J. OPERATOR THEORY
59:1(2008), 53–68

CONTRACTIVE PERTURBATIONS IN C^*-ALGEBRAS

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Communicated by Kenneth R. Davidson

ABSTRACT. We characterize various objects in a C^*-algebra $\mathcal{A}$ in terms of the size and the location of the contractive perturbations. We prove that if $\mathcal{S}$ is a precompact subset of the unit ball of $\mathcal{A}$, there exists a faithful representation π of $\mathcal{A}$ such that $\pi(a)$ is compact for each $a \in \mathcal{S}$ if and only if $\mathrm{cp}^2(\lambda\mathcal{S})$ is compact, for each $0 < \lambda < 1$. We provide a geometric characterization of the hereditary C^*-subalgebras and the essential ideals of $\mathcal{A}$, as well as of any separable C^*-algebra within its multiplier algebra. We present examples showing that the notion of contractive perturbations is not appropriate for the description of compact operators on a general Banach space.

KEYWORDS: *C^*-algebra, compact operator, contractive perturbation, hereditary C^*-subalgebra, essential ideal, face.*

MSC (2000): Primary 46L05; Secondary 47C15, 47L10.

1. INTRODUCTION

The geometric structure of the unit ball of a C^*-algebra has been an object of interest from the beginning of the theory of operator algebras. In his study of isometries between C^*-algebras R. Kadison characterized the extreme points of the unit ball of a unital C^*-algebra [17]. From his result it follows that the extreme points of the unit ball of $\mathcal{B}(\mathcal{H})$ are the isometries and the co-isometries.

In [4], the n-th contractive perturbations of a subset $\mathcal{S}$ of the unit ball of a C^*-algebra $\mathcal{A}$, denoted $\mathrm{cp}^n(\mathcal{S})$, were introduced. An element a in the unit ball of $\mathcal{A}$ is an extreme point precisely when $\mathrm{cp}^2(a) \stackrel{\mathrm{def}}{=} \mathrm{cp}^2(\{a\})$ is the whole unit ball. On the other hand, if $\mathcal{H}$ is a Hilbert space, a bounded contraction A on $\mathcal{H}$ is a compact (respectively finite rank) operator if and only if $\mathrm{cp}^2(A)$ is a compact (respectively finite dimensional) subset of $\mathcal{B}(\mathcal{H})$. The main result of [4] is that if a is an element of the unit ball of a C^*-algebra $\mathcal{A}$ then the image of a under some faithful representation of $\mathcal{A}$ is compact if and only if $\mathrm{cp}^2(a)$ is compact.

In [5] and [6] the contractive perturbations are used in the study of compact and finite rank operators in a nest algebra and in the algebra of adjointable operators on a Hilbert module, and in [18] in the study of the facial structure of the unit ball of an operator algebra. An analogous approach is used in [3] and [13] to give a geometric characterization of partial isometries in a C^*-algebra and of tripotents in a JB*-triple. The set $\mathrm{cp}^1(\mathcal{S})$ is also related to the notion of M-orthogonality introduced and studied in [12].

In this work we extend the study of contractive perturbations in several directions. We are interested in characterizing various objects in C^*-algebras in terms of the size and the location of the contractive perturbations. In Section 3 we examine the joint second contractive perturbations of a precompact subset $\mathcal{S}$ of the unit ball of a C^*-algebra $\mathcal{A}$ and show that there exists a faithful representation π of $\mathcal{A}$ such that $\pi(a)$ is compact for each $a \in \mathcal{S}$ if and only if $\mathrm{cp}^2(\lambda\mathcal{S})$ is compact, for each $0 < \lambda < 1$. This result extends Theorem 2.2 of [4]. As a corollary, we obtain a characterization of compact elementary operators.

In Section 4 we provide a geometric characterization of the hereditary C^*-subalgebras and the essential ideals of a C^*-algebra $\mathcal{A}$. We also show, using a result of L. Brown, that if $\mathcal{A}$ is a separable C^*-algebra and $\mathcal{M}(\mathcal{A})$ its multiplier algebra, then an element a of the unit ball of $\mathcal{M}(\mathcal{A})$ belongs to $\mathcal{A}$ if and only if $\mathrm{cp}^2(a)$ is separable.

In Section 5 we show that a compact face of the unit ball of a C^*-algebra is necessarily finite dimensional. We also determine the affine hull of the smallest face containing a fixed element of the unit ball of a C^*-algebra, in the case it is finite dimensional.

It is natural to ask if the notion of contractive perturbations may be used to describe compact operators on a general Banach space $\mathcal{X}$. In Section 6 we present some examples which show that this cannot be achieved. We show that if $\mathcal{X}$ is any of the spaces l^p, $1 \leqslant p < +\infty$, $p \neq 2$, c_0 or $C(K)$, then there exists a rank one contraction A acting on $\mathcal{X}$ such that $\mathrm{cp}^2(A)$ is not compact. We also show that if A is a compact contraction on c_0, then $\mathrm{cp}^2(A)$ is compact in the weak operator topology, but not vice versa.

We introduce some notation. If $\mathcal{X}$ is a Banach space we will denote by $\mathcal{X}_1$ the closed unit ball of $\mathcal{X}$. If $\mathcal{S} \subseteq \mathcal{X}$, we denote by $[\mathcal{S}]$ the linear span of $\mathcal{S}$ and by $\overline{\mathcal{S}}$ the closure of $\mathcal{S}$.

If $\mathcal{H}$ is a Hilbert space, $\mathcal{B}(\mathcal{H})$ denotes the C^*-algebra of all bounded linear operators on $\mathcal{H}$ and $\mathcal{K}(\mathcal{H})$ the closed ideal of all compact operators. For $A \in \mathcal{B}(\mathcal{H})$ we denote by $|A|$ the square root of A^*A. If P is an (orthogonal) projection on $\mathcal{H}$, we set $P^\perp = I - P$.

Let $\mathcal{A}$ be a C^*-algebra. The *reduced atomic representation* of $\mathcal{A}$ is the direct sum $(\bigoplus_i \pi_i, \bigoplus_i \mathcal{H}_i)$ of a maximal family of pairwise inequivalent irreducible representations $(\pi_i, \mathcal{H}_i)$ of $\mathcal{A}$. We let $\mathcal{A}^{**}$ be the enveloping von Neumann algebra of $\mathcal{A}$, which can be identified with the second dual of $\mathcal{A}$. We denote by $\mathcal{M}(\mathcal{A})$ the multiplier algebra of $\mathcal{A}$. If $a, b \in \mathcal{M}(\mathcal{A})$ we let $M_{a,b}$ be the operator on $\mathcal{A}$ given by $M_{a,b}(x) = axb$, $x \in \mathcal{A}$.

Some of the results in this paper arose from questions that were posed to us by V. S. Shulman. We would like to thank him for his interest in our work.

