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ON TOEPLITZ OPERATORS SIMILAR TO ISOMETRIES

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ABSTRACT. Let T be a Toeplitz operator on the Hardy space H^2 on the unit circle, and let the symbol of T be of the form $\frac{\varphi}{\psi}$, where φ is inner function, ψ is a finite Blaschke product, and $\deg\psi \leqslant \deg\varphi$. D.N. Clark proved that such T is similar to an isometry. In this paper, we find this isometry.

KEYWORDS: *Toeplitz operators, similarity.*

MSC (2000): 47B35.

1. INTRODUCTION

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the unit disk, m the normalized Lebesgue measure on $\mathbb{T}$. Let $L^2 = L^2(\mathbb{T}, m)$ be the Lebesgue space, $H^2 = H^2(\mathbb{T})$ the Hardy space, and P_+ the orthogonal projection from L^2 onto H^2. The Toeplitz operator T_Φ with the symbol Φ, $\Phi \in L^\infty(\mathbb{T}, m)$, acts on H^2 by the formula $T_\Phi h = P_+\Phi h$, $h \in H^2$. For some classes of symbols Φ there exist results on T_Φ concerning their spectral theory. One of the first such results is a concrete spectral representation of selfadjoint Toeplitz operators T_Φ, that is, for real-valued symbols Φ, see [13]. Further, it was proved in [12] that T_Φ with $\Phi = \frac{f}{\overline{f}}$, where $f, \frac{1}{f} \in H^\infty$, is similar to a unitary operator, and a concrete spectral representation of this unitary operator was given in [5], [6] for a large subclass of such symbols. There exist many results of the similarity of T_Φ with smooth symbols Φ to some simpler operators.

For example, in [16], [3], [2] it is proved that T_Φ is similar to the direct sum of a normal operator and analytic and antianalytic Toeplitz operators (that is, operators of the form T_f and $T_{\overline{f}}$, where $f \in H^\infty$) for some subclass of the class of smooth symbols Φ. In [17], [18] it is proved that T_Φ is similar to the operator of multiplication by the independent variable on a specially constructed functional space for some subclass of the class of smooth symbols Φ, and in [19] a linearly similar model for some class of operators is constructed and its application to the similarity of Toeplitz operators with smooth symbols is given.

On the other hand, in [5] it is proved that if Φ has the form

$$\Phi = \frac{\omega f}{\overline{f}}, \tag{1.1}$$

where ω is inner, $f, \frac{1}{f} \in H^\infty$, then T_Φ is similar to an isometry. Any isometry is equal to $U \oplus S$, where U is unitary and S is a unilateral shift, and if T_Φ with Φ from (1.1) is similar to $U \oplus S$, then U is absolutely continuous and the multiplicity of S is equal to the degree of ω. But nothing more about U is known in the general case.

In this paper we consider the Toeplitz operator T_Φ with $\Phi = \frac{\varphi}{\psi}$, where ψ is a finite Blaschke product, $\deg \psi = N < \infty$, and φ is an inner function, $\deg \varphi \geqslant N$ (the degree of an inner function φ is the number of Blaschke factors, if φ is a finite Blaschke product, and ∞ in any other case). It is easy to see (Lemma 2.1 below) that such Φ can be represented in the form (1.1), therefore T_Φ is similar to an isometry $U \oplus S$, where the multiplicity of a unilateral shift S is equal to $\deg \varphi - N$. In this paper, we find the unitary summand U. *But no intertwining operator that gives the similarity of T_Φ and an isometry $U \oplus S$ is constructed.*

If $\deg \varphi < \infty$, then T_Φ (with a little additional assumption on the symbol Φ) is a particular case of Toeplitz operator regarded in [3], [1]. The investigation of T_Φ with $\Phi = \frac{\varphi}{\psi}$, where φ is inner and ψ is a Blaschke factor ($N = 1$), was begun in [4]. In this paper, we follow [1], [4] where it is appropriated.

The main results of the paper are the following theorems.

THEOREM 1.1. *Suppose ψ is a finite Blaschke product, $\deg \psi = N < \infty$, φ is inner, $N \leqslant \deg \varphi \leqslant \infty$, φ and ψ are relatively prime, and $\Phi = \frac{\varphi}{\psi}$. Let $T_\Phi \colon H^2 \to H^2$ be the Toeplitz operator with symbol Φ. We put $\mathcal{I}_n = \{\lambda \in \mathbb{T} : \Phi - \lambda$ has at least n zeros in $\mathbb{D}\}$, and $\sigma_n = \mathbb{T} \setminus \mathcal{I}_n$, $n = 1, \dots, N$. Then T_Φ is similar to $U \oplus S$, where S is a unilateral shift of multiplicity $\deg \varphi - N$ and U is the operator of multiplication by the independent variable in the space $\bigoplus_{n=1}^{N} L^2(\sigma_n, m)$.*

THEOREM 1.2. *Suppose ψ is a finite Blaschke product, $\deg \psi = N < \infty$, φ is inner, $N \leqslant \deg \varphi \leqslant \infty$, φ and ψ are relatively prime, and $\Phi = \frac{\varphi}{\psi}$. Then Φ has a meromorphic continuation to $\mathbb{D}$ given by the formula $\Phi(z) = \frac{\varphi(z)}{\psi(z)}$, $z \in \mathbb{D}$. We put $\Omega_\Phi = \{z \in \mathbb{D} : |\Phi(z)| > 1\}$. Further, let $T_\Phi \colon H^2 \to H^2$ be the Toeplitz operator with symbol Φ. Then T_Φ is similar to a unilateral shift if and only if $m(\partial\Omega_\Phi \cap \mathbb{T}) = 0$.*

The paper is organized as follows. In Section 2 we introduce the notation and collect simple facts concerning operators similar to isometries and functions Φ of the form $\Phi = \frac{\varphi}{\psi}$, where φ is inner and ψ is a finite Blaschke product. Also, in this section we deduce Theorem 1.2 from Theorem 1.1 and properties of Φ. In Section 3 we give a description of the subspace $\mathcal{K}$ such that the restriction $T_\Phi|_\mathcal{K}$ is similar to a unitary summand U and construct a transformation Z from $\mathcal{K}$ to

$\bigoplus_{n=0}^{N-1} L^1$, which intertwines $(T_\Phi|_{\mathcal{K}})^{-1}$ with the multiplication by $\overline{\zeta}$ in $\bigoplus_{n=0}^{N-1} L^1$, where ζ is the independent variable. In the next section, we use this transformation Z to prove Theorem 1.1. Section 5 contains some examples and remarks, which illustrate Theorems 1.1 and 1.2 and show the difficulties that arise in the present approach to the subject.

2. PRELIMINARIES

LEMMA 2.1. *Suppose φ is an inner function, ψ is a finite Blaschke product,* $\deg\psi = N < \infty$*, and* $N \leqslant \deg\varphi \leqslant \infty$*. Then $\Phi = \frac{\varphi}{\psi}$ can be represented in the form* (1.1) *with* $\deg\omega = \deg\varphi - N$.

Proof. We take a number $a \in \mathbb{D}$ such that $B_a = \frac{\varphi - a}{1-\overline{a}\varphi}$ is a Blaschke product (cf. Theorem II.6.4 in [10]), then we have $\deg B_a = \deg\varphi \geqslant N$, therefore $B_a = B\psi_1$, where B, ψ_1 are Blaschke products, $\deg\psi_1 = N$, and $\deg B = \deg\varphi - N$. Let $\{\delta_i\}_{i=1}^N$ and $\{\gamma_i\}_{i=1}^N$ be zeros of ψ and ψ_1, respectively, counted with multiplicities. We put $f_1(z) = \prod_{i=1}^N \frac{1-\overline{\gamma}_i z}{1-\overline{\delta}_i z}$, $z \in \mathbb{T}$. We have $\frac{\psi}{\psi_1} = \frac{\overline{f}}{f_1}$ on $\mathbb{T}$ and $\frac{\varphi}{B_a} = \frac{1-\overline{a}\varphi}{1-a\overline{\varphi}}$ on $\mathbb{T}$. Thus, $\Phi = \frac{\varphi}{\psi} = \frac{\varphi}{B_a}\frac{\psi_1}{\psi}B$ has the form (1.1) with $f = (1-\overline{a}\varphi)f_1$ and $\omega = B$. ∎

COROLLARY 2.2. *If φ, ψ and Φ are as in Lemma* 2.1*, then T_Φ is similar to $U \oplus S$, where U is a unitary absolutely continuous operator, S is a unilateral shift, and the multiplicity of S is* $\deg\varphi - N$.

Proof. First, we note the following evident fact. Let T be a completely nonunitary contraction similar to an isometry. Then this isometry is equal to $U \oplus S$, where U is an absolutely continuous unitary operator, S is a unilateral shift, and the multiplicity of S is equal to $\dim\ker T^*$. Any Toeplitz operator that is a contraction is a completely nonunitary contraction [11]. Let ω be inner and $f, \frac{1}{f} \in H^\infty$, then $T_{\overline{f}/f}$ is similar to a unitary operator [5], [12], and, in particular, is invertible. Therefore $T^*_{\omega f/\overline{f}} = T_{\overline{\omega}\overline{f}/f} = T_{\overline{\omega}}T_{\overline{f}/f}$ and $\dim\ker T^*_{\omega f/\overline{f}} = \dim\ker T_{\overline{\omega}} = \deg\omega$. The corollary is a consequence of this facts. ∎

LEMMA 2.3 ([4]). *Let T be an operator on a Hilbert space $\mathcal{H}$, and let T be similar to an isometry $U \oplus S$, where U is a unitary operator and S is a unilateral shift. We put $\mathcal{K} = \bigcap_{\lambda\in\mathbb{D}} (T-\lambda I)\mathcal{H}$. Then $\mathcal{K} = \bigcap_{\lambda\in\Lambda} (T-\lambda I)\mathcal{H}$ for any $\Lambda \subset \mathbb{D}$ such that Λ has a cluster point in $\mathbb{D}$. Moreover, the restriction $T|_{\mathcal{K}}$ of the operator T on its invariant subspace $\mathcal{K}$ is similar to U.*

LEMMA 2.4 (see [5]). *Let T be an operator on a Hilbert space $\mathcal{K}$, which is similar to an absolutely continuous unitary operator U. Then*

(i) *for* $k, \ell \in \mathcal{K}$ *the function* $F_{k,\ell}(\lambda) = ((T-\lambda)^{-1}k, \ell)$, $\lambda \in \widehat{\mathbb{C}} \setminus \mathbb{T}$, *has the following representation, where* $f_{k,\ell} \in L^1(\mathbb{T}, m)$*:*

$$F_{k,\ell}(\lambda) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} \frac{f_{k,\ell}(\zeta)\mathrm{d}\zeta}{\zeta - \lambda}, \quad \lambda \in \widehat{\mathbb{C}} \setminus \mathbb{T}.$$

(ii) *for a fixed vector* $\ell \in \mathcal{K}$ *the mapping* $k \to f_{k,\ell}$, $\mathcal{K} \to L^1(\mathbb{T}, m)$ *is a linear bounded transformation.*

Proof. Denote by $\mathcal{H}$ the space in which U acts. By assumption, there exists a linear bounded invertible transformation $X \colon \mathcal{K} \to \mathcal{H}$ such that $XT = UX$. For any $k, \ell \in \mathcal{K}$ there exist $x, y \in \mathcal{H}$ such that $k = X^{-1}x$, $\ell = X^* y$, and we have $((T-\lambda)^{-1}k, \ell) = ((T-\lambda)^{-1}X^{-1}x, X^*y) = (X(T-\lambda)^{-1}X^{-1}x, y) = ((U-\lambda)^{-1}x, y)$. Therefore in the conclusion of the Lemma k, ℓ and T can be replaced by x, y and U. Now the result follows from the spectral representation of an absolutely continuous operator U. ∎

LEMMA 2.5. *Suppose* φ *is an inner function,* ψ *is a finite Blaschke product, and* φ *and* ψ *are relatively prime. We put* $\Lambda = \{\lambda \in \mathbb{D}$ *: the inner factor of* $\varphi - \lambda\psi$ *is a Blaschke product with simple zeros*$\}$*. Then the logarithmic capacity of* $\mathbb{D} \setminus \Lambda$ *is zero.*

Proof. We define the set Λ_1 in the same way as the set Λ, but without the assumption of simplicity of zeros. The proof of the fact that the logarithmic capacity of $\mathbb{D} \setminus \Lambda_1$ is zero is almost the same as the proof of the Frostman's theorem, see Theorem II.6.4 in [10]. The set $\{\lambda \in \Lambda_1 : \varphi - \lambda\psi$ has a multiple zero$\}$ is no more than countable, and the result follows. ∎

LEMMA 2.6. *Suppose* φ *is an inner function,* ψ *is a finite Blaschke product,* φ *and* ψ *are relatively prime, and* $\deg \psi = N$*. Then for any* $\lambda \in \mathbb{C}$, $|\lambda| > 1$*, the equation* $\varphi - \lambda\psi = 0$ *has exactly* N *zeros in* $\mathbb{D}$ *(counted with multiplicities).*

This lemma is a consequence of the Rouché theorem.

In the following theorem we collect properties of Φ that are needed to prove Theorem 1.2.

