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QUASIHYPONORMAL TOEPLITZ OPERATORS

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ABSTRACT. Motivated by a question on subnormal Toeplitz subnormal operators raised in 1970 by P.R. Halmos, we show that there exist quasihyponormal Toeplitz operators which are neither hyponormal nor analytic. In addition, for $\varphi \in L^\infty(\mathbb{T})$ and letting $\varphi = f + \overline{g}$, where f and g are in H^2, we show that the Toeplitz operator T_φ is quasihyponormal if and only if $P(g\overline{f}) = c + T_{\overline{h}} f \overline{f}$ for some constant c and some function $h \in H^\infty(\mathbb{D})$ with $\|h\|_\infty \leqslant 1$. Finally, we also show that the problem of quasihyponormality for Toeplitz operators with (trigonometric) polynomial symbols can be reduced to the classical Schur's algorithm in function theory.

KEYWORDS: *Toeplitz operators, Hankel operators, quasihyponormal operators, hyponormal operators, trigonometric polynomial, Schur's algorithm.*

MSC (2000): Primary 47B20, 47B35.

1. INTRODUCTION

Let H be an arbitrary complex Hilbert space. A bounded linear operator T on H is called *quasihyponormal* if $T^*(T^*T - TT^*)T \geqslant 0$ or equivalently

$$\|T^*Tx\| \leqslant \|TTx\| \quad \text{for all } x \text{ in } H.$$

For more details on quasihyponormal Toeplitz operators one can refer to [22] and references therein. $L^2 = L^2(\mathbb{T})$ denotes the space of all complex-valued Lebesgue measurable functions on unit circle $\mathbb{T}$ such that $\int\limits_{\mathbb{T}} |f|^2 < \infty$. This space has a canonical orthonormal basis $\langle e_n \rangle$ given by $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, $\mathbb{Z}$ being the set of integers. The Hardy space $H^2 = H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \ldots\}$. An element $f \in L^2$ is said to be analytic if $f \in H^2$ and co-analytic if $f \in L^2 \ominus H^2$. If P denotes the orthogonal projection from L^2 to H^2, then for every $\varphi \in L^\infty$ the operators T_φ and H_φ on H^2 defined by

$$T_\varphi f = P(\varphi f) \quad \text{and} \quad H_\varphi f = (I - P)(\varphi f), \quad \text{for each } f \in H^2$$

are called the *Toeplitz* and *Hankel operators*, respectively, with symbol φ. Also

$$H_\varphi f = J(I-P)(\varphi f),$$

where J is the unitary operator from $H^{2^\perp}$ onto H^2 defined as

$$J(\mathrm{e}^{-\mathrm{i}n\theta}) = \mathrm{e}^{\mathrm{i}(n-1)\theta}.$$

Another way of representing the operator H_φ [18] is an operator on H^2 defined by

$$\langle zuv, \overline{\varphi}\rangle = \langle H_\varphi u, v^*\rangle, \tag{1.1}$$

for all $u \in H^2, v \in H^\infty$ where $v^*(\mathrm{e}^{\mathrm{i}\theta}) = \overline{v(\mathrm{e}^{-\mathrm{i}\theta})}$.

This paper has been divided into three sections. The first section is devoted to a problem initiated by P.R. Halmos [12], [13] in the year 1970. The second section characterizes quasihyponormal Toeplitz operators while the third section adopts K. Zhu's [23] method based on the classical interpolation theorems of I. Schur [21] to obtain an abstract characterization of the trigonometric polynomials which yield quasihyponormal Toeplitz operators.

2. A PROBLEM BY HALMOS

In the year 1970, P.R. Halmos initiated the following problem:

Is every subnormal Toeplitz operator either normal or analytic?

This was answered in the negative by C. Cowen and J. Long [6]. The question has been answered [16] in the affirmative for a certain class of functions φ and for quasinormal Toeplitz operators [2]. In 1976, M.B. Abrahamse [1] gave a general sufficient condition to obtain the answer of the problem to be affirmative. If the condition of subnormality is weakened to hyponormality then this question has been answered in the negative by Ito and Wong [16]. We further ask the following

Is every quasihyponormal Toeplitz operator either hyponormal or analytic?

In this section we show that the answer to this question is also in the negative. In fact we show that there exist quasihyponormal Toeplitz operators which are neither hyponormal nor analytic.

EXAMPLE 2.1. Consider the Toeplitz operator $2U^* - U$ where U denotes the unilateral shift on the space H^2. We have the following:

$$\begin{aligned}(2U^*-U)^*(2U^*-U) &- (2U^*-U)(2U^*-U)^* \\ &= (2U-U^*)(2U^*-U) - (2U^*-U)(2U-U^*) = 3(UU^*-I).\end{aligned}$$

Now $3(UU^*-I)e_0 = -3e_0 < 0$ and for each $n \neq 0, 3(UU^*-I)e_n = 0$. Hence $(2U^*-U)$ is not hyponormal, rather it is a cohyponormal operator as $[2U^*-U] \leqslant 0$.