2. THE CONTRACTIVE PERTURBATIONS

We start with some basic definitions and general facts concerning the notion of contractive perturbations. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and $s \in \mathcal{X}_1$. We set

$$\mathrm{cp}(s) = \{x \in \mathcal{X} : \|s \pm x\| \leqslant 1\}.$$

For a subset $\mathcal{S} \subseteq \mathcal{X}_1$, we let

$$\mathrm{cp}(\mathcal{S}) = \mathrm{cp}^1(\mathcal{S}) = \bigcap_{s\in\mathcal{S}} \mathrm{cp}(s).$$

We call an $x \in \mathrm{cp}(\mathcal{S})$ a contractive perturbation of $\mathcal{S}$. The n-th contractive perturbations of $\mathcal{S}$ are defined as

$$\mathrm{cp}^n(\mathcal{S}) = \mathrm{cp}(\mathrm{cp}^{n-1}(\mathcal{S})), \quad n = 2, 3, \ldots.$$

Clearly, the set $\mathrm{cp}(\mathcal{S})$ is a closed convex subset of $\mathcal{X}$, $\mathcal{S} \subseteq \mathrm{cp}^2(\mathcal{S})$ and if $\mathcal{S} \subseteq \mathcal{T}$ then $\mathrm{cp}(\mathcal{T}) \subseteq \mathrm{cp}(\mathcal{S})$. It follows that $\mathrm{cp}^3(\mathcal{S}) = \mathrm{cp}(\mathcal{S})$ and thus the consideration of the n-th contractive perturbations for $n > 2$ is of no interest. If $\mathcal{X}$ is a subspace of a Banach space $\mathcal{Y}$ and $\mathcal{S} \subseteq \mathcal{X}_1$ we will denote by $\mathrm{cp}^n_{\mathcal{X}}(\mathcal{S})$ and $\mathrm{cp}^n_{\mathcal{Y}}(\mathcal{S})$ the n-th contractive perturbations, computed with respect to $\mathcal{X}$ and $\mathcal{Y}$, respectively. If $\mathcal{X}$ is a normed linear space and $\mathcal{S} \subseteq \mathcal{X}$ we denote by $\overline{\mathrm{co}}(\mathcal{S})$ the closed convex hull of $\mathcal{S}$.

LEMMA 2.1. *Let $\mathcal{X}$ be a Banach space and $\mathcal{S} \subseteq \mathcal{X}_1$. Then* $\mathrm{cp}(\mathcal{S}) = \mathrm{cp}(\overline{\mathrm{co}}(\mathcal{S}))$.

Proof. Let $x \in \mathrm{cp}(\mathcal{S})$. Then $\mathcal{S} \subseteq \mathrm{cp}(x)$. Since $\mathrm{cp}(x)$ is closed and convex, we have that $\overline{\mathrm{co}}(\mathcal{S}) \subseteq \mathrm{cp}(x)$, which implies that $x \in \mathrm{cp}(\overline{\mathrm{co}}(\mathcal{S}))$. Thus $\mathrm{cp}(\mathcal{S}) \subseteq \mathrm{cp}(\overline{\mathrm{co}}(\mathcal{S}))$. The converse inclusion is trivial. ∎

The following theorem, which was proved by R. Moore and T. Trent ([19], Theorem 3), is useful for the study of the set $\mathrm{cp}(A)$ where A is a contraction on a Hilbert space $\mathcal{H}$.

THEOREM 2.2. *Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})_1$. Then we have* $(I - |A^*|)^{1/2}\mathcal{B}(\mathcal{H})_1(I - |A|)^{1/2} \subseteq \mathrm{cp}(A)$.

The following result follows from Proposition 1.2. of [4] and exhibits non-trivial second contractive perturbations of an element in the unit ball of a C^*-algebra.

PROPOSITION 2.3. *Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}_1$. If $x \in \mathcal{A}$ and $\|x\| \leqslant \frac{1}{2}$ then $axa \in \mathrm{cp}^2(a)$.*

DEFINITION 2.4. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and $x \in \mathcal{X}_1$. We say that x is *geometrically compact* if $\mathrm{cp}^2(x)$ is a compact subset of $\mathcal{X}$. We say that x is of *finite geometric rank* if the dimension of the linear space generated by $\mathrm{cp}^2(x)$ is finite.

The notions defined above were introduced in [4], where the following result was also proved.

THEOREM 2.5. *Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}_1$. The following conditions are equivalent:*

(i) *The element a is geometrically compact.*

(ii) *There exists a faithful representation π of $\mathcal{A}$ such that $\pi(a)$ is compact.*

We will also need a result due to K. Ylinen [23]:

THEOREM 2.6. *Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}_1$. There exists a faithful representation π of $\mathcal{A}$ such that the following conditions are equivalent, for an element $a \in \mathcal{A}_1$:*

(i) *The operator $\pi(a)$ is compact.*

(ii) *The operator $x \to axa$ on $\mathcal{A}$ is compact.*

3. JOINT CONTRACTIVE PERTURBATIONS

The main result of this section is a geometric characterization of the subsets of the unit ball of a C^*- algebra $\mathcal{A}$ which can be faithfully represented as precompact sets of compact operators. First we introduce some notation.

Let $\mathcal{H}$ be a Hilbert space. For $e, f \in \mathcal{H}$ we will denote by $e^* \otimes f$ the rank one operator given by $e^* \otimes f(x) = \langle x, e\rangle f$. We denote by $(s_{n,A})_{n=1}^{\infty}$ the sequence of s-numbers of a compact operator A and we fix a Schmidt expansion $\sum\limits_{n=1}^{\infty} s_{n,A} e_{n,A}^* \otimes f_{n,A}$ of A. We let $E_{k,A}$ be the projection onto the span of $\{e_{i,A} : i \leqslant k\}$, $F_{k,A}$ the projection onto the span of $\{f_{i,A} : i \leqslant k\}$ and $R_{k,A}$ the projection onto the span of $\{e_{i,A}, f_{i,A} : i \leqslant k\}$. We set $E_{0,A} = F_{0,A} = R_{0,A} = 0$ and $s_{0,A} = 0$. For each $\varepsilon > 0$, we let $R_{\varepsilon,A} = R_{k-1,A}$, where k is the smallest positive integer with $s_{k,A} \leqslant \varepsilon$.

LEMMA 3.1. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a non-degenerate C^*-algebra and $\mathcal{S} \subseteq \mathcal{A}_1 \cap \mathcal{K}(\mathcal{H})$. Let $0 < \varepsilon < 1$ and $R \in \mathcal{A}$ be a projection with $R \geqslant R_{\varepsilon,A}$, for all $A \in \mathcal{S}$. Then for each $Y \in \mathrm{cp}^2(\mathcal{S})$, we have*

$$\max\{\|R^{\perp}Y\|, \|YR^{\perp}\|\} \leqslant \sqrt{1-(1-\varepsilon)^2}.$$

Proof. Let $X \in \mathcal{A}_1$. We claim that

$$(1-\varepsilon)R^{\perp}XR^{\perp} \in \mathrm{cp}(\mathcal{S}).$$

Fix $A \in \mathcal{S}$. Let k be the smallest positive integer with $s_{k,A} \leqslant \varepsilon$. We consider orthonormal families $\{x_{i,A}\}$ and $\{y_{j,A}\}$ such that $\{e_{n,A}, x_{i,A} : n, i\}$ and $\{f_{n,A}, y_{j,A} :$

$n, j\}$ are orthonormal bases for $\mathcal{H}$. Let

$$Y_{k,A} = \sum_{l=k}^{\infty} \frac{(1-s_{k,A})^{1/2}}{(1-s_{l,A})^{1/2}} f_{l,A}^* \otimes f_{l,A} + (1-s_{k,A})^{1/2} \sum_j y_{j,A}^* \otimes y_{j,A},$$

$$Z_{k,A} = \sum_{l=k}^{\infty} \frac{(1-s_{k,A})^{1/2}}{(1-s_{l,A})^{1/2}} e_{l,A}^* \otimes e_{l,A} + (1-s_{k,A})^{1/2} \sum_i x_{i,A}^* \otimes x_{i,A},$$

where the sums converge in the strong operator topology. We have that $\|Y_{k,A}\| \leqslant 1$, $\|Z_{k,A}\| \leqslant 1$ and, by simple computations,

$$(I - |A^*|)^{1/2} Y_{k,A} = (1-s_{k,A})^{1/2} F_{k-1,A}^{\perp}, \quad Z_{k,A}(I-|A|)^{1/2} = (1-s_{k,A})^{1/2} E_{k-1,A}^{\perp}.$$

Let $W \in \mathcal{A}_1$. We have

$$(1-s_{k,A}) F_{k-1,A}^{\perp} W E_{k-1,A}^{\perp} = (I-|A^*|)^{1/2} Y_{k,A} W Z_{k,A} (I-|A|)^{1/2}$$

and hence by Theorem 2.2,

$$(1-s_{k,A}) F_{k-1,A}^{\perp} W E_{k-1,A}^{\perp} \in \mathrm{cp}(A).$$

Set $W = \frac{1-\varepsilon}{1-s_{k,A}} R^{\perp} X R^{\perp}$. We obtain

$$(1-\varepsilon) R^{\perp} X R^{\perp} = (1-s_{k,A}) F_{k-1,A}^{\perp} W E_{k-1,A}^{\perp} \in \mathrm{cp}(A).$$

The claim follows.