THEOREM 2.7 ([8]). *Suppose* ψ *is a finite Blaschke product,* $\deg \psi = N < \infty$*,* φ *is inner,* $N \leqslant \deg \varphi \leqslant \infty$*,* φ *and* ψ *are relatively prime, and* $\Phi = \frac{\varphi}{\psi}$*. Then* Φ *has meromorphic continuation to* $\mathbb{D}$ *given by the formula* $\Phi(z) = \frac{\varphi(z)}{\psi(z)}$*,* $z \in \mathbb{D}$*. We put* $\Omega_\Phi = \{z \in \mathbb{D} : |\Phi(z)| > 1\}$ *and* $\mathcal{I}_N = \{\lambda \in \mathbb{T} : \Phi - \lambda$ *has* N *zeros in* $\mathbb{D}\}$*. Then* Φ *has angular boundary value at every point of* $\partial\Omega_\Phi \cap \mathbb{T}$*. We define* Φ *on* $\partial\Omega_\Phi \cap \mathbb{T}$ *by its angular boundary values. Then* Φ *is continuous on* $\operatorname{clos}\Omega_\Phi$*,* $\mathcal{I}_N = \mathbb{T} \setminus \Phi(\partial\Omega_\Phi \cap \mathbb{T})$*, and* $m(\Phi(\partial\Omega_\Phi \cap \mathbb{T})) = 0$ *if and only if* $m(\partial\Omega_\Phi \cap \mathbb{T}) = 0$*.*

Proof of Theorem 1.2. By Theorem 1.1, T_Φ is similar to a unilateral shift if and only if $m(\sigma_N) = 0$, where $\sigma_N = \mathbb{T} \setminus \mathcal{I}_N$. By Theorem 2.7, $\sigma_N = \Phi(\partial\Omega_\Phi \cap \mathbb{T})$, and $m(\Phi(\partial\Omega_\Phi \cap \mathbb{T})) = 0$ if and only if $m(\partial\Omega_\Phi \cap \mathbb{T}) = 0$. ∎

NOTATION. In the remaining part of the paper, we will use the following notation. N is a natural number, ψ is a finite Blaschke product, $\deg \psi = N < \infty$, $\psi(z) = \prod_{i=1}^{N} \frac{z-\delta_i}{1-\overline{\delta}_i z}$, $\alpha(z) = \prod_{i=1}^{N}(z-\delta_i)$, $\beta(z) = \prod_{i=1}^{N}(1-\overline{\delta}_i z)$, $z \in \mathbb{D}$, $\{\delta_i\}_{i=1}^{N} \subset \mathbb{D}$. Further, φ is an inner function, $N \leqslant \deg \varphi \leqslant \infty$, φ and ψ are relatively prime, $\Phi = \frac{\varphi}{\psi}$ is a function meromorphic in $\mathbb{D}$, and $\Phi \in L^\infty(\mathbb{T}, m)$, T_Φ is the Toeplitz operator on H^2 with symbol Φ, $\mathcal{K} = \bigcap_{\lambda \in \mathbb{D}} (T_\Phi - \lambda I)H^2$, $T = T_\Phi|_{\mathcal{K}}$. By Corollary 2.2, T_Φ is similar to an isometry, we denote by U the unitary summand of this isometry. By Lemma 2.3 we have that T is similar to U.

3. THE SUBSPACE $\mathcal{K}$

In this section we give a description of the subspace $\mathcal{K} = \bigcap_{\lambda \in \mathbb{D}} (T_\Phi - \lambda I)H^2$. By Lemma 2.3 the restriction $T = T_\Phi|_{\mathcal{K}}$ is similar to the unitary summand U of the isometry to which T_Φ is similar. Then we construct a linear bounded transformation $Z\colon \mathcal{K} \to \bigoplus_{n=0}^{N-1} L^1$, such that $\ker Z = \{0\}$ and Z intertwines T^{-1} and the multiplication by $\overline{\zeta}$ in $\bigoplus_{n=0}^{N-1} L^1$, where ζ is the independent variable. We shall use this transformation Z to find U in the next section.

LEMMA 3.1. *Let $F_n \colon \widehat{\mathbb{C}} \setminus \mathbb{T} \to \mathbb{C}$, $n = 0, \ldots, N-1$, be analytic functions such that $\sum_{n=0}^{N-1} (F_n \circ \Phi)(z) z^n = 0$ for all $z \in \mathbb{D}$ such that $|\Phi(z)| \neq 1$. Then $F_n \equiv 0$, $n = 0, \ldots, N-1$.*

Proof. Let $\lambda \in \widehat{\mathbb{C}} \setminus \operatorname{clos} \mathbb{D}$ be such that $\Phi - \lambda$ has N simple zeros $z_1, \ldots, z_N$ in $\mathbb{D}$. We have $\sum_{n=0}^{N-1} F_n(\lambda) z_j^n = 0$, $j = 1, \ldots, N$, therefore $F_n(\lambda) = 0$, $n = 0, \ldots, N-1$. Since the set of points $\lambda \in \widehat{\mathbb{C}} \setminus \operatorname{clos} \mathbb{D}$ such that $\Phi - \lambda$ has N simple zeros in $\mathbb{D}$ is dense in $\widehat{\mathbb{C}} \setminus \operatorname{clos} \mathbb{D}$ and F_n are analytic, we conclude that $F_n \equiv 0$ in $\widehat{\mathbb{C}} \setminus \operatorname{clos} \mathbb{D}$, $n = 0, \ldots, N-1$. Further, let $\lambda \in \Lambda$, where Λ is the set defined in Lemma 2.5. Then $\Phi - \lambda$ has infinite number of simple zeros. Taking N different zeros of $\Phi - \lambda$ and acting as above we obtain that $F_n \equiv 0$ in $\mathbb{D}$, $n = 0, \ldots, N-1$. ∎

LEMMA 3.2 (see [4]). *Let $k \in H^2$, and let $\lambda \in \mathbb{C}$. Then $k \in (T_\Phi - \lambda)H^2$ if and only if there exists $k_\lambda \in H^2$ such that $p_{\lambda,k} = \alpha k - \beta(\varphi - \lambda\psi)k_\lambda$ is a polynomial and* $\deg p_{\lambda,k} \leqslant N-1$. *If these conditions are fulfilled, then $k = (T_\Phi - \lambda)k_\lambda$.*

Proof. The "only if" part. We have $k = (T_\Phi - \lambda)k_\lambda = P_+(\varphi\overline{\psi} - \lambda)k_\lambda$ for some $k_\lambda \in H^2$, therefore $(\varphi\overline{\psi} - \lambda)k_\lambda = k + \overline{zh_\lambda}$ for some $h_\lambda \in H^2$, and $(\varphi - \lambda\psi)k_\lambda = \psi k + \psi\overline{zh_\lambda}$. We see that $\psi\overline{zh_\lambda} \in H^2 \cap \psi\overline{zH^2} = H^2 \ominus \psi H^2$ and it is well-known that

$H^2 \ominus \psi H^2 = \frac{1}{\beta}\mathcal{P}_{N-1}$, where $\mathcal{P}_{N-1}$ is the set of polynomials of degree no more that $N-1$. Therefore $\psi\overline{zh_\lambda} = -\frac{1}{\beta}p_{\lambda,k}$, where $p_{\lambda,k} \in \mathcal{P}_{N-1}$, and the result follows. To prove the "if" part the same reasons must be used in the inverse order. ∎

THEOREM 3.3. *Let k be a function in $\mathbb{D}$. The following are equivalent:*
(i) $k \in \mathcal{K}$;
(ii) $k \in H^2$ *and there exist vanishing at infinity functions F_n analytic on $\widehat{\mathbb{C}} \setminus \mathbb{T}$ and such that the next equality holds for all $z \in \mathbb{D}$ such that $|\Phi(z)| \neq 1$:*

$$(\alpha k)(z) = \sum_{n=0}^{N-1} (F_n \circ \Phi)(z) z^n. \tag{3.1}$$

Moreover, if $k \in \mathcal{K}$ and F_n are defined by k in (3.1), *then*

$$F_n(\lambda) = (\alpha k - \beta(\varphi - \lambda\psi)(T-\lambda)^{-1}k, z^n), \quad \lambda \notin \mathbb{T}, \tag{3.2}$$

and F_n has the following representation, where $f_n \in L^1(\mathbb{T}, m)$, $n = 0, \dots, N-1$:

$$F_n(\lambda) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}} \frac{f_n(\zeta)\mathrm{d}\zeta}{\zeta - \lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{T}. \tag{3.3}$$

REMARK. By Lemma 3.1 we have that F_n are defined by k in the only possible way.

Proof. Let $k \in \mathcal{K}$. Then, since $T - \lambda$ is invertible for any $\lambda \in \mathbb{C} \setminus \mathbb{T}$, by Lemma 3.2 we have $p_{\lambda,k} = \alpha k - \beta(\varphi - \lambda\psi)(T-\lambda)^{-1}k$. We put $F_n(\lambda) = (p_{\lambda,k}, z^n)$. Then (3.2) is fulfilled and F_n are analytic functions on $\mathbb{C}\setminus\mathbb{T}$. The equality $\lim_{\lambda\to\infty} F_n(\lambda) = 0$ easily follows from the representation $p_{\lambda,k} = (\alpha T - \beta\varphi)(T-\lambda)^{-1}k$. Now, let $z \in \mathbb{D}$, let $\lambda = \Phi(z)$, and let $\lambda \notin \mathbb{T}$. Then $(\alpha k)(z) = \sum_{n=0}^{N-1} F_n(\lambda) z^n + \beta(z)(\varphi - \lambda\psi)(z)((T-\lambda)^{-1}k)(z)$, but $\Phi = \frac{\varphi}{\psi}$, therefore $(\varphi - \lambda\psi)(z) = 0$ and (3.1) follows. The representation (3.3) is a consequence of (3.2), the fact that T is similar to U and the part (i) of Lemma 2.4.

Now let $k \in H^2$, and let k has a representation (3.1). We take $\lambda \in \Lambda$, where Λ is the set from Lemma 2.5, and we put $p_{\lambda,k}(z) = \sum_{n=0}^{N-1} F_n(\lambda) z^n$. We shall show that $k \in (T_\Phi - \lambda)H^2$, and by Lemma 3.2 it is sufficient to show that

$$\frac{\alpha k - p_{\lambda,k}}{\beta(\varphi - \lambda\psi)} \in H^2. \tag{3.4}$$

Let $\varphi - \lambda\psi = \omega_\lambda g_\lambda$ be the inner-outer factorization of $\varphi - \lambda\psi$, where ω_λ is inner, g_λ is outer. Since $|\lambda| < 1$ and φ, ψ are inner, we have $\frac{1}{g_\lambda} \in H^\infty$, and, evidently, $\frac{1}{\beta} \in H^\infty$. Therefore (3.4) can be rewritten as $\alpha k - p_{\lambda,k} \in \omega_\lambda H^2$. By assumption on λ, the function ω_λ is a Blaschke product with simple zeros, therefore $\alpha k - p_{\lambda,k} \in \omega_\lambda H^2$ if and only if $(\alpha k)(z) = p_{\lambda,k}(z)$ for any zero z of ω_λ. But z is a zero of ω_λ

if and only if $\Phi(z) = \lambda$, and the equality $(\alpha k)(z) = p_{\lambda,k}(z)$ is a consequence of (3.1). Thus, we proved that $k \in (T_\Phi - \lambda)H^2$ for any $\lambda \in \Lambda$. By Lemma 2.5, the set Λ has a cluster point in $\mathbb{D}$, and by Lemma 2.3, $\bigcap\limits_{\lambda\in\Lambda} (T_\Phi - \lambda I)H^2 = \mathcal{K}$. ∎

In Theorem 3.4 below we shall introduce and study the spaces $\mathcal{E}(K)$ for compact sets K, which turn out to be spectral subspaces of T. We shall see it in the next section. In Corollary 3.5 below we shall introduce and study the mapping Z. We shall use this mapping in the next section. It is convenient to formulate properties of $\mathcal{E}(K)$ and Z separately, but to prove them together.

THEOREM 3.4. *Let $K \subset \mathbb{T}$ be a compact set. We put $\mathcal{E}(K) = \{k \in \mathcal{K}$: the functions F_n defined by k in* (3.1) *have analytic continuation to $\mathbb{C} \setminus K$ for all $n = 0, \dots, N-1\}$. Then $\mathcal{E}(K)$ is an invariant subspace of T and $\sigma(T|_{\mathcal{E}(K)}) \subset K$. Moreover, let $\lambda_0 \in \mathbb{C} \setminus K$, let $k \in \mathcal{E}(K)$, and let $G_n(\lambda) = \frac{F_n(\lambda)-F_n(\lambda_0)}{\lambda-\lambda_0}$, where $\lambda \in \mathbb{C} \setminus K$, $n = 0, \dots, N-1$. Then the equality* (3.1) *holds with k replaced by $(T|_{\mathcal{E}(K)} - \lambda_0)^{-1}k$ and F_n replaced by G_n.*

COROLLARY 3.5. *The mapping*

$$Z : \mathcal{K} \to \bigoplus_{n=0}^{N-1} L^1(\mathbb{T}, m), \quad Zk = \{f_n\}_{n=0}^{N-1},$$

where f_n are defined by k in (3.2) *and* (3.3)*, is a linear bounded transformation,* $\ker Z = \{0\}$*, and $ZT^{-1} = VZ$. Here*

$$V : \bigoplus_{n=0}^{N-1} L^1(\mathbb{T}, m) \to \bigoplus_{n=0}^{N-1} L^1(\mathbb{T}, m)$$

is an operator defined by the next formula, where $\{f_n\}_{n=0}^{N-1} \in \bigoplus\limits_{n=0}^{N-1} L^1(\mathbb{T}, m)$, $\zeta \in \mathbb{T}$:

$$(V\{f_n\}_{n=0}^{N-1})(\zeta) = \{\overline{\zeta} f_n(\zeta)\}_{n=0}^{N-1}.$$

Proof of Theorem 3.4 *and Corollary* 3.5. The boundedness and injectivity of Z is a consequence of Theorem 3.3 and the part (ii) of Lemma 2.4. Now, we shall prove Theorem 3.4. First, the linearity of $\mathcal{E}(K)$ follows from (3.2). Further, let $\{k_j\}_{j=1}^\infty \subset \mathcal{E}(K)$, and let $k_j \to_j k$, where $k \in \mathcal{K}$. We need to check that $k \in \mathcal{E}(K)$. We put $Zk_j = \{f_{nj}\}_{n=0}^{N-1}$ and $Zk = \{f_n\}_{n=0}^{N-1}$, by already proved part of Corollary 3.5 we have

$$f_{nj} \xrightarrow[j]{L^1(\mathbb{T})} f_n, \quad n = 0, \dots, N-1. \tag{3.5}$$

By the assumption, the functions F_{nj}, defined by f_{nj} as in (3.3), are analytic on $\mathbb{C} \setminus K$. Therefore $f_{nj} = 0$ a.e. on $\mathbb{T} \setminus K$, and the same is true for f_n by the convergence (3.5). Now, from (3.3) we conclude that $k \in \mathcal{E}(K)$. We proved that $\mathcal{E}(K)$ is a closed set.

Further, let $\lambda_0 \notin K$, let $k \in \mathcal{E}(K)$, and let G_n be defined as above. We put $p(z) = \sum\limits_{n=0}^{N-1} F_n(\lambda_0)z^n$, $z \in \mathbb{D}$, and

$$h = \frac{\alpha k - p}{\varphi - \lambda_0\psi} = \frac{1}{\psi}\frac{\alpha k - p}{\Phi - \lambda_0}. \tag{3.6}$$

From (3.6) we conclude that h is analytic on $\mathbb{D} \setminus \Phi^{-1}(\lambda_0)$ and that

$$(\psi h)(z) = \sum_{n=0}^{N-1} (G_n \circ \Phi)(z)z^n \tag{3.7}$$

for all $z \in \mathbb{D}$ such that $\Phi(z) \notin K \cup \{\lambda_0\}$. But the right part of (3.7) is analytic on a neighbourhood of any point z such that $\Phi(z) = \lambda_0$. Thus, h is analytic on a neighbourhood of $\Phi^{-1}(\lambda_0)$, and we have that h is analytic on $\mathbb{D}$.