As $U = T_{e_1}, U^* = T_{e_{-1}}$ are Toeplitz operators and since the set of Toeplitz operators [3] is a self adjoint vector space containing the identity we have that $2U^* - U$ is a Toeplitz operator. We also observe that $2U^* - U$ is a quasihyponormal Toeplitz operator as

$$(2U^* - U)^*[2U^* - U](2U^* - U) = (2U - U^*)3(UU^* - I)(2U^* - U).$$

Now

$$(2U - U^*)3(UU^* - I)(2U^* - U)e_0 = (2U - U^*)3(UU^* - I)(-e_1) = 0,$$

and for $n \neq 0$

$$(2U - U^*)3(UU^* - I)(2U^* - U)e_n = (2U - U^*)3(UU^* - I)(2e_{n-1} - e_{n+1}) = 0.$$

Hence

$$(2U^* - U)^*[2U^* - U](2U^* - U) = 0.$$

Also $2U^* - U$ is not analytic since $2U^* - U = T_{2e_{-1} - e_1}$.

The above Example 2.1 can be generalized in the following way.

THEOREM 2.2. *The Toeplitz operator* $(aU^{*^{n_0}} - bU^{n_0})$ *with* $|a| > |b| > 0$ *and* $n_0 > 0$ *is a quasihyponormal Toeplitz operator.*

REMARK 2.3. The operator in Theorem 2.2 above is not normal when $|a| \neq |b|$. For if $|a| = |b|$, then

$$[aU^{*^{n_0}} - bU^{n_0}] = |a|^2 U^{n_0} U^{*^{n_0}} - |a|^2 + |b|^2 - |b|^2 U^{n_0} U^{*^{n_0}} = 0.$$

Motivated by Proposition 11 of [4], we observe the following:

THEOREM 2.4. *If* T_φ *is quasihyponormal then*

$$\mathrm{Ker}(H_\varphi T_\varphi) \cap \mathrm{Ker}(T_\varphi^*[T_\varphi]T_\varphi) = \mathrm{Ker}(H_{\overline{\varphi}} T_\varphi).$$

Proof. For $f \in H^2$,

$$\langle f, T_\varphi^*(T_\varphi^* T_\varphi - T_\varphi T_\varphi^*)T_\varphi f\rangle = \langle T_\varphi T_\varphi f, T_\varphi T_\varphi f\rangle - \langle T_\varphi^* T_\varphi f, T_\varphi^* T_\varphi f\rangle.$$

Now

$$\begin{aligned}\langle T_\varphi T_\varphi f, T_\varphi T_\varphi f\rangle &= \langle \varphi T_\varphi f, P(\varphi T_\varphi f)\rangle = \|\varphi T_\varphi f\|^2 - \langle P^\perp \varphi T_\varphi f, P^\perp \varphi T_\varphi f\rangle \\ &= \|\varphi T_\varphi f\|^2 - \|H_\varphi T_\varphi f\|^2.\end{aligned}$$

Similarly

$$\langle T_\varphi^* T_\varphi f, T_\varphi^* T_\varphi f\rangle = \|\overline{\varphi} T_\varphi f\|^2 - \|H_{\overline{\varphi}} T_\varphi f\|^2.$$

Consequently

$$\langle f, T_\varphi^*[T_\varphi]T_\varphi f\rangle = \|H_{\overline{\varphi}} T_\varphi f\|^2 - \|H_\varphi T_\varphi f\|^2.$$

Now a vector $f \in H^2$ is in $\mathrm{Ker}(H_{\overline{\varphi}} T_\varphi)$ if and only if $H_{\overline{\varphi}} T_\varphi f = 0$. So that $\|H_\varphi T_\varphi\|^2 = 0$ and hence

$$\langle f, T_\varphi^*[T_\varphi]T_\varphi f\rangle + \|H_\varphi T_\varphi f\|^2 = 0.$$

Therefore the equation holds if and only if $f \in \operatorname{Ker}(H_\varphi T_\varphi) \cap \operatorname{Ker}(T_\varphi^*[T_\varphi]T_\varphi)$. ∎

T_φ is hyponormal [4] if the analytic part of φ dominates the conjugate analytic part. Similar computations yield the following:

THEOREM 2.5. *If χ is inner and $\varphi \in L^\infty$ is such that T_φ is hyponormal, then $T_{\chi\varphi}$ is quasihyponormal.*

3. CHARACTERIZATION

The goal of this section is to characterize quasihyponormality of Toeplitz operators T_φ on the space H^2 by properties of the symbol $\varphi \in L^\infty$ following an elegant theorem of C. Cowen ([5], Theorem 1).