Let $Y \in \mathrm{cp}^2(\mathcal{S})$ and $\{G_\alpha\}$ be a contractive approximate unit of $\mathcal{A}$ consisting of positive operators. Then $\|Y \pm (1-\varepsilon) R^{\perp} G_\alpha R^{\perp}\| \leqslant 1$ for each α. Since $\mathcal{A}$ is non-degenerate, $G_\alpha \to I$ strongly and hence $\|Y \pm (1-\varepsilon) R^{\perp}\| \leqslant 1$. Therefore,

$$Y^*Y + (1-\varepsilon)^2 R^{\perp} + (1-\varepsilon)(Y^* R^{\perp} + R^{\perp} Y) \leqslant I,$$

and

$$Y^*Y + (1-\varepsilon)^2 R^{\perp} - (1-\varepsilon)(Y^* R^{\perp} + R^{\perp} Y) \leqslant I,$$

and taking the arithmetic mean of these inequalities we see that $Y^*Y \leqslant I - (1-\varepsilon)^2 R^{\perp}$ and so

$$(YR^{\perp})^*(YR^{\perp}) = R^{\perp} Y^* Y R^{\perp} \leqslant (1-(1-\varepsilon)^2) R^{\perp}.$$

Hence, $\|YR^{\perp}\| \leqslant \sqrt{1-(1-\varepsilon)^2}$.

Similarly, we conclude that $\|R^{\perp} Y\| \leqslant \sqrt{1-(1-\varepsilon)^2}$. ∎

PROPOSITION 3.2. *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a C^*-algebra and $\mathcal{S} = \{A_n : n \in \mathbb{N}\}$, where $A_n \in \mathcal{A}_1 \cap \mathcal{K}(\mathcal{H})$ and $\lim_{n\to\infty} A_n = 0$. Then $\mathrm{cp}^2(\overline{\mathrm{co}}\mathcal{S})$ is compact.*

Proof. By Lemma 2.1, it suffices to show that $\mathrm{cp}^2(\mathcal{S})$ is totally bounded. Fix an $\varepsilon > 0$ and let $\delta = 1 - \sqrt{1 - \frac{\varepsilon^2}{16}}$. Since $A_n \to 0$, there exist only finitely many A_j for which $R_{\delta, A_j} \neq 0$. Thus the smallest projection R dominating all R_{δ, A_n}, $n \in \mathbb{N}$, has finite rank and belongs to $\mathcal{A}$. By Lemma 3.1, for every $Y \in \mathrm{cp}^2(\mathcal{S})$ we have

$$\max\{\|R^{\perp} Y\|, \|YR^{\perp}\|\} \leqslant \frac{\varepsilon}{4}.$$

There exists an $\frac{\varepsilon}{2}$-net $(S_i)_{i=1}^N$ for $R\mathcal{B}(\mathcal{H})_1R$. We show that $(S_i)_{i=1}^N$ is an ε-net for $\mathrm{cp}^2(\mathcal{S})$. Let $Y \in \mathrm{cp}^2(\mathcal{S})$ and i be such that $\|RYR - S_i\| < \frac{\varepsilon}{2}$. Then

$$\|Y - S_i\| = \|RYR - S_i + RYR^\perp + R^\perp Y\| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \quad \blacksquare$$

Let $\mathcal{X}$ be a Banach space and $\mathcal{K}\subseteq\mathcal{X}_1$. It follows from a result of Grothendieck [14] that if $\mathcal{K}$ is precompact then $\mathcal{K}$ is in the closed convex hull of a null sequence (x_n). It is easily seen that (x_n) cannot always be chosen in $\mathcal{X}_1$. However, we have the following:

PROPOSITION 3.3. *Let $\mathcal{K}$ be a precompact subset of the unit ball $\mathcal{X}_1$ of a Banach space $\mathcal{X}$. Then for every $\lambda > 1$ there exists a null sequence $(x_n)_{n=1}^\infty \subseteq \lambda\mathcal{X}_1$ such that $\mathcal{K} \subseteq \overline{\mathrm{co}}((x_n)_{n=1}^\infty)$.*

Proof. For every $n = 1, 2, \dots$ we set $\varepsilon_n = \frac{(\lambda-1)^n}{(n+1)\lambda^{n+1}}$ and $\lambda_n = \frac{\lambda^n}{(\lambda-1)^{n-1}}$. Let $\mathcal{F}_n \subseteq \mathcal{K}$ be a finite ε_n-net of $\mathcal{K}$. We set $\mathcal{G}_1 = \lambda\mathcal{F}_1 \cup \{0\}$ and for $n \geqslant 2$, $\mathcal{G}_n = \lambda_n(\mathcal{F}_n - \mathcal{F}_{n-1}) \cap \frac{1}{n}\mathcal{X}_1$. It is clear that the elements of $\bigcup\limits_{n=1}^\infty \mathcal{G}_n$ can be arranged to form a null sequence in $\lambda\mathcal{X}_1$. Since $\bigcup\limits_{n=1}^\infty \mathcal{F}_n$ is a dense subset of $\mathcal{K}$ it is enough to show that $\bigcup\limits_{n=1}^\infty \mathcal{F}_n \subseteq \mathrm{co}\Big(\bigcup\limits_{n=1}^\infty \mathcal{G}_n\Big)$. Let $y_n \in \mathcal{F}_n$ and choose $y_{n-1}, \dots, y_1$ such that $y_i \in \mathcal{F}_i$ and $\|y_i - y_{i-1}\| \leqslant \varepsilon_{i-1}$ for every $i = 2, \dots n$. Then $y_n = \frac{1}{\lambda}(\lambda y_1) + \sum\limits_{i=2}^n \frac{1}{\lambda_i}(\lambda_i(y_i - y_{i-1}))$ and $\frac{1}{\lambda} + \sum\limits_{i=2}^n \frac{1}{\lambda_i} \leqslant 1$. $\blacksquare$

The main result of this section is the following:

THEOREM 3.4. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{S} \subseteq \mathcal{A}_1$. The following are equivalent:*

(i) *$\mathcal{S}$ is precompact and there exists a faithful representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\pi(\mathcal{S})$ consists of compact operators.*

(ii) *The set $\mathrm{cp}^2(\lambda\mathcal{S})$ is compact for each $\lambda \in (0, 1)$.*

(iii) *There exists $\lambda \in (0, 1)$ such that the set $\mathrm{cp}^2(\lambda\mathcal{S})$ is compact.*

Moreover, if one of the conditions above holds then $\pi(\mathrm{cp}^2(\lambda\mathcal{S})) \subseteq \mathcal{K}(\mathcal{H})$, for each $\lambda \in (0, 1)$, where π is any representation satisfying (i).

