'$, and*

$E_T(\overline{\Omega''}\backslash\overline{\mathbb{R}})$ is the Riesz-Dunford projection corresponding to $\sigma(T) \cap \overline{\Omega''}\backslash\overline{\mathbb{R}}$ and if we set $E := E_T(\Delta) + E_T(\overline{\Omega''}\backslash\overline{\mathbb{R}})$, then the following minimality condition is fulfilled:

$$E\mathcal{G} = \mathrm{clsp}\,\{(1 + (\lambda - \lambda_0)(T - \lambda)^{-1})E\gamma x : \lambda \in \rho(T) \cap \Omega',\ x \in \mathbb{C}^n\}.$$

(iv) *Any finite union Δ of open connected subsets of $\overline{\mathbb{R}}$, $\overline{\Delta} \subset \Omega' \cap \overline{\mathbb{R}}$, is of positive type (negative type, type π_+, type π_-) with respect to τ if and only if Δ is of positive type (respectively negative type, type π_+, type π_-) with respect to T.*

If τ and T are as in Theorem 1.8 we shall say that T is an *Ω'-minimal representing relation* for τ.

2. FINITE RANK PERTURBATIONS OF LOCALLY DEFINITIZABLE SELF-ADJOINT RELATIONS IN KREIN SPACES

In [26] P. Jonas and H. Langer proved that a self-adjoint definitizable operator remains definitizable after a finite-dimensional perturbation in resolvent sense if the perturbed operator is self-adjoint and the unperturbed and perturbed operator have a common point in their resolvent sets. In this section we prove that this holds also for locally definitizable operators and relations. The methods we apply here differ from those used in the proof of Theorem 1 in [26], where a definitizing polynomial for the perturbed operator was constructed. The essential ingredients in the proof of Theorem 2.2 below are Theorem 2.5 in [1] which states that the inverse of a matrix-valued locally definitizable function is also locally definitizable, Theorem 1.8 on the representation of locally definitizable functions and a variant of Theorem 2.4 in [4] on the stability of intervals of type π_+ and type π_- under compact perturbations (see Theorem 2.1).

Let $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space. The set of compact operators and finite rank operators defined on $\mathcal{K}$ with values in $\mathcal{K}$ will be denoted by $\mathfrak{S}_\infty$ and $\mathcal{F}$, respectively. Let, as in Section 1.2, Ω be some domain in $\overline{\mathbb{C}}$ symmetric with respect to the real axis such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections of Ω with the upper and lower open half-planes are simply connected.

The following theorem is a simple modification of Theorem 2.4 in [4] (see also Theorem 29 in [2] and Theorem 5.1 in [34] for bounded operators).

THEOREM 2.1. *Let A and B be self-adjoint relations in the Krein space $\mathcal{K}$, let $\rho(A) \cap \rho(B) \cap \Omega \neq \emptyset$ and assume that*

$$(B - \lambda_0)^{-1} - (A - \lambda_0)^{-1} \in \mathfrak{S}_\infty.$$

holds for some $\lambda_0 \in \rho(A) \cap \rho(B)$. Then A is definitizable over Ω, $\Omega \cap \overline{\mathbb{R}}$ is of type π_+ (type π_-) with respect to A and $\sigma(A) \cap (\Omega\backslash\overline{\mathbb{R}}) \subset \sigma_{\mathrm{p,norm}}(A)$ if and only if B is definitizable over Ω, $\Omega \cap \overline{\mathbb{R}}$ is of type π_+ (respectively type π_-) with respect to B and $\sigma(B) \cap (\Omega\backslash\overline{\mathbb{R}}) \subset \sigma_{\mathrm{p,norm}}(B)$.

The next theorem is the main result of this paper.

THEOREM 2.2. *Let A and B be self-adjoint relations in the Krein space $\mathcal{K}$, let $\rho(A) \cap \rho(B) \cap \Omega \neq \emptyset$ and assume that*

$$(B-\lambda_0)^{-1} - (A-\lambda_0)^{-1} \in \mathcal{F}$$

holds for some $\lambda_0 \in \rho(A) \cap \rho(B)$. Then A is definitizable over Ω if and only if B is definitizable over Ω.

Moreover, if A is definitizable over Ω and $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ is an open interval with endpoint $\mu \in \Omega \cap \overline{\mathbb{R}}$ and Δ is of positive type (negative type) with respect to A, then there exists an open interval Δ', $\Delta' \subset \Delta$, with endpoint μ such that Δ' is of positive type (respectively negative type) with respect to B.

Proof. **1.** Assume that A is a self-adjoint relation in $\mathcal{K}$ which is definitizable over Ω. Let $\lambda_0 \in \rho(A) \cap \rho(B)$ and let $e_1, \dots, e_n, f_1, \dots, f_n$ be vectors in $\mathcal{K}$ such that

$$(B-\lambda_0)^{-1} - (A-\lambda_0)^{-1} = \sum_{i=1}^{n} [\cdot, e_i] f_i. \tag{2.1}$$

It is no restriction to assume that the system $\{f_1, \dots, f_n\}$ as well as the system $\{e_1, \dots, e_n\}$ is linearly independent. For $\lambda \in \rho(A) \cap \rho(B)$ the assumption that λ_0 belongs to $\rho(A) \cap \rho(B)$ implies that the vectors

$$(1 + (\lambda - \lambda_0)(A-\lambda)^{-1}) f_i, \quad i = 1, \dots, n, \tag{2.2}$$

as well as the vectors $(1 + (\lambda - \lambda_0)(B-\lambda)^{-1}) e_i, \quad i = 1, \dots, n$, and

$$(1 + (\lambda - \overline{\lambda}_0)(B-\lambda)^{-1}) e_i, \quad i = 1, \dots, n, \tag{2.3}$$

are also linearly independent. From

$$\begin{aligned} &(B-\lambda)^{-1} - (A-\lambda)^{-1} \\ &= (1+(\lambda-\lambda_0)(A-\lambda)^{-1})((B-\lambda_0)^{-1} - (A-\lambda_0)^{-1})(1+(\lambda-\lambda_0)(B-\lambda)^{-1}) \end{aligned} \tag{2.4}$$

we obtain that

$$(B-\lambda)^{-1} - (A-\lambda)^{-1} \in \mathcal{F}$$

holds for every $\lambda \in \rho(A) \cap \rho(B)$. Hence it is no restriction to assume in the following that $\lambda_0 \in \rho(A) \cap \rho(B)$ belongs to Ω.

Since A is definitizable over Ω the set $\sigma(A) \cap (\Omega \backslash \overline{\mathbb{R}})$ is discrete, consists of poles of $\lambda \mapsto (A-\lambda)^{-1}$ and does not accumulate to points in $\Omega \cap \overline{\mathbb{R}}$. Well known perturbation results imply

$$(\rho(A) \cup \sigma_{\mathrm{p,norm}}(A)) \cap (\Omega \backslash \overline{\mathbb{R}}) = (\rho(B) \cup \sigma_{\mathrm{p,norm}}(B)) \cap (\Omega \backslash \overline{\mathbb{R}}),$$

see e.g. [17], and furthermore each point $\nu \in \sigma(A) \backslash \sigma_{\mathrm{p,norm}}(A)$ in $\Omega \backslash \overline{\mathbb{R}}$ is also an accumulation point of $\rho(B)$.

Inserting (2.1) in (2.4) and using the self-adjointness of A and B yields

$$
\begin{aligned}
&(B-\lambda)^{-1}-(A-\lambda)^{-1} \\
&= \sum_{i=1}^{n}[\cdot,(1+(\overline{\lambda}-\overline{\lambda}_0)(B-\overline{\lambda})^{-1})e_i](1+(\lambda-\lambda_0)(A-\lambda)^{-1})f_i \\
&= \sum_{i=1}^{n}[\cdot,(1+(\overline{\lambda}-\lambda_0)(A-\overline{\lambda})^{-1})f_i](1+(\lambda-\overline{\lambda}_0)(B-\lambda)^{-1})e_i
\end{aligned} \tag{2.5}
$$

for all $\lambda \in \rho(A)\cap\rho(B)$. Replacing λ and λ_0 in (2.4) by $\overline{\lambda}$ and $\overline{\lambda}_0$, respectively, and inserting the adjoint in (2.1) gives

$$
(B-\lambda)^{-1}-(A-\lambda)^{-1}=\sum_{i=1}^{n}[\cdot,(1+(\overline{\lambda}-\overline{\lambda}_0)(A-\overline{\lambda})^{-1})e_i](1+(\lambda-\lambda_0)(B-\lambda)^{-1})f_i \tag{2.6}
$$

for all $\lambda \in \rho(A)\cap\rho(B)$. Define $\mathcal{K}'$ as

$$
\mathcal{K}' := \operatorname{clsp}\{(1+(\lambda-\lambda_0)(A-\lambda)^{-1})f_i \mid i=1,\dots,n,\ \lambda\in\rho(A)\cap\rho(B)\cap\Omega\}. \tag{2.7}
$$

By (2.6) we get

$$
\mathcal{K}' = \operatorname{clsp}\{(1+(\lambda-\lambda_0)(B-\lambda)^{-1})f_i \mid i=1,\dots,n,\ \lambda\in\rho(A)\cap\rho(B)\cap\Omega\}. \tag{2.8}
$$

