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A BEURLING THEOREM FOR NONCOMMUTATIVE L^p

DAVID P. BLECHER and LOUIS E. LABUSCHAGNE

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ABSTRACT. We extend Beurling's invariant subspace theorem, by characterizing subspaces K of the noncommutative L^p spaces which are invariant with respect to Arveson's maximal subdiagonal algebras, sometimes known as noncommutative H^∞. We show that a certain subspace, and a certain quotient, of K are $L^p(\mathcal{D})$-modules in the recent sense of Junge and Sherman, and therefore have a nice decomposition into cyclic submodules. This is used, together with earlier results of Nakazi and Watatani, to give our Beurling theorem. We also give general inner-outer factorization formulae for elements in the noncommutative L^p.

KEYWORDS: *Subdiagonal operator algebra, noncommutative Hardy spaces, finite von Neumann algebras, simply invariant subspaces, Beurling's theorem, factorization.*

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1. INTRODUCTION

The starting point of this paper is Beurling's invariant subspace theorem, stating that a certain class of invariant subspaces of L^2 may be characterized as exactly those spaces of the form uH^2, where u is a unimodular function, and H^2 is the Hilbertian Hardy space. Many generalizations of this theorem have appeared over the decades (e.g. [10], [11], [13], [23]), and our paper is concerned with generalizations appropriate to Arveson's noncommutative *subdiagonal algebra* generalization of the Hardy spaces of the disk.

Throughout this paper, M is a finite von Neumann algebra possessing a faithful normal tracial state τ, and A is a *tracial subalgebra* of M. That is, A is a weak* closed unital subalgebra A of M for which the restriction to A of the unique faithful normal conditional expectation Φ from M onto $\mathcal{D} \stackrel{\text{def}}{=} A \cap A^*$ satisfying $\tau = \tau \circ \Phi$, is a homomorphism. Here A^* denotes the set of adjoints of elements in A. Two simple examples which may help the reader understand the setting, are (1) the subalgebra of the $n \times n$ matrices consisting of the upper triangular matrices, and (2) the classical H^∞ space of bounded analytic functions on the disk $\mathbb{D}$ (here $\tau = \Phi$ is just the Haar integral on $L^\infty(\mathbb{T})$). A tracial subalgebra of M is called *maximal subdiagonal* if $A + A^*$ is weak* dense in M. This form of the definition is due to Arveson [1] and Exel [6]. Sometimes such an A is called a *noncommutative* H^∞. (We remark that there are other, more recent, very important operator algebraic generalizations of H^∞ which are currently being intensively studied by many prominent researchers. See e.g. [25] and references therein. However these are only formally related to subdiagonal algebras.) If M is commutative and $\mathcal{D}$ is one dimensional then A is a *weak* Dirichlet algebra* [28]. In an earlier paper we gave a list of many disparate looking conditions which a tracial algebra A might satisfy, which turn out to be equivalent to each other, and equivalent to A being maximal subdiagonal [3] (see also [4]).

We write $L^p(M)$ for $L^p(M,\tau)$, the noncommutative L^p-space associated to the pair (M,τ), in the sense of e.g. Nelson [22] as a certain space of operators affiliated to M. In the present paper we study the structure of (right) A-invariant subspaces of $L^p(M)$, and give general inner-outer factorization formulae of Beurling-Nevanlinna type for elements of $L^p(M)$. These results generalize important classical results (e.g. see references in [28]), and should be useful in the future development of the noncommutative H^p theory. They also constitute a natural occurrence of the "L^p-modules" and "L^p-column sums" due recently to Junge and Sherman [14]. In addition, our results characterize maximal subdiagonal algebras, allowing us to supplement the list given in our earlier paper of criteria equivalent to maximal subdiagonality.

We will use the notation of [3], to which the reader is also referred for further explanations and details. For any set $\mathcal{S}$ of operators, $\mathcal{S}^*$ will denote the set of adjoints of elements in $\mathcal{S}$. We write $[\mathcal{S}]_p$ for the closure of $\mathcal{S}$ in $L^p(M)$. If A is a maximal subdiagonal algebra, then $[A]_p$ is often called a *noncommutative Hardy space*, and written as H^p. We write A_∞ for $[A]_2 \cap M$ (which equals A if A is maximal subdiagonal, as may be seen using a maximality argument ([1], Section 2)), and A_0 for $A \cap \mathrm{Ker}(\Phi)$. We say that A satisfies L^2-density, if $A + A^*$ is L^2-dense in $L^2(M)$. Clearly A satisfies L^2-density if and only if the same is true of A_∞. We say that A has the *unique normal state extension property* if whenever $g \in L^1(M)_+$ with $\tau(gA_0) = 0$, then $g \in L^1(\mathcal{D})$. All maximal subdiagonal algebras have these latter two properties. In fact, it is shown in [3] that maximal subdiagonal algebras are exactly the tracial algebras possessing these two properties.

We recall that a *(right) invariant subspace* of $L^p(M)$ is a closed subspace K of $L^p(M)$ such that $KA \subset K$. For consistency, we will not consider left invariant subspaces at all, leaving the reader to verify that entirely symmetric results pertain in the left invariant case. An invariant subspace is called *simply invariant* if in addition the closure of KA_0 is properly contained in K. It is the latter class of subspaces to which the generalized Beurling theorem applies, e.g. for weak* Dirichlet algebras. In the literature there are several invariant subspace theorems, inspired by the Beurling result and its classical extensions, and associated factorization results, for maximal subdiagonal algebras (see e.g. [16], [18], [19], [20],

[21], [24], [26], [31], or in the work of Ji, Ohwada and Saito). We mention just two which we shall use: Saito showed in [26] that any A-invariant subspace of $L^p(M)$ is the closure of the bounded elements which it contains. Nakazi and Watatani showed in [21] that in the case that the center of M contains the center of $\mathcal{D}$, every "type 1" (defined below) invariant subspace of $L^2(M)$ is of the form $u[A]_2$ for a partial isometry u. Compelling examples of invariant subspaces exhibiting interesting structure may be found in [19], [31].

A (right-) *wandering vector* is a vector $f \in L^2(M)$ with $f \perp [fA_0]_2$. Wandering vectors for maximal subdiagonal algebras were characterized by Arveson in [1] as the vectors f which have a factorization $f = uh$ for $h \in [\mathcal{D}]_2$ and u a partial isometry in $[fM]_2$ with $u^*uh = h$ (in fact, one can also ensure that $u^*u \in \mathcal{D}$). These vectors play a significant role in that paper, and in later work. Arveson was inspired in part here by some of the then emerging prediction theory and operator function theory, where Beurling type decomposition theorems are related to wandering vectors and subspaces by the famous *Wold decomposition*. See e.g. Chapter 1 of [23] for a succinct discussion of these relationships. For example, and to motivate the definitions below, suppose that u is an isometry on a Hilbert space H, and that K is a subspace of H such that $uK \subset K$. If A is the unital operator algebra generated by u, and A_0 the nonunital operator algebra generated by u, then K is A-invariant. (The reader may keep in mind the case where u is multiplication by the monomial z on the circle $\mathbb{T}$; in this case $A = H^\infty(\mathbb{D})$, and A_0 is the algebra of functions vanishing at 0.) The subspace $W = K \ominus uK = K \ominus [A_0K]$ is *wandering* in the classical sense that $u^nW \perp u^mW$ for unequal nonnegative integers n, m. The Wold decomposition allows us to write $K = K_1 \oplus K_2$, an orthogonal direct sum, where K_1 is defined in terms of the wandering subspace W; and u is unitary on K_2 whereas the restriction of u to subspaces of K_1 is never unitary (see e.g. Lemma 1.5.1 of [23]). We have that $uK_2 = K_2$, which is equivalent to $[A_0K_2]_2 = K_2$, a criterion which matches what is called "type 2" below. Also, $[AW]_2 = K_1$, a condition which matches what is called "type 1" below. Indeed the match with the definitions below is exact in the case where $A = H^\infty(\mathbb{D})$.

The Wold decomposition plays a critical role in Helson and Lowdenslager's geometric approach to invariant subspaces, which is fundamental to the subject. Not surprisingly, a variant of the "wandering" concept is key to our decomposition theorems. If K is a right A-invariant subspace of $L^2(M)$, we follow Nakazi and Watatani's decomposition results [21]. We define the *right wandering subspace* of K to be the space $W = K \ominus [KA_0]_2$; and we say that K is *type 1* if W generates K as an A-module (that is, $K = [WA]_2$). We will say that K is *type* 2 if $W = (0)$. (The last notation conflicts with that of [21], where instead of this class of subspaces they have two further subclasses which they call type II and type III.) For any p, one may define the *wandering quotient* to be $K/[KA_0]_p$, and say that K is type 2 if this is trivial. In our paper if $p = \infty$, we take $[\cdot]_p$ to be the weak* closure. It turns out that the wandering quotient is an $L^p(\mathcal{D})$-module in the sense of [14], and it is isometric to a canonically defined subspace of K which can be called the *right* *wandering subspace of* K. We say that K is type 1 if this subspace generates K as an A-module. If $1 \leqslant p < 2$ (respectively $p > 2$) then we will show that K is type 1 if and only if $K \cap L^2(M)$ (respectively $[K]_2$) is type 1 in the sense of the L^2 case above.