Normal Toeplitz operators were characterized [3] by a property of their symbols in the early 1960's by A. Brown and P.R. Halmos. Twenty five years later the exact nature of relationship between the symbol $\varphi \in L^\infty$ and the positivity of the self commutator $[T_\varphi^*, T_\varphi]$ was understood via Cowen's theorem [5] which requires one to solve a certain functional equation in unit ball of H^∞. Motivated by Cowen's theorem we frame the operator theoretic problem of quasihyponormality of Toeplitz operators, into the problem of finding a solution with specified properties to a certain functional equation in unit ball of H^∞ involving the symbol φ. The basis of the proof is this Cowen's theorem again and hence a Dilation theorem [19]. The proof uses standard results about Hankel operators [5], [18]. However it still remains open to characterize subnormality of Toeplitz operators in terms of their symbols.

THEOREM 3.1. *If φ is in L^∞, where $\varphi = f + \overline{g}$ for f and g in H^2, then T_φ is quasihyponormal if and only if*

$$P(g\overline{f}) = c + T_{\overline{h}} f\overline{f},$$

for some constant c and some function $h \in H^\infty(\mathbb{D})$ with $\|h\|_\infty \leqslant 1$.

Proof. Let $\varphi = f + \overline{g}$ where $f, g \in H^2$. Now if p is any polynomial, then

$$\begin{aligned}
&\langle T_\varphi^*[T_\varphi]T_\varphi p, p\rangle \\
&= \langle T_\varphi^*(T_\varphi^* T_\varphi - T_\varphi T_\varphi^*)T_\varphi p, p\rangle \\
&= \langle T_\varphi(T_\varphi p), T_\varphi(T_\varphi p)\rangle - \langle T_\varphi^*(T_\varphi p), T_\varphi^*(T_\varphi p)\rangle \\
&= \langle P f T_\varphi p + P\overline{g} T_\varphi p, P f T_\varphi p + P\overline{g} T_\varphi p\rangle - \langle P\overline{f} T_\varphi p + P g T_\varphi p, P\overline{f} T_\varphi p + P g T_\varphi p\rangle \\
&= \langle P f T_\varphi p, P f T_\varphi p\rangle + \langle P\overline{g} T_\varphi p, P\overline{g} T_\varphi p\rangle - \langle P\overline{f} T_\varphi p, P\overline{f} T_\varphi p\rangle - \langle P g T_\varphi p, P g T_\varphi p\rangle \\
&= \langle \overline{f} T_\varphi p, \overline{f} T_\varphi p\rangle + \langle \overline{g} T_\varphi p, P\overline{g} T_\varphi p\rangle - \langle \overline{f} T_\varphi p, P\overline{f} T_\varphi p\rangle - \langle \overline{g} T_\varphi p, \overline{g} T_\varphi p\rangle \\
&= \langle (I-P)(\overline{f} f + \overline{f} P\overline{g})p, (I-P)(\overline{f} f + \overline{f} P\overline{g})p\rangle - \langle (I-P)(\overline{g} f + \overline{g} P\overline{g})p, (I-P)(\overline{g} f + \overline{g} P\overline{g})p\rangle \\
&= \|H_{\overline{f} f + \overline{f} P\overline{g}} p\|^2 - \|H_{\overline{g} f + \overline{g} P\overline{g}} p\|^2 = \|H_{\overline{f} f} p\|^2 - \|H_{\overline{g} f} p\|^2.
\end{aligned}$$

Since the analytic polynomials are dense in H^2 and the Hankel and Toeplitz operators involved are bounded, we see that T_φ is quasihyponormal if and only if

$$\|H_{\overline{g}f}u\| \leqslant \|H_{\overline{f}f}u\|, \quad \text{for all } u \in H^2. \tag{3.1}$$

Let K denote the closure of range of $H_{\overline{f}f}$ and S denote the compression of the unilateral shift U to K. Since K is invariant for U^*, the operator S^* is the restriction of U^* to K.