If $\mu_0 \in \rho(A)\cap\rho(B)\cap\Omega$, then $(A-\mu_0)^{-1}\mathcal{K}' \subset \mathcal{K}'$ and $(A-\mu_0)^{-1}\mathcal{K}'^{[\perp]} \subset \mathcal{K}'^{[\perp]}$. For $x \in \mathcal{K}'^{[\perp]}$ relation (2.7) implies

$$
[x,(1+(\overline{\lambda}-\lambda_0)(A-\overline{\lambda})^{-1})f_i] = 0
$$

for all $\lambda \in \rho(A)\cap\rho(B)\cap\Omega$ and $i=1,\dots,n$. Therefore (2.5) yields

$$
(A-\lambda)^{-1}|\mathcal{K}'^{[\perp]} = (B-\lambda)^{-1}|\mathcal{K}'^{[\perp]}, \quad \lambda\in\rho(A)\cap\rho(B)\cap\Omega. \tag{2.9}
$$

2. In this part of the proof we show that there exists an invertible $\mathcal{L}(\mathbb{C}^n)$-valued function α and mappings $\Gamma_{\lambda_0}, \widetilde{\Gamma}_{\lambda_0} \in \mathcal{L}(\mathbb{C}^n,\mathcal{K})$ such that

$$
\begin{aligned}
-\alpha(\lambda)^{-1} &= \operatorname{Re}(-\alpha(\lambda_0)^{-1}) + \Gamma_{\lambda_0}^{+}((\lambda-\operatorname{Re}\lambda_0)+(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(A-\lambda)^{-1})\Gamma_{\lambda_0}, \\
\alpha(\lambda) &= \operatorname{Re}\alpha(\lambda_0) + \widetilde{\Gamma}_{\lambda_0}^{+}((\lambda-\operatorname{Re}\lambda_0)+(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(B-\lambda)^{-1})\widetilde{\Gamma}_{\lambda_0},
\end{aligned}
$$

holds for all $\lambda \in \rho(A)\cap\rho(B)$. Some of the following calculations can be found in Proof of Theorem 5.1 in [28] and Proof of Proposition 2.1 in [36]. For the convenience of the reader we present the details.

By (2.2) and (2.3) the vectors

$$
\begin{aligned}
&(1+(\lambda-\lambda_0)(A-\lambda)^{-1})f_j, \quad j=1,\dots,n, \text{ and} \\
&(1+(\lambda-\overline{\lambda}_0)(B-\lambda)^{-1})e_i, \quad i=1,\dots,n,
\end{aligned}
$$

are linearly independent for every $\lambda \in \rho(A)\cap\rho(B)$. Hence for $\lambda \in \rho(A)\cap\rho(B)$ there exists an invertible matrix

$$
\alpha(\lambda) = (\alpha_{ij}(\lambda))_{i,j=1}^{n}
$$

such that

$$(1+(\lambda-\overline{\lambda}_0)(B-\lambda)^{-1})e_i=\sum_{j=1}^{n}\alpha_{ji}(\lambda)(1+(\lambda-\lambda_0)(A-\lambda)^{-1})f_j$$

holds for all $i=1,\dots,n$. Let $\Gamma_{\lambda_0}:\mathbb{C}^n\to\mathcal{K}$, $(c_1,\dots,c_n)^\top\mapsto\sum_{i=1}^{n}c_if_i$ and define

$$\Gamma_\lambda:=(1+(\lambda-\lambda_0)(A-\lambda)^{-1})\Gamma_{\lambda_0}$$

for all $\lambda\in\rho(A)$. Then $\Gamma_{\overline{\lambda}}^+:\mathcal{K}\to\mathbb{C}^n$ is given by

$$x\mapsto\begin{pmatrix}[x,(1+(\overline{\lambda}-\lambda_0)(A-\overline{\lambda})^{-1})f_1]\\ \vdots\\ [x,(1+(\overline{\lambda}-\lambda_0)(A-\overline{\lambda})^{-1})f_n]\end{pmatrix}$$

and we can rewrite (2.5) in the form

$$(B-\lambda)^{-1}-(A-\lambda)^{-1}=\Gamma_\lambda\alpha(\lambda)\Gamma_{\overline{\lambda}}^+. \tag{2.10}$$

Replacing λ by $\overline{\lambda}$ and taking adjoints in (2.10) we obtain $\Gamma_\lambda\alpha(\lambda)\Gamma_{\overline{\lambda}}^+=\Gamma_\lambda\alpha(\overline{\lambda})^*\Gamma_{\overline{\lambda}}^+$. From $(\operatorname{ran}\Gamma_{\overline{\lambda}}^+)^\perp=\ker\Gamma_{\overline{\lambda}}^-$ and the fact that $\Gamma_{\overline{\lambda}}^-$ and Γ_λ are injective we conclude

$$\alpha(\lambda)=\alpha(\overline{\lambda})^*,\quad \lambda\in\rho(A)\cap\rho(B). \tag{2.11}$$

It is straightforward to check that the relation $(\mu-\lambda)((B-\lambda)^{-1}-(A-\lambda)^{-1})((B-\mu)^{-1}-(A-\mu)^{-1})=(1+(\lambda-\mu)(A-\lambda)^{-1})((B-\mu)^{-1}-(A-\mu)^{-1})-((B-\lambda)^{-1}-(A-\lambda)^{-1})(1+(\mu-\lambda)(A-\mu)^{-1})$ holds for all $\lambda,\mu\in\rho(A)\cap\rho(B)$ (cf. Proof of Theorem 5.1 in [28]). Using (2.10) and $\Gamma_\lambda=(1+(\lambda-\mu)(A-\lambda)^{-1})\Gamma_\mu$, $\lambda,\mu\in\rho(A)$, we find

$$(\mu-\lambda)\Gamma_\lambda\alpha(\lambda)\Gamma_{\overline{\lambda}}^+\Gamma_\mu\alpha(\mu)\Gamma_{\overline{\mu}}^+=\Gamma_\lambda\alpha(\mu)\Gamma_{\overline{\mu}}^+-\Gamma_\lambda\alpha(\lambda)\Gamma_{\overline{\mu}}^+.$$

From $\ker\Gamma_\lambda=\ker\Gamma_{\overline{\mu}}=\{0\}$ we obtain $(\mu-\lambda)\alpha(\lambda)\Gamma_{\overline{\lambda}}^+\Gamma_\mu\alpha(\mu)=\alpha(\mu)-\alpha(\lambda)$ and

$$(\mu-\lambda)\Gamma_{\overline{\lambda}}^+\Gamma_\mu=\alpha(\lambda)^{-1}-\alpha(\mu)^{-1},\quad \lambda,\mu\in\rho(A)\cap\rho(B). \tag{2.12}$$

In particular we have

$$(\operatorname{Im}\lambda_0)\,\Gamma_{\lambda_0}^+\Gamma_{\lambda_0}=\operatorname{Im}(-\alpha(\lambda_0)^{-1}). \tag{2.13}$$

It is easy to see that the function τ defined for all $\lambda\in\rho(A)$ by

$$\lambda\mapsto\tau(\lambda):=\Gamma_{\lambda_0}^+((\lambda-\operatorname{Re}\lambda_0)+(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(A-\lambda)^{-1})\Gamma_{\lambda_0} \tag{2.14}$$

fulfills

$$\tau(\lambda)-\tau(\mu)=(\lambda-\mu)\Gamma_{\overline{\lambda}}^+\Gamma_\mu\quad\text{and}\quad\tau(\lambda)=\tau(\overline{\lambda})^* \tag{2.15}$$

for all $\lambda,\mu\in\rho(A)$. The relations (2.11), (2.12), (2.13), (2.14) and (2.15) imply that the function

$$\lambda\mapsto\alpha(\lambda)^{-1}+\tau(\lambda)$$

is equal to the self-adjoint constant $\mathrm{Re}(\alpha(\lambda_0)^{-1}) \in \mathcal{L}(\mathbb{C}^n)$. Therefore (2.10) can be written in the form $(B-\lambda)^{-1} = (A-\lambda)^{-1} + \Gamma_\lambda(\mathrm{Re}(\alpha(\lambda_0)^{-1}) - \tau(\lambda))^{-1}\Gamma_{\overline{\lambda}}^{+}$ and (2.13) and (2.14) imply

$$\begin{aligned} -\alpha(\lambda)^{-1} &= \mathrm{Re}(-\alpha(\lambda_0)^{-1}) + \Gamma_{\lambda_0}^{+}((\lambda - \mathrm{Re}\,\lambda_0) + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(A-\lambda)^{-1})\Gamma_{\lambda_0} \\ &= (-\alpha(\lambda_0)^{-1})^* + (\lambda-\overline{\lambda}_0)\Gamma_{\lambda_0}^{+}(1+(\lambda-\lambda_0)(A-\lambda)^{-1})\Gamma_{\lambda_0} \\ &= (-\alpha(\lambda_0)^{-1})^* + (\lambda-\overline{\lambda}_0)\Gamma_{\lambda_0}^{+}(1+(\lambda_0-\lambda)(A-\lambda_0)^{-1})^{-1}\Gamma_{\lambda_0}. \end{aligned} \tag{2.16}$$

