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COMPACT PERTURBATIONS AND STABILITY OF THE ESSENTIAL SPECTRUM OF SINGULAR DIFFERENTIAL OPERATORS

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ABSTRACT. We establish criteria which ensure that the difference of the resolvents of two unbounded operators acting in Banach modules is compact. The applications cover operators acting on sections of vector fiber bundles over non-smooth manifolds or locally compact abelian groups, in particular differential operators of any order with complex measurable coefficients on $\mathbb{R}^n$, singular Dirac operators, and Laplace-Beltrami operators on Riemannian manifolds with measurable metrics.

KEYWORDS: *Elliptic operators, essential spectrum, compactness criteria, Banach modules, Riemannian manifolds, locally compact abelian groups, Dirac operators.*

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INTRODUCTION

If A, B are closed operators acting in a Banach space $\mathscr{H}$ and if there is $z \in \rho(A) \cap \rho(B)$ such that $(A-z)^{-1} - (B-z)^{-1}$ is a compact operator, then we say that B *is a compact perturbation of* A (see the end of this section for notations). If this is the case, then the difference $(A-z)^{-1} - (B-z)^{-1}$ is a compact operator for all $z \in \rho(A) \cap \rho(B)$. In particular, *if B is a compact perturbation of A, then A and B have the same essential spectrum*, the essential spectrum of a closed operator S being the set of complex numbers λ such that $S - \lambda$ is not Fredholm.

We are mainly interested in the case when A and B are differential operators with complex measurable coefficients which are equal at infinity in a rather weak sense. An important point then is that only a generalized version of the "quadratic form domain" of the operator is explicitly known and one has not much information about the domain of the operator.

To avoid this problem we shall work with operators constructed as follows. Let $\mathscr{G}, \mathscr{H}, \mathscr{K}$ be reflexive Banach spaces such that $\mathscr{G} \subset \mathscr{H} \subset \mathscr{K}$ continuously and densely and let A_0, B_0 be continuous maps $\mathscr{G} \to \mathscr{K}$. We assume that there is a complex number z such that $A_0 - z$ and $B_0 - z$ are bijective maps $\mathscr{G} \to \mathscr{K}$ and we define A and B as the restrictions of A_0 and B_0 to $\mathcal{D}(A) = A_0^{-1}\mathscr{H}$ and $\mathcal{D}(B) = B_0^{-1}\mathscr{H}$ respectively, considered as operators on $\mathscr{H}$. It is easy to see that these are closed densely defined operators on $\mathscr{H}$ such that $z \in \rho(A) \cap \rho(B)$ and our purpose is to find criteria such that $(A-z)^{-1} - (B-z)^{-1}$ be a compact operator.

We summarize now a standard way of proving such a fact. We assume, without loss of generality, that $z = 0$. Then $A_0^{-1} - B_0^{-1} = A_0^{-1}(B_0 - A_0)B_0^{-1}$ holds in $\mathcal{B}(\mathscr{K}, \mathscr{G})$ and we get $A^{-1} - B^{-1} = A_0^{-1}(B_0 - A_0)B^{-1}$ in $\mathcal{B}(\mathscr{H})$. Thus if $A_0 - B_0 : \mathscr{G} \to \mathscr{K}$ is compact then B is a compact perturbation of A (but much more is true, in fact $A_0^{-1} - B_0^{-1} : \mathscr{K} \to \mathscr{G}$ is also compact). Unfortunately this assumption is never fulfilled if A_0, B_0 are differential operators with distinct principal part (for the natural choices of $\mathscr{G}, \mathscr{K}$). This also excludes singular lower order perturbations, e.g. the important case of Dirac Hamiltonian's with Coulomb potentials.

If we require that $A_0 - B_0 : \mathcal{D}(B) \to \mathscr{K}$ be compact we get a more general compactness criterion ($\mathcal{D}(B)$ is equipped with the graph topology and we have $\mathcal{D}(B) \subset \mathscr{G}$ continuously and densely). Now perturbations of the principal part of a differential operator and singular lower order terms are not a priori excluded, cf. [17] for the Dirac case. However, in order to be able to use this criterion one must have some information about $\mathcal{D}(B)$ which is quite difficult to get if B is a singular differential operator.