Proof. (i) $\Rightarrow$ (ii) We identify $\mathcal{A}$ with $\pi(\mathcal{A})$. Let $0 < \lambda < 1$. By Proposition 3.3, there exists a null sequence $(x_n) \subseteq \frac{1}{\lambda}\mathcal{A}_1 \cap \mathcal{K}(\mathcal{H})$ such that $\mathcal{S} \subseteq \overline{\mathrm{co}}((x_n))$. But then $\lambda\mathcal{S} \subseteq \overline{\mathrm{co}}((\lambda x_n))$ and $(\lambda x_n) \subseteq \mathcal{A}_1 \cap \mathcal{K}(\mathcal{H})$. By Proposition 3.2, $\mathrm{cp}^2(\lambda\mathcal{S})$ is compact.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) Since $\lambda\mathcal{S} \subseteq \mathrm{cp}^2(\lambda\mathcal{S})$ we have that $\lambda\mathcal{S}$ is precompact and hence $\mathcal{S}$ is precompact. Let $a \in \mathcal{S}$. Then $\mathrm{cp}^2(\lambda a) \subseteq \mathrm{cp}^2(\lambda\mathcal{S})$ and hence $\mathrm{cp}^2(\lambda a)$ is compact. By Proposition 2.3, $\frac{1}{2}\lambda^2(a\mathcal{A}_1 a) \subseteq \mathrm{cp}^2(\lambda a)$ and hence the operator $x \to axa$ on $\mathcal{A}$ is compact. It follows from Theorem 2.6 that there exists a faithful representation π of $\mathcal{A}$ such that $\pi(a)$ is compact for each $a \in \mathcal{S}$. $\blacksquare$

REMARK 3.5. It follows from the proof of Theorem 3.4 that if $\mathcal{S}$ is a finite set, then condition (ii) is satisfied with $\lambda \in (0,1]$.

COROLLARY 3.6. *Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})_1$. The following are equivalent:*
(i) *$\mathcal{S}$ is a precompact set of compact operators.*
(ii) *The set $\mathrm{cp}^2(\lambda\mathcal{S})$ is compact for each $\lambda \in (0,1)$.*
(iii) *There exists $\lambda \in (0,1)$ such that the set $\mathrm{cp}^2(\lambda\mathcal{S})$ is compact.*

Proof. (i)⇒(ii) follows from Theorem 3.4, while (ii)⇒(iii) is trivial.
(iii)⇒(i) Assume $\mathrm{cp}^2(\lambda\mathcal{S})$ is compact, for some $\lambda \in (0,1)$. Then clearly $\mathcal{S}$ is precompact. For each $A \in \mathcal{S}$, $\mathrm{cp}^2(\lambda A) \subseteq \mathrm{cp}^2(\lambda\mathcal{S})$ and hence it is compact. By Proposition 2.3, the operator $X \to AXA$ on $\mathcal{B}(\mathcal{H})$ is compact. It is well-known (see e.g. Section 33 Corollary 5 of [8]) that this implies $A \in \mathcal{K}(\mathcal{H})$. ∎

COROLLARY 3.7. *Let $\mathcal{A}$ be a prime C^*-algebra and $T = \sum_{i=1}^{n} M_{a_i,b_i}$ be an elementary operator whose minimal length is n, where $a_i, b_i \in \mathcal{A}_1$, $i = 1,2,\ldots,n$. The following are equivalent:*
(i) *T is compact.*
(ii) *$\mathrm{cp}^2_{\mathcal{A}}(\{a_1, b_1, \ldots, a_n, b_n\})$ is compact.*

Proof. The proof follows from Theorem 3.4 and Corollary 5.3.26 of [7]. ∎

REMARK 3.8. The implication (ii)⇒(i) of Corollary 3.7 holds for arbitrary C^*-algebras while (i)⇒(ii) does not hold in general. Indeed, let $\mathcal{A} = l^\infty$. Let A and B be infinite disjoint sets with characteristic functions χ_A and χ_B. Since A and B are infinite, $\mathrm{cp}^2(\{\chi_A, \chi_B\})$ is not compact. But M_{χ_A,χ_B} is the zero operator and hence is compact.

As can be seen from the following proposition, condition (i) of Theorem 3.4 does not imply in general that $\mathrm{cp}^2(\mathcal{S})$ is compact.

PROPOSITION 3.9. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. There exists a compact set $\mathcal{S} \subseteq \mathcal{K}(\mathcal{H})_1$ such that $\mathrm{cp}^2(\mathcal{S})$ is not compact.*

Proof. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal set of $\mathcal{H}$ and $f_n = \lambda_n e_1 + \mu_n e_n$, where $|\lambda_n|^2 + |\mu_n|^2 = 1$ for each n, $\lambda_1 = 1$, $|\lambda_n| \neq 1$ for each $n \geqslant 2$ and $\lambda_n \to 1$. Let $A_n = e_1^* \otimes f_n$ and $\mathcal{S} = \{A_n : n = 1,2,\ldots\}$. We have that $f_n \to e_1$ and so $A_n \to A_1$; hence $\mathcal{S}$ is a compact set of compact contractions. Assume that $X \in \mathrm{cp}(\mathcal{S})$. Then $\|X \pm e_1^* \otimes f_n\| \leqslant 1$ for each n. It follows that $XX^* \leqslant I - f_n^* \otimes f_n$ and hence $X^* f_n = 0$. This implies that $X^* e_n = 0$, for each n. Similarly, $X^* X \leqslant I - e_1^* \otimes e_1$ and so $Xe_1 = 0$. It follows that $e_1^* \otimes e_n$ is a contractive perturbation of X, for each n, and thus $\mathrm{cp}^2(\mathcal{S})$ is not compact. ∎

In the following theorem we characterize the subsets $\mathcal{S}$ of the unit ball of a C^*-algebra $\mathcal{A}$ for which $[\mathrm{cp}^2(\mathcal{S})]$ is finite dimensional.

THEOREM 3.10. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{S} \subseteq \mathcal{A}_1$. The following are equivalent:*

(i) $\dim[\mathrm{cp}^2(\mathcal{S})] < \infty$.

(ii) $\dim[\mathcal{S}] < \infty$ *and there exists a faithful representation π of $\mathcal{A}$ such that $\pi(a)$ has finite rank for each $a \in \mathcal{S}$.*

Proof. (i)$\Rightarrow$(ii) Since $[\mathcal{S}] \subseteq [\mathrm{cp}^2(\mathcal{S})]$, it follows that $\dim[\mathcal{S}] < \infty$. For each $a \in \mathcal{S}$, we have that $\mathrm{cp}^2(a) \subseteq \mathrm{cp}^2(\mathcal{S})$ and hence $\dim[\mathrm{cp}^2(a)] < \infty$. By Proposition 2.3, the operator $x \to axa$ on $\mathcal{A}$ has finite rank for each $a \in \mathcal{S}$. It follows from [23] that there exists a faithful representation π of $\mathcal{A}$ such that $\pi(a)$ has finite rank for each $a \in \mathcal{S}$.