Now we shall prove that $h \in H^2$. Let $D(\lambda_0)$ be a closed disk centered at λ_0 and such that F_n are analytic on $D(\lambda_0)$, $n = 0, \dots, N-1$. Let $z \in \mathbb{D}$ and $\Phi(z) = \lambda$. If $\lambda \in D(\lambda_0)$, we use (3.7) to obtain that $|(\psi h)(z)| \leqslant \sum\limits_{n=0}^{N-1} |G_n(\lambda)|\,|z|^n \leqslant \sup\{|G_n(w)|,\ w \in D(\lambda_0),\ n = 0, \dots, N-1\}$. If $\lambda \notin D(\lambda_0)$, we use (3.6) to obtain that $|(\psi h)(z)| \leqslant \frac{|(\alpha k)(z)| + \|p\|_\infty}{|\lambda - \lambda_0|} \leqslant \frac{|(\alpha k)(z)| + \|p\|_\infty}{\varepsilon}$, where ε is the radius of $D(\lambda_0)$. Now it is easy to see that $h \in H^2$.

We put $k_0 = \frac{h}{\beta}$, from (3.7) and (3.1) we conclude that $k_0 \in \mathcal{K}$, and from (3.7) we have $k_0 \in \mathcal{E}(K)$. By Lemma 3.2 and by (3.6) we have $(T_\Phi - \lambda_0)k_0 = k$.

Thus, for arbitrary $k \in \mathcal{E}(K)$ there exists $k_0 \in \mathcal{E}(K)$ such that $(T_\Phi - \lambda_0)k_0 = k$, that is, $(T - \lambda_0)\mathcal{E}(K) \supset \mathcal{E}(K)$ for $\lambda_0 \notin K$. In particular, $(T - \lambda)\mathcal{E}(K) \supset \mathcal{E}(K)$ for all $\lambda \in \mathbb{C} \setminus \operatorname{clos}\mathbb{D}$, in the other words, $\mathcal{E}(K) \supset (T - \lambda)^{-1}\mathcal{E}(K)$, therefore $T\mathcal{E}(K) \subset \mathcal{E}(K)$. Now, $(T - \lambda_0)\mathcal{E}(K) \subset \mathcal{E}(K)$, and we have $(T - \lambda_0)\mathcal{E}(K) = \mathcal{E}(K)$. Since $\ker(T - \lambda_0) = \{0\}$, we conclude that $T|_{\mathcal{E}(K)} - \lambda_0$ is invertible.

Finally, the intertwining property of Z is a consequence of Theorem 3.4 (where $K = \mathbb{T}$). ∎

4. PROOF OF THEOREM 1.1

In this section we shall find the unitary summand U of the isometry to which T_Φ is similar. Since U is absolutely continuous, it is sufficient to find a local spectral multiplicity function $\mathbf{n}_U$ of U. To do it, we shall use the transformation Z, which was defined in Corollary 3.5, and the similarity of $T = T_\Phi|_{\mathcal{K}}$ and U. We can consider U as the operator of multiplication by ζ in the space $\bigoplus\limits_{n=1}^{\infty} L^2(\tau_n, m)$, where τ_n are measurable subsets of $\mathbb{T}$ such that $\mathbb{T} \supset \tau_1 \supset \cdots \supset \tau_n \supset \tau_{n+1} \supset \cdots$. Then $\mathbf{n}_U(\zeta) = \max\{n : \zeta \in \tau_n\}$ for a.e. $\zeta \in \mathbb{T}$. Let $X : \bigoplus\limits_{n=1}^{\infty} L^2(\tau_n, m) \to \mathcal{K}$ be a transformation which gives the similarity of T and U. Then the transformation

$$ZX : \bigoplus_{n=1}^{\infty} L^2(\tau_n, m) \to \bigoplus_{n=0}^{N-1} L^1(\mathbb{T}, m)$$

intertwines the operators U^{-1} and V, where V is the multiplication by $\overline{\zeta}$ in the space $\bigoplus_{n=0}^{N-1} L^1(\mathbb{T}, m)$, and $\ker ZX = \{0\}$. The space $\operatorname{clos} ZX \bigoplus_{n=1}^{\infty} L^2(\tau_n, m)$ is an invariant subspace of V, therefore there exists a measurable family of spaces $\mathbf{S}(\zeta) \subset \mathbb{C}^N$, $\zeta \in \mathbb{T}$, such that

$$\operatorname{clos} ZX \bigoplus_{n=1}^{\infty} L^2(\tau_n, m) = \Big\{ \mathbf{f} \in \bigoplus_{n=0}^{N-1} L^1(\mathbb{T}, m) = L^1(\mathbb{T}, m; \mathbb{C}^N) : \mathbf{f}(\zeta) \in \mathbf{S}(\zeta) \text{ for a.e. } \zeta \in \mathbb{T} \Big\},$$

and $\mathbf{n}_U(\zeta) = \dim \mathbf{S}(\zeta)$ for a.e. $\zeta \in \mathbb{T}$. The conclusion of Theorem 1.1 concerning U can be rewritten as $\mathbf{n}_U(\zeta) = N + 1 - \min\{n : 1 \leqslant n \leqslant N,\ \zeta \in \sigma_n\}$ for a.e. $\zeta \in \mathbb{T}$. To prove Theorem 1.1 we shall show that $\dim \mathbf{S}(\zeta) \geqslant N + 1 - \min\{n : 1 \leqslant n \leqslant N,\ \zeta \in \sigma_n\}$ and $\dim \mathbf{S}(\zeta) \leqslant N + 1 - \min\{n : 1 \leqslant n \leqslant N,\ \zeta \in \sigma_n\}$. Theorem 4.2 shows that $\dim \mathbf{S}(\zeta) \geqslant N - \nu$ for a.e. $\zeta \in \mathcal{I}_\nu \setminus \mathcal{I}_{\nu+1} = \sigma_{\nu+1} \setminus \sigma_\nu$. To prove the inverse inequality we will use Lemma 4.6.

Before formulating Theorem 4.2 we need to prove Lemma 4.1 and to introduce the notation.

LEMMA 4.1. *Let $\mathcal{I}$ be an open subset of $\mathbb{T}$. We put $\mathfrak{E}(\mathcal{I}) = \{E \subset \mathbb{T} \setminus \mathcal{I} : E$ is separated from $(\mathbb{T} \setminus \mathcal{I}) \setminus E\}$, $\mathfrak{E}_0(\mathcal{I}) = \{E \in \mathfrak{E}(\mathcal{I}) : m(E) = 0\}$, and $\mathfrak{J}(\mathcal{I}) = \mathcal{I} \cup \bigcup_{E \in \mathfrak{E}_0(\mathcal{I})} E$. Then:*

(i) *$\mathfrak{J}(\mathcal{I})$ is an open subset of $\mathbb{T}$;*

(ii) *$m(\mathfrak{J}(\mathcal{I})) = m(\mathcal{I})$;*

(iii) *$\mathbb{T} \setminus \mathfrak{J}(\mathcal{I})$ does not contain (nonempty) separated subsets of zero measure;*

(iv) *if $a_0, b_0 \in \mathbb{T}$, $a_0 \neq b_0$, $I = [a_0, b_0]$ is a closed subarc of $\mathbb{T}$ and E is a subset of $I \setminus \mathfrak{J}(\mathcal{I})$ such that E is separated from $(I \setminus \mathfrak{J}(\mathcal{I})) \setminus E$ and $m(E) = 0$, then $E \subset \{a_0, b_0\}$; in particular, if $(a_0, b_0) \setminus \mathfrak{J}(\mathcal{I}) \neq \varnothing$, then $m(I \setminus \mathfrak{J}(\mathcal{I})) > 0$.*

Proof. First, we will show that $\mathfrak{E}(\mathcal{I})$ is no more than countable. We denote by $\mathfrak{O}$ the collection of finite unions of open subarcs of $\mathbb{T}$ which have rational endpoints. We note that $E \in \mathfrak{E}(\mathcal{I})$ if and only if there exists $\mathcal{O} \in \mathfrak{O}$ such that $\operatorname{clos} E \subset \mathcal{O}$ and $\mathcal{O} \cap ((\mathbb{T} \setminus \mathcal{I}) \setminus E) = \varnothing$. From the inclusion $\operatorname{clos} E \subset \mathbb{T} \setminus \mathcal{I}$ we conclude that $\operatorname{clos} E = E$. We have that for every $E \in \mathfrak{E}(\mathcal{I})$ there exists $\mathcal{O}_E \in \mathfrak{O}$ such that $E \subset \mathcal{O}_E$ and $\mathcal{O}_E \cap ((\mathbb{T} \setminus \mathcal{I}) \setminus E) = \varnothing$. If $E_1, E_2 \in \mathfrak{E}(\mathcal{I})$ and $E_1 \neq E_2$, from the last equality we conclude that $\mathcal{O}_{E_1} \neq \mathcal{O}_{E_2}$. But $\mathfrak{O}$ is countable, therefore $\mathfrak{E}(\mathcal{I})$ is no more than countable set. Now (ii) is proved and (i) follows from the representation $\mathfrak{J}(\mathcal{I}) = \mathcal{I} \cup \bigcup_{E \in \mathfrak{E}_0(\mathcal{I})} \mathcal{O}_E$.

Parts (iii) and (iv) of the lemma can be proved in a similar way, so their proofs are omitted. ∎

Let us recall that Φ is a function meromorphic in $\mathbb{D}$, and $\Phi = \frac{\varphi}{\psi}$, where φ is inner, ψ is a finite Blaschke product, $\deg\psi = N < \infty$, and $N \leqslant \deg\varphi \leqslant \infty$. For $n = 1, \dots, N$ we introduced the sets $\mathcal{I}_n = \{\lambda \in \mathbb{T} : \Phi - \lambda$ has at least n zeros in $\mathbb{D}$ (counted with multiplicities)$\}$. Also, we put $\mathcal{I}_0 = \mathbb{T}$ and $\mathcal{I}_{N+1} = \emptyset$. It is evident that $\mathcal{I}_n$ is an open subset of $\mathbb{T}$. Now, we put $\mathcal{J}_n = \mathfrak{J}(\mathcal{I}_n)$, where $\mathfrak{J}(\mathcal{I})$ for an open subset $\mathcal{I}$ of $\mathbb{T}$ is defined in Lemma 4.1. Also, we put $A = \{\lambda \in \mathbb{T} :$ there exists $z \in \mathbb{D}$ such that $\Phi(z) = \lambda$ and $\Phi'(z) = 0\}$. Evidently, A is no more than countable. We recall that the subspace $\mathcal{E}(K)$ for a compact set $K \subset \mathbb{T}$ was defined in Theorem 3.4.

THEOREM 4.2. *We use the notation introduced above. Let* $0 \leqslant \nu \leqslant N-1$, *and let* $\lambda_0 \in \mathcal{I}_\nu \setminus (A \cup \mathcal{J}_{\nu+1})$. *Then there exists a compact set* $K(\lambda_0) \subset \mathbb{T}$ *such that* $\lambda_0 \in K(\lambda_0)$, *the set* $K(\lambda_0)$ *does not contain (nonempty) separated subsets of zero measure, and there exists a linear bounded transformation*

$$Y : \mathcal{E}(K(\lambda_0)) \to \bigoplus_{n=\nu}^{N-1} L^1(K(\lambda_0), m)$$

such that $\ker Y = \{0\}$, $\operatorname{clos} Y\mathcal{E}(K(\lambda_0)) = \bigoplus\limits_{n=\nu}^{N-1} L^1(K(\lambda_0), m)$ *and* $YT^{-1} = VY$, *where* V *acts in* $\bigoplus\limits_{n=\nu}^{N-1} L^1(K(\lambda_0), m)$ *by the formula*

$$(V\{f_n\}_{n=\nu}^{N-1})(\zeta) = \{\overline{\zeta} f_n(\zeta)\}_{n=\nu}^{N-1}.$$

To prove Theorem 4.2 we need the following lemmas.

LEMMA 4.3. *Let* $K \subset \mathbb{T}$ *be a compact set, let* $m(K) > 0$, *and let* ν *be a number,* $0 \leqslant \nu \leqslant N$. *Let* $\eta_j : K \to \mathbb{C}$ *be continuous functions such that* $\eta_j(K) \cap \eta_\ell(K) = \emptyset$ *for* $j \neq \ell$, $j, \ell = 1, \dots, \nu$. *Further, let*

$$\mathcal{M} = \begin{pmatrix} 1 & \eta_1 & \dots & \eta_1^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \eta_\nu & \dots & \eta_\nu^{N-1} \end{pmatrix} \tag{4.1}$$

be a matrix-function. We put

$$\mathcal{L} = \mathcal{L}(K, \mathcal{M}) = \Big\{ \{f_n\}_{n=0}^{N-1} \in \bigoplus_{n=0}^{N-1} L^1(K, m) : \mathcal{M}\{f_n\}_{n=0}^{N-1} = \mathbb{O} \Big\},$$

and we define the mapping $W = W(K, \mathcal{M}) : \mathcal{L} \to \bigoplus\limits_{n=\nu}^{N-1} L^1(K, m)$ *by the formula* $W\{f_n\}_{n=0}^{N-1} = \{f_n\}_{n=\nu}^{N-1}$.

Then W *is a linear bounded invertible transformation, and for*

$$\{f_n\}_{n=\nu}^{N-1} \in \bigoplus_{n=\nu}^{N-1} L^1(K, m)$$

the elements of $W^{-1}\{f_n\}_{n=\nu}^{N-1}$ with numbers $0,\dots,\nu-1$ are given by the formula $\{f_n\}_{n=0}^{\nu-1} = -\mathcal{M}_1^{-1}\mathcal{M}_2\{f_n\}_{n=\nu}^{N-1}$, where the matrix-functions $\mathcal{M}_1$ and $\mathcal{M}_2$ consist of the columns of $\mathcal{M}$ with numbers $1,\dots,\nu$ and $\nu+1,\dots,N$, respectively.

If $\nu = N$, then the claim of the lemma is that $\mathcal{L} = \{0\}$.

The proof of Lemma 4.3 consists in easy computations. We mention only that the determinant of $\mathcal{M}_1$ is a Wandermonde determinant, which is not zero because $\eta_j(K)\cap\eta_\ell(K)=\emptyset$, if $j\neq\ell$.