In this paper we develop a new method for proving compactness of the difference $A^{-1} - B^{-1}$ which is efficient in situations where we have really no information concerning the domains of A and B (besides the fact that they are subspaces of $\mathscr{G}$). The case when A, B are second order elliptic operators with measurable complex coefficients acting in $\mathscr{H} = L^2(\mathbb{R}^n)$ has been studied by Ouhabaz and Stollmann in [18] and, as far as we know, this is the only paper where the "unperturbed" operator is not smooth. Their approach consists in proving that the difference $A^{-k} - B^{-k}$ is compact for some $k \geqslant 2$ (which implies the compactness of $A^{-1} - B^{-1}$). In order to prove this, they take advantage of the fact that $\mathcal{D}(A^k)$ is a subset of the Sobolev space $W^{1,p}$ for some $p > 2$, which means that we have a certain gain of local regularity. Of course, L^p techniques from the theory of partial differential equations are required for their methods to work.

Our approach to these questions is quite different, we explain here the main idea in the case of uniformly elliptic operators of order $2m$ in divergence form on $\mathbb{R}^n$. Let $\mathscr{H} = L^2$ and let $\mathscr{G}$ and $\mathscr{K}$ be the Sobolev spaces $\mathscr{H}^m$ and its adjoint $\mathscr{H}^{-m}$. We take $A_0 = D^*aD = \sum_{|\alpha|,|\beta| \leqslant m} P^\alpha a_{\alpha\beta} P^\beta$ (where $P_k = -\mathrm{i}\partial_k$) and $B_0 = D^*bD$ of a similar form. Assume for definiteness that $a_{\alpha\beta}$ are bounded measurable functions and that $\operatorname{Re}\langle u, A_0 u\rangle \geqslant \mu\|u\|^2_{\mathscr{H}^m}$ for some constant $\mu > 0$ and similarly for B_0. Then

$$R \equiv A^{-1} - B^{-1} = A_0^{-1} D^*(b-a) D B^{-1} \equiv S(b-a) D B^{-1}.$$

If $\mathscr{L} = \bigoplus_{|\alpha|\leqslant m} \mathscr{H}$ then $DB^{-1} : \mathscr{H} \to \mathscr{L}$ is bounded so it suffices to find a condition which ensures that $S(b-a) : \mathscr{L} \to \mathscr{H}$ is compact. Now $b-a$ is a bounded operator on $\mathscr{L}$ and S cannot be compact unless $b-a$ decays in some sense at infinity. So assume that we can factorize $b - a = \xi(Q)U$ where $U \in \mathcal{B}(\mathscr{L})$ and $\xi(Q)$ is the operator of multiplication by a function $\xi \in B_0(\mathbb{R}^n)$ (bounded Borel functions which vanish at infinity in the usual sense; but we stress that a much weaker sense is natural in this context, and this will be the case of main interest for us). Since $S : \mathscr{L} \to \mathscr{H}^m$ is bounded one can easily prove that $S\xi(Q) : \mathscr{L} \to \mathscr{H}$ is compact if and only if one can write $S\xi(Q) = \varphi(Q)T$ for some $\varphi \in B_0(\mathbb{R}^n)$ and $T \in \mathcal{B}(\mathscr{L}, \mathscr{H})$.

To conclude, R will be compact if the operator $S \in \mathcal{B}(\mathscr{L}, \mathscr{H})$ has the following property: for each $\xi \in B_0(\mathbb{R}^n)$ there are $\varphi \in B_0(\mathbb{R})$ and $T \in \mathcal{B}(\mathscr{L}, \mathscr{H})$ such that $S\xi(Q) = \varphi(Q)T$. An operator S with this property will be called *quasilocal*, or *decay preserving*, with respect to the notion of decay specified by the algebra B_0. Thus R is compact if S preserves decay. The main point of our approach is that this property of S holds under very general assumptions on A and for notions of decay at infinity more general than that specified by the algebra B_0.

An abstract formulation of these ideas allows one to treat situations of a very general nature: pseudo-differential operators on finite dimensional vector spaces over a local field, Laplace operators on manifolds with locally L^∞ Riemannian metrics, operators acting on sections of vector bundles over locally compact spaces. Sections 3, 4, 7 and 8 are devoted to such applications.