(ii)$\Rightarrow$(i) We identify $\mathcal{A}$ with $\pi(\mathcal{A})$. Since $\mathcal{S}$ is finite dimensional and every element of $\mathcal{S}$ has finite rank, there exists a projection $p \in \mathcal{A}$ of finite rank such that $\mathcal{S} \subseteq p\mathcal{A}p$. It follows from Theorem 4.1 below (see also Proposition 2.1 of [6]) that $\mathrm{cp}^2(\mathcal{S}) \subseteq p\mathcal{A}p$, hence $\dim[\mathrm{cp}^2(\mathcal{S})] < \infty$. ∎

4. CHARACTERIZATION RESULTS

Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})_1$. It is not hard to see that if $T \in \mathcal{K}(\mathcal{H})_1$ then $\mathrm{cp}^2(T) \subseteq \mathcal{K}(\mathcal{H})$. This leads to the following problem: Find the C^*-subalgebras $\mathcal{B}$ of a C^*-algebra $\mathcal{A}$ such that $b \in \mathcal{B}_1$ implies $\mathrm{cp}^2(b) \subseteq \mathcal{B}$. In the following theorem we show that these are precisely the hereditary C^*-subalgebras of $\mathcal{A}$.

THEOREM 4.1. *Let $\mathcal{B}$ be a C^*-subalgebra of a C^*-algebra $\mathcal{A}$. The following are equivalent:*

(i) *If $b \in \mathcal{B}_1$ then $\mathrm{cp}^2(b) \subseteq \mathcal{B}$.*

(ii) *$\mathcal{B}$ is a hereditary C^*-subalgebra of $\mathcal{A}$.*

Proof. (i)$\Rightarrow$(ii) Let b be a positive element in $\mathcal{B}_1$ and assume that $a \in \mathcal{A}$ satisfies $0 \leqslant a \leqslant b$. It follows from Proposition 2.3 that $b\mathcal{A}b \subseteq [\mathrm{cp}^2(b)]$. By our hypothesis, $b\mathcal{A}b \subseteq \mathcal{B}$ and so $\overline{b\mathcal{A}b} \subseteq \mathcal{B}$. Now, $\overline{b\mathcal{A}b}$ is a hereditary C^*-subalgebra of $\mathcal{A}$ containing b ([20], Corollary 3.2.4) and hence $a \in \overline{b\mathcal{A}b}$. We conclude that $a \in \mathcal{B}$.

(ii)$\Rightarrow$(i) Represent $\mathcal{A}$ faithfully and non-degenerately on a Hilbert space $\mathcal{H}$. Fix $b \in \mathcal{B}_1$. Let $\{u_\alpha\}$ be an approximate unit of $\mathcal{A}$ consisting of positive contractions, $x = (1 - |b|)^{1/2}$ and $y = (1 - |b^*|)^{1/2}$. It is easy to see that $yu_\alpha x \in \mathcal{A}$. By Theorem 2.2, $yu_\alpha x \in \mathrm{cp}(b)$. Thus, if $c \in \mathrm{cp}^2(b)$, then $\|c \pm yu_\alpha x\| \leqslant 1$. This implies that

$$cc^* \leqslant 1 - yu_\alpha x^2 u_\alpha y.$$

But $1 - yu_\alpha x^2 u_\alpha y \to 1 - yx^2 y$ weakly, and hence $cc^* \leqslant 1 - yx^2 y$. On the other hand, $1 - yx^2 y$ is positive and belongs to $\mathcal{B}$. Since $\mathcal{B}$ is hereditary in $\mathcal{A}$, we have that $cc^* \in \mathcal{B}$. Similarly, we show that $c^*c \in \mathcal{B}$. Now it follows from Theorem 3.2.1 of [20] that $c \in \mathcal{B}$. ∎

REMARK 4.2. Let $\mathcal{S}$ be a finite subset of $\mathcal{B}(\mathcal{H})$. If follows from Remark 3.5 that if $\mathcal{S} \subseteq \mathcal{K}(\mathcal{H})_1$, then $\mathrm{cp}^2(\mathcal{S}) \subseteq \mathcal{K}(\mathcal{H})$. The following example shows that this is not true if we replace $\mathcal{B}(\mathcal{H})$ by a C^*-algebra $\mathcal{A}$ and $\mathcal{K}(\mathcal{H})$ by a hereditary C^*-subalgebra of $\mathcal{A}$. Note that, by Theorem 4.1, this inclusion is valid if $\mathcal{S}$ is a singleton.

EXAMPLE 4.3. Let $\mathcal{H}$ be a Hilbert space and P, Q be projections on $\mathcal{H}$ in generic position forming a zero angle (see [15]). Let $\mathcal{B}$ be the C^*-algebra generated by P and Q; then $\mathcal{B}$ is non-unital [22]. Let $\mathcal{A} = \mathcal{M}(\mathcal{B})$. Then every contractive perturbation of P in $\mathcal{A}$ is of the form $P^\perp A P^\perp$, and every contractive perturbation of Q in $\mathcal{A}$ is of the form $Q^\perp B Q^\perp$. Since $P^\perp\mathcal{H} \cap Q^\perp\mathcal{H} = \{0\}$, we have that $\mathrm{cp}(\{P,Q\}) = \{0\}$ and $\mathrm{cp}^2(\{P,Q\}) = \mathcal{A}_1$.

COROLLARY 4.4. *Let $\mathcal{A}$ be a unital C^*-algebra and $a \in \mathcal{A}$ be a positive contraction. Then $\overline{a\mathcal{A}a} = \overline{[\mathrm{cp}^2(a)]}$. Thus, if $\mathcal{A}$ is separable then its hereditary C^*-subalgebras are precisely the subsets of the form $\overline{[\mathrm{cp}^2(a)]}$, where a is a positive contraction.*

Proof. It follows from Proposition 2.3 that

$$\overline{a\mathcal{A}a} \subseteq \overline{[\mathrm{cp}^2(a)]}.$$

On the other hand, $\overline{a\mathcal{A}a}$ is a hereditary subalgebra of $\mathcal{A}$ containing a. By Theorem 4.1, $\overline{[\mathrm{cp}^2(a)]} \subseteq \overline{a\mathcal{A}a}$. ∎

The following theorem shows that if $\mathcal{A}$ is a separable C^*-algebra then the elements of $\mathcal{A}_1$ can be characterized among the elements of $\mathcal{M}(\mathcal{A})$ in terms of the size of their second contractive perturbations.

THEOREM 4.5. *Let $\mathcal{A}$ be a separable C^*-algebra and $a \in \mathcal{M}(\mathcal{A})_1$. Then $a \in \mathcal{A}$ if and only if $\mathrm{cp}^2_{\mathcal{M}(\mathcal{A})}(a)$ is separable.*

Proof. Assume that $a \in \mathcal{A}$. By Theorem 4.1, $\mathrm{cp}^2_{\mathcal{M}(\mathcal{A})}(a)$ is contained in $\mathcal{A}$ and hence is separable. Conversely, assume that $a \notin \mathcal{A}$ and $\mathrm{cp}^2_{\mathcal{M}(\mathcal{A})}(a)$ is separable. By Proposition 2.3, $a\mathcal{M}(\mathcal{A})a$ is separable. Since $\mathcal{A}$ is a hereditary C^*-subalgebra of $\mathcal{M}(\mathcal{A})$, it follows from Theorem 3.2.1 in [20] that $aa^* \notin \mathcal{A}$ or $a^*a \notin \mathcal{A}$. Without loss of generality we may assume that $aa^* \notin \mathcal{A}$. We have that $aa^*\mathcal{M}(\mathcal{A})aa^* \subseteq a\mathcal{M}(\mathcal{A})aa^*$ and hence $aa^*\mathcal{M}(\mathcal{A})aa^*$ is separable. But $\overline{aa^*\mathcal{M}(\mathcal{A})aa^*}$ is the hereditary subalgebra of $\mathcal{M}(\mathcal{A})$ generated by aa^* ([20], Corollary 3.2.4); it follows from Corollary 7 in [9] that it is non-separable, a contradiction. ∎