Making use in (2.13) it follows that

$$U := 1 - (\lambda_0 - \overline{\lambda}_0)\Gamma_{\lambda_0}\alpha(\lambda_0)^*\Gamma_{\lambda_0}^{+} \in \mathcal{L}(\mathcal{K})$$

is unitary, i.e. $U^+ = U^{-1}$, and that $U\Gamma_{\lambda_0}\alpha(\lambda_0) = \Gamma_{\lambda_0}\alpha(\lambda_0)^*$ holds. With the help in (2.10) and the relation $(\overline{\lambda}_0-\lambda_0)\Gamma_{\lambda_0}\alpha(\lambda_0)^*\Gamma_{\lambda_0}^{+}\Gamma_{\lambda_0}\alpha(\lambda_0)\Gamma_{\overline{\lambda}_0}^{+} = \Gamma_{\lambda_0}\alpha(\lambda_0)^*\Gamma_{\overline{\lambda}_0}^{+} - \Gamma_{\lambda_0}\alpha(\lambda_0)\Gamma_{\overline{\lambda}_0}^{+}$, which follows easily from (2.13), we obtain that $U(1+(\lambda_0-\lambda)(B-\lambda_0)^{-1})$ coincides with

$$\begin{aligned} &1 + (\lambda_0-\lambda)(A-\lambda_0)^{-1} - (\lambda_0-\overline{\lambda}_0)\Gamma_{\lambda_0}\alpha(\lambda_0)^*\Gamma_{\lambda_0}^{+} \\ &\quad - (\lambda_0-\overline{\lambda}_0)(\lambda_0-\lambda)\Gamma_{\lambda_0}\alpha(\lambda_0)^*\Gamma_{\lambda_0}^{+}(A-\lambda_0)^{-1} + (\lambda_0-\lambda)\Gamma_{\lambda_0}\alpha(\lambda_0)^*\Gamma_{\overline{\lambda}_0}^{+}. \end{aligned} \tag{2.17}$$

From $\Gamma_{\overline{\lambda}_0}^{+} = \Gamma_{\lambda_0}^{+}(1+(\lambda_0-\overline{\lambda}_0)(A-\lambda_0)^{-1})$ and (2.17) we get $U(1+(\lambda_0-\lambda)(B-\lambda_0)^{-1}) = 1 + (\lambda_0-\lambda)(A-\lambda_0)^{-1} + (\overline{\lambda}_0-\lambda)\Gamma_{\lambda_0}\alpha(\lambda_0)^*\Gamma_{\lambda_0}^{+}$. In particular the right hand side is a boundedly invertible operator with an everywhere defined inverse. Now we can apply Lemma 3.1 in [33] to (2.16) and we obtain $\alpha(\lambda) = \alpha(\lambda_0)^* + (\lambda-\overline{\lambda}_0)\alpha(\lambda_0)^*\Gamma_{\lambda_0}^{+}(1+(\lambda_0-\lambda)(A-\lambda_0)^{-1} + (\overline{\lambda}_0-\lambda)\Gamma_{\lambda_0}\alpha(\lambda_0)^*\Gamma_{\lambda_0}^{+})^{-1}\Gamma_{\lambda_0}\alpha(\lambda_0)^*$. Let $\widetilde{\Gamma}_{\lambda_0} := \Gamma_{\lambda_0}\alpha(\lambda_0)$. Then we have

$$\widetilde{\Gamma}_{\lambda_0} = U^{-1}\Gamma_{\lambda_0}\alpha(\lambda_0)^* \quad \text{and} \quad (\mathrm{Im}\,\lambda_0)\,\widetilde{\Gamma}_{\lambda_0}^{+}\widetilde{\Gamma}_{\lambda_0} = \mathrm{Im}\,\alpha(\lambda_0)$$

and therefore

$$\begin{aligned} \alpha(\lambda) &= \alpha(\lambda_0)^* + (\lambda-\overline{\lambda}_0)\widetilde{\Gamma}_{\lambda_0}^{+}(1+(\lambda_0-\lambda)(B-\lambda_0)^{-1})^{-1}\widetilde{\Gamma}_{\lambda_0} \\ &= \alpha(\lambda_0)^* + (\lambda-\overline{\lambda}_0)\widetilde{\Gamma}_{\lambda_0}^{+}(1+(\lambda-\lambda_0)(B-\lambda)^{-1})\widetilde{\Gamma}_{\lambda_0} \\ &= \mathrm{Re}\,\alpha(\lambda_0) + \widetilde{\Gamma}_{\lambda_0}^{+}((\lambda-\mathrm{Re}\,\lambda_0) + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(B-\lambda)^{-1})\widetilde{\Gamma}_{\lambda_0}. \end{aligned} \tag{2.18}$$

3. In this part of the proof we show that every point $\mu \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighbourhood I_μ in $\overline{\mathbb{R}}$ such that both components of $I_\mu \backslash \{\mu\}$ are of definite type with respect to B.

Let $\mu \in \Omega \cap \overline{\mathbb{R}}$ and assume e.g. that $\Delta_+, \overline{\Delta}_+ \subset \Omega \cap \overline{\mathbb{R}}$, is an open interval with endpoint μ such that Δ_+ is of positive type with respect to A. From Theorem 1.7 we obtain that the function

$$-\alpha(\lambda)^{-1} = \mathrm{Re}(-\alpha(\lambda_0)^{-1}) + \Gamma_{\lambda_0}^{+}((\lambda - \mathrm{Re}\,\lambda_0) + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(A-\lambda)^{-1})\Gamma_{\lambda_0}$$

is definitizable in Ω and Δ_+ is of positive type with respect to this function. By Theorem 2.5 in [1] the function $\lambda \mapsto \alpha(\lambda)$ is also definitizable in Ω and there exists a (in general smaller) smaller open interval Δ'_+, $\Delta'_+ \subset \Delta_+$, with endpoint μ, which is of positive type with respect to α.

Let Ω_+ be a domain with the same properties as Ω such that $\overline{\Omega_+} \subset \Omega$ and $\Delta'_+ = \Omega_+ \cap \mathbb{R}$. As A is definitizable over Ω_+, $\rho(A) \cap \rho(B) \cap \Omega_+$ is nonempty and $\Omega_+ \cap \mathbb{R}$ is of positive type with respect to A we can apply Theorem 2.1. It follows that B is definitizable over Ω_+ and Δ'_+ is of type π_+ with respect to B. Let δ_+ be an open interval such that $\overline{\delta}_+ \subset \Delta'_+$ and $E_B(\delta_+)$ is defined. Then $(E_A(\delta_+)\mathcal{K}, [\cdot,\cdot])$ is a Hilbert space and $(E_B(\delta_+)\mathcal{K}, [\cdot,\cdot])$ is a Pontryagin space with finite rank of negativity.

In the following we will show that $E_B(\delta_+)\mathcal{K}$ equipped with the indefinite inner product $[\cdot,\cdot]$ is a Hilbert space. This will be done in four steps.

(i) Let Ω' be a domain with the same properties as Ω such that $\overline{\Omega'} \subset \Omega$, $\overline{\Delta'_+} \subset \Omega' \cap \overline{\mathbb{R}}$ and $\lambda_0 \in \Omega'$ holds. As the function α is definitizable in Ω and Δ'_+ is of positive type with respect to α we obtain from Theorem 1.8 that there exists a Krein space $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}})$, a self-adjoint relation T in $\mathcal{G}$ definitizable over Ω' and a mapping $\Gamma \in \mathcal{L}(\mathbb{C}^n, \mathcal{G})$ such that $\rho(T) \cap \Omega' = \mathfrak{h}(\alpha) \cap \Omega'$ and

$$\alpha(\lambda) = \operatorname{Re}\alpha(\lambda_0) + \Gamma^+((\lambda - \operatorname{Re}\lambda_0) + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(T-\lambda)^{-1})\Gamma \tag{2.19}$$

holds for all $\lambda \in \rho(T) \cap \Omega'$. Note that by (2.18) the function α is holomorphic at λ_0 and therefore λ_0 belongs to $\rho(T)$. According to Theorem 1.8 we can assume that T is chosen Ω'-minimal and that Δ'_+ is of positive type with respect to T. Then the spectral projection $E_T(\delta_+)$ of T corresponding to the open interval δ_+ is defined, $E_T(\delta_+)\mathcal{G}$ equipped with the inner product $[\cdot,\cdot]_{\mathcal{G}}$ is a Hilbert space and the following minimality condition is fulfilled:

$$E_T(\delta_+)\mathcal{G} = \operatorname{clsp}\{(1+(\lambda-\lambda_0)(T-\lambda)^{-1})E_T(\delta_+)\Gamma x \,|\, \lambda \in \rho(T)\cap\Omega', x \in \mathbb{C}^n\}. \tag{2.20}$$

From (2.18) and (2.19) we obtain

$$\widetilde{\Gamma}^+_{\lambda_0}\widetilde{\Gamma}_{\lambda_0} = \Gamma^+\Gamma \quad \text{and} \quad \widetilde{\Gamma}^+_{\lambda_0}(B-\lambda)^{-1}\widetilde{\Gamma}_{\lambda_0} = \Gamma^+(T-\lambda)^{-1}\Gamma$$

for all $\lambda \in \rho(T) \cap \rho(B) \cap \Omega'$. Therefore the relation $V \subset \mathcal{G} \times \mathcal{K}$ defined by

$$V := \left\{ \begin{pmatrix} \sum_{k=1}^{l} (1+(\lambda_k-\lambda_0)(T-\lambda_k)^{-1})\Gamma x_k \\ \sum_{k=1}^{l} (1+(\lambda_k-\lambda_0)(B-\lambda_k)^{-1})\widetilde{\Gamma}_{\lambda_0} x_k \end{pmatrix} : \begin{matrix} \lambda_k \in \rho(T)\cap\rho(B)\cap\Omega' \\ x_k \in \mathbb{C}^n,\ k=1,\dots,l \end{matrix} \right\}$$

is linear and isometric and the same holds for its closure $\overline{V} \in \widetilde{\mathcal{C}}(\mathcal{G},\mathcal{K})$.