Now suppose first that T_φ is quasihyponormal. Define an operator A on the range of $H_{\overline{f}f}$ by

$$A(H_{\overline{f}f}u) = H_{\overline{g}f}u.$$

A is well defined because if $H_{\overline{f}f}u_1 = H_{\overline{f}f}u_2$ then $H_{\overline{f}f}(u_1 - u_2) = 0$. The inequality (3.1) implies that $H_{\overline{g}f}(u_1 - u_2) = 0$ too and it follows $H_{\overline{g}f}u_1 = H_{\overline{g}f}u_2$. Also, inequality (3.1) implies that $\|A\| \leqslant 1$, so A has an extension to K, which will also be denoted by A, with the same norm. Now by the intertwining formula for Hankel operators $H_\varphi U = U^* H_\varphi$ and the fact that K is invariant for U^*, we have

$$H_{\overline{g}f}U = AH_{\overline{f}f}U = AU^*H_{\overline{f}f} = AS^*H_{\overline{f}f}$$

and also

$$H_{\overline{g}f}U = U^*H_{\overline{g}f} = U^*AH_{\overline{f}f} = S^*AH_{\overline{f}f}.$$

Since the range of $H_{\overline{f}f}$ is dense in K, we find that $AS^* = S^*A$ on K, or taking adjoints, that $SA^* = A^*S$. In the following cases we use the fact: *Either H_φ is one-one or* $\mathrm{Ker}(H_\varphi) = \chi H^2$ *where χ is an inner function. The closure of the range of H_φ is H^2 in the former case and $(\chi^* H^2)^\perp$ in the later.*

Case 1. $H_{\overline{f}f}$is one-one. Then $K = H^2$ so that A^* is an operator on H^2 commuting with U. Hence by Theorem 1 of [19], (or by the usual theory of the unilateral shift if $K = H^2$), there is a function k in $H^\infty(\mathbb{D})$ with

$$\|k\|_\infty = \|A^*\| = \|A\| \leqslant 1, \quad \text{and } A^* = T_k.$$

Case 2. $H_{\overline{f}f}$ is not one-one, then $K = (\chi^* H^2)^\perp = H^2 \ominus \chi^* H^2$ and A^* is an operator on K commuting with S, so that by Theorem 1 of [19], there is a function k in $H^\infty(\mathbb{D})$ with

$$\|k\|_\infty = \|A^*\| = \|A\|,$$

such that A^* is the compression of T_k to K .

Since K is invariant for $T_{k^*} = T_{\overline{k}}$, this means that A is the restriction of $T_{\overline{k}}$ to K and

$$H_{\overline{g}f} = T_{\overline{k}}H_{\overline{f}f}. \tag{3.2}$$

Conversely, if equation (3.2) holds for some k in $H^\infty(\mathbb{D})$ with $\|k\|_\infty \leqslant 1$, then clearly inequality (3.1) holds for all u, and T_φ is quasihyponormal. The proof will be completed by analyzing the relationship given by (3.2). Using the formulation (1.1), equation (3.2) holds if and only if for all functions $u, v \in H^\infty$

$$\begin{aligned}\langle zuv, g\overline{f}\rangle &= \langle H_{\overline{g}f}u, v^*\rangle = \langle T_{\overline{k}}H_{\overline{f}f}u, v^*\rangle = \langle H_{\overline{f}f}u, kv^*\rangle = \langle zuk^*v, f\overline{f}\rangle \\ &= \langle zuv, \overline{k}^* f\overline{f}\rangle = \langle zuv, T_{\overline{k}^*}f\overline{f}\rangle.\end{aligned}$$

Since the closed span of $\{zuv : u, v \in H^\infty\}$ is zH^2, this means that equation (3.2) holds if and only if $(g\overline{f} - T_{\overline{k}^*}f\overline{f})$ is in $H^{2^\perp}$, so that $P(g\overline{f}) - T_{\overline{k}^*}f\overline{f}$ is in $H^{2^\perp}$, and hence

$$P(g\overline{f}) = c + T_{\overline{h}}f\overline{f},$$

for some constant c and $h = k^*$, with $\|h\|_\infty = \|k^*\|_\infty = \|k\|_\infty \leqslant 1$. ∎

4. MATRICIAL REPRESENTATION

We discuss Kehe Zhu's computing process [23] by way of Schur's functions so that we can determine any given polynomial φ such that T_φ is quasihyponormal. We reformulate the characterization of quasihyponormality in the case of a trigonometric polynomial φ, so that the quasihyponormality of T_φ can be decided by applying Schur's algorithm to the Schur function Φ_N. This approach has been put to use in the works [8], [10], [11], [15], [17] and [23] to study hyponormal Toeplitz operators on H^2.

If φ is a *trigonometric polynomial* of the form

$$\varphi(z) = \sum_{n=-m}^{N} a_n z^n,$$

where a_{-m} and a_N are non-zero, then the non-negative integers N and m denote the analytic and co-analytic degrees of φ.

Theorem 3.1 says that if f_1 and f_2 are functions in H^2 with $f = f_1 + \overline{f}_2$ in L^∞, then T_f is quasihyponormal if and only if there exist a constant c and a function g in $(H^\infty)_1$ (that is $g \in H^\infty, \|g\|_\infty \leqslant 1$) such that $P(f_2\overline{f}_1) = c + T_{\overline{g}}f_1\overline{f}_1$. It is clear that the above holds if and only if $P(f_2\overline{f}_1 - \overline{g}f_1\overline{f}_1) = c$ if and only if $(\overline{f}_2 f_1 - g\overline{f}_1 f_1)$ is in H^2.