LEMMA 4.4. *Let $0\leqslant\nu\leqslant N-1$, let $K\subset\mathbb{T}$ be a compact set, and let η_j, $j=1,\dots,\nu$, be functions analytic in a neighbourhood $\mathcal{D}$ of K and such that for K and η_j, $j=1,\dots,\nu$, the assumption of Lemma 4.3 holds. Further, let $\nu\leqslant\kappa\leqslant N-1$, and let f_κ be a function on K such that F_κ is bounded on $\widehat{\mathbb{C}}\setminus K$. We put $f_n=0$, $n\neq\kappa$, $n=\nu,\dots,N-1$, $\{f_n\}_{n=0}^{N-1} = W^{-1}\{f_n\}_{n=\nu}^{N-1}$, and $g_{\kappa j}(\lambda) = \sum\limits_{n=0}^{N-1} F_n(\lambda)(\eta_j(\lambda))^n$, $\lambda\in\mathcal{D}\setminus K$, $j=1,\dots,\nu$, where F_n are defined by f_n in* (3.3)*, $n=0,\dots,N-1$.*

Then $g_{\kappa j}$ can be extended to functions analytic in $\mathcal{D}$, $j=1,\dots,\nu$.

Proof. We shall use the notation from Lemma 4.3. The elements a_{jn} of the matrix $-\mathcal{M}_1^{-1}\mathcal{M}_2$ are functions analytic and bounded on some neighbourhood $\mathcal{D}_1\subset\mathcal{D}$ of K, and $f_n = a_{n\kappa}f_\kappa$, $n=0,\dots,\nu-1$. Further,

$$\begin{aligned} g_{\kappa j}(\lambda) &= \frac{1}{2\pi\mathrm{i}}\int_K \frac{\Big(\sum\limits_{n=0}^{\nu-1} a_{n\kappa}(\zeta)f_\kappa(\zeta)\Big)\eta_j^n(\lambda) + f_\kappa(\zeta)\eta_j^\kappa(\lambda)}{\zeta-\lambda}\mathrm{d}\zeta \\ &= \frac{1}{2\pi\mathrm{i}}\int_K \frac{f_\kappa(\zeta)}{\zeta-\lambda}\Big(\sum_{n=0}^{\nu-1} a_{n\kappa}(\zeta)\eta_j^n(\lambda)+\eta_j^\kappa(\lambda)\Big)\mathrm{d}\zeta \end{aligned}$$

for $\lambda\in\mathcal{D}_1\setminus K$. Further, $\sum\limits_{n=0}^{\nu-1} a_{n\kappa}\eta_j^n$ is the element of $\mathcal{M}_1(-\mathcal{M}_1^{-1}\mathcal{M}_2) = -\mathcal{M}_2$ on the intersection of the jth row and κth column, that is, $\sum\limits_{n=0}^{\nu-1} a_{n\kappa}\eta_j^n = -\eta_j^\kappa$, and

$$\sum_{n=0}^{\nu-1}a_{n\kappa}(\zeta)\eta_j^n(\lambda)+\eta_j^\kappa(\lambda)=\sum_{n=0}^{\nu-1}a_{n\kappa}(\zeta)\eta_j^n(\lambda)-\sum_{n=0}^{\nu-1}a_{n\kappa}(\lambda)\eta_j^n(\lambda)=\sum_{n=0}^{\nu-1}(a_{n\kappa}(\zeta)-a_{n\kappa}(\lambda))\eta_j^n(\lambda),$$

therefore

$$g_{\kappa j}(\lambda) = \frac{1}{2\pi\mathrm{i}}\sum_{n=0}^{\nu-1}\eta_j^n(\lambda)\int_K\frac{a_{n\kappa}(\zeta)-a_{n\kappa}(\lambda)}{\zeta-\lambda}f_\kappa(\zeta)\mathrm{d}\zeta.$$

We put

$$\xi_n(\lambda) = \int_K \frac{a_{n\kappa}(\zeta)-a_{n\kappa}(\lambda)}{\zeta-\lambda}f_\kappa(\zeta)\mathrm{d}\zeta, \quad \lambda\in\mathcal{D}_1\setminus K.$$

We recall that $f_\kappa \in L^\infty(K,m)$ and $a_{n\kappa}$ are analytic on $\mathcal{D}_1$. It is well-known (and easy to see by using the Morera theorem) that ζ_n are analytic on $\mathcal{D}_1$. We have $g_{\kappa j} = \frac{1}{2\pi \mathrm{i}} \sum\limits_{n=0}^{\nu-1} \eta_j^n \zeta_n$ on $\mathcal{D}_1$, therefore $g_{\kappa j}$ are analytic on $\mathcal{D}_1$, and, consequently, on $\mathcal{D}$. ∎

Before formulating next lemmas we introduce some more notation. Let $0 \leqslant \nu \leqslant N$, and let $\lambda_0 \in \mathcal{I}_\nu \setminus A$. Then all zeros of $\Phi - \lambda_0$ are simple, and $\Phi - \lambda_0$ has at least ν zeros $z_1, \ldots, z_\nu$ in $\mathbb{D}$. There exist a closed disk $D(\lambda_0)$ centered at λ_0 and open sets $\mathcal{V}_1, \ldots, \mathcal{V}_\nu$, such that $z_j \in \mathcal{V}_j$, $\mathcal{V}_j \cap \mathcal{V}_\ell = \emptyset$, $j \neq \ell$, Φ is univalent on $\mathcal{V}_j$ and $D(\lambda_0) \subset \Phi(\mathcal{V}_j)$, $j, \ell = 1, \ldots, \nu$. We put

$$\eta_j = \Phi^{-1}|_{D(\lambda_0)} : D(\lambda_0) \to \mathbb{D}, \quad j = 1, \ldots, \nu.$$

Then, evidently, η_j are analytic functions, $\Phi(\eta_j(\lambda)) = \lambda$, $\lambda \in D(\lambda_0)$, $\eta_j(\Phi(z)) = z$, $z \in \eta_j(D(\lambda_0))$, and $\eta_j(D(\lambda_0)) \subset \mathcal{V}_j$, therefore $\eta_j(D(\lambda_0)) \cap \eta_\ell(D(\lambda_0)) = \emptyset$ if $j \neq \ell$, $j, \ell = 1, \ldots, \nu$.

We shall say that *the local branches η_j of the inverse function to Φ are well-defined on $D(\lambda_0)$*. Of course, η_j depend on λ_0, and, if $\Phi - \lambda_0$ has more than ν zeros, on the choice of zeros $z_1, \ldots, z_\nu$.

In what follows, we shall substitute $z = \eta_j(\lambda)$, $j = 1, \ldots, \nu$, into (3.1) and study the linear relations that are obtained in this way. Also, Lemmas 4.3 and 4.4, which are formulated for arbitrary functions η_j, really will be applied to local branches of the inverse function to Φ.

LEMMA 4.5. *Let $0 \leqslant \nu \leqslant N$, and let $\lambda_0 \in \mathcal{I}_\nu \setminus (A \cup \mathcal{J}_{\nu+1})$. Then there exist a closed arc $I(\lambda_0) \subset \mathbb{T}$ and a closed disk $D(\lambda_0)$ centered at λ_0 such that $\lambda_0 \in I(\lambda_0)$, $(I(\lambda_0) \setminus \mathcal{J}_{\nu+1}) \cap A = \emptyset$, $I(\lambda_0) \setminus \mathcal{J}_{\nu+1}$ does not contain (nonempty) separated subsets of zero measure, $I(\lambda_0) \subset \operatorname{int} D(\lambda_0)$, and local branches of the inverse function to Φ are well-defined on $D(\lambda_0)$.*

Proof. By the assumption, $\Phi - \lambda_0$ has exactly ν simple zeros $z_1, \ldots, z_\nu$ in $\mathbb{D}$. We take closed disks $D_1(\lambda_0)$, $D(\lambda_0)$ centered at λ_0 such that local branches of the inverse function to Φ are well-defined on $D(\lambda_0)$, $D_1(\lambda_0) \subset D(\lambda_0)$, $D_1(\lambda_0) \neq D(\lambda_0)$, and we put $I(\lambda_0) = D_1(\lambda_0) \cap \mathbb{T}$. By the construction, $\Phi - \lambda$ has exactly ν simple zeros in $\mathbb{D}$ for any $\lambda \in I(\lambda_0) \setminus \mathcal{J}_{\nu+1}$. In particular, $(I(\lambda_0) \setminus \mathcal{J}_{\nu+1}) \cap A = \emptyset$.

We denote by a_0 and b_0 the ends of the arc $I(\lambda_0)$. Let E be a set such that $E \subset I(\lambda_0) \setminus \mathcal{J}_{\nu+1}$, $m(E) = 0$, and E is separated from $(I(\lambda_0) \setminus \mathcal{J}_{\nu+1}) \setminus E$. Then, by the part (iv) of Lemma 4.1, $E \subset \{a_0, b_0\}$, that is, $E = \emptyset$, or $E = \{a_0\}$, or $E = \{b_0\}$, or $E = \{a_0, b_0\}$. For definiteness, let $E = \{a_0\}$. Since $\{a_0\}$ is separated from $(I(\lambda_0) \setminus \mathcal{J}_{\nu+1}) \setminus \{a_0\}$, there exists a point $\zeta_0 \in \mathbb{T}$ such that the arc $(a_0, \zeta_0]$ is a subset of $I(\lambda_0)$ and $(a_0, \zeta_0] \subset \mathcal{J}_{\nu+1}$. In this case we replace $I(\lambda_0) = [a_0, b_0]$ by $[\zeta_0, b_0]$. If $\{b_0\}$ is separated from $(I(\lambda_0) \setminus \mathcal{J}_{\nu+1}) \setminus \{b_0\}$, too, we change $I(\lambda_0)$ once again. Now, the endpoints of the new (changed) arc $I(\lambda_0)$ belong to $\mathcal{J}_{\nu+1}$. By the part (iv) of Lemma 4.1 we conclude that $I(\lambda_0) \setminus \mathcal{J}_{\nu+1}$ does not contain separated subsets of zero measure. ∎

LEMMA 4.6. *Let* $0 \leqslant \nu \leqslant N$, *let* $\lambda_0 \in \mathcal{I}_\nu \setminus A$, *and let* $D(\lambda_0)$ *be a closed disk such that local branches* η_j, $j = 1, \dots, \nu$, *of the inverse function to* Φ *are well-defined on* $D(\lambda_0)$. *Further, let* $K \subset D(\lambda_0) \cap \mathbb{T}$ *be a compact set such that* $m(K) > 0$, *and let* $\mathcal{M}$ *be the matrix-function defined in* (4.1) *by* $\eta_j|_K$, $j = 1, \dots, \nu$. *Then* $Z\mathcal{E}(K) \subset \mathcal{L}(K, \mathcal{M})$.

We recall that Z is defined in Corollary 3.5, and $\mathcal{E}(K)$ for a compact set K is defined in Theorem 3.4.

Proof. Let $k \in \mathcal{E}(K)$, let F_n be defined by k in (3.1), and let $\{f_n\}_{n=0}^{N-1} = Zk$. Then for F_n and f_n (3.3) holds, therefore F_n has angular boundary values F_{n+} and F_{n-} from inside of $\mathbb{D}$ and from outside of $\mathbb{D}$, respectively, and $f_n = F_{n+} - F_{n-}$, $n = 0, \dots, N-1$. From the definition of $\mathcal{E}(K)$ we have $f_n = 0$ on $\mathbb{T} \setminus K$. Further, αk is analytic on $\mathbb{D}$ and η_j are analytic in $D(\lambda_0)$, $j = 1, \dots, \nu$, therefore $(\alpha k) \circ \eta_j$ are analytic in $D(\lambda_0)$, $j = 1, \dots, \nu$. From (3.1) we have $((\alpha k) \circ \eta_j)(\lambda) = \sum_{n=0}^{N-1} (F_n \circ \Phi)(\eta_j(\lambda))(\eta_j(\lambda))^n$ for all $\lambda \in D(\lambda_0)$ such that $|\Phi(\eta_j(\lambda))| \neq 1$. By the definition of η_j we have $\Phi(\eta_j(\lambda)) = \lambda$, that is,

$$((\alpha k) \circ \eta_j)(\lambda) = \sum_{n=0}^{N-1} F_n(\lambda)(\eta_j(\lambda))^n \tag{4.2}$$

for all $\lambda \in D(\lambda_0) \setminus \mathbb{T}$. But from the analyticity of $(\alpha k) \circ \eta_j$ on $D(\lambda_0)$ we have that $\lim\limits_{\lambda \to \zeta, |\lambda| < 1} ((\alpha k) \circ \eta_j)(\lambda) = \lim\limits_{\lambda \to \zeta, |\lambda| > 1} ((\alpha k) \circ \eta_j)(\lambda)$ for all $\zeta \in K \subset D(\lambda_0) \cap \mathbb{T}$, and from the last equality and (4.2) we have

$$\sum_{n=0}^{N-1} F_{n+}(\zeta)(\eta_j(\zeta))^n = \sum_{n=0}^{N-1} F_{n-}(\zeta)(\eta_j(\zeta))^n$$

for $\zeta \in K$. Therefore, $\sum_{n=0}^{N-1} f_n \eta_j^n = 0$ on K, $j = 1, \dots, \nu$. ∎

Proof of Theorem 4.2. We apply Lemma 4.5 to the number ν and the point λ_0. We put $K(\lambda_0) = I(\lambda_0) \setminus \mathcal{J}_{\nu+1}$. We apply Lemma 4.3 to the set $K(\lambda_0)$ and to the local branches η_j of the inverse function to Φ. Thus, we have the matrix-function $\mathcal{M}$ and the mapping W. We put $Y = WZ$, where Z was defined in Corollary 3.5. By Lemma 4.6 we have $Z\mathcal{E}(K(\lambda_0)) \subset \mathcal{L}(K(\lambda_0), \mathcal{M})$. Now we shall use Lemma 4.4 to prove that $\operatorname{clos} WZ\mathcal{E}(K(\lambda_0)) = \bigoplus_{n=\nu}^{N-1} L^1(K(\lambda_0), m)$.

By Theorem VIII.2.2 in [9], for any compact set $K \subset \mathbb{T}$ such that $m(K) > 0$ there exists a function F analytic and bounded on $\widehat{\mathbb{C}} \setminus K$ and such that $f \neq 0$ a.e. on K, where $F(\lambda) = \frac{1}{2\pi \mathrm{i}} \int_K \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - \lambda}$, $\lambda \notin K$; evidently, $f \in L^\infty(K, m)$. For the compact set $K(\lambda_0)$ we denote these functions by F_* and f_*.