(ii) Now we show that $\overline{V}$ is reduced by $E_T(\delta_+)\mathcal{G} \times E_B(\delta_+)\mathcal{K}$, i.e. we verify that $\overline{V}$ can be written as

$$\overline{V} \cap (E_T(\delta_+)\mathcal{G} \times E_B(\delta_+)\mathcal{K}) \dotplus \overline{V} \cap ((I-E_T(\delta_+))\mathcal{G} \times (I-E_B(\delta_+))\mathcal{K}).$$

Let $\binom{f}{g} \in \overline{V}$ and choose a sequence $\binom{f_m}{g_m} \in V$ such that $\binom{f_m}{g_m} \to \binom{f}{g}$ for $m \to \infty$. Let us assume first that the endpoints d_1 and d_2 of the bounded open interval $\delta_+ = (d_1, d_2)$ are no eigenvalues of T and B.

We fix some $\eta > 0$ such that the rectangle

$$Q := \{z \in \mathbb{C} : d_1 \leqslant \operatorname{Re} z \leqslant d_2,\ -\eta \leqslant \operatorname{Im} z \leqslant \eta\}$$

has the property $Q\backslash\mathbb{R} \subset \rho(T) \cap \rho(B)$. Let the boundary $\mathcal{C}_\infty$ of Q be oriented in the mathematical positive sense and let the curves

$$\mathcal{C}_k := \mathcal{C}_\infty \cap \{z \in \mathbb{C} : |\operatorname{Im} z| \geqslant k^-1\}, \quad k > \eta^{-1},$$

be oriented as $\mathcal{C}_\infty$.

As $\binom{f_m}{g_m} \in V$, $m = 1, 2, \ldots$, we obtain $\binom{(T-\lambda)^{-1} f_m}{(B-\lambda)^{-1} g_m} \in V$ for all $\lambda \in \rho(T) \cap \rho(B) \cap \Omega'$ and $m = 1, 2, \ldots$. Therefore the elements

$$\begin{pmatrix} -\frac{1}{2\pi \mathrm{i}} \int\limits_{\mathcal{C}_k} (T-\lambda)^{-1}\, \mathrm{d}\lambda\, f_m \\ -\frac{1}{2\pi \mathrm{i}} \int\limits_{\mathcal{C}_k} (B-\lambda)^{-1}\, \mathrm{d}\lambda\, g_m \end{pmatrix}, \quad m \in \mathbb{N},\ k > \eta^{-1},$$

belong to $\overline{V}$. Since

$$E_T(\delta_+) f_m = \lim_{k\to\infty} -\frac{1}{2\pi \mathrm{i}} \int\limits_{\mathcal{C}_k} (T-\lambda)^{-1} \mathrm{d}\lambda f_m, \quad E_B(\delta_+) g_m = \lim_{k\to\infty} -\frac{1}{2\pi \mathrm{i}} \int\limits_{\mathcal{C}_k} (B-\lambda)^{-1} \mathrm{d}\lambda g_m$$

we conclude $\binom{E_T(\delta_+) f_m}{E_B(\delta_+) g_m} \in \overline{V}$ and this implies

$$\begin{pmatrix} E_T(\delta_+) f \\ E_B(\delta_+) g \end{pmatrix} \in \overline{V} \quad \text{and} \quad \begin{pmatrix} (I - E_T(\delta_+)) f \\ (I - E_B(\delta_+)) g \end{pmatrix} \in \overline{V}.$$

Thus $\overline{V}$ is reduced by $E_T(\delta_+)\mathcal{G} \times E_B(\delta_+)\mathcal{K}$.

If d_1 or d_2 is an eigenvalue of T or B it follows from the strong σ-additivity of the local spectral function and the case considered above that $\overline{V}$ is reduced by $E_T(\delta_+)\mathcal{G} \times E_B(\delta_+)\mathcal{K}$.

(iii) We prove that

$$\overline{V}_{\delta_+} := \overline{V} \cap (E_T(\delta_+)\mathcal{G} \times E_B(\delta_+)\mathcal{K})$$

is an operator and that $(E_B(\delta_+)\mathcal{K}', [\cdot,\cdot])$ (cf. (2.7) and (2.8)) is a Hilbert space.

The relation $\overline{V}_{\delta_+}$ is isometric and by the definition of Γ_{λ_0} and $\widetilde{\Gamma}_{\lambda_0}$ we have $\operatorname{ran} \widetilde{\Gamma}_{\lambda_0} = \operatorname{ran} \Gamma_{\lambda_0} = \operatorname{sp}\{f_i : i = 1, \ldots, n\}$. As the elements

$$\begin{pmatrix} \sum\limits_{k=1}^{l} (1 + (\lambda_k - \lambda_0)(T - \lambda_k)^{-1}) E_T(\delta_+) \Gamma x_k \\ \sum\limits_{k=1}^{l} (1 + (\lambda_k - \lambda_0)(B - \lambda_k)^{-1}) E_B(\delta_+) \widetilde{\Gamma}_{\lambda_0} x_k \end{pmatrix},$$

$\lambda_k \in \rho(T) \cap \rho(B) \cap \Omega'$, $x_k \in \mathbb{C}^n$, $k = 1, \ldots, l$, belong to $\overline{V}_{\delta_+}$ we conclude from (2.8) and (2.20) that $\operatorname{dom} \overline{V}_{\delta_+}$ and $\operatorname{ran} \overline{V}_{\delta_+}$ are dense in $E_T(\delta_+)\mathcal{G}$ and $E_B(\delta_+)\mathcal{K}'$, respectively. From the fact that $(E_T(\delta_+)\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}})$ is a Hilbert space we conclude that $\operatorname{ran} \overline{V}_{\delta_+}$ and $E_B(\delta_+)\mathcal{K}'$ are nonnegative subspaces of the Pontryagin space $(E_B(\delta_+)\mathcal{K}, [\cdot,\cdot])$.

Let us show that

$$\mathcal{L}_0 := \{x \in E_B(\delta_+)\mathcal{K}' : [x,x] = 0\}$$

is trivial. As $E_B(\delta_+)\mathcal{K}'$ is nonnegative $\mathcal{L}_0[\perp]E_B(\delta_+)\mathcal{K}'$ and $\mathcal{L}_0[\perp](I - E_B(\delta_+))\mathcal{K}'$ holds, and therefore $\mathcal{L}_0 \subset \mathcal{K}'^{[\perp]}$. In view in (2.9)

$$(A-\lambda)^{-1}|\mathcal{L}_0 = (B-\lambda)^{-1}|\mathcal{L}_0$$

holds for all $\lambda \in \rho(A) \cap \rho(B) \cap \Omega$. Hence for $x_0 \in \mathcal{L}_0$ and $\delta_+ = (d_1, d_2)$ we conclude that

$$E_A(\delta_+)x_0 = \lim_{\eta \searrow 0} \lim_{\varepsilon \searrow 0} -\frac{1}{2\pi \mathrm{i}} \int_{d_1+\eta}^{d_2-\eta} ((A-(\lambda+\mathrm{i}\varepsilon))^{-1} - (A-(\lambda-\mathrm{i}\varepsilon))^{-1})x_0 \,\mathrm{d}\lambda$$

and $E_B(\delta_+)x_0$ coincide. As $E_A(\delta_+)\mathcal{L}_0 = E_B(\delta_+)\mathcal{L}_0$ and $(E_A(\delta_+)\mathcal{K}, [\cdot,\cdot])$ is a Hilbert space for $x \in E_B(\delta_+)\mathcal{L}_0$, $x \neq 0$, we find $[x,x] \neq 0$. As $E_B(\delta_+)\mathcal{L}_0 = \mathcal{L}_0$ we conclude $\mathcal{L}_0 = \{0\}$.

The fact $\mathcal{L}_0 = \{0\}$ implies that the multivalued part

$$\operatorname{mul} \overline{V}_{\delta_+} = \left\{ x \in E_B(\delta_+)\mathcal{K} : \begin{pmatrix} 0 \\ x \end{pmatrix} \in \overline{V}_{\delta_+} \right\}$$

of $\overline{V}_{\delta_+}$ is trivial. Hence $\overline{V}_{\delta_+}$ is a densely defined closed isometric operator. We claim that $\operatorname{ran} \overline{V}_{\delta_+}$ is closed. As in the proof of Theorem 6.2 in [18] one verifies that $\overline{V}_{\delta_+}$ is a bounded operator and therefore $\operatorname{dom} \overline{V}_{\delta_+} = E_T(\delta_+)\mathcal{G}$ holds. Moreover, from $\mathcal{L}_0 = \{0\}$ we also obtain that $\overline{V}_{\delta_+}$ is injective and another application of Theorem 6.2 in [18] shows that the closed isometric operator $\overline{V}_{\delta_+}^{-1}$ is bounded. Thus $\operatorname{dom} \overline{V}_{\delta_+}^{-1} = \operatorname{ran} \overline{V}_{\delta_+}$ is closed.