REMARK 4.1. Thus the quasihyponormality of the Toeplitz operator T_f is equivalent to the solvability of the functional equation

$$\overline{f}_2 f_1 - g\overline{f}_1 f_1 = h, \quad \text{for } g \in (H^\infty)_1 \text{ and } h \in H^2.$$

By Zhu [23] and in view of the above observation we deduce the following:

COROLLARY 4.2. *Suppose $f = p + \overline{q}$ where p is an analytic polynomial of degree n and $q \in H^\infty$. If T_f is quasihyponormal, then q must be an analytic polynomial of degree less than or equal to n.*

Proof. Write $p(z) = \sum_{k=0}^{n} a_k z^k, a_n \neq 0$. The quasihyponormality of T_f implies that there exist functions $g \in (H^\infty)_1$ and $h \in H^2$ such that $(\overline{q}p - g\overline{p}p) = h$. Now for $z \in \mathbb{T}$, we have

$$\begin{aligned} g(z)\overline{p(z)}p(z) = z^{-n}g(z)\Big[\Big(\sum_{i=1}^{n}\overline{a}_i a_{i-1}\Big)z^{n-1} + \Big(\sum_{i=2}^{n}\overline{a}_i a_{i-2}\Big)z^{n-2} + \cdots \\ + \overline{a}_n a_0 z_0 + \Big(\sum_{i=n-1}\overline{a}_i a_{i-(n-1)}\Big)z + \Big(\sum_{i=n-2}\overline{a}_i a_{i-(n-2)}\Big)z^2 + \cdots \\ + \sum_{i=0}^{n}|a_i|^2 z^n + \Big(\sum_{i=0}^{n-1}\overline{a}_i a_{i+1}\Big)z^{n+1} + \Big(\sum_{i=0}^{n-2}\overline{a}_i a_{i+2}\Big)z^{n+2} + \cdots + \overline{a}_0 a_n z^{2n}\Big]. \end{aligned}$$

Thus $z^n\overline{q(z)}p(z) = z^n g(z)\overline{p(z)}p(z) + h(z)z^n$ is analytic and hence $z^{-n}q(z)\overline{p(z)}$ is co-analytic. This implies that q is an analytic polynomial of degree $\leqslant n$. ∎

Given two analytic polynomials $p(z) = \sum_{k=0}^{n} a_k z^k, q(z) = \sum_{k=0}^{n} b_k z^k, a_n\overline{a}_0 \neq 0$, we proceed to determine, in terms of the coefficients of p and q the following:

When is the Toeplitz operator $T_f, f = p + \overline{q}$, quasihyponormal?

By the Remark 4.1, we are trying to determine whether or not the functional equation

$$\overline{q}p - g\overline{p}p = h,$$

has solutions (g, h) in $(H^\infty)_1 \times H^2$. Write

$$g(z) = \sum_{k=0}^{\infty} c_k z^k, \quad \text{and} \quad h(z) = \sum_{k=0}^{\infty} h_k z^k.$$

Comparing Taylor coefficients, we find that the equation $\overline{q}p - g\overline{p}p = h$, has solutions (g, h) in $(H^\infty)_1 \times H^2$ if and only if

$$\begin{aligned} b_n\overline{a}_0 &= \overline{c}_0 a_n\overline{a}_0 \\ b_n\overline{a}_1 + b_{n-1}\overline{a}_0 &= \overline{c}_1 a_n\overline{a}_0 + \overline{c}_0 a_n\overline{a}_1 + \overline{c}_0 a_{n-1}\overline{a}_0 \\ b_n\overline{a}_2 + b_{n-1}\overline{a}_1 + b_{n-2}\overline{a}_0 &= \overline{c}_2 a_n\overline{a}_0 + \overline{c}_1 a_n\overline{a}_1 + \overline{c}_0 a_n\overline{a}_0 + \overline{c}_1 a_{n-1}\overline{a}_0 \\ &\quad + \overline{c}_0 a_{n-1}\overline{a}_1 + \overline{c}_0 a_{n-2}\overline{a}_0 \\ &\;\;\vdots \\ b_n\overline{a}_{n-2} + b_{n-1}\overline{a}_{n-3} + \cdots + b_2\overline{a}_0 &= \overline{c}_{n-2} a_n\overline{a}_0 + \overline{c}_{n-3} a_n\overline{a}_1 + \cdots + \overline{c}_0 a_n\overline{a}_{n-2} + \overline{c}_{n-3} a_{n-1}\overline{a}_0 \\ &\quad + \overline{c}_{n-4} a_{n-1}\overline{a}_1 + \cdots + \overline{c}_0 a_{n-1}\overline{a}_{n-3} + \cdots + \overline{c}_0 a_2\overline{a}_0 \end{aligned}$$