In the classical case, or more generally whenever $\mathcal{D}$ is one dimensional, there is a dichotomy: in this case type 1 is the same thing as being simply invariant, and any invariant subspace which is not type 1 is type 2. In the general case, being type 2 is the same as being not simply invariant; and any nontrivial type 1 subspace is simply invariant. However the "simply invariant" condition no longer plays a very significant role for us. Moreover, there is no longer a dichotomy between types 1 and 2. Instead, there is a direct sum decomposition. We use the *column L^p-sum* recently studied by Junge and Sherman [14] to investigate this: If X is a subspace of $L^p(M)$, and if $\{X_i : i \in I\}$ is a collection of subspaces of X, which together densely span X, with the property that $X_i^* X_j = \{0\}$ if $i \neq j$, then we say that X is the *internal column L^p-sum* $\bigoplus_i^{\mathrm{col}} X_i$. If $p = \infty$ we also assume that X and X_i are weak* closed, and the word "densely" above is taken with regard to the weak* topology. Our main result, which builds on earlier ideas and results from [21], [19], is as follows:

THEOREM 1.1. *If A is a maximal subdiagonal subalgebra of M, if $1 \leqslant p \leqslant \infty$, and if K is a closed (indeed weak* closed, if $p = \infty$) right A-invariant subspace of $L^p(M)$, then:*

(i) *K may be written uniquely as an (internal) L^p-column sum $K_1 \oplus^{\mathrm{col}} K_2$ of a type* 1 *and a type* 2 *invariant subspace of $L^p(M)$, respectively.*

(ii) *If $K \neq (0)$ then K is type* 1 *if and only if $K = \bigoplus_i^{\mathrm{col}} u_i H^p$, for u_i partial isometries with mutually orthogonal ranges and $|u_i| \in \mathcal{D}$.*

(iii) *The right wandering subspace W of K is an $L^p(\mathcal{D})$-module in the sense of Junge and Sherman, and in particular $W^* W \subset L^{p/2}(\mathcal{D})$.*

Conversely, if A is a tracial subalgebra of M such that every A-invariant subspace of $L^2(M)$ satisfies (i) *and* (ii) *(respectively* (ii) *and* (iii)*), then A is maximal subdiagonal.*

Note that this theorem immediately implies the classical Beurling theorem, and its generalization to simply invariant subspaces of weak* Dirichlet algebras. Indeed if $\mathcal{D}$ is one-dimensional, and if K is a simply invariant subspace of L^p, then K is not type 2, and so $K_1 \neq (0)$. Thus there is a nonzero partial isometry u with $|u| \in \mathcal{D} = \mathbb{C}1$. Hence $u^*u = 1$, and since M is finite u is a unitary in M. Thus $K_1 = uH^p$. Since $K_1^* K_2 = (0)$, we have $K_2 = (0)$, and so $K = uH^p$, as desired.

EXAMPLE. To help the reader's intuition, it may be worthwhile to take the time to consider a very simple example. Suppose that A is the upper triangular 3×3 matrices. Let K be the upper triangular 3×3 matrices supported in the 1-3, 2-2, and 2-3 entries, and with the $L^p(M_3)$-norm. Here $\mathcal{D}$ consists of the diagonal matrices, and A_0 is the "strictly upper triangular matrices". The wandering quotient $W = K/[KA_0]$ may be identified with the subspace of K whose 2-3 entry is also 0. If $u_1 = E_{13}, u_2 = E_{22}$ then the latter subspace is $u_1[\mathcal{D}]_p \oplus^{\mathrm{col}} u_2[\mathcal{D}]_p$. The decomposition in Theorem 1.1 (1) becomes $K = u_1[A]_p \oplus^{\mathrm{col}} u_2[A]_p$. Of course the $u_k[A]_p$ piece is just the kth row of K. More generally, if z is any matrix then zA is an invariant subspace for the upper triangular algebra A. If z is invertible, then the wandering subspace here is just $Q\mathcal{D}$, where Q is the unitary occurring in the well known "QR-factorization" (from matrix theory) of z.

An element ξ of a $\mathcal{D}$-module is called *separating* if the map $d \mapsto \xi d$ is one-to-one on $\mathcal{D}$; and *cyclic* if $\xi\mathcal{D}$ is dense. If $p = \infty$ we mean dense in the weak* topology.

COROLLARY 1.2. *If A and K are as in Theorem* 1.1, *then K is of the form uH^p for a unitary $u \in M$, if and only if the right wandering subspace of K is a "standard" representation of $\mathcal{D}$, that is, it has a nonzero separating and cyclic vector for the right action of $\mathcal{D}$. This is equivalent to the right wandering quotient having a separating and cyclic vector.*

We say that $f \in L^2(M)$ is *Beurling-Nevanlinna factorizable* (or *BN-factorizable*), if $f = uh$, for a unitary u in M and an h with $[hA]_2 = [A]_2$. An h with $[hA]_2 = [A]_2$ will be called *outer*, as in the classical theory. The content of the assertion in this definition is that $h \in [A]_2$, and $1 \in [hA]_2$. The unitary u here is called *inner*. We will say that $f \in L^2(M)$ is *partially BN-factorizable* if $f = uh$, for a partial isometry u in M with $u^*u \in \mathcal{D}$, and an h with $[hA]_2 = (u^*u)[A]_2$. That is, $h = (u^*u)h \in [A]_2$, and $u^*u \in [hA]_2$. We remark that the first noncommutative BN-factorization of this type was Arveson's characterization of right wandering vectors mentioned above. Our invariant subspace theorem above leads immediately, as in the classical case, to Beurling-Nevanlinna factorization results, as we shall see in Section 3. It will be clear in Section 4 that these results extend to L^p for $p \neq 2$.

It is worth noting that if we assume that our tracial algebra is *antisymmetric* (that is, $\mathcal{D}$ is one-dimensional, which forces $\Phi(\cdot) = \tau(\cdot)1$), then almost all of the classical results about generalized Hardy spaces found in [27], [28], for example, and their proofs, seem to transfer almost verbatim and without difficulty. To illustrate this, we now discuss a special case of our main results. First we will need a simple lemma:

LEMMA 1.3. *Let A be a tracial subalgebra of M which satisfies $[A]_1 = \{x \in L^1(M) : \tau(xA_0) = 0\}$. Then $A = A_\infty$, and $A = \{f \in M : \tau(fA_0) = 0\}$.*

Proof. This follows just as in lemma, p. 816 of [27]. Following that proof we find an $f \in L^1(M)$ with $\tau(fA) = 0$. By our hypothesis, $f \in [A]_1$. For $d \in \mathcal{D}$ we have

$$0 = \tau(fd) = \tau(\Phi(fd)) = \tau(\Phi(f)d).$$

It follows that $\Phi(f) = 0$, and so $f \in [A_0]_1$. As in the last cited reference we conclude that $[A]_1 \cap M = A$, which implies the last assertion of the lemma. Also, $A_\infty = [A]_2 \cap M \subset [A]_1 \cap M = A$, giving the other assertion. ∎

PROPOSITION 1.4. *Let A be a tracial subalgebra of M. Consider the conditions:*
(i) *every simply right invariant subspace of $L^2(M)$ is of the form $u[A]_2$, for a unitary u in M;*
(ii) *whenever $f \in L^2(M)$ with $f \notin [fA_0]_2$, then f is* BN-*factorizable;*
(iii) *A is maximal subdiagonal.*
Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). *If A is antisymmetric, then the conditions are all equivalent.*

Proof. That (i) $\Rightarrow$ (ii) follows just as in [28], for example, with only minor modifications. One needs to use the fact that left invertibility implies invertibility in a finite von Neumann algebra. Similarly for the implication (iii) $\Rightarrow$ (i) in the antisymmetric case (see also [16]), or note that this is proved in the paragraph after Theorem 1.1. Similarly, supposing (ii), it also follows just as in [28], that A_∞ is maximal subdiagonal. We next claim that $f \in L^1(M)$ and $\tau(fA_0) = 0$, if and only if $f \in [A]_1$. This may be proved as in Corollary 2.3 of [27] (note that Theorem 2 of [27] follows easily from condition (ii)). By Lemma 1.3, $A_\infty = A$. So A is maximal subdiagonal. ∎

If A is not antisymmetric, then (iii) in this last proposition need not imply (i) or (ii). This may be seen by considering the example of M a two-dimensional von Neumann algebra, and $A = M$. Also, for the upper triangular matrices both (i) and (ii) fail. In fact, it is not hard to see that if $\mathcal{D}$ is not a factor, then the conditions cannot be equivalent. Below, we will find the appropriate generalizations of the statements of the simply invariant subspace theorem, and the Beurling-Nevanlinna factorization result, and we will show that the new statements are each equivalent to maximal subdiagonality.

We will not say very much about the structure of type 2 invariant subspaces in this paper. Indeed such structure is not well understood at the present time, aside from the classical theorem of Wiener ([23], Theorem 1.2.1), and generalizations of this to relatively concrete function-operator theoretic settings. For example not much seems to be known even in the commutative case of weak* Dirichlet algebras. Certain noncommutative algebras, such as the upper triangular matrix algebra, clearly have no type 2 invariant subspaces (see also (i) in Remarks after Theorem 2.1). Nonetheless, the following examples may help the reader to picture type 2 invariant subspaces. We suppose that A is a maximal subdiagonal algebra such that A_0 contains an isometry. Simple examples where this happens are $H^\infty(\mathbb{D})$ or, more generally, algebras constructed from ordered groups as in the work of Helson and Lowdenslager. Arveson already considered in Section 3 of [1] several interesting noncommutative examples of the latter. More specifically, let G be an ordered discrete countable group, and let M be the group von

Neumann algebra (the second commutant of the image of the left regular representation $g \mapsto U_g$) with its usual trace. We take A to be the weak* closure of the span of $\{U_g : g \in G, g \geqslant 0\}$; combining Arveson's arguments from 3.2.2 of [1] with Exel's result [6] we see that A is an antisymmetric maximal subdiagonal algebra. Let $\mathcal{S} = \{U_g : g \in G, g > 0\}$. We claim that A_0 equals E, the weak* closure of the span of $\mathcal{S}$. Since the trace annihilates $\mathcal{S}$, clearly A_0 contains E. Conversely, since $E + \mathbb{C}\,1$ is weak* closed, it equals A, the weak* closure of the span of $\{U_g : g \in G, g \geqslant 0\}$. Thus $E = A_0$. Hence A_0 is actually generated by unitaries.