In Section 1 we present an algebraic formalism which allows us to treat in a unified way operators acting on sections of vector fiber bundles over a locally compact space. This framework allows one to study differential operators in L^p or more general Banach spaces. Since these extensions are rather obvious and the examples are not particularly interesting, we shall not consider explicitly such situations. The class of decay improving (or vanishing at infinity) operators is defined through an a priori given algebra of operators on a Banach space $\mathscr{H}$, the *multiplier algebra of* $\mathscr{H}$, and this allows us to define the notion of decay preserving operator in a natural and general context, that of Banach modules. Examples of multiplier algebras are given in Sections 3, 6 and 8. Section 2 contains abstract compactness criteria which formalize in the context of Banach modules the ideas involved in the example discussed above.

In Section 3 we construct Banach modules associated to representations of locally compact abelian groups and consider compact perturbations of a general class of operators of arbitrary order, for example the Dirac operators on $\mathbb{R}^n$. In Section 4 we discuss operators in divergence form on $\mathbb{R}^n$, hence of order $2m$ with

$m \geqslant 1$ integer, with coefficients of a rather general form (e.g. they do not have to be functions).

In Section 7 we present several results concerning the case when the coefficients of the operator $A - B$ vanish at infinity in a generalized sense (this question has been studied before, for example in [11], [14], [18], [21]). Theorem 7.4 is one of the main applications of our formalism: we prove a compactness result for irregular operators of order $2m$ in divergence form assuming that the difference between their coefficients weakly vanishes at infinity. Such results were known before only in the case $m = 1$, see especially Theorem 2.1 in [18]. We assume that the coefficients of the higher order terms are bounded, thus their Theorem 3.1 is not covered unless we add an implicit assumption, as is done in [18] (or in our Theorems 8.8 and 8.9). In fact, our main abstract compactness result (Theorem 2.1) is stated such as to apply to situations when the coefficients of the principal part of the operators are locally unbounded, as in [4], [5], but we have not developed this idea here.

We present the notion of weakly vanishing at infinity functions in terms of filters finer than the Fréchet filter, a natural idea in our context being to extend the standard notion of neighborhood of infinity. If X is a locally compact space, it is usual to define the filter of neighborhoods of infinity as the family of subsets of X with relatively compact complement; this is the Fréchet filter. If $\mathcal{F}$ is a filter on X finer than the Fréchet filter then a function $\varphi : X \to \mathbb{C}$ such that $\lim_{\mathcal{F}} \varphi = 0$ can naturally be thought as convergent to zero at infinity in a generalized sense (recall that $\lim_{\mathcal{F}} \varphi = 0$ means that for each $\varepsilon > 0$ the set of points x such that $|\varphi(x)| < \varepsilon$ belongs to $\mathcal{F}$). In Subsection 6 we consider several such filters and describe corresponding classes of decay preserving operators, see for example Theorem 6.1 and Theorem 6.5.

Theorem 6.5 is a consequence of a factorization theorem that we prove in Section 5 and which involves tools from the modern theory of Banach spaces. In fact, Theorem 5.6, the main result of Section 5, is a version of the "strong factorization theorem" of B. Maurey (see Theorem 5.1) which does not seem to be covered by the results existing in the literature. We also use Maurey's theorem directly to prove some of our main results, for example Theorems 8.8 and 8.9 which depend on Theorem 6.1.

In Section 8 we study perturbations of the Laplace operator on a Riemannian manifold with locally L^∞ metric. We consider an abstract model of this situation which fits naturally in our algebraic framework and covers the case of Lipschitz manifolds with measurable metrics. We consider in more detail the case when the manifold is C^1 (but the metric is only locally L^∞) and establish stability of the essential spectrum under certain perturbations of the metric, see Theorems 8.5, 8.8 and 8.9.

We mention that the preprint version [10] of this paper contains an alternative, sometimes more detailed, presentation of the topics considered here.