$$\begin{aligned} b_n\overline{a}_{n-1}+b_{n-1}\overline{a}_{n-2}+\cdots+b_1\overline{a}_0=\overline{c}_{n-1}a_n\overline{a}_0+\overline{c}_{n-2}a_n\overline{a}_1+\cdots+\overline{c}_0a_n\overline{a}_{n-1}+\overline{c}_{n-2}a_{n-1}\overline{a}_0 \\ +\overline{c}_{n-3}a_{n-1}\overline{a}_1+\cdots+\overline{c}_0a_{n-1}\overline{a}_{n-2} \\ +\cdots+\overline{c}_1a_2\overline{a}_0+\overline{c}_0a_2\overline{a}_1+\overline{c}_0a_1\overline{a}_0, \end{aligned}$$

where $c_0, c_1, \ldots, c_{n-1}$ are the first n coefficients of a function in $(H^\infty)_1$. In matrix form the above equations become

$$B_nX_n = A_nY_n,$$

where

$$B_n = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-2} & b_{n-1} & b_n \\ b_2 & b_3 & b_4 & \cdots & b_{n-1} & b_n & 0 \\ b_3 & b_4 & b_5 & \cdots & b_n & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ b_{n-1} & b_n & 0 & \cdots & 0 & 0 & 0 \\ b_n & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$A_n = \begin{bmatrix} a_1\overline{a}_0+\cdots+a_n\overline{a}_{n-1} & a_2\overline{a}_0+\cdots+a_n\overline{a}_{n-2} & \cdots & \cdots & a_n\overline{a}_1+a_{n-1}\overline{a}_0 & a_n\overline{a}_0 \\ a_2\overline{a}_0+\cdots+a_n\overline{a}_{n-2} & a_3\overline{a}_0+\cdots+a_n\overline{a}_{n-3} & \cdots & \cdots & a_n\overline{a}_0 & 0 \\ a_3\overline{a}_0+\cdots+a_n\overline{a}_{n-3} & a_4\overline{a}_0+\cdots+a_n\overline{a}_{n-4} & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ a_{n-2}\overline{a}_0+\cdots+a_n\overline{a}_2 & a_{n-1}\overline{a}_0+a_n\overline{a}_1 & a_n\overline{a}_0 & \cdots & 0 & 0 \\ a_{n-1}\overline{a}_0+a_n\overline{a}_1 & a_n\overline{a}_0 & 0 & \cdots & 0 & 0 \\ a_n\overline{a}_0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$X_n = \begin{bmatrix} \overline{a}_0 & \overline{a}_1 & \overline{a}_2 & \cdot & \cdot & \cdot & \overline{a}_{n-2} & \overline{a}_{n-1} \end{bmatrix}',$$

and

$$Y_n = \begin{bmatrix} \overline{c}_0 & \overline{c}_1 & \overline{c}_2 & \cdot & \cdot & \cdot & \overline{c}_{n-2} & \overline{c}_{n-1} \end{bmatrix}'.$$

Note that A_n', the transpose of the $n \times n$ matrix A_n, is equal to A_n. Since $a_n\overline{a}_0 \neq 0$, the matrix A_n is invertible. Thus the first n coefficients of the function g in $(H^\infty)_1$ are uniquely determined by the coefficients of p and q.

Summarizing, we obtain the following:

THEOREM 4.3. *Suppose $p(z) = \sum_{k=0}^{n} a_kz^k$ with $a_n\overline{a}_0 \neq 0, q(z) = \sum_{k=0}^{n} b_kz^k$ and $f = p + \overline{q}$. With the notation as above, let*

$$Y_n = A_n^{-1}B_nX_n.$$

Then T_f is quasihyponormal if and only if there exists a function g in $(H^\infty)_1$, whose first n Taylor coefficients are $c_0, c_1, \ldots, c_{n-1}$ in that order.

The problem of determining the quasihyponormality of T_f, $f = p + \overline{q}$ with $\deg(p) = n$ and $\deg(q) \leqslant n$, is now equivalent to the problem of characterizing the first n Taylor coefficients of function in $(H^\infty)_1$. We adopt Kehe Zhu's method [23] based on Schur's solution to this coefficient problem [21] to obtain an abstract characterization of those trigonometric polynomial symbols that correspond to quasihyponormal Toeplitz operators.