In any case, if A is a maximal subdiagonal algebra such that A_0 contains an isometry u, and if e is any projection in M, then the A-invariant subspace $K = eL^2(M)$ is type 2, since $K = Ku^*u \subset KA_0$. For a second example, suppose that A_0 is generated by a subset $\mathcal{S}$ with the property that $\mathcal{S} \subset \mathcal{S}^2$ (e.g. the ordered group example above in the case that the group is the rational numbers — this example is actually weak* Dirichlet). Then $A_0 \subset [A_0 A_0]_2$, and so $[A_0]_2$ is a type 2 A-invariant subspace. Indeed it is then clear that $[fA_0]_2$ is type 2 for any $f \in M$.

2. INVARIANT SUBSPACES OF $L^2(M)$

We begin with some general observations about the structure of invariant subspaces. There is a substantial overlap between (v) below, and results in [21] (see particularly Lemma 2.4 and Theorem 2.14 there).

THEOREM 2.1. *Let A be a tracial algebra.*

(i) *Suppose that X is a subspace of $L^2(M)$ of the form $X = Z \oplus^{\mathrm{col}} [YA]_2$ where Z, Y are closed subspaces of X, with Z a type 2 invariant subspace, and $\{y^*x : y, x \in Y\} = Y^*Y \subset L^1(\mathcal{D})$. Then X is simply right A-invariant if and only if $Y \neq \{0\}$.*

(ii) *If X is as in* (i), *then $[Y\mathcal{D}]_2 = X \ominus [XA_0]_2$ (and $X = [XA_0]_2 \oplus [Y\mathcal{D}]_2$).*

(iii) *If X is as described in* (i), *then that description also holds if Y is replaced by $[Y\mathcal{D}]_2$. Thus (after making this replacement) we may assume that Y is a $\mathcal{D}$-submodule of X.*

(iv) *The subspaces $[Y\mathcal{D}]_2$ and Z in the decomposition in* (i) *are uniquely determined by X. So is Y if we take it to be a $\mathcal{D}$-submodule (see* (iii)*).*

(v) *If A is maximal subdiagonal, then any right A-invariant subspace X of $L^2(M)$ is of the form described in* (i), *with Y the right wandering subspace of X.*

Proof. (i) Let X be of the form described above. We show that $Y \perp [XA_0]_2$ from which it follows that X is simply right A-invariant if $Y \neq \{0\}$. To see this it is enough to show that $y \perp (z + xa)$ for any $z \in Z$, $a \in A_0$, and $x, y \in Y$. From the hypotheses $Y^*Z = \{0\}$ and $Y^*Y \subset L^1(\mathcal{D})$, it now easily follows that

$$\tau(y^*(z + xa)) = \tau((y^*x)a) = \tau(\Phi((y^*x)a)) = \tau((y^*x)\Phi(a)) = 0$$

as required. The converse is obvious.

(ii) We saw in the proof of (i) that $Y \perp [XA_0]_2$. Since $[XA_0]_2 = [XA_0]_2\mathcal{D}$, we have $[Y\mathcal{D}]_2 \perp [XA_0]_2$. Since $A = \mathcal{D} + A_0$, it therefore follows that $[YA]_2 =$

$[Y\mathcal{D}]_2 \oplus [YA_0]_2$. Now since $Z = [ZA_0]_2$, and

$$[YA_0]_2 \subset [XA_0]_2 = [(Z \oplus [YA]_2)A_0]_2 \subset Z \oplus [YA_0]_2,$$

it is clear that $[XA_0]_2 = Z \oplus [YA_0]_2$. The result therefore follows, since

$$X = Z \oplus [YA]_2 = Z \oplus ([YA_0]_2 \oplus [Y\mathcal{D}]_2) = [XA_0]_2 \oplus [Y\mathcal{D}]_2.$$

(iii) Since $Y^*Z = \{0\}$ and $Y^*Y \subset L^1(\mathcal{D})$, we have $\mathcal{D}^*Y^*Z = \{0\}$ and $\mathcal{D}^*Y^*Y\mathcal{D} \subset L^1(\mathcal{D})$. Hence if $\widetilde{Y} = [Y\mathcal{D}]_2$ then $\widetilde{Y}^*Z = \{0\}$ and $\widetilde{Y}^*\widetilde{Y} \subset L^1(\mathcal{D})$. Finally, notice that since $\mathcal{D}A = A$ and $\mathcal{D}A_0 = A_0$, it is easy to see that $[\widetilde{Y}A]_2 = [[Y\mathcal{D}]_2A]_2 = [YA]_2$ and $[\widetilde{Y}A_0]_2 = [[Y\mathcal{D}]_2A_0]_2 = [YA_0]_2$.

(iv) This follows from (ii), and the fact that $Z = X \ominus [YA]_2$.

(v) Assume that A is maximal subdiagonal and that X is right A-invariant. Set $Y = X \ominus [XA_0]_2$. The subspace Y will clearly be nontrivial if X is simply right A-invariant. We show that $Y^*Y \subset L^1(\mathcal{D})$. Let $y, x \in Y$ be given. Since $[xA_0]_2 \subset [XA_0]_2$ we clearly have $y \perp [xA_0]_2$, and hence that

$$\tau(y^*xa) = \tau(y^*x(a - \Phi(a))) + \tau(y^*x\Phi(a)) = \tau(y^*x\Phi(a)) = \tau(\Phi(y^*x)a)$$

for all $a \in A$. In the last line we have used several properties of Φ which are obvious for Φ considered as a map on M, and which are easily verified for the extension of Φ to $L^1(M)$. (See for example 3.10 of [18].) On swapping the roles of x and y and noting that the extension of Φ preserves adjoints on $L^1(M)$, we get

$$\tau(y^*xa^*) = \overline{\tau(ax^*y)} = \overline{\tau(a\Phi(x^*y))} = \tau(\Phi(y^*x)a^*)$$

for all $a \in A$. So $y^*x - \Phi(y^*x) \perp A + A^*$ which forces $y^*x = \Phi(y^*x) \in L^1(\mathcal{D})$.

Next let $Z = X \ominus [YA]_2$ and let $y \in Y$ and $z \in Z$ be given. Now by construction $z \perp [yA]_2$ and $y \perp [zA_0]_2$, whence

$$0 = \tau((ya)^*z) = \tau((y^*z)a^*) \quad \text{for all } a \in A$$

and

$$0 = \tau(y^*(za)) = \tau((y^*z)a) \quad \text{for all } a \in A_0.$$

So $y^*z \perp A + (A_0)^* = A + A^*$, which forces $y^*z = 0$.

To see that $ZA \subset Z$, notice that for any $z \in Z$, $y \in Y$ and $a, b \in A$ it follows from what we just proved that $(yb)^*(za) = 0$ and hence that

$$za \in ([XA]_2 \ominus [YA]_2) \subset (X \ominus [YA]_2) = Z.$$

Finally, let $V = Z \ominus [ZA_0]_2$. Then since $V \subset Z$, we have $Y^*V = \{0\}$, which ensures that $V \perp [YA_0]_2$. But by construction $V \perp [ZA_0]_2$, and so

$$V \perp [ZA_0 + YA_0]_2 \supset [(Z \oplus [YA]_2)A_0]_2 = [XA_0]_2.$$

But then $V \subset Y = X \ominus [XA_0]_2$ which by what we noted earlier, forces $V^*V = \{0\}$. Clearly $V = \{0\}$, or in other words $Z = [ZA_0]_2$. ∎

REMARKS. (i) Let X be as in (i) of Theorem 2.1. Then $Z = [ZA_0]_2 \subset [XA_0]_2$. Proceeding inductively we conclude that $Z \subset \bigcap_{n \geqslant 1} [XA_0^n]_2$. From this last fact it follows, for example, that $Z = (0)$ if A is the upper triangular matrices.

Indeed, for certain maximal subdiagonal algebras, any type 2 invariant subspace is automatically (0), and thus every closed A-invariant subspace X is type 1.

(ii) There are well-known variants of Beurling's theorem characterizing A-invariant subspaces of H^2. Some of these variants follow easily from the above result. Consider for example Theorem 5.5 of [13], where Hoffman proves (in our notation) that if K is an A-invariant subspaces of H^2 such that τ is nontrivial (i.e. nonzero) on K, then $K = uH^2$ for a unitary u in A. In his case, $\mathcal{D}$ is one-dimensional, so that the hypothesis here is saying that K is not perpendicular to $\mathcal{D}$. One may deduce this theorem from Theorem 2.1 in the case that A is maximal subdiagonal and antisymmetric as follows: Under these hypotheses, the right wandering space W is nontrivial, for otherwise $K = [KA_0]_2 \subset [A_0]_2 \perp \mathcal{D} = \mathbb{C}\,1$. As in the proof immediately after the statement of Theorem 1.1, this implies that $K = uH^2$ for a unitary $u \in A$. For non-antisymmetric algebras the situation appears to be quite complicated, but at least one can say for A-invariant subspaces of H^2 that in addition to the other conclusions of Theorem 2.1 we have that Z is orthogonal to $\mathcal{D}$. That is, $Z \subset [A_0]_2$. Indeed this is quite obvious since in this case $Z = [ZA_0]_2 \subset [A_0]_2 \perp \mathcal{D}$. Thus there can be no type 2 A-invariant subspaces of H^2 which are not contained in $[A_0]_2$. In the case of $H^\infty(\mathbb{D})$, the type 2 invariant subspaces were characterized by Wiener, and none of them are contained in H^2. However as we said in the introduction, there certainly do exist maximal subdiagonal algebras with type 2 A-invariant subspaces of H^2.