We fix a number κ, $\nu \leqslant \kappa \leqslant N-1$. We construct the functions F_n, $n = 0, \dots, N-1$, by Lemma 4.4 with $f_\kappa = f_*$. We define a function h by the formula

$$h(z) = \sum_{n=0}^{N-1} F_n(\Phi(z)) z^n$$

for all $z \in \mathbb{D}$ such that $\Phi(z) \notin K(\lambda_0)$. We shall show that h can be continued into $\mathbb{D}$ so that this continuation will be from $H^\infty(\mathbb{D})$. By Lemma 4.5, we have $\Phi^{-1}(K(\lambda_0)) \subset \bigcup_{j=1}^{\nu} \eta_j(D(\lambda_0))$, and $\eta_j(D(\lambda_0)) \cap \eta_\ell(D(\lambda_0)) = \emptyset$, if $j \neq \ell$. We fix j, $1 \leqslant j \leqslant \nu$, and we write $h = (h \circ \eta_j) \circ \Phi$ on $\eta_j(D(\lambda_0) \setminus K(\lambda_0))$, and we have $h \circ \eta_j = g_{\kappa j}$, where $g_{\kappa j}$ is a function from Lemma 4.4. The function $g_{\kappa j}$ has an analytic continuation to $D(\lambda_0)$, therefore h has an analytic continuation to $\eta_j(D(\lambda_0))$. Thus, h has an analytic continuation in $\mathbb{D}$. Now we will check that h is bounded on $\mathbb{D}$. Since F_n are bounded on $\widehat{\mathbb{C}} \setminus D(\lambda_0)$, we have that h is bounded on $\mathbb{D} \setminus \bigcup_{j=1}^{\nu} \eta_j(D(\lambda_0))$. The set $\bigcup_{j=1}^{\nu} \eta_j(D(\lambda_0))$ is a compact set ($D(\lambda_0)$ is a closed disk), and we conclude that h is bounded on $\mathbb{D}$. Of course, $F_n(\infty) = 0$, $n = 0, \dots, N-1$, therefore h can be represented in the form $h = \alpha k$, where $k \in H^\infty \subset H^2$. From (3.1) we conclude that $k \in \mathcal{K}$. By the construction, $k \in \mathcal{E}(K(\lambda_0))$ and

$$WZk = \begin{pmatrix} 0 \\ \vdots \\ f_* \\ \vdots \\ 0 \end{pmatrix},$$

where the number of f_* in this column, numerated from ν to $N-1$, is κ.

Now we use the intertwining property of the transformations W and Z. We shall denote by the same letter V the multiplication by $\overline{\zeta}$ in the spaces $\bigoplus_{\ell=n}^{N-1} L^1(K, m)$ for all numbers n, $0 \leqslant n \leqslant N-1$, and all (measurable) sets $K \subset \mathbb{T}$. The equality $ZT^{-1} = VZ$ was proved in Corollary 3.5, the equality $WV = VW$ is evident from the definition of W, and the inclusion $T^{-1}\mathcal{E}(K) \subset \mathcal{E}(K)$ for a compact set K is proved in Theorem 3.4. Now we write

$$WZT^{-n}k = \begin{pmatrix} 0 \\ \vdots \\ \overline{\zeta}^n f_* \\ \vdots \\ 0 \end{pmatrix}, \quad n = 0, 1, 2, \dots.$$

Since $K(\lambda_0) \neq \mathbb{T}$ and $f_* \neq 0$ a.e. on $K(\lambda_0)$, we have $\bigvee_{n=0}^{\infty} \overline{\zeta}^n f_* = L^1(K(\lambda_0), m)$. Now we conclude that

$$\begin{pmatrix} 0 \\ \vdots \\ L^1(K(\lambda_0), m) \\ \vdots \\ 0 \end{pmatrix} \subset \operatorname{clos} WZ\mathcal{E}(K(\lambda_0)),$$

where the number of $L^1(K(\lambda_0), m)$ in this column, numerated from ν to $N-1$, is κ.

Since κ was arbitrary number from ν to $N-1$, we can conclude that

$$\bigoplus_{n=\nu}^{N-1} L^1(K(\lambda_0), m) \subset \operatorname{clos} WZ\mathcal{E}(K(\lambda_0)).$$

Thus, we achieve the equality $\operatorname{clos} Y\mathcal{E}(K(\lambda_0)) = \bigoplus_{n=\nu}^{N-1} L^1(K(\lambda_0), m)$. The equalities $\ker Y = \{0\}$ and $YT^{-1} = VY$ are evident consequences of the construction of Y. ∎

Proof of Theorem 1.1. We recall that U is an absolutely continuous unitary operator which is similar to $T = T_\Phi|_{\mathcal{K}}$. We can consider U as the operator of multiplication by ζ in the space $\bigoplus_{n=1}^{\infty} L^2(\tau_n, m)$, where τ_n are measurable subsets of $\mathbb{T}$ such that $\mathbb{T} \supset \tau_1 \supset \cdots \supset \tau_n \supset \tau_{n+1} \supset \cdots$. Further, there exists a linear bounded invertible transformation $X : \bigoplus_{n=1}^{\infty} L^2(\tau_n, m) \to \mathcal{K}$ such that $XU = TX$. From Corollary 3.5 we have that the linear bounded transformation $ZX : \bigoplus_{n=1}^{\infty} L^2(\tau_n, m) \to \bigoplus_{n=0}^{N-1} L^1(\mathbb{T}, m)$ intertwines U^{-1} and V and that $\ker ZX = \{0\}$. From this facts we obtain that $m(\tau_n) = 0$ for $n > N$, that is, $\bigoplus_{n=1}^{\infty} L^2(\tau_n, m) = \bigoplus_{n=1}^{N} L^2(\tau_n, m)$.

Let ν and λ_0 be as in Theorem 4.2, and let $\mathcal{F}_0 = X^{-1}\mathcal{E}(K(\lambda_0))$. Then $\mathcal{F}_0$ is a subspace of $\bigoplus_{n=1}^{N} L^2(\tau_n, m)$ such that $U\mathcal{F}_0 \subset \mathcal{F}_0$ and $U^{-1}\mathcal{F}_0 \subset \mathcal{F}_0$. Therefore $U|_{\mathcal{F}_0}$ is a unitary operator which is unitarily equivalent to the multiplication by ζ in $\bigoplus_{n=1}^{N} L^2(\tau_{0n}, m)$, where $\mathbb{T} \supset \tau_{01} \supset \cdots \supset \tau_{0N}$, and, evidently, $\tau_{0n} \subset \tau_n$, $n = 1, \ldots, N$. By Theorem 4.2, there exists a linear bounded transformation

$$Y_0 : \bigoplus_{n=1}^{N} L^2(\tau_{0n}, m) \to \bigoplus_{n=\nu}^{N-1} L^1(K(\lambda_0), m)$$

such that

$$\ker Y_0 = \{0\}, \quad \operatorname{clos} Y_0\Big(\bigoplus_{n=1}^{N} L^2(\tau_{0n}, m)\Big) = \bigoplus_{n=\nu}^{N-1} L^1(K(\lambda_0), m), \quad Y_0\overline{\zeta} = \overline{\zeta}Y_0.$$

From this fact we conclude that $\tau_{0n} = K(\lambda_0)$, $n = 1, \dots, N-\nu$, and $\tau_{0n} = \varnothing$, $n = N-\nu+1, \dots, N$. From the inclusions $\tau_{0n} \subset \tau_n$, $n = 1, \dots, N$, we conclude that $K(\lambda_0) \subset \tau_{N-\nu}$. Since ν was an arbitrary number between 0 and $N-1$ and λ_0 was an arbitrary point from $\mathcal{I}_\nu \setminus (A \cup \mathcal{J}_{\nu+1})$, we conclude that $\mathcal{I}_n \setminus \mathcal{J}_{n+1} \subset \tau_{N-n}$, $n = 0, \dots, N-1$. From the equalities $m(\mathcal{J}_{n+1}) = m(\mathcal{I}_{n+1})$ and $\sigma_n = \mathbb{T} \setminus \mathcal{I}_n$ we conclude that $\sigma_{n+1} \setminus \sigma_n \subset \tau_{N-n}$, and, using the inclusions $\sigma_n \subset \sigma_{n+1}$ and $\tau_{n+1} \subset \tau_n$, we obtain $\sigma_{n+1} \subset \tau_{N-n}$, $n = 0, \dots, N-1$.

Now let us suppose that for some ν, $1 \leqslant \nu \leqslant N$, the set $\tau_{N-\nu+1} \setminus \sigma_\nu$ has positive measure. Then there exists a compact set K_0 such that $K_0 \subset \tau_{N-\nu+1} \setminus \sigma_\nu$, $m(K_0) > 0$ and K_0 does not contain separated subsets of zero measure. We have $K_0 \subset \mathcal{I}_\nu$, therefore $(\mathcal{I}_\nu \setminus A) \cap K_0 \neq \varnothing$. Let $\lambda_0 \in (\mathcal{I}_\nu \setminus A) \cap K_0$, and let $D(\lambda_0)$ be a closed disk such that local branches η_j, $j = 1, \dots, \nu$, of the inverse function to Φ are well-defined on $D(\lambda_0)$. We put $K = K_0 \cap D(\lambda_0)$. By the part (iv) of Lemma 4.1 applied to $\mathbb{T} \setminus K_0 = \mathfrak{J}(\mathbb{T} \setminus K_0)$, we have $m(K) > 0$. We put $\mathcal{F} = \bigoplus_{n=1}^{N-\nu+1} L^2(K, m)$ and $\mathcal{E} = X\mathcal{F}$. Then $\mathcal{E} \subset \mathcal{K}$, the subspace $\mathcal{E}$ is invariant for T, and $\sigma(T|_{\mathcal{E}}) \subset K$. Let $k \in \mathcal{E}$, and let $\{F_n\}_{n=0}^{N-1}$ be the functions defined by k in (3.2). From the equality $(T-\lambda)^{-1}|_{\mathcal{E}} = (T|_{\mathcal{E}} - \lambda)^{-1}$, $\lambda \notin \mathbb{T}$, we conclude that F_n has an analytic continuation to $\widehat{\mathbb{C}} \setminus K$, $n = 0, \dots, N-1$, that is, $k \in \mathcal{E}(K)$. We have $\mathcal{E} \subset \mathcal{E}(K)$. By Lemma 4.6 $Z\mathcal{E}(K) \subset \mathcal{L}(K, \mathcal{M})$, where the matrix $\mathcal{M}$ is constructed by the functions η_j, $j = 1, \dots, \nu$, as in Lemma 4.3. Thus, by Lemmas 4.3 and 4.6 we have $WZ\mathcal{E}(K) \subset \bigoplus_{n=\nu}^{N-1} L^1(K, m)$. Consequently, $WZX\mathcal{F} \subset \bigoplus_{n=\nu}^{N-1} L^1(K, m)$. The linear bounded transformation WZX acts from $\bigoplus_{n=1}^{N-\nu+1} L^2(K, m)$ to $\bigoplus_{n=\nu}^{N-1} L^1(K, m)$, has zero kernel and intertwines the multiplications by $\overline{\zeta}$ in these spaces. From this fact we must conclude that the number of summands in $\bigoplus_{n=1}^{N-\nu+1} L^2(K, m)$ is no more than the number of summands in $\bigoplus_{n=\nu}^{N-1} L^1(K, m)$, which is a contradiction. We obtain $\sigma_n = \tau_{N-n+1}$, $n = 1, \dots, N$. ∎

5. EXAMPLES AND REMARKS

Let us recall that we consider functions Φ of the form $\Phi = \frac{\varphi}{\psi}$, where φ is inner, ψ is a finite Blaschke product, $\deg \psi = N < \infty$, $N \leqslant \deg \varphi \leqslant \infty$, φ and ψ are relatively prime, and we put $\Omega_\Phi = \{z \in \mathbb{D} : |\Phi(z)| > 1\}$.

5.1. SOME TECHNICAL LEMMAS.

LEMMA 5.1. *Let μ be a positive atomic measure on $\mathbb{T}$ with the weight (denoted by the same letter μ) at the point 1 only. Let $\varphi_\mu(z) = \exp\Big(-\mu\frac{1+z}{1-z}\Big)$, $\psi(z) = z$, $z \in \mathbb{D}$, and let $\Phi_\mu = \frac{\varphi_\mu}{\psi}$. Further, let $0 < \mu < 2$. Then $\arg \Phi_\mu(\mathrm{e}^{\mathrm{i}t})$ increases on $(0, t(\mu)) \cup (2\pi - t(\mu), 2\pi)$ and decreases on $(t(\mu), 2\pi - t(\mu))$, where $t(\mu) = \arccos(1-\mu)$. Thus, $\partial\Omega_{\Phi_\mu} \cap \mathbb{T} = \{\mathrm{e}^{\mathrm{i}t} : t \in [t(\mu), 2\pi - t(\mu)]\}$. Further, $m(\partial\Omega_{\Phi_\mu} \cap \mathbb{T})$ decreases from 1 to 0, when μ ranges over $(0,2)$. If $\mu = 2$, then $\partial\Omega_{\Phi_\mu} \cap \mathbb{T} = \{-1\}$, and if $\mu > 2$, then $\partial\Omega_{\Phi_\mu} \cap \mathbb{T} = \varnothing$.*

The proof of Lemma 5.1 consists of a trivial computation and is omitted.

The following two lemmas describe some properties of functions that are compositions with Φ.