As an immediate consequence, we obtain the following:

COROLLARY 4.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be separable C^*-algebras. If $\phi : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{B})$ is a surjective isometry then $\phi(\mathcal{A}) = \mathcal{B}$.*

We will need some facts about open projections and hereditary subalgebras. We refer the reader to [21]. Let $\mathcal{A}$ be a C^*-algebra. If $p \in \mathcal{A}^{**}$ is a projection, we write $\mathcal{B}^p = \mathcal{A} \cap p\mathcal{A}^{**}p$. The set $\mathcal{B}^p$ is a hereditary C^*-subalgebra of $\mathcal{A}$. Conversely, if $\mathcal{B}$ is a hereditary C^*-subalgebra of $\mathcal{A}$, there exists a unique open projection $p \in \mathcal{A}^{**}$ such that $\mathcal{B} = \mathcal{B}^p$. If $p \in \mathcal{A}^{**}$ is any projection, we write $\mathrm{int}(p)$ for the largest open projection dominated by p and $\overline{p}$ for the smallest closed projection which dominates p. It is easy to see that if $p \in \mathcal{A}^{**}$ then $\mathcal{B}^p = \mathcal{B}^{\mathrm{int}(p)}$.

PROPOSITION 4.7. *Let $\mathcal{A}$ be a unital C^*-algebra. If $p \in \mathcal{A}^{**}$ is an open projection then $\mathrm{cp}(\mathcal{B}^p_1) = \mathcal{B}_1^{\mathrm{int}(p^\perp)}$.*

Proof. It suffices to show that $\mathrm{cp}(\mathcal{B}^p_1) = \mathcal{B}_1^{p^\perp}$. It is clear that $\mathcal{B}_1^{p^\perp} \subseteq \mathrm{cp}(\mathcal{B}^p_1)$. Let $y \in \mathcal{A}$ and assume that $y \in \mathrm{cp}(\mathcal{B}^p_1)$. We consider $\mathcal{A}^{**}$ as the weak closure of the image of $\mathcal{A}$ under its universal representation. Since p is open, there exists an increasing net of positive contractions in $\mathcal{B}^p$ whose weak limit is p. Since the set of contractive perturbations of any set is weakly closed, we conclude that $\|y \pm p\| \leqslant 1$, which implies that $y \in \mathcal{B}^{p^\perp}$. ∎

The last proposition has the following corollaries.

COROLLARY 4.8. *Let $\mathcal{A}$ be a unital C^*-algebra and $p \in \mathcal{A}^{**}$ be an open projection. The following are equivalent:*

(i) $\mathrm{cp}^2(\mathcal{B}^p_1) = \mathcal{B}^p_1$.

(ii) $\overline{\mathrm{int}(p^\perp)} = p^\perp$.

Proof. It is immediate from Proposition 4.7 that condition (i) is equivalent to the condition $\mathrm{int}(\mathrm{int}(p^\perp)^\perp) = p$. The equivalence with (ii) follows from the identity $\mathrm{int}(q^\perp) = \overline{q}^\perp$, which holds for any open projection q, by letting $q = \mathrm{int}(p^\perp)$. ∎

Let $\mathcal{H}$ be a Hilbert space. It is easy to see that $\mathrm{cp}_{\mathcal{B}(\mathcal{H})}(\mathcal{K}(\mathcal{H})_1) = \{0\}$. It follows from Proposition 4.7 that this property characterizes the essential ideals of a unital C^*-algebra. Recall that an ideal $\mathcal{J}$ is essential if and only if it is of the form $\mathcal{B}^p$ with p open, dense and central.

COROLLARY 4.9. (i) *Let $\mathcal{A}$ be a unital C^*-algebra. An ideal $\mathcal{J} \subseteq \mathcal{A}$ is essential if and only if $\mathrm{cp}_{\mathcal{A}}(\mathcal{J}_1) = \{0\}$.*

(ii) *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ a surjective isometry. Assume that $\mathcal{J}$ is an ideal of $\mathcal{A}$ and that $\phi(\mathcal{J})$ is an ideal of $\mathcal{B}$. Then $\mathcal{J}$ is essential if and only if $\phi(\mathcal{J})$ is essential.*

5. SMALL FACES IN C^*-ALGEBRAS

The faces of the unit ball of a C^*-algebra were examined in detail by Akemann and Pedersen in [2]; in this section we obtain some results concerning the finite dimensional ones.

We introduce some notation (see also [1]). The segment joining the points x, y of a Banach space $\mathcal{X}$ is the set

$$\operatorname{seg}[x,y] = \{\lambda x + (1-\lambda)y : \lambda \in [0,1]\}.$$

We let $\operatorname{seg}(x,y) = \operatorname{seg}[x,y] \setminus \{x,y\}$. Let $\mathcal{K}$ be a convex subset of $\mathcal{X}$. A convex subset $\mathcal{F}$ of $\mathcal{K}$ is called a *face* of $\mathcal{K}$ if for any $x, y \in \mathcal{K}$ and $v \in \mathcal{F}$ such that $v \in \operatorname{seg}(x,y)$ we have that $x, y \in \mathcal{F}$. A face $\mathcal{F}$ is called finite dimensional if the linear subspace generated by $\mathcal{F}$ is finite dimensional. The real subspace spanned by a subset $\mathcal{S}$ of $\mathcal{X}$ is denoted by $[\mathcal{S}]_{\mathbb{R}}$.

If $\mathcal{H}$ is a separable infinite dimensional Hilbert space then the unit ball of $\mathcal{B}(\mathcal{H})$ contains finite dimensional faces of arbitrarily large dimension. On the other hand the unit ball of a unital C^*-algebra has no infinite dimensional compact faces as the following proposition shows. In the sequel we shall consider $\mathcal{A}$ as a subalgebra of its enveloping von Neumann algebra $\mathcal{A}^{**}$.

PROPOSITION 5.1. *Let $\mathcal{F}$ be a compact face of the unit ball of a unital C^*-algebra $\mathcal{A}$. Then $\mathcal{F}$ is finite dimensional and there exists a partial isometry v in $\mathcal{A}$ such that*

$$\mathcal{F} = v + (1-vv^*)\mathcal{A}_1(1-v^*v).$$

Proof. It follows from Theorem 5.6 of [2] that the weak closure $\overline{\mathcal{F}}^{\mathrm{w}}$ of $\mathcal{F}$ is a weakly closed face of the unit ball of the von Neumann algebra $\mathcal{A}^{**}$ and hence by Theorem 4.4 of [2] there exists a partial isometry v in $\mathcal{A}^{**}$ such that

$$\overline{\mathcal{F}}^{\mathrm{w}} = v + (1-vv^*)(\mathcal{A}^{**})_1(1-v^*v).$$

Since $\mathcal{F}$ is compact, $\mathcal{F} = \overline{\mathcal{F}}^{\mathrm{w}}$. Hence the unit ball of the Banach space $(1-vv^*)\mathcal{A}^{**}(1-v^*v)$ is compact and this implies that $(1-vv^*)\mathcal{A}^{**}(1-v^*v)$ is finite dimensional. We conclude that $\mathcal{F}$ is finite dimensional.