Combining these thoughts with (i) in Remarks suggests that one could isolate various classes of algebras with "shift-like structure" for which every A-invariant subspace of H^2 is type 1; in which n-fold products of terms in A_0 "converge to zero" in some sense. Perhaps the examples in [19], [31] might be helpful here.

(iii) For maximal subdiagonal algebras, it is proved in [21] that an invariant subspace X is of type 1 if and only if $[XM]_2 = [YM]_2$, where $Y = X \ominus [XA_0]_2$. The same result holds for spaces of the form in Theorem 2.1 (i).

(iv) It is interesting to note that if A is maximal subdiagonal, then given $f, g \in L^2(M)$ we have that $f^*g = 0$ if and only if $[fA]_2 \perp [gA]_2$. The "only if"' part is obvious, and hence suppose that $[fA]_2 \perp [gA]_2$. It is an easy exercise to show that this forces $f^*g \perp A + A^*$. Since A is maximal subdiagonal, this yields $f^*g = 0$.

PROPOSITION 2.2. *Suppose that X is as in Theorem 2.1, and that W is the right wandering subspace of X. Then W may be decomposed as an orthogonal direct sum $\bigoplus_i^2 u_i[\mathcal{D}]_2$, where u_i are partial isometries in $W \cap M$ with $u_i^* u_i \in \mathcal{D}$, and $u_j^* u_i = 0$ if*

$i \neq j$. If W has a cyclic vector for the $\mathcal{D}$-action, then we need only one partial isometry in the above.

Proof. By the theory of representations of a von Neumann algebra (see e.g. the discussion at the start of Section 3 in [14]), any normal Hilbert $\mathcal{D}$-module is an L^2 direct sum of cyclic Hilbert $\mathcal{D}$-modules, and if K is a normal cyclic Hilbert $\mathcal{D}$-module, then K is spatially isomorphic to $[e\mathcal{D}]_2$, for an orthogonal projection $e \in \mathcal{D}$. Suppose that the latter isomorphism is implemented by a unitary $\mathcal{D}$-module map φ. If in addition $K \subset W$, let $\varphi(e) = u \in W$. Then $\tau(d^*u^*ud) = \|\varphi(ed)\|_2^2 = \tau(d^*ed)$, for each $d \in \mathcal{D}$. By Theorem 2.1, $u^*u \in L^1(\mathcal{D})$, and so $u^*u = e$. Hence u is a partial isometry. Note that $u[\mathcal{D}]_2 \subset [u\mathcal{D}]_2$ clearly. However, $u\mathcal{D} \subset u[\mathcal{D}]_2$, and the latter space is easily seen to be closed, so that $[u\mathcal{D}]_2 \subset u[\mathcal{D}]_2$. Thus $[u\mathcal{D}]_2 = u[\mathcal{D}]_2$. Putting these facts together, we see that W is of the desired form. Note that $u_j^*u_i = 0$ if $i \neq j$, since $u_j^*u_i \in \mathcal{D}$, but $\tau(u_j^*u_id) = 0$ for any $d \in \mathcal{D}$. ∎

COROLLARY 2.3. *If X is an invariant subspace of the form described in Theorem 2.1, then X is type 1 if and only if $X = \bigoplus_i^{\mathrm{col}} u_i[A]_2$, for u_i as in Proposition 2.2.*

Proof. If X is type 1, then $X = [WA]_2$ where W is the right wandering space, and so one assertion follows from Proposition 2.2. If $X = \bigoplus_i^{\mathrm{col}} u_i[A]_2$, for u_i as above, then $[XA_0]_2 = \bigoplus_i^{\mathrm{col}} u_i[A_0]_2$, and from this it is easy to argue that $W = \bigoplus_i^{\mathrm{col}} u_i[\mathcal{D}]_2$. Thus $X = [WA]_2 = \bigoplus_i^{\mathrm{col}} u_i[A]_2$. ∎

PROPOSITION 2.4. *Let X be a closed A-invariant subspace of $L^2(M)$, where A is a tracial subalgebra of M.*

(i) *If $X = Z \oplus [YA]_2$ as in Theorem 2.1, then Z is type 2, and $[YA]_2$ is type 1.*

(ii) *If A is maximal subdiagonal algebra, and if $X = K_1 \oplus^{\mathrm{col}} K_2$ where K_1 and K_2 are types 1 and 2 respectively, then K_1 and K_2 are respectively the unique spaces Z and $[YA]_2$ in Theorem 2.1.*

(iii) *If A and X are as in (ii), and if X is type 1 (respectively type 2), then the space Z of Theorem 2.1 for X is (0) (respectively $Z = X$).*

(iv) *If $X = K_1 \oplus^{\mathrm{col}} K_2$ where K_1 and K_2 are types 1 and 2 respectively, then the right wandering subspace for X equals the right wandering subspace for K_1.*

Proof. (i) Clearly in this case Z is type 2. To see that $[YA]_2$ is type 1, note that since $Y \perp XA_0$, we must have $Y \perp YA_0$. Thus $Y \subset [YA]_2 \ominus [YA_0]_2$, and consequently $[YA]_2 = [([YA]_2 \ominus [YA_0]_2)A]_2$.

(ii) Suppose that $X = K_1 \oplus^{\mathrm{col}} K_2$ where K_1 and K_2 are types 1 and 2 respectively. Let Y be the right wandering space for K_1. By Theorem 2.1 we have $Y^*Y \subset L^1(\mathcal{D})$. So $X = [YA]_2 \oplus^{\mathrm{col}} K_2$, and by the uniqueness assertion in Theorem 2.1, K_2 is the space Z in Theorem 2.1 for X, and $K_1 = K_2^{\perp} = [YA]_2$.

(iii) This is obvious from Theorem 2.1.

(iv) If $K = K_1 \oplus^{\mathrm{col}} K_2$ as above, then $K_2 = [K_2 A_0]_2 \subset [KA_0]_2$, and so $K \ominus [KA_0]_2 \subset K \ominus K_2 = K_1$. Thus $K \ominus [KA_0]_2 \subset K_1 \ominus [K_1 A_0]_2$. Conversely, if $\eta \in K_1 \ominus [K_1 A_0]_2$, then $\eta \perp KA_0$ since $\eta^* K_2 = (0)$. So $\eta \in K \ominus [KA_0]_2$. ∎

In the case $p = 2$, items (i)–(iii) in Theorem 1.1 follow from the last results.

Proof of Corollary 1.2. If $p = 2$: if $K = uH^2$ for a unitary u, then as in the proof of Corollary 2.3, $K \ominus [KA_0]_2 = u[\mathcal{D}]_2$, which has separating and cyclic vector u. Conversely, if the right wandering subspace W has a separating and cyclic vector v for $\mathcal{D}$, then the proof of Proposition 2.2, with $e = 1$ in that proof, shows that $u^*u = 1$. So u is unitary. Since $K_2^* u = (0)$ we have $K_2 = (0)$. Thus $K = [WA]_2 = uH^2$. ∎

The equivalences in the next result generalize, and give as an immediate consequence the equivalence of (i) and (iii) in Proposition 1.4, in the antisymmetric case.

COROLLARY 2.5. *Let A be a tracial algebra. The following are equivalent:*

(i) *A is maximal subdiagonal.*

(ii) *For every right A-invariant subspace X of $L^2(M)$, the right wandering subspace W of X satisfies $W^*W \subset L^1(\mathcal{D})$, and $W^*(X \ominus [WA]_2) = (0)$.*

(iii) *Every right invariant subspace of $L^2(M)$ satisfies* (i) *and* (iii) *of Theorem* 1.1.

Proof. The fact that (i) implies (ii) is proved in Theorem 2.1. The fact that (ii) implies (i) may be proved via the later Beurling-Nevanlinna type factorization Theorem 3.3, along the same lines as the proof of Proposition 1.4. We choose to also give a direct proof. To see that (ii) implies (i), first set $X = L^2 \ominus [A_0^*]_2$. We will deduce that A satisfies L^2-density. That is, $L^2(M)$ is the closure of $A + A^* = A + A_0^*$, or equivalently that $X = [A]_2$. To this end, note that X is right A-invariant. It is easy to see that $1 \in W$, which forces $X \ominus [WA]_2 = (0)$, and $W \subset W^*W \subset L^1(\mathcal{D})$. Thus $W \subset L^2(M) \cap L^1(\mathcal{D}) = L^2(\mathcal{D})$. So $X = [WA]_2 \subset [A]_2$. The converse inclusion $[A]_2 \subset X$ follows from the fact that $[A]_2$ is orthogonal to $[A_0^*]_2$.