LEMMA 5.2. *Let s_1, s_2 be numbers such that $-\pi < s_1 < s_2 < \pi$, let $J = \{\mathrm{e}^{\mathrm{i}s} : s \in [s_1, s_2]\}$, and let $w(\mathrm{e}^{\mathrm{i}s}) = ((s - s_1)(s_2 - s))^{-1/2}$, where $s_1 < s < s_2$. Further, let $\omega \colon \widehat{\mathbb{C}} \setminus J \to \mathbb{D}$ be a conformal mapping. Then there exist a constant $C > 0$ such that*

$$\|F \circ \omega^{-1}\|_{H^2} \leqslant C \|f\|_{L^2(J, w\mathrm{d}m)} \tag{5.1}$$

for every $f \in L^2(J, w\mathrm{d}m)$, where

$$F(\lambda) = \frac{1}{2\pi\mathrm{i}} \int_J \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - \lambda}, \quad \lambda \in \widehat{\mathbb{C}} \setminus J. \tag{5.2}$$

Proof. First, we extend w on $\mathbb{T}$ by the formula $w(\mathrm{e}^{\mathrm{i}s}) = (|s - s_1|\,|s - s_2|)^{-1/2}$, where $-\pi < s < \pi$, and we show that w satisfies Helson–Szegö condition. The function $h(z) = \log(1 + \mathrm{e}^{\mathrm{i}(\pi - s_1)}z) + \log(1 + \mathrm{e}^{\mathrm{i}(\pi - s_2)}z)$, $z \in \mathbb{D}$, is analytic in $\mathbb{D}$, and $-\frac{\mathrm{i}}{2}h = v + \mathrm{i}\widetilde{v}$ on $\mathbb{T}$, where

$$v(\mathrm{e}^{\mathrm{i}s}) = \begin{cases} \frac{1}{2}\Big(s - \frac{s_1+s_2}{2} + \pi\Big) & -\pi < s < s_1, \\ \frac{1}{2}\Big(s - \frac{s_1+s_2}{2}\Big) & s_1 < s < s_2, \\ \frac{1}{2}\Big(s - \frac{s_1+s_2}{2} - \pi\Big) & s_2 < s < \pi. \end{cases}$$

We put

$$u(\mathrm{e}^{\mathrm{i}s}) = -\frac{1}{2}\log\frac{|s - s_1|\,|s - s_2|}{4\sin\frac{|s-s_1|}{2}\sin\frac{|s-s_2|}{2}}.$$

We have $\log w = u + \widetilde{v}$, $u, v \in L^\infty(\mathbb{T}, m)$ and $\|v\|_\infty < \frac{\pi}{2}$. Thus, w satisfies Helson–Szegö condition. Therefore there exist a constant $C_1 > 0$ such that

$$\|F_\pm\|_{L^2(\mathbb{T}, w\mathrm{d}m)} \leqslant C_1 \|f\|_{L^2(J, w\mathrm{d}m)}$$

for every $f \in L^2(J, w\mathrm{d}m)$, where F is defined by f in (5.2) and F_+, F_- are angular boundary values of F from inside of $\mathbb{D}$ and from outside of $\mathbb{D}$, respectively.

Further, it is sufficient to check the estimate (5.1) for functions f smooth on $\mathbb{T}$ and vanishing on $\mathbb{T}\setminus J$ only. We have that $F\circ\omega^{-1}\in H^\infty$ for smooth f. By usual change of variables

$$\int_{\mathbb{T}}|F\circ\omega^{-1}|^2\mathrm{d}m\asymp\int_J|F_+|^2w\mathrm{d}m+\int_J|F_-|^2w\mathrm{d}m\leqslant 2C_1^2\|f\|^2_{L^2(J,w\mathrm{d}m)}$$

(where $a\asymp b$ denotes, as usual, that there exist positive constants c,C such that $ca\leqslant b\leqslant Ca$). Lemma is proved. ∎

COROLLARY 5.3. *Suppose that the conditions of Lemma* 5.2 *are fulfilled, and* $\Phi\colon\mathbb{D}\to\widehat{\mathbb{C}}\setminus J$ *is a meromorphic function. Then there exist a constant* $C>0$ *such that*

$$\|F\circ\Phi\|_{H^2}\leqslant C\|f\|_{L^2(J,w\mathrm{d}m)} \tag{5.3}$$

for every $f\in L^2(J,w\mathrm{d}m)$, *where* F *is defined by* f *in* (5.2).

Proof. We apply the Littlewood subordination theorem (see, for example, Theorem 1.7 in [7]) to the analytic function $\omega\circ\Phi\colon\mathbb{D}\to\mathbb{D}$. We obtain that there exists a constant C_1 (which depends on $\omega\circ\Phi$) such that

$$\|h\circ(\omega\circ\Phi)\|_{H^2}\leqslant C_1\|h\|_{H^2} \tag{5.4}$$

for any $h\in H^2$. We write $F\circ\Phi=(F\circ\omega^{-1})\circ(\omega\circ\Phi)$. The estimate (5.3) follows from the last equality and the estimates (5.1) and (5.4). ∎

LEMMA 5.4. *Let* $\mathcal{I}\subset\mathbb{T}$ *be an open arc, and let* $\mathcal{G}\subset\mathbb{D}$ *be a simple Jordan domain such that* $\partial\mathcal{G}\cap\mathbb{T}=\operatorname{clos}\mathcal{I}$. *Further, let* $\Phi\colon\mathbb{D}\to\widehat{\mathbb{C}}$ *be a meromorphic function such that* Φ *has angular boundary values* $\Phi(\zeta)\in\mathbb{T}$ *at a.e.* $\zeta\in\mathcal{I}$ *and* $\Phi(\operatorname{clos}\mathcal{G}\cap\mathbb{D})\subset\mathbb{D}$. *Finally, let* $\mathcal{I}_0$ *be a closed arc such that* $\mathcal{I}_0\subset\mathcal{I}$. *Then there exists a constant* C *such that* $\int_{\mathcal{I}_0}|F\circ\Phi|\mathrm{d}m\leqslant C\int_{\mathbb{T}}|F|\mathrm{d}m$ *for any* $F\in L^1(\mathbb{T},m)$.

Proof. Let $\omega\colon\mathbb{D}\to\mathcal{G}$ be a conformal mapping. By well-known properties of conformal mappings, ω can be extended to a continuous function (denoted by the same letter) $\omega\colon\ \operatorname{clos}\mathbb{D}\to\operatorname{clos}\mathcal{G}$, and ω^{-1} is continuous, too. We put $\Gamma=\omega^{-1}(\mathcal{I})$ and $\Gamma_0=\omega^{-1}(\mathcal{I}_0)$. We have that ω is analytic on Γ and $|\omega'|\asymp 1$ on Γ_0.

Further, if $\zeta\in\mathcal{I}$ and Φ has angular boundary value $\Phi(\zeta)$ at ζ, then $\Phi\circ\omega$ has an angular boundary value $\Phi(\zeta)$ at $\omega^{-1}(\zeta)$. Let $\sigma\subset\mathbb{T}$ be a measurable set. We put $\Phi^{-1}(\sigma)=\{\zeta\in\mathcal{I}_0:\Phi$ has angular boundary value $\Phi(\zeta)$ at ζ and $\Phi(\zeta)\in\sigma\}$ and $(\Phi\circ\omega)^{-1}(\sigma)=\{\xi\in\Gamma_0:\Phi\circ\omega$ has angular boundary value $(\Phi\circ\omega)(\xi)$ at ξ and $(\Phi\circ\omega)(\xi)\in\sigma\}$. We put $\mu(\sigma)=m(\Phi^{-1}(\sigma))$. We have $\omega((\Phi\circ\omega)^{-1}(\sigma))=\Phi^{-1}(\sigma)\cup e$, where $e\subset\mathcal{I}_0$ and Φ has no angular boundary values on e. By assumption, $m(e)=0$, hence, $\mu(\sigma)=m(\omega((\Phi\circ\omega)^{-1}(\sigma)))$. By usual change of variables,

$$m(\omega((\Phi\circ\omega)^{-1}(\sigma)))=\int_{(\Phi\circ\omega)^{-1}(\sigma)}|\omega'|\mathrm{d}m\leqslant C_1 m((\Phi\circ\omega)^{-1}(\sigma)),$$

where C_1 is from the estimate $|\omega'| \asymp 1$ on Γ_0. Now we apply Theorem VIII.30 from [15] to the function $\Phi \circ \omega \colon \mathbb{D} \to \mathbb{D}$. We obtain that there exists a constant C_2 such that $m((\Phi \circ \omega)^{-1}(\sigma)) \leqslant C_2 m(\sigma)$ for any measurable set $\sigma \subset \mathbb{T}$. Thus, $\mu(\sigma) \leqslant C_1 C_2 m(\sigma)$, in particular, μ is a measure absolutely continuous with respect to m. By the Radon–Nikodym theorem, there exists a function $h \in L^\infty(\mathbb{T}, m)$, $h \geqslant 0$, such that $\mu(\sigma) = \int\limits_\sigma h \, \mathrm{d}m$ for any measurable set $\sigma \subset \mathbb{T}$. By the change of variable theorem, $\int\limits_{\mathcal{I}_0} F \circ \Phi \, \mathrm{d}m = \int\limits_{\mathbb{T}} F h \mathrm{d}m$ for any measurable function F, and the lemma is proved. ∎

5.2. SUBARCS OF $\partial\Omega_\Phi \cap \mathbb{T}$ AND ANALYTICITY OF Φ ON SUBARCS OF $\mathbb{T}$. Let $I \subset \mathbb{T}$ be an open arc, and suppose Φ has an analytic continuation throughout I. From well-known properties of analytic functions it is easy to see that $I \cap \partial\Omega_\Phi$ consists of no more than a countable union of mutually separated points and subarcs of I, closed as subsets of I. Moreover, if J is an open arc such that $J \subset I \cap \partial\Omega_\Phi$, then $\arg \Phi$ decreases on J. On the other hand, if $I \subset \mathbb{T}$ is an open arc such that $I \subset \partial\Omega_\Phi$, then Φ has an analytic continuation throughout I [8].

However, there exist functions Φ such that $m(\partial\Omega_\Phi \cap \mathbb{T}) > 0$ and $\partial\Omega_\Phi \cap \mathbb{T}$ does not contain open arcs. An example of such Φ with $\psi(z) = z$, $z \in \mathbb{D}$, is given in [8].

5.3. THE CASE $N = 1$. In this subsection, we consider the functions Φ of the form $\Phi = \frac{\varphi}{\psi}$, where φ is inner and ψ is a Blaschke factor, that is, $\deg \psi = 1$.

LEMMA 5.5 ([8]). *Let $\Phi = \frac{\varphi}{\psi}$, where φ is inner, ψ is a Blaschke factor, and let $\Omega_\Phi = \{z \in \mathbb{D} : |\Phi(z)| > 1\}$. Then Ω_Φ is a simple Jordan domain and $\Phi|_{\Omega_\Phi}$ is a conformal mapping of Ω_Φ onto $\widehat{\mathbb{C}} \setminus \operatorname{clos} \mathbb{D}$, in particular, $\Phi|_{\partial\Omega_\Phi}$ is a homeomorphism of $\partial\Omega_\Phi$ and $\mathbb{T}$.*

REMARK. As was mentioned in the previous subsection, there exists a function Φ such that $m(\partial\Omega_\Phi \cap \mathbb{T}) > 0$ and $\partial\Omega_\Phi \cap \mathbb{T}$ does not contain open arcs. By Theorems 1.1 and 2.7 T_Φ is similar to $U \oplus S$, where S is a unilateral shift of infinite multiplicity, and U is the operator of the multiplication by independent variable in the space $L^2(\Phi(\partial\Omega_\Phi \cap \mathbb{T}), m)$. By Theorem 2.7, $m(\Phi(\partial\Omega_\Phi \cap \mathbb{T})) > 0$, and by Lemma 5.5, $\Phi(\partial\Omega_\Phi \cap \mathbb{T})$ does not contain open arcs.

The following lemma shows that in the case $N = 1$ the space L^1 in Corollary 3.5 can be replaced by L^2.

LEMMA 5.6. *Let $\Phi = \frac{\varphi}{\psi}$, where φ is inner and ψ is a Blaschke factor. We recall that the space $\mathcal{K}$ was defined in the end of Section 2, and the transformation $Z \colon \mathcal{K} \to L^1(\mathbb{T}, m)$ was defined in Corollary 3.5. We have that $Z\mathcal{K} \subset L^2(\mathbb{T}, m)$ and $Z \colon \mathcal{K} \to L^2(\mathbb{T}, m)$ is a linear bounded transformation.*

Proof. We recall that $\alpha(z) = z - \delta$, where δ is the only zero of ψ. By Theorem 3.3, $\mathcal{K} = \{k \in H^2$: there exists a vanishing at infinity function F analytic on $\widehat{\mathbb{C}} \setminus \mathbb{T}$ such that $(\alpha k)(z) = (F \circ \Phi)(z)$ for all $z \in \mathbb{D}$ such that $|\Phi(z)| \neq 1\}$. But αk is analytic on $\mathbb{D}$, therefore F must be analytic on $\Phi(\mathbb{D})$. It is evident that $k \in H^2$ if and only if $F \circ \Phi \in H^2$. Further, if $F \circ \Phi \in H^2$, then, acting in the same way as in the proof of Proposition 3.3 of [14], we obtain that $F|_{\mathbb{D}} \in H^2(\mathbb{D})$ and $F|_{\widehat{\mathbb{C}}\setminus \operatorname{clos}\mathbb{D}} \in H^2(\widehat{\mathbb{C}} \setminus \operatorname{clos}\mathbb{D})$. Therefore angular boundary values F_+, F_- of F from inside of $\mathbb{D}$ and from outside of $\mathbb{D}$ belong to $L^2(\mathbb{T}, m)$, hence, $f = F_+ - F_- \in L^2(\mathbb{T}, m)$. Thus, $Zk = f \in L^2(\mathbb{T}, m)$. By the closed graph theorem, $Z \colon \mathcal{K} \to L^2(\mathbb{T}, m)$ is bounded. ∎

In the following lemma we consider a particular case of functions Φ, and we construct a linear bounded transformation which gives a similarity of $T = T_\Phi|_{\mathcal{K}}$ and the unitary summand U of isometry to which T_Φ is similar.

LEMMA 5.7. *Let $\Phi = \frac{\varphi}{\psi}$, where φ is inner, ψ is a Blaschke factor, and let $\Omega_\Phi = \{z \in \mathbb{D} : |\Phi(z)| > 1\}$. Further, let $I \subset \mathbb{T}$ be a closed subarc such that $I \subset \partial\Omega_\Phi$ and there is no arc $\mathcal{I}$ such that I is a proper subset of $\mathcal{I}$ and $\mathcal{I} \subset \partial\Omega_\Phi$. We put $J = \Phi(I)$. We have that J is a closed arc and $J \neq \mathbb{T}$. For convenience, we suppose that $J = \{\mathrm{e}^{\mathrm{i}s} : s \in [s_1, s_2]\}$, where $-\pi < s_1 < s_2 < \pi$. Further, let Φ be analytic on a neighbourhood of I. We put $w(\mathrm{e}^{\mathrm{i}s}) = ((s - s_1)(s_2 - s))^{-1/2}$, where $s_1 < s < s_2$. Then $Z \colon \mathcal{E}(J) \to L^2(J, w\mathrm{d}m)$ is a bounded invertible transformation, which gives the similarity of $T^{-1}|_{\mathcal{E}(J)}$ and the multiplication by $\overline{\zeta}$ on $L^2(J, w\mathrm{d}m)$, where ζ is the independent variable.*

We recall that $\mathcal{E}(K)$ for $K \subset \mathbb{T}$ was defined in Theorem 3.4, and the transformation Z was defined in Corollary 3.5.