Since $\mathcal{F} = v + (1-vv^*)(\mathcal{A}^{**})_1(1-v^*v)$, we see that $v \in \mathcal{A}$. Now it follows from Kaplansky's Density Theorem that $(1-vv^*)\mathcal{A}_1(1-v^*v)$ is strongly dense in $(1-vv^*)(\mathcal{A}^{**})_1(1-v^*v)$. Since $(1-vv^*)(\mathcal{A}^{**})_1(1-v^*v)$ is finite dimensional, we have

$$(1-vv^*)\mathcal{A}_1(1-v^*v) = (1-vv^*)(\mathcal{A}^{**})_1(1-v^*v)$$

and so $\mathcal{F} = v + (1-vv^*)\mathcal{A}_1(1-v^*v)$. ∎

Let $\mathcal{K}$ be a convex set and $x \in \mathcal{K}$. The minimal face of $\mathcal{K}$ containing x is denoted by $\mathcal{F}(x)$. It can be shown that $\mathcal{F}(x)$ is the union of all segments $\operatorname{seg}[y,z]$ such that $y, z \in \mathcal{K}$ and $x \in \operatorname{seg}(y,z)$ (see Theorem 1.2 of [1] for a proof). If $\mathcal{F}$ is a face and $x \in \mathcal{F}$, then the subspace $[x - \mathcal{F}]_{\mathbb{R}}$ does not depend on x. We set $S(\mathcal{F}) = [x - \mathcal{F}]_{\mathbb{R}}$.

Let $\mathcal{A}$ be a unital C^*-algebra. It is shown in [18] that if $\mathcal{F}$ is a finite dimensional face of $\mathcal{A}_1$ then $\mathcal{F} = (a + S(\mathcal{F})) \cap \mathcal{A}_1$, where a is any point of $\mathcal{F}$. Let $a \in \mathcal{A}$, $\|a\| = 1$. It is proved in [18] that $S(\mathcal{F}(a)) = [\mathrm{cp}(a)]_{\mathbb{R}}$. In the sequel we completely determine $[\mathrm{cp}(a)]_{\mathbb{R}}$ in case it is finite dimensional.

We shall need the following fact whose proof is left to the reader.

LEMMA 5.2. *Let E and F be finite rank operators on a Hilbert space $\mathcal{H}$. Then*

$$E\mathcal{B}(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})F = E\mathcal{B}(\mathcal{H})F.$$

THEOREM 5.3. *Let $\mathcal{A}$ be a unital C^*-algebra and $a \in \mathcal{A}_1$. Assume that* $\dim[\mathrm{cp}(a)]_{\mathbb{R}} < \infty$ *or* $\dim(1 - |a^*|^2)^{1/2}\mathcal{A}(1 - |a|^2)^{1/2} < \infty$. *Then*

$$[\mathrm{cp}(a)]_{\mathbb{R}} = (1 - |a^*|^2)^{1/2}\mathcal{A}(1 - |a|^2)^{1/2}.$$

Proof. We set $b = (1 - |a^*|^2)^{1/2}$ and $c = (1 - |a|^2)^{1/2}$. It follows from Theorem 2.2 that $b\mathcal{A}c \subseteq [\mathrm{cp}(a)]_{\mathbb{R}}$, and hence to prove the theorem it suffices to prove that if $\dim b\mathcal{A}c < \infty$ then $\mathrm{cp}(a) \subseteq b\mathcal{A}c$.

Let $\pi = (\oplus_i \pi_i, \oplus_i \mathcal{H}_i)$ be the reduced atomic representation of $\mathcal{A}$. Then the subspace $\pi(b)\pi(\mathcal{A})\pi(c)$ is finite dimensional and so it is equal to its strong closure $\bigoplus_i \pi_i(b)\mathcal{B}(\mathcal{H}_i)\pi_i(c)$. Put $J = \{i : \pi_i(b) \neq 0\} \cap \{i : \pi_i(c) \neq 0\}$. It follows that the set J is finite and that $\pi_i(b), \pi_i(c)$ are finite rank operators for every $i \in J$.

Let $x \in \mathrm{cp}(a)$. Then $\pi_i(x) \in \mathrm{cp}(\pi_i(a))$ and Lemma 1 of [19] implies that $\pi_i(x) \in \pi_i(b)B(\mathcal{H}_i) \cap \mathcal{B}(\mathcal{H}_i)\pi_i(c)$. By Lemma 5.2, $\pi_i(b)B(\mathcal{H}_i) \cap \mathcal{B}(\mathcal{H}_i)\pi_i(c) = \pi_i(b)\mathcal{B}(\mathcal{H}_i)\pi_i(c)$ which is equal to $\pi_i(b)\pi(\mathcal{A})\pi_i(c)$. Finally we have

$$\pi(x) \in \bigoplus_{i \in J} \pi_i(b)\pi(\mathcal{A})\pi_i(c) = \pi(b)\pi(\mathcal{A})\pi(c) = \pi(b\mathcal{A}c)$$

and so $x \in b\mathcal{A}c$. ∎

6. CONTRACTIVE PERTURBATIONS IN $\mathcal{B}(\mathcal{X})$

In this section we present some results showing that the notion of geometric compactness is not appropriate for the description of the compact operators on a general Banach space. All Banach spaces considered in this section are assumed to be real.

Let $\mathcal{X}$ be a Banach space. We denote by $\mathcal{B}(\mathcal{X})$ the Banach algebra of all bounded linear operators on $\mathcal{X}$ and by $\mathcal{K}(\mathcal{X})$ the ideal of compact operators on $\mathcal{X}$. If $x \in \mathcal{X}$ and x^* is in the dual $\mathcal{X}^*$ of $\mathcal{X}$, we denote by $x^* \otimes x$ the rank one operator on $\mathcal{X}$ defined by $x^* \otimes x(y) = x^*(y)x$. We denote by $\{e_n\}_{n \in \mathbb{N}}$ the canonical basis of c_0 (respectively ℓ^p, $p \geqslant 1$) and by e_n^* the linear functional on c_0 (respectively ℓ^p, $p \geqslant 1$) defined by $e_n^*(e_m) = 0$ for $n \neq m$ and $e_n^*(e_n) = 1$.

In the next three propositions we show that if $\mathcal{X}$ is any of the spaces l^p, $1 \leqslant p < +\infty$, $p \neq 2$, or $C(K)$, there are rank one operators in $\mathcal{B}(\mathcal{X})_1$ which are extreme points of $\mathcal{B}(\mathcal{X})_1$. Note that in Proposition 4.4 of [18] Katsoulis provides an example of an operator algebra $\mathcal{A}$ acting on a Hilbert space H, and a rank one operator $a \in \mathcal{A}_1$ which is an extreme point of $\mathcal{A}_1$.

PROPOSITION 6.1. *Let x^* be an extreme point of the unit ball of ℓ^∞ and y be an extreme point of the unit ball of ℓ^1. Then the rank one operator $A = x^* \otimes y$ is an extreme point of $\mathcal{B}(\ell^1)_1$.*

Proof. Let $T \in \mathcal{B}(\ell^1)_1$ be such that $\|A \pm T\| \leqslant 1$. Then

$$\|Az \pm Tz\| \leqslant 1, \quad \forall z \in \ell^1, \ \|z\| = 1.$$

We conclude that if $x^*(z) = \pm 1$ then Tz must be 0. But the set

$$\{z \in \ell^1 : \ \|z\| = 1, \ x^*(z) = \pm 1\}$$

contains the canonical basis of ℓ^1 and hence $T = 0$. ∎

Next we show that an analogous result holds for the space $\mathcal{B}(\ell^p)$, $p > 1$, $p \neq 2$. We recall a lemma of J. Hennefeld [16]. If $x \in \ell^p$ we set

$$\operatorname{supp}(x) = \{n \in \mathbb{N} : e_n^*(x) \neq 0\}.$$

LEMMA 6.2. *Let $T \in \mathcal{B}(\ell^p)$, $p > 2$ and suppose that for some n, m with $n \neq m$, we have that $\operatorname{supp}(Te_n) \cap \operatorname{supp}(Te_m) \neq \emptyset$. Then*

$$\|T\| > \max\{\|Te_n\|, \|Te_m\|\}.$$

PROPOSITION 6.3. *There exists a rank one operator in $\mathcal{B}(\ell^p)$ for $1 < p < \infty$, $p \neq 2$, which is an extreme point of $\mathcal{B}(\ell^p)_1$.*