We now prove that A possesses the unique normal state extension property; so that A is maximal subdiagonal. To this end, let $g \in L^1(M)_+$ with $\tau(gA_0) = 0$. We may assume that $g \neq 0$. Let $h = g^{1/2} \in L^2(M)$, and set $X = [hA]_2$. Note that $h \perp [hA_0]_2$ since if $a_n \in A_0$ with $ha_n \to k$ in L^2-norm, then $\tau(h^*k) = \lim_n \tau(h^*ha_n) = 0$. In particular, the fact that $h \perp [hA_0]_2$ ensures that $h \in X \ominus [XA_0]_2 = W$. By hypothesis, $h^*h = g \in L^1(\mathcal{D})$.

We have seen already that (i) implies (iii). If (iii) holds then by Proposition 2.4 (iv) the wandering subspace $W \subset K_1$, and $K \ominus [WA]_2 = K \ominus K_1 = K_2$. Thus $W^*K_2 \subset K_1^*K_2 = (0)$, and $W^*W \subset L^1(\mathcal{D})$. This is (ii). ∎

REMARK. The conditions in the last result are also equivalent to: (iv) every simply right A-invariant subspace X of $L^2(M)$ is of the form $X = Z \oplus^{\text{col}} [YA]_2$ where Y, Z are closed subspaces of X with Z type 2, and $(0) \neq Y^*Y \subset L^1(\mathcal{D})$; and to: (v) every right A-invariant subspace of $L^2(M)$ satisfies (i) and (ii) of Theorem 1.1. This is fairly obvious from the proof of Corollary 2.5.

3. NONCOMMUTATIVE BEURLING-NEVANLINNA FACTORIZATION

We now discuss several general inner-outer, or Beurling-Nevanlinna, factorization theorems each of which turns out to characterize maximal subdiagonal algebras. One of these theorems (Theorem 3.3 (iv)), has the classical equivalence with the Beurling-Nevanlinna factorization theorem as an immediate consequence, or special case (namely that (ii) is equivalent to (iii) in Proposition 1.4 in the classical situation). This will be clear from an argument in the present paragraph. Our other Beurling-Nevanlinna factorization theorems (namely Theorem 3.3 (ii) or (iii)) have the advantage of having more attractive hypotheses, which are perhaps easier to check. To understand the hypothesis of these theorems, suppose for a moment that $\mathcal{D}$ is one dimensional as in the classical case, and that $f \in L^2(M)$ is such that $f \notin [fA_0]_2$. Thus $[fA]_2$ is type 1. The right wandering subspace $[fA]_2 \ominus [fA_0]_2$ in this case is also one dimensional (since $fA \subset [fA_0]_2 + f\mathcal{D} \subset [fA]_2$, and since $[fA_0]_2 + f\mathcal{D}$ is closed, $[fA_0]_2$ has codimension one in $[fA]_2$). In particular the wandering subspace has a separating and cyclic vector. In the general case, having such a vector is a necessary condition for BN-factorizability:

LEMMA 3.1. *Suppose that A is a tracial algebra, and that $f \in L^2(M)$ is* BN-*factorizable (respectively partially* BN-*factorizable). Then the right wandering subspace of $[fA]_2$ has a separating and cyclic vector (respectively a cyclic vector) for the right $\mathcal{D}$-action. In fact, that vector may be taken to be a unitary (respectively partial isometry) in M. Conversely, if A is maximal subdiagonal, and if the right wandering subspace of $[fA]_2$ has a separating and cyclic vector, then f is* BN-*factorizable.*

Proof. If $f = uh$ is a BN-factorization, then $[fA]_2 = [uhA]_2 = u[hA]_2 = u[A]_2$. Similarly, $[fA_0]_2 = u[hA_0]_2 = u[A_0]_2$, the latter since $hA_0 \subset [A]_2A_0 \subset [A_0]_2$ and

$$A_0 = AA_0 \subset [A]_2A_0 \subset [hA]_2A_0 \subset [hA_0]_2.$$

Thus $[fA]_2 \ominus [fA_0]_2 = u([A]_2 \ominus [A_0]_2) = u[\mathcal{D}]_2$. Similar, but slightly more cumbersome, arguments work in the partial isometry case.

Suppose that A is maximal subdiagonal and the right wandering subspace of $[fA]_2$ has a separating and cyclic vector, then by Corollary 1.2, $[fA]_2 = u[A]_2$ for a unitary u. We may write $f = uh$ for $h \in [A]_2$. Clearly $u \in M \cap [fA]_2$, and

$$[hA]_2 = u^*u[hA]_2 = u^*[fA]_2 = [A]_2.$$

This completes the proof. ∎

Recall that a *wandering vector* is a vector $f \in L^2(M)$ with $f \perp [fA_0]_2$. Examples include the partial isometries u_i in the previous section. Note that if f is a wandering vector, then $[f\mathcal{D}]_2 \perp [fA_0]_2$, and so one easily sees that $[f\mathcal{D}]_2$ is the right wandering subspace of $[fA]_2$. Moreover, $[([fA]_2 \ominus [fA_0]_2)A]_2 = [[f\mathcal{D}]_2A]_2 = [fA]_2$, so that $[fA]_2$ is type 1. In this case, to say that the right wandering subspace of $[fA]_2$ has a cyclic separating vector is equivalent to saying that $[f\mathcal{D}]_2$ has such a vector. By Exercise 9.6.2 of [15], this is also equivalent to saying that f is separating; that is, the map $d \mapsto fd$ on $\mathcal{D}$ is one-to-one.

PROPOSITION 3.2. *Suppose that A is a tracial algebra. If $0 \neq f \in L^2(M)$, consider the following conditions:*
(i) *$f \in M^{-1}$;*
(ii) *f is a wandering vector, and the map $d \mapsto fd$ on $\mathcal{D}$ is one-to-one;*
(iii) *the right wandering subspace of $[fA]_2$ has a nonzero separating and cyclic vector for the right action of $\mathcal{D}$.*
Then any one of conditions (i) *or* (ii) *imply* (iii)*.*

Proof. Suppose that (ii) holds. In this case, $[f\mathcal{D}]_2$ is the right wandering subspace of $[fA]_2$, as remarked above, and so (iii) holds.

If (i) holds, then $[fA]_2 = f[A]_2$ and $[fA_0]_2 = f[A_0]_2$. Thus as purely algebraic $\mathcal{D}$-modules,

$$[fA]_2 \ominus [fA_0]_2 \cong \frac{[fA]_2}{[fA_0]_2} \cong \frac{[A]_2}{[A_0]_2} \cong [\mathcal{D}]_2.$$

The latter module has a nonzero separating and cyclic vector. Since these isomorphisms are also continuous, so does $[fA]_2 \ominus [fA_0]_2$. So (iii) holds. ∎

THEOREM 3.3 (Beurling-Nevanlinna factorization for tracial algebras). *For a tracial subalgebra A of M, the following are equivalent:*
(i) *A is maximal subdiagonal.*
(ii) *Every nonzero $f \in L^2(M)$ satisfying either* (i) *or* (ii) *in* 3.2 *is* BN*-factorizable.*
(iii) *If $f \in M^{-1} \cap M_+$, or if f is a wandering vector, then f is partially* BN*-factorizable.*
(iv) *Every $f \in L^2(M)$ satisfying* (iii) *in Proposition* 3.2 *is* BN*-factorizable.*
Indeed, every $f \in M^{-1} \cap M_+$ is BN*-factorizable, if and only if A_∞ is maximal subdiagonal. Also, A has the unique normal state extension property if and only if every wandering vector is partially* BN*-factorizable.*

Proof. The proof of Lemma 2.4.3 in [27] demonstrates one direction of the penultimate "if and only if", and we leave the converse as an exercise.

For the equivalences between (i), (ii), and (iv), by Proposition 3.2 and Lemma 3.1 it remains to show that (ii) implies (i). If (ii) holds, then by the last statement A_∞ is maximal subdiagonal. Hence A satisfies L^2-density. If therefore we can show that A also satisfies the unique normal state extension property we are done (as in the proof of Corollary 2.5). Suppose that $g \in L^1(M)_+$ satisfies $\tau(gA_0) = 0$. We need to show that $g \in L^1(\mathcal{D})_+$. Since $\tau((g+1)A_0) = 0$, we can replace g with $g+1$ if necessary. Let $f = g^{1/2} \in L^2(M)$. Then $f \perp [fA_0]_2$. Obviously, f is a cyclic vector for $[f\mathcal{D}]_2$. If $fd = 0$ then $d = 0$ since $\tau(d^*d) \leqslant \tau(d^*f^*fd)$. So f is separating. By hypothesis, $f = uh$ for an outer $h \in [A]_2$ and some unitary u in M. Since $h = u^*f \perp [hA_0]_2 = [A_0]_2$, and $h \in [A]_2$, it follows that $h \in [\mathcal{D}]_2$. Thus $g \in [\mathcal{D}]_1 = L^1(\mathcal{D})$.

An easy modification of the last argument gives one direction of the last assertion of the theorem. For the other direction, suppose that A has the unique normal state extension property, and $f \in L^2(M) \ominus [fA_0]_2$. By the remarks preceding Proposition 3.2, $[fA]_2$ is a type 1 invariant subspace. Following the first half of the proof of Theorem 2.1 (v), we see that $f^*f \subset L^1(\mathcal{D})$, and so $[fA]_2$ has a decomposition of the type considered in Theorem 2.1 (here $Z = (0)$). Since $Y = [f\mathcal{D}]_2$ is cyclic, by Proposition 2.2 there is a partial isometry u with $[fA]_2 = u[A]_2$ and $u^*u \in \mathcal{D}$. Clearly $u \in [fA]_2 \cap M$. We may write $f = uh$ with $h \in (u^*u)[A]_2 \subset [A]_2$. Also, $u[hA]_2 = [fA]_2 = u[A]_2$, so that $[hA]_2 = (u^*u)[hA]_2 = (u^*u)[A]_2$. Thus f is partially BN-factorizable.