Proof. We recall that $\alpha(z) = z - \delta$, where δ is the only zero of ψ. By Lemma 5.5 $\Phi(\mathbb{D}) \subset \widehat{\mathbb{C}} \setminus J$, therefore it is easy to see that $\mathcal{E}(J) = \{k \in H^2$: there exists a vanishing at infinity function F analytic on $\widehat{\mathbb{C}} \setminus J$ such that $\alpha k = F \circ \Phi\}$.

We define the mapping $\mathcal{Z} \colon L^2(J, w\mathrm{d}m) \to H^2$ by the formula $\mathcal{Z}f = F \circ \Phi$, where F is defined by f in (5.2). By Corollary 5.3 $\mathcal{Z}$ is bounded. We define the mapping $\mathcal{Z}_1 \colon L^2(J, w\mathrm{d}m) \to H^2$ by the formula $\mathcal{Z}_1 f = \frac{1}{\alpha} F \circ \Phi$, where F is defined by f in (5.2). Since $\mathcal{Z}$ is bounded, we have $\mathcal{Z}_1$ is bounded.

By Lemma 5.6 we have that $Z\mathcal{E}(J) \subset L^2(J, m)$, therefore to show that $Z\mathcal{E}(J) \subset L^2(J, w\mathrm{d}m)$ we need to check only that $Z\mathcal{E}(J) \subset L^2(J_1 \cup J_2, w\mathrm{d}m)$, where J_1 and J_2 are some closed arcs such that $J_1, J_2 \subset J$, $J_1 \cap J_2 = \emptyset$ and $\mathrm{e}^{\mathrm{i}s_j} \in J_j$, $j = 1, 2$. We denote the ends of I by $\mathrm{e}^{\mathrm{i}t_j}$, then $\Phi(\mathrm{e}^{\mathrm{i}t_j}) = \mathrm{e}^{\mathrm{i}s_j}$, $j = 1, 2$. By the assumption, we can choose closed arcs I_1, I_2 such that $I_j \cap I = \mathrm{e}^{\mathrm{i}t_j}$, $j = 1, 2$, and Φ is analytic in a neighbourhood of $I \cup I_1 \cup I_2$. By the assumption, $\arg\Phi$ decreases on I and increases on I_1 and on I_2, therefore $\Phi'(\mathrm{e}^{\mathrm{i}t_j}) = 0$, and $\Phi(I_j) \subset J$, if I_j is sufficiently small, $j = 1, 2$. We request that $\Phi(I_1) \cap \Phi(I_2) = \emptyset$, too. We put $J_j = \Phi(I_j)$, $j = 1, 2$. From the form of Φ we conclude that $\Phi''(\mathrm{e}^{\mathrm{i}t_j}) \neq 0$, $j = 1, 2$.

Indeed, if $\Phi''(\mathrm{e}^{\mathrm{i}t_j}) = 0$, it would imply that $\Phi - \lambda$ has two zeros in $\mathbb{D}$ for λ from a neighbourhood of $\mathrm{e}^{\mathrm{i}s_j}$, which contradicts to Lemma 2.6. Now let F be a function analytic on $\widehat{\mathbb{C}} \setminus J$ and such that $F \circ \Phi \in H^2$. Then

$$\int_{\mathbb{T}} |F \circ \Phi|^2 \mathrm{d}m \geqslant \int_{I \cup I_1 \cup I_2} |F \circ \Phi|^2 \mathrm{d}m$$
$$= \int_J |F_-|^2 \frac{\mathrm{d}m}{|\Phi' \circ \Phi_I^{-1}|} + \int_{J_1} |F_+|^2 \frac{\mathrm{d}m}{|\Phi' \circ \Phi_{I_1}^{-1}|} + \int_{J_2} |F_+|^2 \frac{\mathrm{d}m}{|\Phi' \circ \Phi_{I_2}^{-1}|},$$

where $\Phi_{\mathcal{I}}^{-1}$ denote the branch of Φ^{-1} with values on $\mathcal{I}$. From analyticity of Φ on a neighbourhood of $I \cup I_1 \cup I_2$ we obtain that $w \asymp \frac{1}{|\Phi' \circ \Phi_{\mathcal{I}}^{-1}|}$ for $\mathcal{I} = I, I_1, I_2$. Thus, $F_\pm \in L^2(J_1 \cup J_2, w\mathrm{d}m)$, and $f = F_+ - F_- \in L^2(J_1 \cup J_2, w\mathrm{d}m)$. We proved that $Z\mathcal{E}(J) \subset L^2(J, w\mathrm{d}m)$. Now we write $Z^{-1} = \mathcal{Z}_1$, and Lemma 5.7 is proved. ∎

5.4. ON THE SPACE $\mathcal{E}(K)$ FOR $K \subset \mathbb{T} \setminus \Phi(\mathbb{D})$. Let K be a compact set, let $K \subset \mathbb{T} \setminus \Phi(\mathbb{D})$, and let $m(K) > 0$. For $n = 0, \dots, N-1$ we put

$$\mathcal{E}_n(K) = \{k \in \mathcal{E}(K) : f_j = 0 \text{ for } j \neq n, j = 0, \dots, N-1\}.$$

It is easy to see from propositions proved in Sections 3 and 4 that $\mathcal{E}_n(K) \neq \{0\}$, $\mathcal{E}_n(K)$ is a linear closed subset of $\mathcal{E}(K)$, $T^{-1}\mathcal{E}_n(K) = \mathcal{E}_n(K)$ and $\bigvee_{n=0}^{N-1} Z\mathcal{E}_n(K) = \operatorname{clos} Z\mathcal{E}(K) = \bigoplus_{n=0}^{N-1} L^1(K, m)$. From these equalities and the fact that $T|_{\mathcal{E}(K)}$ is similar to the multiplication by ζ on $\bigoplus_{n=0}^{N-1} L^2(K, m)$ one can obtain that

$$\bigvee_{n=0}^{N-1} \mathcal{E}_n(K) = \mathcal{E}(K).$$

Indeed, let $X\colon \bigoplus_{n=0}^{N-1} L^2(K, m) \to \mathcal{E}(K)$ be a linear bounded invertible transformation such that $X\overline{\zeta} = T^{-1}X$, let $\mathcal{F}_n = X^{-1}\mathcal{E}_n(K)$, and let $\mathcal{F} = \bigvee_{n=0}^{N-1} \mathcal{F}_n$. We have $ZX\mathcal{F} = \bigvee_{n=0}^{N-1} ZX\mathcal{F}_n = \bigvee_{n=0}^{N-1} Z\mathcal{E}_n(K) = \bigoplus_{n=0}^{N-1} L^1(K, m)$ and $ZX\overline{\zeta} = \overline{\zeta}ZX$, hence, $\mathcal{F} = \bigoplus_{n=0}^{N-1} L^2(K, m)$. Thus, $\bigvee_{n=0}^{N-1} \mathcal{E}_n(K) = \mathcal{E}(K)$.

In general, $\mathcal{E}_0(K) + \dots + \mathcal{E}_{N-1}(K) \neq \mathcal{E}(K)$. To show this, we consider the following example.

EXAMPLE 5.8. Let Ψ be a function with the following properties:

(i) $\Psi = \frac{\varphi}{\psi}$, where φ is inner and ψ is a Blaschke factor ($\deg \psi = 1$);

(ii) $I = \partial\Omega_\Psi \cap \mathbb{T}$ is a closed arc, and Ψ is analytic in sufficiently large neighbourhood of I;

(iii) $m(\Psi(I)) > \frac{1}{2}$ and, for convenience, the center of the arc $\Psi(I)$ is 1, that is, $\Psi(I) = \{e^{it} : |t| \leqslant \frac{\pi}{2} + s_0\}$, where $0 < s_0 < \frac{\pi}{2}$.

The example of such function Ψ is $\Psi(z) = -\frac{\exp(-\mu\frac{1+z}{1-z})}{z}$, $z \in \mathbb{D}$, with appropriate choice of μ, see Lemma 5.1.

We put $\Phi = \Psi^2$ and $K = \{e^{is} : \pi - 2s_0 \leqslant s \leqslant \pi\}$. We have $K \subset J = \{e^{is} : \pi - 2s_0 \leqslant s \leqslant \pi \text{ or } -\pi \leqslant s \leqslant 2s_0 - \pi\} = \mathbb{T} \setminus \Phi(\mathbb{D})$. We shall show that $\mathcal{E}_0(K) + \mathcal{E}_1(K) \neq \mathcal{E}(K)$.

First, we introduce the notation. $J_0 = \{e^{is} : |s| \leqslant \pi - 2s_0\}$, e^{it_1} and e^{it_2} are the ends of I, the sets I_1, I_2, I_3, I_4 are the closed subarcs of $\mathbb{T}$ with the following properties: $I_3, I_4 \subset I$, $e^{it_1} \in I_3$, $e^{it_2} \in I_4$, $I_j \cap I = e^{it_j}$, $j = 1,2$, $\Psi(I_2) = \Psi(I_4) = \{e^{is} : |s - \frac{\pi}{2}| \leqslant s_0\}$, $\Psi(I_1) = \Psi(I_3) = \{e^{is} : |s + \frac{\pi}{2}| \leqslant s_0\}$, and Ψ is analytic on a neighbourhood of $I \cup I_1 \cup I_2$. Assumption (iii) allows us to choose such arcs I_1, I_2, I_3, I_4. We have $\Phi(I_j) = J$, $j = 1,2,3,4$. We put $\eta_j = (\Phi|_{I_j})^{-1}$, $\eta_j \colon J \to I_j$, $j = 1,2,3,4$, and $\eta_0 = (\Phi|_{I\setminus(I_3\cup I_4)})^{-1}$, $\eta_0 \colon J_0 \to I \setminus (I_3 \cup I_4)$. Further, we put $w_1(e^{is}) = |s - (\pi - 2s_0)|^{-1/2}$, $w_2(e^{is}) = |s - (2s_0 - \pi)|^{-1/2}$, $s \in (-\pi, \pi)$, and $w = w_1 w_2$. We have the following estimates:

$$\frac{1}{|\Phi' \circ \eta_j|} \asymp w_1, \quad j = 1,3; \qquad \frac{1}{|\Phi' \circ \eta_j|} \asymp w_2, \quad j = 2,4; \tag{5.5}$$

$$|\eta_j - e^{it_1}| \asymp \frac{1}{w_1}, \quad j = 1,3; \qquad |\eta_j - e^{it_2}| \asymp \frac{1}{w_2}, \quad j = 2,4. \tag{5.6}$$

Let $f_0, f_1 \in L^1(J, m)$, let F_0 and F_1 be defined by f_0 and f_1 as in (5.2), and let $g(z) = (F_0 \circ \Phi)(z) + z(F_1 \circ \Phi)(z)$, $z \in \mathbb{D}$. Using the change of variables and (5.5) we have

$$\begin{aligned}
\int_{I_1\cup I_3} |g|^2 \mathrm{d}m &= \int_{I_1} |(F_{0+} \circ \Phi)(\zeta) + \zeta(F_{1+} \circ \Phi)(\zeta)|^2 \mathrm{d}m(\zeta) \\
&\quad + \int_{I_3} |(F_{0-} \circ \Phi)(\zeta) + \zeta(F_{1-} \circ \Phi)(\zeta)|^2 \mathrm{d}m(\zeta) \\
&= \int_J (|F_{0+} + \eta_1 F_{1+}|^2 \frac{1}{|\Phi' \circ \eta_1|} + |F_{0-} + \eta_3 F_{1-}|^2 \frac{1}{|\Phi' \circ \eta_3|} \mathrm{d}m \\
&\asymp \int_J (|F_{0+} + \eta_1 F_{1+}|^2 + |F_{0-} + \eta_3 F_{1-}|^2) w_1 \mathrm{d}m.
\end{aligned}$$

Thus, we have the estimate

$$\int_{I_1\cup I_3} |g|^2 \mathrm{d}m \asymp \int_J (|F_{0+} + \eta_1 F_{1+}|^2 + |F_{0-} + \eta_3 F_{1-}|^2) w_1 \mathrm{d}m, \tag{5.7}$$

and, similarly, the estimate

$$\int\limits_{I_2\cup I_4} |g|^2 \mathrm{d}m \asymp \int\limits_J (|F_{0+} + \eta_2 F_{1+}|^2 + |F_{0-} + \eta_4 F_{1-}|^2) w_2 \mathrm{d}m, \tag{5.8}$$

where F_{n+} and F_{n-} are angular boundary values of F_n from inside of $\mathbb{D}$ and from outside of $\mathbb{D}$, respectively, $n = 0, 1$.

Now we shall show that $Z\mathcal{E}(J) \subset L^2(J,m) \oplus L^2(J,m)$, and that for $k \in \mathcal{E}(J)$ $f_0 + \mathrm{e}^{\mathrm{i}t_j} f_1 \in L^2(J, w_j \mathrm{d}m)$, $j = 1, 2$, where $\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = Zk$. Let $k \in \mathcal{E}(J)$, we put $g(z) = (F_0 \circ \Phi)(z) + z(F_1 \circ \Phi)(z)$, where F_0, F_1 are defined by k in (3.1). By (5.7) and (5.8) $\int\limits_{\mathbb{T}} |g|^2 \mathrm{d}m \geqslant \int\limits_{\cup_{j=1}^4 I_j} |g|^2 \mathrm{d}m \asymp \int\limits_J (|F_{0+} + \eta_1 F_{1+}|^2 w_1 + |F_{0+} + \eta_2 F_{1+}|^2 w_2 + |F_{0-} + \eta_3 F_{1-}|^2 w_1 + |F_{0-} + \eta_4 F_{1-}|^2 w_2) \mathrm{d}m$, where F_{n+} and F_{n-} are angular boundary values of F_n from inside of $\mathbb{D}$ and from outside of $\mathbb{D}$, respectively, $n = 0, 1$. We obtain that $F_{0+} + \eta_j F_{1+} \in L^2(J,m)$, $j = 1, 2$, and $F_{0-} + \eta_j F_{1-} \in L^2(J,m)$, $j = 3, 4$. From these inclusions we have $(\eta_1 - \eta_2)F_{1+}, (\eta_3 - \eta_4)F_{1-} \in L^2(J,m)$. Further, $\frac{1}{\eta_1 - \eta_2}, \frac{1}{\eta_3 - \eta_4} \in L^\infty(J,m)$, thus, $F_{1+}, F_{1-} \in L^2(J,m)$, and we conclude that $f_1 = F_{1+} - F_{1-} \in L^2(J,m)$, $f_0 = F_{0+} - F_{0-} \in L^2(J,m)$. Also we have $f_0 + \eta_1 F_{1+} - \eta_3 F_{1-} \in L^2(J, w_1 \mathrm{d}m)$, and $f_0 + \eta_1 F_{1+} - \eta_3 F_{1-} = f_0 + (\eta_1 - \eta_3)F_{1+} + \eta_3 f_1 = f_0 + ((\eta_1 - \mathrm{e}^{\mathrm{i}t_1}) + (\mathrm{e}^{\mathrm{i}t_1} - \eta_3))F_{1+} + (\eta_3 - \mathrm{e}^{\mathrm{i}t_1}) f_1 + \mathrm{e}^{\mathrm{i}t_1} f_1$. By (5.6) we have $\int\limits_J |(\eta_j - \mathrm{e}^{\mathrm{i}t_1})h|^2 w_1 \mathrm{d}m \leqslant C \int\limits_J |h|^2 \frac{1}{w_1} \mathrm{d}m \leqslant C_1 \int\limits_J |h|^2 \mathrm{d}m$ for $h \in L^2(J,m)$, $j = 1, 3$. From the inclusion $F_{1+}, f_1 \in L^2(J,m)$ we conclude that $f_0 + \mathrm{e}^{\mathrm{i}t_1} f_1 \in L^2(J, w_1 \mathrm{d}m)$. The inclusion $f_0 + \mathrm{e}^{\mathrm{i}t_2} f_1 \in L^2(J, w_2 \mathrm{d}m)$ can be proved in a similar way.