We first review *Schur's algorithm* [21]: Suppose $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is a general function in $(H^\infty)_1$. Let $f_0 := f$. Define by induction a sequence $\{f_n\}$ of functions in $(H^\infty)_1$ as follows:

$$f_{n+1} = \frac{f_n(z) - f_n(0)}{z(1 - \overline{f_n(0)} f_n(z))}, \quad n \geqslant 0, |z| < 1.$$

Then $f_n(0)$ only depends on the coefficients $c_0, c_1, \ldots, c_n$. Thus we can write

$$f_n(0) = \Phi_n(c_0, c_1, \ldots, c_n), \quad n \geqslant 0,$$

where Φ_n is a function of $n+1$ complex variables. We call the $\Phi_n' s$ *Schur's functions*. By straight calculations we can see that:

$$\Phi_0(c_0) = c_0, \quad \Phi_1(c_0, c_1) = \frac{c_1}{1 - |c_0|^2}, \quad \Phi_2(c_0, c_1, c_2) = \frac{c_2(1 - |c_0|^2) + \bar{c}_0 c_1^2}{(1 - |c_0|^2)^2 - |c_1|^2}.$$

THEOREM 4.4 ([21]). *Given $n+1$ complex numbers $c_0, c_1, \ldots, c_n$, they are the first $n+1$ Taylor coefficients (in the given order) of a function g in $(H^\infty)_1$ if and only if*

$$|\Phi_k(c_0, c_1, \ldots, c_k)| \leqslant 1, \quad 0 \leqslant k \leqslant n.$$

As a consequence we obtain the following:

THEOREM 4.5. *Suppose*

$$f(z) = \sum_{k=0}^{n} a_k z^k + \overline{\sum_{k=0}^{n} b_k z^k}, \quad a_n \bar{a}_0 \neq 0.$$

As in Theorem 4.3, let

$$Y_n = A_n^{-1} B_n X_n.$$

Then T_f is quasihyponormal if and only if

$$|\Phi_k(c_0, c_1, \ldots, c_k)| \leqslant 1, \quad 0 \leqslant k \leqslant n-1.$$

We use this characterization to give explicit necessary and sufficient condition for quasihyponormality in terms of the coefficients of the polynomial φ when $n \leqslant 3$. It is interesting to note that the conditions coincide with those for hyponormality as calculated by Kehe Zhu [23].

EXAMPLE 4.6. If $f(z) = a_{-1}\bar{z} + a_0 + a_1 z$, then it is easily checked that T_f is quasihyponormal if and only if $|a_{-1}| \leqslant |a_1|$.

EXAMPLE 4.7. If $f(z) = a_0 + a_1 z + a_2 z^2 + \overline{g(z)}$ with $a_2 \bar{a}_0 \neq 0$ and $g \in H^\infty$, then T_f is quasihyponormal if and only if $g(z) = b_0 + b_1 z + b_2 z^2$ with

$$|b_2|^2 + |a_2 b_1 - a_1 b_2| \leqslant |a_2|^2.$$

Proof. Let $Y_2 = A_2^{-1}B_2X_2$ then $Y_2 = Z_2$, where

$$Z_2 = \begin{bmatrix} \frac{b_2}{a_2} \\ \frac{a_2b_1 - a_1b_2}{a_2^2} \end{bmatrix},$$

which turns out to be same matrix as obtained by Kehe Zhu [23] for hyponormality of T_f and thus by Corollary 4.2 and Theorem 4.5, T_f is quasihyponormal if and only if $g(z) = b_0 + b_1z + b_2z^2$ with $|b_2|^2 + |a_2b_1 - a_1b_2| \leqslant |a_2|^2$. ∎

Thus by Example 6 of [23] and Example 4.7, we conclude T_f is quasihyponormal if and only if is hyponormal.

EXAMPLE 4.8. Suppose $f(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \overline{g(z)}$ with $a_3\bar{a}_0 \neq 0$ and $g \in H^\infty$. Then

(i) T_f is quasihyponormal if and only if $g(z) = b_0 + b_1z + b_2z^2 + b_3z^3$ with $|b_3| \leqslant |a_3|$ and

$$\begin{aligned} &|(|a_3|^2 - |b_3|^2)[a_3^2b_1 - a_2a_3b_2 + (a_2^2 - a_1a_3)b_3] + a_3\bar{b}_3(a_3b_2 - a_2b_3)^2| \\ &\qquad \leqslant |a_3|[(|a_3|^2 - |b_3|^2)^2 - |a_3b_2 - a_2b_3|^2]. \end{aligned}$$

(ii) T_f is quasihyponormal if and only if $g(z) = b_0 + b_1z + b_2z^2$ with

$$|b_2|^2 + |a_3b_1 - a_2b_2| \leqslant |a_3|^2.$$

Proof. Let $Y_3 = A_3^{-1}B_3X_3$ then $Y_3 = Z_3$, where

$$Z_3 = \begin{bmatrix} \frac{b_3}{a_3} \\ \frac{a_3b_2 - a_2b_3}{a_3^2} \\ \frac{a_3^2b_1 - a_2a_3b_2 + (a_2^2 - a_1a_3)b_3}{a_3^2} \end{bmatrix}.$$