The equivalence of (i) with (iii) follows from the last two if and only if's of our theorem's statement, the fact that if A_∞ is maximal subdiagonal then A satisfies L^2-density, and the fact that $f \in M^{-1}$ is partially BN-factorizable if and only if it is BN-factorizable. Indeed if f is invertible, with partial BN-factorization $f = uh$, then $h = u^*f$ is bounded, and so both u and h are also invertible since we are in a finite von Neumann algebra. Hence u is a unitary. ∎

At the end of Section 4 we give a very general "inner-outer factorization" result.

4. THE CASE OF L^p FOR $p \neq 2$

To discuss the L^p-version of some of the results above, we use the "column L^p-sum" from [14]. Suppose that $1 \leqslant p < \infty$ and that $\{X_i : i \in I\}$ is a collection of closed subspaces of $L^p(M)$. We then define the *external* column L^p-sum $\bigoplus_i^{\mathrm{col}} X_i$ to be the closure of the restricted algebraic sum in the norm $\|(x_i)\|_p \stackrel{\mathrm{def}}{=} \tau\Big(\Big(\sum_i x_i^* x_i\Big)^{p/2}\Big)^{1/p}$. That this is a norm for $1 \leqslant p < \infty$ is verified in [14]. If X is a subspace of $L^p(M)$, and if $\{X_i : i \in I\}$ is a collection of subspaces of X, which together densely span X, with the property that $X_i^* X_j = \{0\}$ if $i \neq j$, then we say that X is the *internal* column L^p-sum $\bigoplus_i^{\mathrm{col}} X_i$. Note that if J is a finite subset of I, and if $x_i \in X_i$ for all $i \in J$, then we have that

$$\tau\Big(\Big|\sum_{i\in J} x_i\Big|^p\Big)^{1/p} = \tau\Big(\Big(\Big|\sum_{i\in J} x_i\Big|^2\Big)^{p/2}\Big)^{1/p} = \tau\Big(\Big(\sum_{i\in J} x_i^* x_i\Big)^{p/2}\Big)^{1/p}.$$

This shows that X is then isometrically isomorphic to the external column L^p-sum $\bigoplus_i^{\rm col} X_i$. Since the projections onto the summands are clearly contractive, it follows by routine arguments (or by Lemma 2.4 of [14]) that if $(x_i) \in \bigoplus_i^{\rm col} X_i$, then the net $\left(\sum_{j\in J} x_j\right)$, indexed by the finite subsets J of I, converges in norm to (x_i).

We will need a couple of technical results:

LEMMA 4.1. *If $0 < p < \infty$, and if e is a projection in M, then $v \in L^p(M)_+$ satisfies $\tau((d^*vd)^p) = \tau((d^*ed)^p)$ for all $d \in M$, if and only if $v = e$.*

Proof. We regard v as an unbounded positive operator affiliated with M, then $v^p \in L^1(M)_+$. Choosing $d = e^\perp$ shows that $e^\perp v e^\perp = 0$. It is easy to see, as in the bounded case, that these imply that $ve^\perp = e^\perp v = 0$ too. So $v = eve$. Replacing M by eMe, we can assume that $e = 1$. If E is a projection in the minimal von Neumann algebra M_0 generated by v (see e.g. p. 349 in [15]), then $(EvE)^p = Ev^p$, and so by hypothesis $\tau(E(v^p - 1)) = 0$. If E is the spectral projection for v corresponding to $[0,1]$, then $Ev^p \leqslant E$ and so we have $E = Ev^p$ since τ is faithful. On the other hand, $E^\perp v^p \geqslant E^\perp$ so that $E^\perp = E^\perp v^p$. Thus $v^p = 1$. ∎

LEMMA 4.2. *Let A be maximal subdiagonal and let K be an A-invariant subspace of $L^p(M)$, for $2 \leqslant p \leqslant \infty$. Then $[K]_1 \cap L^p(M) = [K]_2 \cap L^p(M) = [K]_p$. (The last symbol, if $p = \infty$, is always taken in this paper to mean the weak* closure.)*

Proof. In this proof we will denote polars taken with respect to the dual pair $(L^1(M), M)$ by $\perp$, and polars taken with respect to the dual pair $(L^p(M), L^q(M))$ by $\circ$. Here $1/p + 1/q = 1$. We assume that $p < \infty$, and leave the case $p = \infty$ to the reader. Note that if K is a right A-invariant subspace of $L^p(M)$ then K° is a closed right A^*-invariant subspace of $L^q(M)$. To see this note that we clearly have

$$0 = \tau(y^*(xa)) = \tau((ya^*)^*x), \quad x \in K, a \in A, y \in K^\circ,$$

since $xa \in K$. But then $ya^* \in K^\circ$ as required.

Since K° is a closed A^*-invariant subspace of $L^q(M)$, by the result of Saito mentioned in our introduction $K^\circ \cap M$ is norm dense in K°. Regarding K as a subspace of $L^1(M)$ we have $K^\perp = K^\circ \cap M$. Hence by the bipolar theorem $(K^\circ \cap M)_\perp = [K]_1$. Clearly $(K^\circ \cap M)_\perp \cap L^p(M) = (K^\circ \cap M)^\circ$. Hence

$$[K]_1 \cap L^p(M) = (K^\circ \cap M)_\perp \cap L^p(M) = (K^\circ \cap M)^\circ = (K^\circ)^\circ = [K]_p.$$

The other assertion now follows from the fact that $K \subset [K]_2 \cap L^p(M) \subset [K]_1 \cap L^p(M)$. If $p = \infty$ it is easy to check that $[K]_2 \cap M$ is weak* closed. So in all cases, $[K]_p \subset [K]_2 \cap L^p(M)$, which gives the result. ∎

COROLLARY 4.3. *If A is maximal subdiagonal then for any $1 \leqslant p \leqslant q \leqslant \infty$ there is a lattice isomorphism between the closed (weak*-closed, if $q = \infty$) right A-invariant subspaces of $L^p(M)$ and $L^q(M)$.*

Proof. We may take p or q to be 2. The isomorphism (respectively its inverse) of course is the map taking K to its closure (respectively intersection with the appropriate L^p space). This follows easily from the aforementioned result of Saito, and the lemma (or a tiny variant of it). ∎

DEFINITION 4.4. We define the right wandering subspace of K, if $1 \leqslant p \leqslant 2$ (respectively $p \geqslant 2$) to be the L^p-closure of the right wandering subspace of $K \cap L^2(M)$ (respectively to be the intersection of $L^p(M)$ with the right wandering subspace of $[K]_2$).

THEOREM 4.5. *Let A be a maximal subdiagonal subalgebra of M, and suppose that K is a closed A-invariant subspace of $L^p(M)$, for $1 \leqslant p \leqslant \infty$. (For $p = \infty$ we assume that K is weak* closed.)*

(i) *K may be written as a column L^p-sum $K = Z \oplus^{\text{col}} \left(\bigoplus_i^{\text{col}} u_i H^p\right)$, where Z is a closed (indeed weak* closed if $p = \infty$) type 2 subspace of $L^p(M)$, and where u_i are partial isometries in $M \cap K$ with $u_j^* u_i = 0$ if $i \neq j$, and with $u_i^* u_i \in \mathcal{D}$. Moreover, for each i, $u_i^* Z = (0)$, left multiplication by the $u_i u_i^*$ are contractive projections from K onto the summands $u_i[A]_p$, and left multiplication by $1 - \sum_i u_i u_i^*$ is a contractive projection from K onto Z.*

(ii) *The wandering quotient $K/[KA_0]_p$ is isometrically $\mathcal{D}$-isomorphic to the right wandering subspace of K; and the latter also equals $\bigoplus_i^{\text{col}} u_i[\mathcal{D}]_p$, where u_i are from* (i). *(Here $[\cdot]_\infty$ is the weak* closure as usual.)*

(iii) *K is type 1 if and only if $K \cap L^2(M)$ (respectively $[K]_2$) is type 1, and if and only if $Z = (0)$ in* (i). *If $1 \leqslant p \leqslant 2$ (respectively $p \geqslant 2$), then K is type 2 if and only if $K \cap L^2(M)$ (respectively $[K]_2$) is type 2, and if and only if $K = Z$.*

Proof. (i) First suppose that $p \leqslant 2$. By Saito's theorem mentioned in the introduction, K is the L^p-closure of $K \cap L^2(M)$. Theorem 2.1 gives a decomposition $K \cap L^2(M) = Z' \oplus [YA]_2$, where Z' is type 2, and Y is the right wandering subspace of $K \cap L^2(M)$. Let Z be the L^p-closure of Z'. Note that Z is type 2. Indeed $[ZA_0]_p = [[Z']_p A_0]_p = [Z' A_0]_p$, as is easy to check. Thus

$$[ZA_0]_p = [Z'A_0]_p = [[Z'A_0]_2]_p = [Z']_p = Z.$$

We leave it as an exercise that $Z \cap L^2(M) = Z'$.

Since $z^* y = 0$ for $y \in Z'$, $y \in YA$, it follows using (4.8) in [14] that $(0) = Z^*[YA]_p$. From this it is easy to see that $K = Z \oplus^{\text{col}} K_1$ where $K_1 = [YA]_p$. By Corollary 2.3, $[YA]_2 = \bigoplus_i^{\text{col}} u_i[A]_2$, for u_i as above. Thus the L^p-closure of $\sum_i u_i A$ is all of K_1. On the other hand, $(u_j[A]_p)^*(u_i[A]_p) = (0)$ if $i \neq j$. So $K_1 = \bigoplus_i^{\text{col}} u_i[A]_p$.