Proof. Assume first that $p > 2$. Let $a = (a_1, a_2, \ldots)$ be a vector in ℓ^p such that $\|a\| = 1$ and $a_i \neq 0$, $\forall i \in \mathbb{N}$. Put $T = e_1^* \otimes a$. We show that T is an extreme point of the unit ball of $B(l^p)$. Let $S \in \mathcal{B}(\ell^p)_1$ be such that $\|T \pm S\| \leqslant 1$. We have $\|Te_1 \pm Se_1\| \leqslant 1$ and since Te_1 is an extreme point of the unit ball of l^p, $Se_1 = 0$. Assume that $Se_m \neq 0$ for some $m \neq 1$. It follows from Lemma 6.2 that

$$\|S + T\| > \max\{\|(S+T)e_1\|, \|(S+T)e_m\|\} = \max\{1, \|Se_m\|\} = 1$$

a contradiction. So, $Se_m = 0$ and hence T is an extreme point of the unit ball of $\mathcal{B}(l^p)$.

For $1 < p < 2$ the assertion follows by duality. ∎

PROPOSITION 6.4. *Let K be a compact metrizable Hausdorff space. There exists a rank one operator on $C(K)$ which is an extreme point of $\mathcal{B}(C(K))_1$.*

Proof. Let k_0 be in K and λ be a continuous function on K satisfying $|\lambda(k)| = 1, \forall k \in K$. Then the operator T on $\mathcal{B}(C(K))$ defined by

$$Tf(k) = \lambda(k) f(k_0)$$

is a rank-one operator and it is easy to see that it is an extreme point of the unit ball of $\mathcal{B}(C(K))$. (For a complete description of the extreme points of $\mathcal{B}(C(K))$ we refer the reader to [10]). ∎

The situation for $\mathcal{B}(\mathcal{X})$ when $\mathcal{X} = c_0$ is somewhat different. Indeed, by a result of J. Hennefeld [16], $\mathcal{K}(\mathcal{X})_1$ is the norm closed convex hull of its extreme points when $\mathcal{X} = \ell^p$ with $p \geqslant 1$, $p \neq 2$. On the other hand it is known that for $\mathcal{X} = c_0$ or $\mathcal{X} = \ell^2$ the unit ball of $\mathcal{K}(\mathcal{X})$ has no extreme points. However, we shall see (Proposition 6.5) that there exist compact operators in $\mathcal{B}(c_0)$ which are not geometrically compact.

We shall need a description of the operators in $\mathcal{B}(c_0)$ which follows from Theorem VII.2.1 in [11]. An operator $T \in \mathcal{B}(c_0)$ has the form

$$T = \sum_{i=1}^{\infty} a_i \otimes e_i, \quad \text{where } a_i \in \ell^1, \text{w*-}\lim_{i\to\infty} a_i = 0,$$

and its norm is given by

$$\|T\| = \sup_{i\in\mathbb{N}} \|a_i\|_{\ell^1}.$$

The operator T is compact if and only if $\lim_{i\to\infty} a_i = 0$.

PROPOSITION 6.5. *Let* $A = e_1^* \otimes e_1 \in \mathcal{B}(c_0)$. *Then*

$$\mathrm{cp}^2(A) = \{x \otimes e_1 : x \in \ell^1, \|x\|_{\ell^1} \leqslant 1\}.$$

Proof. The statement follows from the fact that

$$\mathrm{cp}(A) = \Big\{T = \sum_{i=2}^{\infty} a_i \otimes e_i : a_i \in \ell^1, \text{w*-}\lim_i a_i = 0, \|a_i\|_{\ell^1} \leqslant 1\Big\}. \quad \blacksquare$$

We next show that if T is in $\mathcal{K}(c_0)_1$ then $\mathrm{cp}^2(T)$ is a compact subset of $\mathcal{B}(c_0)$ with respect the weak operator topology. Recall that if $\mathcal{X}$ is a Banach space then the *weak operator topology (wot)* on $\mathcal{B}(\mathcal{X})$ is defined by the the family of neighborhoods of 0

$$U_{\varepsilon,F,G} = \{T : |x^*(Tx)| < \varepsilon, x^* \in F, x \in G\}$$

where F is a finite subset of $\mathcal{X}^*$, G is a finite subset of $\mathcal{X}$ and $\varepsilon > 0$.

THEOREM 6.6. *Let* $T \in \mathcal{B}(c_0)_1$. *If* T *is compact, then* $\mathrm{cp}^2(T)$ *is* wot *compact.*

Proof. Let $P_n \in \mathcal{B}(c_0)$ be defined by $P_n\Big(\sum_{j=1}^{\infty} x_j e_j\Big) = \sum_{j=1}^{n} x_j e_j$. Fix a wot neighbourhood U of 0 in $\mathcal{B}(c_0)$. To show that $\mathrm{cp}^2(T)$ is wot compact it suffices to show that there exists a finite set F such that $F + U$ covers $\mathrm{cp}^2(T)$. Let V be a wot neighbourhood of 0 in $\mathcal{B}(c_0)$ such that $V + V \subseteq U$. Let $\varepsilon > 0$ be such that the open ball $B(0, \varepsilon)$ with radius ε and centre 0 is contained in V. Let $n \in \mathbb{N}$ be such that $\|T - P_n T\| < \frac{\varepsilon}{2}$.

We show that if $X \in \mathrm{cp}^2(T)$, then $\|(I - P_n)X\| \leqslant \varepsilon$. Let S be an operator such that $(I - P_n)S = S$ and $\|S\| \leqslant 1 - \varepsilon$. Since $\|T - P_n T\| < \frac{\varepsilon}{2}$ we have that $S \in \mathrm{cp}(T)$. Let $X \in \mathrm{cp}^2(T)$. Then $X = P_n X + (I - P_n)X$ and since $\|X \pm S\| \leqslant 1$

we have $\|(I - P_n)X \pm (I - P_n)S\| \leqslant 1$. Letting $S_n = \frac{1-\varepsilon}{\|(I-P_n)X\|}(I - P_n)X$, we conclude that $\|(I - P_n)X\| \leqslant \varepsilon$.

Since $P_n\mathcal{B}(c_0)_1$ is wot-compact, there exists a finite set F such that $P_n\mathcal{B}(c_0)_1 \subseteq F + V$. Let $X \in \mathrm{cp}^2(T)$. Then

$$X = P_n X + (I - P_n)X \in F + V + V \subseteq F + U. \qquad \blacksquare$$

REMARK 6.7. The converse of the last theorem is not true. Indeed, let T be the operator on c_0 defined by:

$$T = \sum_{i=1}^{\infty} \frac{1}{2}(e^*_{2i-1} + e^*_{2i}) \otimes e_i.$$

It is clear that the operator T is not compact. Simple computations show that

$$\mathrm{cp}^2(T) = \Big\{ \sum_{i=1}^{\infty} \frac{1}{2}(a_i e^*_{2i-1} + a_i e^*_{2i}) \otimes e_i,\ |a_i| \leqslant 1,\ \forall i \in \mathbb{N} \Big\}$$

which is a wot compact set.

Acknowledgements. The third-named author was partially supported by Nuffield Foundation Grant NAL/01028/G.

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Received October 15, 2005.