As $\operatorname{ran} \overline{V}_{\delta_+} = E_B(\delta_+)\mathcal{K}'$ is a closed positive subspace of the Pontryagin space $(E_B(\delta_+)\mathcal{K}, [\cdot,\cdot])$ we infer that $E_B(\delta_+)\mathcal{K}'$ is uniformly positive, i.e. $E_B(\delta_+)\mathcal{K}'$ equipped with the inner product $[\cdot,\cdot]$ is a Hilbert space.

(iv) Let $\mathcal{H}$ be the orthogonal complement of $E_B(\delta_+)\mathcal{K}'$ in the Pontryagin space $(E_B(\delta_+)\mathcal{K}, [\cdot,\cdot])$,

$$E_B(\delta_+)\mathcal{K} = E_B(\delta_+)\mathcal{K}'[\dot{+}]\mathcal{H}. \tag{2.21}$$

$\mathcal{H}$ is a Pontryagin space with finite rank of negativity. From $\mathcal{H}[\perp]E_B(\delta_+)\mathcal{K}'$ and $\mathcal{H}[\perp](I - E_B(\delta_+))\mathcal{K}'$ we obtain $\mathcal{H} \subset \mathcal{K}'^{[\perp]}$. By (2.9) the resolvents of A and B restricted to $\mathcal{H}$ coincide and by writing the projections $E_A(\delta_+)$ and $E_B(\delta_+)$ as strong limits of the resolvent of A and B, respectively, we see that $E_A(\delta_+)\mathcal{H}$ and $E_B(\delta_+)\mathcal{H}$ coincide. As above we obtain that $\mathcal{H} = E_B(\delta_+)\mathcal{H}$ is a Hilbert space and from (2.21) we conclude that $(E_B(\delta_+)\mathcal{K}, [\cdot,\cdot])$ is a Hilbert space.

As for any open interval δ_+ in Δ'_+, $\overline{\delta}_+ \subset \Delta'_+$, such that $E_B(\delta_+)$ is defined the spectral subspace $(E_B(\delta_+)\mathcal{K}, [\cdot,\cdot])$ is a Hilbert space it follows that the open interval Δ'_+ is of positive type with respect to B. In fact, let $\xi \in \Delta'_+ \cap \sigma(B)$ and choose an open interval δ_+ with $\xi \in \delta_+$ such that $\overline{\delta}_+ \subset \Delta'_+$ and the boundary points of δ_+ are of positive type with respect to B. If $\binom{x_n}{y_n} \in B$ is a sequence with $\|x_n\| = 1$ and $\|y_n - \xi x_n\| \to 0$ for $n \to \infty$ then

$$(B \cap ((I - E_B(\delta_+))\mathcal{K})^2 - \xi)^{-1} \in \mathcal{L}((I - E_B(\delta_+))\mathcal{K}) \text{ and } \lim_{n\to\infty} \|(I - E_B(\delta_+))(y_n - \xi x_n)\| = 0$$

imply $\|(I - E_B(\delta_+))x_n\| \to 0$ and $\|E_B(\delta_+)x_n\| \to 1$ for $n \to \infty$. As $E_B(\delta_+)\mathcal{K}$ equipped with the inner product $[\cdot,\cdot]$ is a Hilbert space we have

$$\liminf_{n\to\infty} [x_n, x_n] = \liminf_{n\to\infty} [E_B(\delta_+)x_n, E_B(\delta_+)x_n] > 0,$$

that is, ξ is of positive type with respect to B. Hence Δ'_+ is of positive type with respect to B.

Analogously one verifies that an open interval Δ_- with endpoint $\mu \in \Omega \cap \overline{\mathbb{R}}$ which is of negative type with respect to A contains an open interval Δ'_- with endpoint μ which is of negative type with respect to B. Therefore we have shown that for every point $\mu \in \Omega \cap \overline{\mathbb{R}}$ there exists an open connected neighbourhood I_μ in $\overline{\mathbb{R}}$ such that both components of $I_\mu \backslash \{\mu\}$ are of the same sign type with respect to A and B.

4. In order to show that B is definitizable over Ω it remains to verify that B belongs to $S^\infty(\Omega \cap \overline{\mathbb{R}})$.

As α is a definitizable function in Ω and A is definitizable over Ω no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of nonreal poles of α and nonreal spectrum of A in $\Omega \backslash \overline{\mathbb{R}}$. Hence by (2.10) the nonreal spectrum of B in $\Omega \backslash \overline{\mathbb{R}}$ does not accumulate to points in $\Omega \cap \overline{\mathbb{R}}$. Relation (2.10) also implies that each point in $\sigma(B) \cap (\Omega \backslash \overline{\mathbb{R}})$ is an isolated pole of the resolvent of B. Now the growth properties of α (see Definition 1.6) and the resolvent of A imply $B \in S^\infty(\Omega \cap \overline{\mathbb{R}})$. Therefore B is definitizable over Ω and Theorem 2.2 is proved. ∎

3. SELF-ADJOINT EXTENSIONS OF SYMMETRIC OPERATORS AND DIRECT SUMS OF INDEFINITE STURM-LIOUVILLE OPERATORS

In this section we apply the perturbation results from the previous section to self-adjoint extensions of symmetric operators and relations of finite defect. As an example we consider direct sums of symmetric Sturm-Liouville operators with the indefinite weight sgn x. Here the self-adjoint extensions are not definitizable but turn out to be locally definitizable over $\mathbb{C}$. First we recall some necessary definitions and the notion of boundary value spaces and associated Weyl functions for symmetric operators and relations in Krein spaces, cf. e.g. [11].

3.1. SELF-ADJOINT EXTENSIONS OF SYMMETRIC OPERATORS AND RELATIONS OF FINITE DEFECT. Let $\mathcal{K}$ be a separable Krein space, let J be a corresponding fundamental symmetry and let $S \in \widetilde{\mathcal{C}}(\mathcal{K})$ be a closed symmetric relation in $\mathcal{K}$. We say that S is of *defect* $m \in \mathbb{N} \cup \{\infty\}$, if both deficiency indices

$$n_\pm(JS) = \dim \ker((JS)^* - \overline{\lambda}), \quad \lambda \in \mathbb{C}^\pm,$$

of the symmetric relation JS in the Hilbert space $(\mathcal{K}, [J\cdot,\cdot])$ are equal to m. With the help of the von Neumann formulas for a closed symmetric relation in a Hilbert space (see e.g. Section 2.3 in [13]) one can verify without difficulty that this is equivalent to the fact that there exists a self-adjoint extension of S in $\mathcal{K}$ and that each self-adjoint extension $\widetilde{A}$ of S in $\mathcal{K}$ satisfies $\dim(\widetilde{A}/S) = m$.

For the description of the self-adjoint extensions of closed symmetric relations we use the so-called boundary value spaces.

DEFINITION 3.1. Let S be a closed symmetric relation in the Krein space $\mathcal{K}$. We say that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a *boundary value space for* S^+ if $(\mathcal{G}, (\cdot,\cdot))$ is a Hilbert space and there exist linear mappings $\Gamma_0, \Gamma_1 : S^+ \to \mathcal{G}$ such that $\Gamma := \binom{\Gamma_0}{\Gamma_1} : S^+ \to \mathcal{G} \times \mathcal{G}$ is surjective, and the relation

$$[f', g] - [f, g'] = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g}) - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g}) \tag{3.1}$$

holds for all $\widehat{f} = \binom{f}{f'}, \widehat{g} = \binom{g}{g'} \in S^+$.

If S is a closed symmetric relation in $\mathcal{K}$ and $\widetilde{A} \in \widetilde{\mathcal{C}}(\mathcal{K})$ is a self-adjoint extension of S with $\rho(\widetilde{A}) \neq \emptyset$, then there exists a boundary value space $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^+ such that $\widetilde{A}$ coincides with $\ker \Gamma_0$ (see [11]).