Now suppose that $2 \leqslant p \leqslant \infty$. We embed K into $L^2(M)$; by the L^2 result $[K]_2 = Z' \oplus^{\rm col} \left(\bigoplus_i^{\rm col} u_i H^2\right)$ as usual, where Z' is type 2. We assume that $p < \infty$ in the arguments below, however the case $p = \infty$ is a simple variant. Let $K_1 = \left(\bigoplus_i^{\rm col} u_i H^2\right) \cap L^p(M)$. Note that $\left[\bigoplus_i^{\rm col} u_i H^p\right]_2 = \bigoplus_i^{\rm col} u_i H^2$, so that by Lemma 4.2 we have $K_1 = \bigoplus_i^{\rm col} u_i H^p$.

Define $Z = Z' \cap L^p(M)$, which is also easily seen to be closed. Since Z' is A-invariant, by Saito's result Z is an A-invariant subspace of $L^p(M)$ which is L^2-dense in Z'. On setting $X_0 = Z \oplus K_1$, it follows that X_0 is L^2-dense in $[K]_2$. Also, X_0 is an invariant subspace of $L^p(M)$, which is easily seen to be closed. Clearly $Z^* K_1 = (0)$. By Lemma 4.2 we have

$$K = [K]_2 \cap L^p(M) = [X_0]_2 \cap L^p(M) = X_0 = Z \oplus^{\rm col} K_1.$$

To see that Z is type 2, by Lemma 4.2, and the fact that $[Z]_2 = Z'$, we have

$$Z = Z' \cap L^p(M) = [Z' A_0]_2 \cap L^p(M) \subset [Z A_0]_2 \cap L^p(M) = [Z A_0]_p.$$

Finally, suppose $1 \leqslant p \leqslant \infty$. Since left multiplication by $u_i u_i^*$ annihilates Z' and $u_j[A]_p$ if $j \neq i$, left multiplication by the $u_i u_i^*$ are contractive projections from K onto the summands $u_i[A]_p$, and left multiplication by $1 - \sum_i u_i u_i^*$ is a contractive projection onto Z.

(ii) The fact that the right wandering subspace equals $\bigoplus_i^{\rm col} u_i [\mathcal{D}]_p$ follows from a slight modification of the argument in (i) that $K_1 = \bigoplus_i^{\rm col} u_i H^p$. We also need the fact that $L^p(M) \cap [\mathcal{D}]_1 = [\mathcal{D}]_p$.

By e.g. [18], Φ induces a contractive "expectation" on every $L^p(M)$. Assume $p \geqslant 2$. Then $\Phi(x)^* \Phi(x) \leqslant \Phi(x^* x)$ for $x \in L^p(M)$, as may be seen by routine continuity arguments. Again we assume $p < \infty$, and leave the variation of the argument in the case $p = \infty$ to the reader. Define a map $\theta : K = K_1 \oplus^{\rm col} K_2 \to K$ by $\theta(w) = \sum_i u_i \Phi(x_i)$ if $w = \sum_i u_i x_i + k_2$ for $x_i \in H^p$ and $k_2 \in K_2$. It is easy to see that $\theta(w)$ equals $\sum_i u_i \Phi(u_i^* w)$, which shows that θ is well defined (and weak* continuous if $p = \infty$). Indeed since $u_i^* u_i \in \mathcal{D}$, we have

$$\tau\Big(\Big(\sum_i \Phi(x_i)^* u_i^* u_i \Phi(x_i)\Big)^{p/2}\Big) \leqslant \tau\Big(\Phi\Big(\sum_i x_i^* u_i^* u_i x_i\Big)^{p/2}\Big) \leqslant \tau\Big(\Big(\sum_i x_i^* u_i^* u_i x_i\Big)^{p/2}\Big),$$

which shows that $\|\theta(w)\|_p^p \leqslant \|w\|_p^p$. (In the first inequality here, we have used the fact that for elements of the type that we are considering, $0 \leqslant S \leqslant T$ implies that $\|S\|_r \leqslant \|T\|_r$, which follows from e.g. Lemma 2.5 (iii), Corollary 2.8 of [7]. The second inequality follows from the $L^{p/2}$-contractivity of Φ, see e.g. 3.10 in [18].) Thus θ is a contractive projection onto its range, and hence induces an isometric $\mathcal{D}$-module map from $K/\mathrm{Ker}(\theta)$ onto $\bigoplus_i^{\rm col} u_i [\mathcal{D}]_p$. Since $x_i A_0 \in [A_0]_p \subset$

$\mathrm{Ker}(\Phi)$ if $x_i \in H^p$, it is clear that $\mathrm{Ker}(\theta)$ contains KA_0, and hence contains $[KA_0]_p$. Conversely if $\theta\Big(\sum_i u_i x_i + k_2\Big) = 0$ then $u_i^* u_i \Phi(x_i) = 0$ for every i. Thus $u_i^* u_i x_i \in \mathrm{Ker}(\Phi) \cap H^p = [A_0]_p$. Hence $u_i x_i \in u_i [A_0]_p$, and so $\sum_i u_i x_i + k_2 \in [KA_0]_p$. So $\mathrm{Ker}(\theta) = [KA_0]_p$.

If $1 \leqslant p < 2$, we follow the same argument with the following modification. Suppose that $1/p + 1/q = 1$, and that F is a finite set of indices. Define θ_F on $L^q(M)$ taking $w \mapsto \sum_{i \in F} u_i \Phi(u_i^* w)$. Since $\sum_F u_i u_i^*$ is a projection, by the arguments in the last paragraph it is easy to see that $\tau\Big(\Big(\sum_F \Phi(u_i^* w)^* u_i^* u_i \Phi(u_i^* w)\Big)^{q/2}\Big)$ is dominated by

$$\tau\Big(\Phi\Big(w^*\Big(\sum_F u_i u_i^*\Big) w\Big)^{q/2}\Big) \leqslant \tau(\Phi(w^* w)^{q/2}) \leqslant \tau((w^* w)^{q/2}) = \tau(|w|^q),$$

which shows that θ_F is a contraction. For $g \in L^p(M), f \in L^q(M)$ we have that

$$\tau(g^* \theta_F(f)) = \sum_{i \in F} \tau(g^* u_i \Phi(u_i^* f)) = \sum_{i \in F} \tau(\Phi(g^* u_i) u_i^* f) = \tau\Big(\Big(\sum_{i \in F} u_i \Phi(u_i^* g)\Big)^* f\Big),$$

where the middle equality follows by e.g. 3.10 of [18]. This shows that the contraction $(\theta_F)_* \in B(L^p(M))$ is precisely the map $w \mapsto \sum_{i \in F} u_i \Phi(u_i^* w)$ on $L^p(M)$. Since this holds for every finite subset F, this implies that the map θ above is a densely defined contraction on K, and thus extends continuously to K. By continuity it follows that this extension has precisely the same formula as before, and now the earlier argument works.

(iii) If $K \cap L^2(M)$ (respectively $[K]_2$) is type 1 then it is obvious from the proof of (i) that $Z' = Z = (0)$. Similarly, if $K \cap L^2(M)$ (respectively $[K]_2$) is type 2 then it is obvious that $K = Z$. It is trivial that if $K = Z$ then K is type 2. Conversely, if K is type 2 then the wandering quotient is (0). Identifying this with the subspace of K described in (ii), all the u_i are zero. Thus $K = Z$. Also, if $p > 2$ then $[K]_2 = [Z]_2 = Z'$ which is type 2; and a similar argument works if $p < 2$. Since $\Big[\Big(\bigoplus_i^{\mathrm{col}} u_i [\mathcal{D}]_p\Big) A\Big]_p = \bigoplus_i^{\mathrm{col}} u_i H^p$, we have that $Z = (0)$ if and only if $\bigoplus_i^{\mathrm{col}} u_i [\mathcal{D}]_p$ generates K. By (ii) this happens if and only if Z is type 1.

If $1 \leqslant p \leqslant 2$, suppose that $K = \bigoplus_i^{\mathrm{col}} u_i H^p$ for partial isometries u_i satisfying the relations in (i). Then $\Big[\bigoplus_i^{\mathrm{col}} u_i H^2\Big]_p = K$, so by Lemma 4.2 we have $K \cap L^2(M) \subset \Big[\bigoplus_i^{\mathrm{col}} u_i H^2\Big]_1 \cap L^2(M) = \bigoplus_i^{\mathrm{col}} u_i H^2$. So $\bigoplus_i^{\mathrm{col}} u_i [A]_2 = K \cap L^2(M)$, which is type 1 by Corollary 1.2.