Now let $k \in \mathcal{E}(K)$, then f_0, f_1 vanish in the neighbourhood of $\mathrm{e}^{\mathrm{i}(2s_0 - \pi)}$, therefore $f_0 + \mathrm{e}^{\mathrm{i}t_1} f_1 \in L^2(K, w \mathrm{d}m)$. By Corollary 5.3, $(F_0 + \mathrm{e}^{\mathrm{i}t_1} F_1) \circ \Phi \in H^2(\mathbb{D})$, hence, $(\mathrm{e}^{\mathrm{i}t_1} - z)(F_1 \circ \Phi)(z) \in H^2(\mathbb{D})$.

We define the mapping $\mathcal{Z} \colon L^2(K,m) \to H^2(\mathbb{D})$ by the formula

$$(\mathcal{Z}f)(z) = (\mathrm{e}^{\mathrm{i}t_1} - z)(F \circ \Phi)(z), \quad z \in \mathbb{D},$$

where F is defined by f as in (5.2), on functions f smooth on $\mathbb{T}$ and vanishing on $\mathbb{T} \setminus K$. We shall prove that there exists a constant C such that $\|\mathcal{Z}f\|_{H^2(\mathbb{D})} \leqslant C\|f\|_{L^2(K,m)}$ for smooth functions f. Let f be a such function. We put $g = \mathcal{Z}f$. By F_+ and F_- we denote angular boundary values of F from inside of $\mathbb{D}$ and from outside of $\mathbb{D}$, respectively. By (5.7) and (5.6) we have

$$\begin{aligned}\int\limits_{I_1\cup I_3} |g|^2 \mathrm{d}m &\asymp \int\limits_J (|\mathrm{e}^{\mathrm{i}t_1} F_+ - \eta_1 F_+|^2 + |\mathrm{e}^{\mathrm{i}t_1} F_- - \eta_3 F_-|^2) w_1 \mathrm{d}m \\ &= \int\limits_J (|\mathrm{e}^{\mathrm{i}t_1} - \eta_1|^2 |F_+|^2 + |\mathrm{e}^{\mathrm{i}t_1} - \eta_3|^2 |F_-|^2) w_1 \mathrm{d}m\end{aligned}$$

$$\leqslant C_1 \int\limits_J \frac{1}{w_1}(|F_+|^2 + |F_-|^2)\mathrm{d}m \leqslant C_2\|f\|^2_{L^2(K,m)},$$

and we obtain the estimate

$$\int\limits_{I_1\cup I_3} |g|^2\mathrm{d}m \leqslant C_2\|f\|^2_{L^2(K,m)}. \tag{5.9}$$

Using (5.8), (5.6) and the fact that $|\mathrm{e}^{\mathrm{i}t_1} - \eta_j|$, $j = 2, 4$, are bounded, we obtain that

$$\begin{aligned}\int\limits_{I_2\cup I_4} |g|^2\mathrm{d}m &\asymp \int\limits_J (|\mathrm{e}^{\mathrm{i}t_1}F_+ - \eta_2 F_+|^2 + |\mathrm{e}^{\mathrm{i}t_1}F_- - \eta_4 F_-|^2)w_2\mathrm{d}m\\ &= \int\limits_J (|\mathrm{e}^{\mathrm{i}t_1} - \eta_2|^2|F_+|^2 + |\mathrm{e}^{\mathrm{i}t_1} - \eta_4|^2|F_-|^2)w_2\mathrm{d}m \leqslant C_3\int\limits_J (|F_+|^2 + |F_-|^2)w_2\mathrm{d}m.\end{aligned}$$

Now we use that $f = 0$ in a neighbourhood $\mathcal{W}$ of $\mathrm{e}^{\mathrm{i}(2s_0-\pi)}$. We have $F_+(\lambda) = F_-(\lambda) = F(\lambda)$ for $\lambda \notin K$, and $|F(\lambda)| \leqslant \int\limits_K \frac{|f(\zeta)|\mathrm{d}m(\zeta)}{|\zeta-\lambda|} \leqslant \int\limits_K \frac{|f(\zeta)|\mathrm{d}m(\zeta)}{\mathrm{dist}(\lambda,K)}$ for $\lambda \in \mathcal{W}$, therefore $|F(\lambda)| \leqslant C_4\Big(\int\limits_K |f|^2\mathrm{d}m\Big)^{1/2} = C_4\|f\|_{L^2(K,m)}$, where C_4 depends on $\mathcal{W}$. We have $\int\limits_J (|F_+|^2 + |F_-|^2)w_2\mathrm{d}m \leqslant 2C_4^2\|f\|^2_{L^2(K,m)} \int\limits_J w_2\mathrm{d}m$. Thus, the estimate

$$\int\limits_{I_2\cup I_4} |g|^2\mathrm{d}m \leqslant C_5\|f\|^2_{L^2(K,m)} \tag{5.10}$$

is proved. Further, $\int\limits_{I\setminus(I_3\cup I_4)} |g|^2\mathrm{d}m = \int\limits_{J_0} |(\mathrm{e}^{\mathrm{i}t_1} - \eta_0)F_-|^2 \frac{1}{|\Phi'\circ\eta_0|}\mathrm{d}m \leqslant C_6 \int\limits_{J_0} |F_-|^2\mathrm{d}m$ $\leqslant C_6\|f\|^2_{L^2(K,m)}$, and we have the estimate

$$\int\limits_{I\setminus(I_3\cup I_4)} |g|^2\mathrm{d}m \leqslant C_6\|f\|^2_{L^2(K,m)}. \tag{5.11}$$

It remains to obtain the estimate

$$\int\limits_{\mathbb{T}\setminus(I\cup I_1\cup I_2)} |g|^2\mathrm{d}m \leqslant C_7\|f\|^2_{L^2(K,m)}. \tag{5.12}$$

To do it, we apply Lemma 5.4. We put $\mathcal{I}_0 = \mathrm{clos}(\mathbb{T} \setminus (I \cup I_1 \cup I_2))$, and by $\mathcal{I}$ we denote an open arc such that $\mathcal{I}_0 \subset \mathcal{I}$ and $\mathrm{clos}\,\mathcal{I} \subset \mathbb{T} \setminus I$. By the assumption, for every $\zeta \in \mathcal{I}$ there exists a disk $D(\zeta)$ centered at ζ and such that $\Phi(D(\zeta)\cap\mathbb{D}) \subset \mathbb{D}$ and $D(\zeta)\cap\mathbb{T} \subset \mathcal{I}$. Therefore there exists a sequence $\{\zeta_n\}_n$ such that $\zeta_n \in \mathcal{I}$ and $\bigcup\limits_n D(\zeta_n) \supset \mathcal{I}$. We put $\mathcal{G} = \bigcup\limits_n D(\zeta_n)\cap\mathbb{D}$. By Lemma 5.4, there exists a constant C_0

such that $\int\limits_{\mathcal{I}_0} |G\circ\Phi| \mathrm{d}m \leqslant C_0 \int\limits_{\mathbb{T}} |G| \mathrm{d}m$ for any $G \in L^1(\mathbb{T}, m)$. Thus,

$$\int\limits_{\mathcal{I}_0} |g|^2 \mathrm{d}m = \int\limits_{\mathcal{I}_0} |(\mathrm{e}^{\mathrm{i}t_1} - \zeta)(F_+\circ\Phi)(\zeta)|^2 \mathrm{d}m \leqslant \sup_{\zeta\in\mathcal{I}_0} |\mathrm{e}^{\mathrm{i}t_1} - \zeta|^2 \int\limits_{\mathcal{I}_0} |F_+\circ\Phi|^2 \mathrm{d}m$$
$$\leqslant 4C_0 \int\limits_{\mathbb{T}} |F_+|^2 \mathrm{d}m \leqslant C\|f\|^2_{L^2(K,m)},$$

and (5.12) is proved.

From (5.9), (5.10), (5.11), and (5.12) we conclude that there exists $C > 0$ such that

$$\int\limits_{\mathbb{T}} |g|^2 \mathrm{d}m \leqslant C\|f\|^2_{L^2(K,m)}$$

for functions f smooth on $\mathbb{T}$ and vanishing on $\mathbb{T}\setminus K$, where $g = \mathcal{Z}f$. Thus, $\mathcal{Z}$ can be extended from smooth functions to the whole space $L^2(K, m)$. We recall that $\alpha(z) = (z-\delta)^2$, $z \in \mathbb{D}$, where δ is the only zero of ψ. We take $f \in L^2(K, m)$ such that $f \notin L^2(K, w\mathrm{d}m)$, and we put $k = \frac{1}{\alpha}\mathcal{Z}f$. Then $k \in \mathcal{E}(K)$. We will show that $k \notin \mathcal{E}_0(K) + \mathcal{E}_1(K)$. Indeed, if we assume that $k \in \mathcal{E}_0(K) + \mathcal{E}_1(K)$, then we must have $F\circ\Phi \in H^2(\mathbb{D})$, and, by (5.7) (where $F_0 = F$ and $F_1 \equiv 0$)

$$\int\limits_{\mathbb{T}} |F\circ\Phi|^2 \mathrm{d}m \geqslant \int\limits_{I_1\cup I_3} |F\circ\Phi|^2 \mathrm{d}m \asymp \int\limits_{J} (|F_+|^2 + |F_-|^2) w_1 \mathrm{d}m,$$

hence, F_+, $F_- \in L^2(J, w_1\mathrm{d}m)$ and $f = F_+ - F_- \in L^2(J, w_1\mathrm{d}m)$. But, by the assumption, f vanishes on $\mathbb{T}\setminus K$, therefore the last inclusion means that $f \in L^2(K, w\mathrm{d}m)$, which is a contradiction. Thus, we proved that $\mathcal{E}_0(K) + \mathcal{E}_1(K) \neq \mathcal{E}(K)$.

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REFERENCES

[1] D.N. CLARK, Following functions of class H^2, in *Harmonic Analysis in Euclidean Spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1*, Proc. Sympos. Pure Math., vol. 35, Amer. Math. Soc., Providence, R.I. 1979, pp. 451–456.

[2] D.N. CLARK, On Toeplitz operators with loops, *J. Operator Theory* **4**(1980), 37–54.

[3] D.N. CLARK, On a similarity theory for rational Toeplitz operators, *J. Reine Angew. Math.* **320**(1980), 6–31.

[4] D.N. CLARK, Sz.-Nagy–Foiaş theory and similarity for a class of Toeplitz operators, in *Spectral Theory (Warsaw, 1977)*, Banach Center Publ., vol. 8, PWN, Warsaw 1982, pp. 221–229.

[5] D.N. CLARK, On Toeplitz operators with unimodular symbols, in *Operators in Indefinite Metric Spaces, Scattering Theory and other Topics (Bucharest, 1985)*, Oper. Theory Adv. Appl., vol. 24, Birkhäuser, Basel 1987, pp. 59–68.

[6] D.N. CLARK, Perturbation and similarity of Toeplitz operators, in *Topics in Operator Theory: Ernst D. Hellinger Memorial Volume*, Oper. Theory Adv. Appl., vol. 48, Birkhäuser, Basel 1990, pp. 235–243.

[7] P.L. DUREN, *Theory of H^p Spaces*, Pure Appl. Math., vol. 38., Academic Press, New York-London 1970.

[8] M.F. GAMAL', On Toeplitz operators similar to unilateral shift, in *Zap. Nauchn. Sem. St. Petersburg. Otdel. Mat. Inst. Steklov*, vol. 345, POMI, 2007, pp. 85–104.

[9] T.W. GAMELIN, *Uniform Algebras*, Prentice-Hall, Inc., Englewood Cliffs, N.J. 1969.

[10] J.B. GARNETT, *Bounded Analytic Functions*, Pure Appl. Math., vol. 96, Academic Press, Inc., New York-London 1981.

[11] R. GOOR, On Toeplitz operators which are contractions, *Proc. Amer. Math. Soc.* **34**(1972), 191–192.

[12] V.V. PELLER, Spectrum, similarity, and invariant subspaces of Toeplitz operators [Russian], *Izv. Akad. Nauk SSSR Ser. Mat.* **50**(1986), 776-787; English *Math. USSR Izv.* **29**(1987), 133–141.

[13] M. ROSENBLUM, J. ROVNYAK, *Hardy Classes and Operator Theory*, Oxford Math. Monographs, Oxford Sci. Publ., The Clarendon Press, Oxford Univ. Press, New York 1985.

[14] K. STEPHENSON, Functions which follow inner functions, *Illinois J. Math.* **23**(1979), 259–266.

[15] M. TSUJI, *Potential Theory in Modern Function Theory*, Maruzen Co., Ltd., Tokyo 1959.

[16] D. WANG, Similarity theory of smooth Toeplitz operators, *J. Operator Theory* **12**(1984), 319–330.

[17] D.V. YAKUBOVICH, Riemann surface models of Toeplitz operator, in *Toeplitz Operators and Spectral Function Theory*, Oper. Theory Adv. Appl., vol. 42, Birkhäuser, Basel 1989, pp. 305–415.

[18] D.V. YAKUBOVICH, On the spectral theory of Toeplitz operators with a smooth symbol [Russian], *Algebra i Analiz* **3**(1991), 208–226; English *St. Petersburg Math. J.* **3**(1992), 903–921.

[19] D.V. YAKUBOVICH, Dual piecewise analytic bundle shift models of linear operators, *J. Funct. Anal.* **136**(1996), 294–330.

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