For basic facts on boundary value spaces and further references see e.g. [10], [11], [12], [13]. We recall only a few important consequences. Let S be a closed symmetric relation and assume that there exists a boundary value space $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for S^+. Then

$$A_0 := \ker \Gamma_0 \quad \text{and} \quad A_1 := \ker \Gamma_1 \tag{3.2}$$

are self-adjoint extensions of A. The mapping $\Gamma = \binom{\Gamma_0}{\Gamma_1}$ induces, via

$$A_\Theta := \Gamma^{-1}\Theta = \{\widehat{f} \in S^+ : \Gamma \widehat{f} \in \Theta\}, \quad \Theta \in \widetilde{\mathcal{C}}(\mathcal{G}), \tag{3.3}$$

a bijective correspondence $\Theta \mapsto A_\Theta$ between $\widetilde{\mathcal{C}}(\mathcal{G})$ and the set of closed extensions $A_\Theta \subset S^+$ of S. In particular (3.3) gives a one-to-one correspondence between the closed symmetric (self-adjoint) extensions of S and the closed symmetric (respectively self-adjoint) relations in $\mathcal{G}$. If Θ is a closed operator in $\mathcal{G}$, then the corresponding extension A_Θ of S is determined by

$$A_\Theta = \ker(\Gamma_1 - \Theta \Gamma_0). \tag{3.4}$$

Let $\mathcal{N}_\lambda := \ker(S^+ - \lambda) = \operatorname{ran}(S - \overline{\lambda})^{[\perp]}$, $\lambda \in r(S)$, be the defect subspace of S and set

$$\widehat{\mathcal{N}}_\lambda := \left\{ \begin{pmatrix} f \\ \lambda f \end{pmatrix} : f \in \mathcal{N}_\lambda \right\}.$$

Now we assume that the self-adjoint relation A_0 in (3.2) has a nonempty resolvent set. Then for $\lambda \in \rho(A_0)$ the adjoint S^+ is the direct sum of A_0 and $\widehat{\mathcal{N}}_\lambda$. Denote by π_1 the orthogonal projection onto the first component of $\mathcal{K} \times \mathcal{K}$. The functions

$$\lambda \mapsto \gamma(\lambda) := \pi_1(\Gamma_0|\widehat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G},\mathcal{K}), \quad \lambda \in \rho(A_0),$$

and

$$\lambda \mapsto M(\lambda) := \Gamma_1(\Gamma_0|\widehat{\mathcal{N}}_\lambda)^{-1} \in \mathcal{L}(\mathcal{G}), \quad \lambda \in \rho(A_0), \tag{3.5}$$

are holomorphic on $\rho(A_0)$ and are called the *γ-field* and *Weyl function* corresponding to S and $\{\mathcal{G},\Gamma_0,\Gamma_1\}$. We remark that for a fixed $\lambda_0 \in \rho(A_0)$ and all $\lambda \in \rho(A_0)$ the Weyl function M can be written in the form

$$M(\lambda) = \mathrm{Re}\, M(\lambda_0) + \gamma(\lambda_0)^+((\lambda - \mathrm{Re}\,\lambda_0) + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(A_0-\lambda)^{-1})\gamma(\lambda_0). \tag{3.6}$$

With the help of the Weyl function the spectral properties of the closed extensions of S can be described. If $\Theta \in \widetilde{\mathcal{C}}(\mathcal{G})$ and A_Θ is the corresponding extension of S via (3.3), then a point $\lambda \in \rho(A_0)$ belongs to $\rho(A_\Theta)$ ($\sigma_i(A_\Theta)$, $i = p,c,r$) if and only if 0 belongs to $\rho(\Theta - M(\lambda))$ (respectively $\sigma_i(\Theta - M(\lambda))$, $i = p,c,r$) and the well-known formula

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\overline{\lambda})^+ \tag{3.7}$$

holds for all $\lambda \in \rho(A_0) \cap \rho(A_\Theta)$ (see e.g. [11]).

In the special case that S is of defect one the self-adjoint extensions A_Θ of S, $A_\Theta \neq \ker \Gamma_0$, in $\mathcal{K}$ can be parametrized with the real numbers $\Theta \in \mathbb{R}$. Therefore, in this case, all self-adjoint extensions of S have a nonempty resolvent set if the (scalar) Weyl function M is not identically equal to a constant. If $M(\lambda) = const.$ for all $\lambda \in \rho(A_0)$, then there can exist one self-adjoint extension with an empty resolvent set.

The following theorem is an immediate consequence of Theorem 2.1, Theorem 2.2 and the fact that the difference of the resolvents of two self-adjoint extensions of a symmetric relation of finite defect is a finite rank operator, see (3.7).

THEOREM 3.2. *Let S be a closed symmetric relation in the Krein space $\mathcal{K}$ and assume that the defect of S is finite. Then the following holds:*

(i) *If there exists a self-adjoint extension A of S in $\mathcal{K}$ which is definitizable over Ω, then every self-adjoint extension $\widetilde{A}$ of S in $\mathcal{K}$ with $\rho(\widetilde{A}) \cap \Omega \neq \emptyset$ is definitizable over Ω.*

(ii) *If A is a self-adjoint extension of S in $\mathcal{K}$ which is definitizable over Ω and $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ is an open interval with endpoint $\mu \in \Omega \cap \overline{\mathbb{R}}$ and Δ is of positive type (negative type) with respect to A, then for every self-adjoint extension $\widetilde{A}$ of A with $\rho(\widetilde{A}) \cap \Omega \neq \emptyset$ there exists an open interval Δ', $\Delta' \subset \Delta$, with endpoint μ such that Δ' is of positive type (respectively negative type) with respect to $\widetilde{A}$.*

(iii) *If there exists a self-adjoint extension A of S in $\mathcal{K}$ which is of type π_+ (type π_-) over Ω, then every self-adjoint extension $\widetilde{A}$ of S in $\mathcal{K}$ with $\rho(\widetilde{A}) \cap \Omega \neq \emptyset$ is of type π_+ (respectively type π_-) over Ω.*

3.2. DIRECT SUMS OF SECOND ORDER DIFFERENTIAL OPERATORS. In this section we investigate the spectral properties of direct sums of regular and singular Sturm-Liouville operators with the indefinite weight $\operatorname{sgn} x$. The following notation will be useful. If $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{K}})$ and $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{H}})$ are Krein spaces, then the elements of $\mathcal{K} \times \mathcal{H}$ will be written in the form $\{k, h\}$, $k \in \mathcal{K}$, $h \in \mathcal{H}$. The direct sum of a linear operator S in $\mathcal{K}$ and a linear operator T in $\mathcal{H}$ will be denoted by $S \times T$. If S and T are symmetric in $\mathcal{K}$ and $\mathcal{H}$, respectively, then $S \times T$ is symmetric in the Krein space $(\mathcal{K} \times \mathcal{H}, [\cdot,\cdot])$, where

$$[\{k,h\},\{\widetilde{k},\widetilde{h}\}] := [k,\widetilde{k}]_{\mathcal{K}} + [h,\widetilde{h}]_{\mathcal{H}}, \quad k,\widetilde{k} \in \mathcal{K},\ h,\widetilde{h} \in \mathcal{H}.$$

Let in the following $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{K}})$ be the Krein space $L^2(\mathbb{R})$ equipped with the inner product

$$[f,g]_{\mathcal{K}} := \int_{-\infty}^{\infty} f(x)\overline{g(x)} \operatorname{sgn} x \,\mathrm{d}x, \quad f,g \in L^2(\mathbb{R}),$$

and let

$$(Sf)(x) := -\operatorname{sgn} x\, f''(x), \quad \operatorname{dom} S := \{f \in W^{2,2}(\mathbb{R}) : f(0) = f'(0) = 0\}.$$

Then S is a densely defined closed symmetric operator in $\mathcal{K}$ of defect two and the adjoint operator S^+ is given by

$$(S^+f)(x) = -\operatorname{sgn} x\, f''(x), \quad \operatorname{dom} S^+ = W^{2,2}(\mathbb{R}^-) \times W^{2,2}(\mathbb{R}^+),$$

where $\mathbb{R}^+ := (0,\infty)$ and $\mathbb{R}^- := (-\infty,0)$. A straightforward calculation shows that $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0\widehat{f} := \begin{pmatrix} f(0+) - f(0-) \\ f'(0+) - f'(0-) \end{pmatrix}, \quad \widehat{f} = \begin{pmatrix} f \\ S^+f \end{pmatrix}, \tag{3.8}$$

$$\Gamma_1\widehat{f} := \frac{1}{2}\begin{pmatrix} f'(0+) + f'(0-) \\ -f(0+) - f(0-) \end{pmatrix}, \quad \widehat{f} = \begin{pmatrix} f \\ S^+f \end{pmatrix}, \tag{3.9}$$

is a boundary value space for S^+ and the self-adjoint extension $A_0 = \ker \Gamma_0$ is the usual second order differential operator with the indefinite weight function $x \mapsto \operatorname{sgn} x$ on $\mathbb{R}$.

Besides the Krein space $\mathcal{K}$ we consider the Krein space

$$\mathcal{H} := (L^2((a,b)), [\cdot,\cdot]_{\mathcal{H}}), \quad -\infty < a < 0 < b < \infty,$$

where $[\cdot,\cdot]_{\mathcal{H}}$ is defined by

$$[h,k]_{\mathcal{H}} := -\int_a^b h(x)\overline{k(x)} \operatorname{sgn} x \,\mathrm{d}x, \quad h,k \in L^2((a,b)).$$

Let $p^{-1}, q \in L^1((a,b))$ be real functions and assume that $p > 0$ is fulfilled. We consider the densely defined closed symmetric operator T,

$$(Th)(x) := \operatorname{sgn} x(-(p(x)h'(x))' + q(x)h(x)),$$
$$\operatorname{dom} T := \left\{ h \in L^2((a,b)) \;\middle|\; \begin{matrix} h, ph' \in AC((a,b)),\ -(ph')' + qh \in L^2((a,b)) \\ h(a) = h(b) = h(0) = (ph')(0) = 0 \end{matrix} \right\},$$

in $\mathcal{H}$. Then T has defect two and the operator T^+ is given by

$$(T^+h)(x) = \operatorname{sgn} x(-(p(x)h'(x))' + q(x)h(x)),$$
$$\operatorname{dom} T^+ = \left\{ h \in L^2((a,b)) \;\middle|\; \begin{matrix} h, ph' \in AC((a,0)) \times AC((0,b)), \\ -(ph')' + qh \in L^2((a,b)),\ h(a) = h(b) = 0 \end{matrix} \right\}.$$

Here we choose $\{\mathbb{C}^2, \Gamma_0', \Gamma_1'\}$, where

$$\Gamma_0'\widehat{h} := \begin{pmatrix} h(0-) \\ h(0+) \end{pmatrix} \quad \text{and} \quad \Gamma_1'\widehat{h} := \begin{pmatrix} (ph')(0-) \\ -(ph')(0+) \end{pmatrix}, \quad \widehat{h} = \begin{pmatrix} h \\ T^+h \end{pmatrix}, \tag{3.10}$$

as a boundary value space for T^+. We remark that the self-adjoint extension B_Φ of T in $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{H}})$ corresponding to the self-adjoint relation $\Phi = \left\{ \begin{pmatrix} (x,x)^\top \\ (y,-y)^\top \end{pmatrix} : x, y \in \mathbb{C} \right\} \in \widetilde{\mathcal{C}}(\mathbb{C}^2)$ via (3.3) is the usual second order differential operator $\operatorname{sgn} x\big(-\frac{\mathrm{d}}{\mathrm{d}x}\big(p\frac{\mathrm{d}}{\mathrm{d}x}\big) + q\big)$ in $L^2((a,b))$ with Dirichlet boundary conditions.