If $2 \leqslant p \leqslant \infty$ and $K = \bigoplus_i^{\text{col}} u_i H^p$ for partial isometries u_i satisfying the relations in (i), then it is easy to argue that $\bigoplus_i^{\text{col}} u_i [A]_2 = [K]_2$. So $[K]_2$ is type 1. (In the case $p = \infty$, note that $\sum_i u_i A \subset [K]_2 \cap M = K$ by Lemma 4.2, so that $\Big[\overline{\sum_i u_i A}^{\text{weak}^*}\Big]_2 \subset [K]_2$.) ∎

We have now proved the existence of the type decomposition in Theorem 1.1 (i). The uniqueness of this type decomposition follows from the following:

COROLLARY 4.6. *If K is a subspace of $L^p(M)$ of the form $K = K_1 \oplus^{\text{col}} K_2$ where K_1 is type 1 and K_2 is type 2, and if $1 \leqslant p \leqslant 2$, then K_1 (respectively K_2) is the L^p closure of the type 1 (respectively type 2) part of $K \cap L^2(M)$. If $2 \leqslant p \leqslant \infty$ then K_1 (respectively K_2) is the intersection of $L^p(M)$ with the type 1 (respectively type 2) part of $[K]_2$.*

Proof. If $1 \leqslant p \leqslant 2$, then clearly $(K_1 \cap L^2(M)) + (K_2 \cap L^2(M)) \subset K \cap L^2(M)$. On the other hand, if $x \in K \cap L^2(M)$, we can write $x = k_1 + k_2$, for $k_i \in K_i$. Since K_1 is type 1, we can write $K_1 = \bigoplus_i^{\text{col}} u_i [A]_p$ for some u_i as above. So $k_1 = \sum_i u_i u_i^* k_1$. Since $u_i \in K_1$ we have $u_i^* k_2 = 0$. Thus since $x = \Big(1 - \sum_i u_i u_i^*\Big) x + \sum_i u_i u_i^* x$, we have $k_1 = \sum_i u_i u_i^* x$, and so $k_2 = \Big(1 - \sum_i u_i u_i^*\Big) x$. This forces $k_1 \in \Big(\sum_i u_i u_i^*\Big) L^2(M) \subset L^2(M)$, similarly $k_2 \in L^2(M)$. Thus we have $K \cap L^2(M) = (K_1 \cap L^2(M)) \oplus^{\text{col}} (K_2 \cap L^2(M))$. Since $K_1 \cap L^2(M)$ is type 1 and $K_2 \cap L^2(M)$ is type 2, we have by the uniqueness of decomposition in the L^2 case, that $K_1 \cap L^2(M) = [(K \cap L^2(M) \ominus [(K \cap L^2(M)) A_0]_2) A]_2$. By Saito's theorem K_1 is the L^p closure of $K_1 \cap L^2(M)$. Since $K_2 = \{z \in K : z^* K_1 = (0)\}$, it is uniquely determined, and so the assertion about K_2 follows by the proof of the previous theorem.

If $2 \leqslant p$, then $K = K_1 \oplus^{\text{col}} K_2$ implies easily that $[K]_2 = [K_1]_2 \oplus^{\text{col}} [K_2]_2$. Since $[K_1]_2$ and $[K_2]_2$ are types 1 and 2 respectively (by (iii) of the theorem), the result follows from Lemma 4.2. ∎

PROPOSITION 4.7. *Suppose that K is a closed (indeed weak* closed, if $p = \infty$) A-invariant subspace of $L^p(M)$, and that A is maximal subdiagonal. The following are equivalent:*

(i) *$K = uH^p$ for a unitary $u \in M$.*

(ii) *The right wandering subspace of K (or equivalently, the right wandering quotient) has a cyclic and separating vector.*

(iii) *The right wandering subspace of $K \cap L^2(M)$ (respectively $[K]_2$) in $L^2(M)$ has a cyclic and separating vector, when $1 \leqslant p \leqslant 2$ (respectively $2 \leqslant p \leqslant \infty$).*

Proof. If $K = u[A]_p$ and $p < 2$, then $K \cap L^2(M) = uH^2$ by an argument in the proof of (iii) of the theorem. It follows that the right wandering subspace of $K \cap L^2(M)$ is $u[\mathcal{D}]_2$. From this it is clear that the right wandering subspace of K is $u[\mathcal{D}]_p$. These both have separating cyclic vectors. If $p > 2$ then this argument is easier (one also uses the simple fact that $L^p(M) \cap L^2(\mathcal{D}) = L^p(\mathcal{D})$).

To prove that (iii) implies (i), note that by Corollary 1.2 in the case $p = 2$, we have that $K \cap L^2(M)$ (respectively $[K]_2$) equals $u[A]_2$ for a unitary $u \in M$. Thus $K = u[A]_p$ (if $p < 2$ use Saito's theorem mentioned in the introduction, whereas if $p > 2$ use Lemma 4.2).

If (ii) holds, then by adapting of an argument from p. 13 in [14] there exists an isometric $\mathcal{D}$-module isomorphism $\psi : L^p(\mathcal{D}) \to W$. (This is a variant of the well known fact that a W^*-module over $\mathcal{D}$ with a cyclic separating vector is unitarily isomorphic to $\mathcal{D}$; indeed the latter is the case $p = \infty$ of the assertion under discussion.) Set $\psi(1) = u$ and set $v = u^*u \in L^{p/2}(\mathcal{D})$. If $p < \infty$ we have

$$\tau((d^*d)^{p/2}) = \|\psi(d)\|_p^p = \|ud\|_p^p = \tau((d^*vd)^{p/2}), \quad d \in \mathcal{D}.$$

By Lemma 4.1 with p replaced by $p/2$, the last identity forces $v = 1$, so that u is unitary (since we are in a finite von Neumann algebra), and $W = u[\mathcal{D}]_p$. Since $K_2^*W = (0)$, we have $K_2 = (0)$, and so $K = uH^p$. A similar argument works if $p = \infty$; here in place of Lemma 4.1 one may use the fact that an isometric $\mathcal{D}$-module isomorphism between C^*-modules is unitary (see e.g. 8.1.5 in [5]). ∎

The following is the analogue of Beurling's characterization of weak* closed ideals of $H^\infty(\mathbb{D})$. The last part uses also the main theorem from [21], which is valid in L^2, but which transfers easily to L^∞ using results above.

COROLLARY 4.8. *If A is maximal subdiagonal, then the type 1 weak* closed right ideals of A are precisely those right ideals of the form $\bigoplus_i^{\mathrm{col}} u_i A$, for partial isometries $u_i \in A$ with mutually orthogonal ranges and $|u_i| \in \mathcal{D}$. If the center of $\mathcal{D}$ is contained in the center of $\mathcal{M}$, then one needs only one partial isometry here.*

REMARK. As is proved in [21] in the case $p = 2$, a closed A-invariant subspace K of $L^p(M)$ has type 1 if and only if $[WM]_p = [KM]_p$ where W is the right wandering subspace. The one direction of this is obvious. For the other note that if K is not type 1 then by our main theorem there exists $\eta \in K$ with $\eta^* W = 0$. Thus $\eta^*[WM]_p = 0$ by e.g. (4.8) in [14], which shows that $[WM]_p \neq [KM]_p$.

COROLLARY 4.9. *Suppose that A is maximal subdiagonal, and $f \in L^p(M)$, for $1 \leqslant p < \infty$. If $[fA]_p$ is a type 1 invariant subspace, then $f = \sum_i u_i h_i$ (a norm convergent sum), where u_i are partial isometries in $[fA]_p \cap M$ with $u_i^* u_i \in \mathcal{D}$, and $u_j^* u_i = 0$ if $i \neq j$, and $h_i \in [A]_p$ with $(u_i^* u_i) h_i = h_i$ and $u_i^* u_i \in [h_i A]_p$.*

Proof. By the results above, $[fA]_p = \bigoplus_i^{\mathrm{col}} u_i [A]_p$, where u_i are partial isometries in $M \cap [fA]_p$ of the correct form. Moreover, left multiplication by the $u_i u_i^*$ are contractive projections from $[fA]_p$ onto the summands $u_i[A]_p$. Thus $f = \sum_i u_i h_i$ for $h_i \in (u_i^* u_i)[A]_p$. We have

$$u_i^* u_i \in (u_i^* u_i)[A]_p = u_i^*[fA]_p = (u_i^* u_i)[h_i A]_p = [h_i A]_p.$$

This completes the proof. ∎

A similar result with almost identical proof holds if $p = \infty$, interpreting closures and convergence in the weak* topology.

The last result is a generalized "inner-outer" factorization. The sum of products can be replaced by a single product if the wandering subspace has a cyclic vector. For example, if $f \in L^p(M)$ and if $K = [fA]_p$ has a wandering subspace which has a cyclic and separating vector, then by Proposition 4.7 we have $K = uH^p$ for a unitary $u \in M$. Thus $f = uh$ for $h \in H^p$, and as in the last lines of the proof of Lemma 3.1, this implies that $[hA]_p = [A]_p$. We take the latter condition as the definition of h being *outer*, as in the classical case.

CLOSING REMARK. In Sections 2 and 3, and in [3], we have been able to generalize almost the entire circle of equivalent characterizations from [28] of (at least weak* closed) weak* Dirichlet algebras. There are two items remaining in that list. The first is known as the Gleason-Whitney theorem, and we have recently been able to demonstrate the equivalence of this condition with the others, at least under some restriction on $\mathcal{D}$ [4]. The second item is the condition that A_∞ is maximal subdiagonal. Unfortunately we were unable to follow the proof given in [28] for the latter equivalence, nor have we been able to find this equivalence mentioned elsewhere in the literature (without additional hypotheses). One sufficient condition under which A_∞ being maximal subdiagonal implies that A is maximal subdiagonal, is that the extension of Φ to $L^1(M)$ be continuous with respect to the topology of *convergence in measure* (see [30], [7], [29], [22] for details). However the latter does not hold for many interesting subdiagonal algebras.

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DAVID P. BLECHER, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TX 77204-3008, USA
E-mail address: dblecher@math.uh.edu

LOUIS E. LABUSCHAGNE, DEPARTMENT OF MATHEMATICAL SCIENCES, P.O. BOX 392, 0003 UNISA, SOUTH AFRICA
E-mail address: labusle@unisa.ac.za

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