These are indeed the Fourier coefficients of the $(H^\infty)_1$ function for hyponormality of T_f as obtained in Example 7 of [23]. The desired result follows from Corollary 4.2 and Theorem 4.5 and previously obtained formulas for Φ_0, Φ_1 and Φ_2. ∎

Again Example 7 of [23] and Example 4.8 conclude that T_f is quasihyponormal if and only if it is hyponormal. Indeed the case of arbitrary trigonometric polynomial φ, though solved in principle by Theorem 3.1 or Schur's algorithm is in practice very complicated.

REFERENCES

[1] M.B. ABRAHAMSE, Subnormal Toeplitz operators and functions of bounded type, *Duke Math. J.* **43**(1976), 597–604.

[2] I. AMEMIYA, T. ITO, T.K. WONG, On quasinormal Toeplitz operators, *Proc. Amer. Math. Soc.* **50**(1975), 254–258.

[3] A. BROWN, P.R. HALMOS, Algebraic properties of Toeplitz operators, *J. Reine Angew. Math.* **213**(1963-64), 89–102.

[4] C.C. COWEN, Hyponormal and subnormal Toeplitz operators, in *Surveys of Some Recent Results in Operator Theory, Vol. 1*, Pitman Res. Notes Math. Ser., vol. 171, Longman 1988, pp. 155–167.

[5] C.C. COWEN, Hyponormality of Toeplitz operators, *Proc. Amer. Math. Soc.* **103**(1988), 809–812.

[6] C.C. COWEN, J.J. LONG, Some subnormal Toeplitz operators, *J. Reine Angew. Math.* **351**(1984), 216–220.

[7] R.E. CURTO, S.H. LEE, W.Y. LEE, Subnormality and 2-hyponormality for Toeplitz operators, *Integral Equations Operator Theory* **44**(2002), 138–148.

[8] R.E. CURTO, W.Y. LEE, Joint hyponormality of Toeplitz pairs, *Mem. Amer. Math. Soc.* **712**(2001).

[9] R. DOUGLAS, *Banach Algebra Techniques in Operator Theory*, Second edition, Springer-Verlag, New York 1998.

[10] D.R. FARENICK, W.Y. LEE, Hyponormality and spectra of Toeplitz operators, *Trans. Amer. Math. Soc.* **348**(1996), 4153–4174.

[11] D.R. FARENICK, W.Y. LEE, On hyponormal Toeplitz operators with polynomial and circulant-type symbols, *Integral Equations Operator Theory* **29**(1997), 202–210.

[12] P.R. HALMOS, Ten problems in Hilbert space, *Bull. Amer. Math. Soc.* **76**(1970), 887–933.

[13] P.R. HALMOS, Ten years in Hilbert space, *Integral Equations Operator Theory* **2**(1979), 529–564.

[14] P.R. HALMOS, *A Hilbert Space Problem Book*, Second edition, Springer-Verlag, New York 1982.

[15] I.S. HWANG, I.H. KIM, W.Y. LEE, Hyponormality of Toeplitz operators with polynomial symbols, *Math. Ann.* **313**(1999), 247–261.

[16] T. ITO, T.K. WONG, Subnormality and quasinormality of Toeplitz operators, *Proc. Amer. Math. Soc.* **34**(1972), 157–164.

[17] T. NAKAZI, K. TAKAHASHI, Hyponormal Toeplitz operators and extremal problems of Hardy spaces, *Trans. Amer. Math. Soc.* **338**(1993), 753–769.

[18] S.C. POWER, *Hankel Operators on Hilbert Space*, Pitman, Boston 1982.

[19] D. SARASON, Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.* **127**(1967), 179–203.

[20] D. SARASON, Holomorphic spaces: a brief and selective survey, in *Holomorphic Spaces (Berkeley, CA 1995)*, Math. Sci. Res. Inst. Publ., vol. 33, Cambidge Univ. Press, Cambidge 1998, pp. 1–34.

[21] I. SCHUR, Über Potenzreihen die im Innern des Einheitskreises beschränkt sind, *J. Reine Angew. Math.* **147**(1917), 205–232.

[22] N.C. SHAH, I.H. SNETH, Some results on quasihyponormal operators, *J. Indian Math. Soc. (N.S.)* **39**(1975), 285–291.

[23] K. ZHU, Hyponormal Toeplitz operators with polynomial symbols, *Integral Equations Operator Theory* **21**(1995), 376–381.

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