THEOREM 3.3. *Let S and T be the symmetric differential operators in the Krein spaces $\mathcal{K}$ and $\mathcal{H}$ from above and let $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ and $\{\mathbb{C}^2, \Gamma_0', \Gamma_1'\}$ be the boundary value spaces from* (3.8)–(3.9) *and* (3.10)*, set $A_0 = \ker \Gamma_0$ and $B_0 = \ker \Gamma_0'$, and denote the corresponding Weyl functions by M and τ, respectively. Then the following assertions* (i)–(iii) *hold:*

(i) *All canonical self-adjoint extensions of S and T in the Krein spaces $\mathcal{K}$ and $\mathcal{H}$, respectively, are definitizable (over $\overline{\mathbb{C}}$).*

(ii) *The self-adjoint operator $A_0 \times B_0$ in the Krein space $\mathcal{K} \times \mathcal{H}$ is definitizable over $\overline{\mathbb{C}}\backslash\{\infty\}$ and $\sigma(A_0 \times B_0)$ coincides with $\mathbb{R}$. The interval $(0,\infty)$ is of type π_+ and the interval $(-\infty,0)$ is of type π_- with respect to $A_0 \times B_0$.*

(iii) *If $\widetilde{\Theta} \in \widetilde{\mathcal{C}}(\mathbb{C}^4)$ is a self-adjoint relation such that $0 \in \rho(\widetilde{\Theta} - M(\lambda) \oplus \tau(\lambda))$ for some $\lambda \in \mathbb{C}\backslash\mathbb{R}$, then the self-adjoint differential operator*

$$\widetilde{A}_{\widetilde{\Theta}} = S^+ \times T^+ \upharpoonright \operatorname{dom} \widetilde{A}_{\widetilde{\Theta}},$$
$$\operatorname{dom} \widetilde{A}_{\widetilde{\Theta}} = \left\{ \{f,h\} \in \mathcal{K} \times \mathcal{H} \;\middle|\; \begin{pmatrix} (\Gamma_0\widehat{f}, \Gamma_0'\widehat{h})^\top \\ (\Gamma_1\widehat{f}, \Gamma_1'\widehat{h})^\top \end{pmatrix} \in \widetilde{\Theta},\ \begin{matrix} \widehat{f} = \binom{f}{S^+f}, \\ \widehat{h} = \binom{h}{T^+h} \end{matrix} \right\},$$

in $\mathcal{K} \times \mathcal{H}$ is definitizable over $\overline{\mathbb{C}}\backslash\{\infty\}$ and the interval $(0,\infty)$ ($(-\infty,0)$) is of type π_+ (respectively type π_-) with respect to $\widetilde{A}_{\widetilde{\Theta}}$.

Proof. (i) As S is a densely defined symmetric operator in $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{K}})$ and A_0 is a nonnegative self-adjoint operator with $\sigma(A_0) = \mathbb{R}$ (see e.g. [9]) it follows from

Proposition 1.1 in [8] that all self-adjoint extensions A_Θ, $\Theta \in \widetilde{\mathcal{C}}(\mathbb{C}^2)$, of S in $\mathcal{K}$ are definitizable (over $\overline{\mathbb{C}}$).

Denote by p_1 (p_2) and q_1 (q_2) the restrictions of the functions p and q onto the interval $(a,0)$ (respectively $(0,b)$). Then the self-adjoint extension $B_0 = \ker \Gamma_0'$ of T in $\mathcal{H}$ is a fundamentally reducible operator as it coincides with the direct sum of the self-adjoint realizations of the regular Sturm-Liouville differential expressions $\frac{\mathrm{d}}{\mathrm{d}x}(p_1 \frac{\mathrm{d}}{\mathrm{d}x}) - q_1$ and $-\frac{\mathrm{d}}{\mathrm{d}x}(p_2 \frac{\mathrm{d}}{\mathrm{d}x}) + q_2$, in $(L^2((a,0)), (\cdot,\cdot))$ and $(L^2((0,b)), (\cdot,\cdot))$ corresponding to Dirichlet boundary conditions. Hence $\sigma(B_0)$ is real and consists only of eigenvalues (with one or two-dimensional eigenspaces) accumulating only to ∞ and $-\infty$. Here the assumptions $p_1, p_2 > 0$ imply that there are only finitely many eigenvalues belonging to

$$\sigma_{++}(B_0) \cap (0,\infty) \quad \text{and} \quad \sigma_{--}(B_0) \cap (-\infty, 0)$$

(cf. [27], [37]). Therefore the hermitian form $[B_0\cdot,\cdot]$ defined on $\operatorname{dom} B_0$ has finitely many positive squares and it follows again from Proposition 1.1 in [8] that all self-adjoint extensions B_Φ, $\Phi \in \widetilde{\mathcal{C}}(\mathbb{C}^2)$, are definitizable.

(ii) Since A_0 and B_0 are definitizable they belong to the class $S^\infty(\overline{\mathbb{R}})$ and therefore $A_0 \times B_0$ is also in the class $S^\infty(\overline{\mathbb{R}})$. From $\sigma(A_0) \cup \sigma(B_0) = \mathbb{R}$ we obtain $\sigma(A_0 \times B_0) = \mathbb{R}$. In order to see that $A_0 \times B_0$ is definitizable over $\overline{\mathbb{C}} \backslash \{\infty\}$ we have to check that for every point $\mu \in \mathbb{R}$ there exists an open interval $I_\mu \subset \mathbb{R}$, $\mu \in I_\mu$, such that both components of $I_\mu \backslash \{\mu\}$ are of definite type with respect to $A_0 \times B_0$. This follows from the nonnegativity of A_0, hence $(0,\infty)$ ($(-\infty,0)$) is of positive type (respectively negative type) with respect to A_0, and the fact that $\sigma(B_0)$ consists of eigenvalues accumulating only to ∞ and $-\infty$.

Let δ be an open interval such that $\overline{\delta} \subset (0,\infty)$ and the boundary points of δ in $\mathbb{R}$ are no eigenvalues of B_0. As the spectral subspace $(E_{A_0}(\delta)\mathcal{K}, [\cdot,\cdot]_{\mathcal{K}})$ is a Hilbert space and $(E_{B_0}(\delta)\mathcal{H}, [\cdot,\cdot]_{\mathcal{H}})$ is a finite dimensional Pontryagin space we conclude that $(0,\infty)$ is of type π_+ with respect to $A_0 \times B_0$. A similar argument shows that $(-\infty,0)$ is of type π_- with respect to $A_0 \times B_0$.

(iii) It is easy to see that

$$\widetilde{\Gamma}_0\{\widehat{f},\widehat{h}\} := \begin{pmatrix} \Gamma_0 \widehat{f} \\ \Gamma_0' \widehat{h} \end{pmatrix} \quad \text{and} \quad \widetilde{\Gamma}_1\{\widehat{f},\widehat{h}\} := \begin{pmatrix} \Gamma_1 \widehat{f} \\ \Gamma_1' \widehat{h} \end{pmatrix}, \; \widehat{f} = \begin{pmatrix} f \\ S^+ f \end{pmatrix}, \; \widehat{h} = \begin{pmatrix} h \\ T^+ h \end{pmatrix},$$

defines a boundary value space $\{\mathbb{C}^4, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for $S^+ \times T^+$ with $\ker \widetilde{\Gamma}_0 = A_0 \times B_0$ and corresponding Weyl function

$$\lambda \mapsto \begin{pmatrix} M(\lambda) & 0 \\ 0 & \tau(\lambda) \end{pmatrix} = M(\lambda) \oplus \tau(\lambda) \in \mathcal{L}(\mathbb{C}^4), \quad \lambda \in \rho(A_0 \times B_0). \tag{3.11}$$

Now assertion (iii) follows from Theorem 3.2. ∎

Let S and T be the symmetric differential operators from above and let $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ and $\{\mathbb{C}^2, \Gamma_0', \Gamma_1'\}$ be the boundary value spaces from (3.8)–(3.9) and (3.10). By (3.6) and Theorem 1.7 the Weyl functions M and τ corresponding to

$\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ and $\{\mathbb{C}^2, \Gamma_0', \Gamma_1'\}$, respectively, are definitizable functions (in $\overline{\mathbb{C}}$) (see Section 1.3 and [23]) and the function (3.11) is definitizable in $\mathbb{C}$. Here M can be calculated explicitely and also the structure of τ is known.

Indeed, if $\sqrt[+]{\cdot}$ ($\sqrt[-]{\cdot}$) denotes the branch of $\sqrt{\cdot}$ defined in $\mathbb{C}$ with a cut along $[0,\infty)$ ($(-\infty,0]$) and fixed by $\operatorname{Im}\sqrt{\lambda} > 0$ for $\lambda \notin [0,\infty)$ and $\sqrt{\lambda} \geqslant 0$ for $\lambda \in [0,\infty)$ (respectively $\operatorname{Re}\sqrt{\lambda} > 0$ for $\lambda \notin (-\infty,0]$ and $\operatorname{Im}\sqrt{\lambda} \geqslant 0$ for $\lambda \in (-\infty,0]$), then for $\lambda \in \mathbb{C}\backslash\mathbb{R}$ the defect subspace $\mathcal{N}_\lambda = \ker(S^+ - \lambda)$ is spanned by the functions

$$f_\lambda(x) := \begin{cases} \exp(\mathrm{i}\sqrt[+]{\lambda}x) & x > 0, \\ 0 & x < 0, \end{cases} \quad \text{and} \quad g_\lambda(x) := \begin{cases} 0 & x > 0, \\ \exp(\sqrt[-]{\lambda}x) & x < 0. \end{cases}$$

Hence with $\widehat{f}_\lambda = \binom{f_\lambda}{\lambda f_\lambda}$ and $\widehat{g}_\lambda = \binom{g_\lambda}{\lambda g_\lambda}$ we have $\Gamma_0\widehat{f}_\lambda = \binom{1}{\mathrm{i}\sqrt[+]{\lambda}}$, $\Gamma_1\widehat{f}_\lambda = \frac{1}{2}\binom{\mathrm{i}\sqrt[+]{\lambda}}{-1}$, $\Gamma_0\widehat{g}_\lambda = \binom{-1}{-\sqrt[-]{\lambda}}$ and $\Gamma_1\widehat{g}_\lambda = \frac{1}{2}\binom{\sqrt[-]{\lambda}}{-1}$, and therefore the Weyl function M corresponding to the boundary value space $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ is given by

$$M(\lambda) = \frac{1}{\mathrm{i}\sqrt[+]{\lambda} - \sqrt[-]{\lambda}} \begin{pmatrix} -\mathrm{i}\sqrt[+]{\lambda}\sqrt[-]{\lambda} & \frac{1}{2}(\mathrm{i}\sqrt[+]{\lambda} + \sqrt[-]{\lambda}) \\ \frac{1}{2}(\mathrm{i}\sqrt[+]{\lambda} + \sqrt[-]{\lambda}) & -1 \end{pmatrix}, \quad \lambda \in \mathbb{C}\backslash\mathbb{R}.$$

Similarly $\ker(T^+ - \lambda)$, $\lambda \in \rho(B_0)$, is spanned by some functions h_λ (k_λ), which vanish on the interval $(0,b)$ (respectively $(a,0)$). It is not difficult to see that there exist scalar Nevanlinna functions N_1 and N_2, $\mathfrak{h}(N_1) \cap \mathfrak{h}(N_2) = \rho(B_0)$, such that the Weyl function τ corresponding to $\{\mathbb{C}^2, \Gamma_0', \Gamma_1'\}$ has the form

$$\tau(\lambda) = \begin{pmatrix} N_1(\lambda) & 0 \\ 0 & -N_2(\lambda) \end{pmatrix}, \quad \lambda \in \rho(B_0).$$

3.3. AN EXAMPLE FOR A SELF-ADJOINT EXTENSION $\widetilde{A}_{\widetilde{\Theta}}$ OF A DIRECT SUM OF DIFFERENTIAL OPERATORS WITH $\sigma_{\mathrm{p}}(\widetilde{A}_{\widetilde{\Theta}}) = \mathbb{C}$. In the following we will give a simple example of a direct sum $S \times T$ of two differential operators S and T where a certain self-adjoint extension has an empty resolvent set.

In the Hilbert space $\mathcal{K} := (L^2((\alpha,\beta)), (\cdot,\cdot))$, $-\infty < \alpha < \beta < \infty$, we consider the symmetric second order differential operator

$$(Sf)(x) := -f''(x),$$
$$\operatorname{dom} S := \{f \in L^2((\alpha,\beta)) : f \in W^{2,2}((\alpha,\beta)),\ f(\alpha) = f'(\alpha) = f(\beta) = 0\},$$

the adjoint operator S^*,

$$(S^*f)(x) = -f''(x), \quad \operatorname{dom} S^* = \{f \in L^2((\alpha,\beta)) : f \in W^{2,2}((\alpha,\beta)),\ f(\beta) = 0\},$$

and we choose $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$, $\Gamma_0\widehat{f} := f(\alpha)$, $\Gamma_1\widehat{f} := f'(\alpha)$, $\widehat{f} = \binom{f}{S^*f}$, as a boundary value space for S^*. Let $\ker(S^* - \lambda) = \operatorname{sp}\{f_\lambda\}$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Then the Weyl function M corresponding to $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is given by

$$M(\lambda) = \frac{\Gamma_1\widehat{f}_\lambda}{\Gamma_0\widehat{f}_\lambda}, \quad \widehat{f}_\lambda = \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix}, \quad \lambda \in \mathbb{C}\backslash\mathbb{R}.$$

We equip $L^2((\alpha,\beta))$ with the negative definite inner product $[\cdot,\cdot]$ defined by $[g,h] := -(g,h)$, $g,h \in L^2((\alpha,\beta))$, and denote the corresponding Krein space by $\mathcal{H}$. The differential operator $(Th)(x) := -h''(x)$, with $\operatorname{dom} T = \operatorname{dom} S$, is symmetric in $\mathcal{H}$ and the adjoint operator is given by $(T^+h)(x) = -h''(x)$, where $\operatorname{dom} T^+ = \operatorname{dom} S^*$. Here $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$, $\Gamma_0'\widehat{h} := h'(\alpha)$, $\Gamma_1'\widehat{h} := h(\alpha)$, $\widehat{h} = \binom{h}{S^+h}$, is a boundary value space for T^+ and the corresponding Weyl function τ has the form

$$\tau(\lambda) = \frac{\Gamma_1'\widehat{f}_\lambda}{\Gamma_0'\widehat{f}_\lambda} = \frac{\Gamma_0\widehat{f}_\lambda}{\Gamma_1\widehat{f}_\lambda} = \frac{1}{M(\lambda)}, \quad \lambda \in \mathbb{C}\backslash\mathbb{R}.$$

As in the proof of Theorem 3.3(iii) we define the boundary value space $\{\mathbb{C}^2, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for $S^* \times T^+$ by

$$\widetilde{\Gamma}_0\{\widehat{f},\widehat{h}\} := \begin{pmatrix} \Gamma_0\widehat{f} \\ \Gamma_0'\widehat{h} \end{pmatrix} \quad \text{and} \quad \widetilde{\Gamma}_1\{\widehat{f},\widehat{h}\} := \begin{pmatrix} \Gamma_1\widehat{f} \\ \Gamma_1'\widehat{h} \end{pmatrix},$$

$\{\widehat{f},\widehat{h}\} := \left\{\binom{f}{S^*f}, \binom{h}{T^+h}\right\}$. Note that the selfadjoint operator $\ker\widetilde{\Gamma}_0$ is definitizable over $\overline{\mathbb{C}}\backslash\{\infty\}$. Now the corresponding Weyl function $\widetilde{M}$ is $\lambda \mapsto \widetilde{M}(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ 0 & \frac{1}{M(\lambda)} \end{pmatrix}, \lambda \in \mathbb{C}\backslash\mathbb{R}$. The self-adjoint extension $\widetilde{A}_{\widetilde{\Theta}}$, $\widetilde{\Theta} := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathcal{L}(\mathbb{C}^2)$, of $S \times T$ in the Krein space $\mathcal{K} \times \mathcal{H}$ via (3.3)–(3.4) is given by

$$\begin{gathered} \widetilde{A}_{\widetilde{\Theta}} = S^* \times T^+ \upharpoonright \operatorname{dom} \widetilde{A}_{\widetilde{\Theta}}, \\ \operatorname{dom} \widetilde{A}_{\widetilde{\Theta}} = \left\{\{f,h\} \in \operatorname{dom} S^* \times \operatorname{dom} T^+ : f(\alpha) = h(\alpha),\, f'(\alpha) = h'(\alpha)\right\}, \end{gathered} \tag{3.12}$$

and we have $\sigma_{\mathrm{p}}(\widetilde{A}_{\widetilde{\Theta}}) = \mathbb{C}$ since the function $\lambda \mapsto \det(\widetilde{M}(\lambda) - \widetilde{\Theta})$ is identically equal to zero.

We note that it can also be checked directly that $\widetilde{A}_{\widetilde{\Theta}}$ in (3.12) is self-adjoint and that $\sigma_{\mathrm{p}}(\widetilde{A}_{\widetilde{\Theta}}) = \mathbb{C}$ holds.

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