NOTATIONS. If $\mathscr{G}$ and $\mathscr{H}$ are Banach spaces then $\mathcal{B}(\mathscr{G},\mathscr{H})$ is the space of bounded linear operators $\mathscr{G} \to \mathscr{H}$, the subspace of compact operators is denoted $\mathcal{K}(\mathscr{G},\mathscr{H})$, and we set $\mathcal{B}(\mathscr{H}) = \mathcal{B}(\mathscr{H},\mathscr{H})$ and $\mathcal{K}(\mathscr{H}) = \mathcal{K}(\mathscr{H},\mathscr{H})$. The domain and the resolvent set of an operator S will be denoted by $\mathcal{D}(S)$ and $\rho(S)$ respectively. The norm of a Banach space $\mathscr{G}$ is denoted by $\|\cdot\|_{\mathscr{G}}$ and we omit the index if the space plays a central rôle. The adjoint space (space of antilinear continuous forms) of a Banach space $\mathscr{G}$ is denoted $\mathscr{G}^*$ and if $u \in \mathscr{G}$ and $v \in \mathscr{G}^*$ then we set $v(u) = \langle u, v\rangle$. The embedding $\mathscr{G} \subset \mathscr{G}^{**}$ is realized by defining $\langle v, u\rangle = \overline{\langle u, v\rangle}$.

If $\mathscr{G},\mathscr{H},\mathscr{K}$ are Banach spaces such that $\mathscr{G} \subset \mathscr{H}$ continuously and densely and $\mathscr{H} \subset \mathscr{K}$ continuously then we have an obvious continuous embedding $\mathcal{B}(\mathscr{H}) \hookrightarrow \mathcal{B}(\mathscr{G},\mathscr{K})$ which will be used without comment later on.

A *Friedrichs couple* $(\mathscr{G},\mathscr{H})$ is a pair of Hilbert spaces $\mathscr{G},\mathscr{H}$ together with a continuous dense embedding $\mathscr{G} \subset \mathscr{H}$. The *Gelfand triplet* associated to it is obtained by identifying $\mathscr{H} = \mathscr{H}^*$ with the help of the Riesz isomorphism and then taking the adjoint of the inclusion map $\mathscr{G} \to \mathscr{H}$. Thus we get $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ with continuous and dense embeddings. Now if $u \in \mathscr{G}$ and $v \in \mathscr{H} \subset \mathscr{G}^*$ then $\langle u, v\rangle$ is the scalar product in $\mathscr{H}$ of u and v and also the action of the functional v on u.

If X is a locally compact topological space then $B(X)$ is the C^*-algebra of bounded Borel complex functions on X, with norm $\sup_{x\in X}|\varphi(x)|$, and $B_0(X)$ is the subalgebra consisting of functions which tend to zero at infinity. Then $C(X)$, $C_0(X)$ and $C_c(X)$ are the spaces of complex functions on X which are continuous, continuous and convergent to zero at infinity, and continuous with compact support respectively. We denote χ_S the characteristic function of a set $S \subset X$.

1. BANACH MODULES

We use the terminology of [9] but with some abbreviations, e.g. a *morphism* is a linear multiplicative map between two algebras, and a $*$*-morphism* is a morphism between two $*$-algebras which commutes with the involutions. An algebra $\mathcal{M}$ of operators on a Banach space $\mathscr{H}$ is *non-degenerate* if $\{Mu : M \in \mathcal{M}, u \in \mathscr{H}\}$ is a total set in $\mathscr{H}$.

A *Banach module* is a couple $(\mathscr{H},\mathcal{M})$ consisting of a Banach space $\mathscr{H}$ and a non-degenerate Banach subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathscr{H})$ which has an approximate unit ([9], p. 404) . If $\mathscr{H}$ is a Hilbert space and $\mathcal{M}$ is a C^*-algebra of operators on $\mathscr{H}$, we say that $\mathscr{H}$ is a *Hilbert module* and we identify $\mathscr{H}^* = \mathscr{H}$. In general we just say that $\mathscr{H}$ is a Banach module (over $\mathcal{M}$). The distinguished subalgebra $\mathcal{M}$ is the *multiplier algebra of* $\mathscr{H}$ and, when required by the clarity of the presentation, it is denoted $\mathcal{M}(\mathscr{H})$. The operators from $\mathcal{M}$ are the prototype of decay improving, or vanishing at infinity, operators (so only the case when $\mathcal{M}$ does not have a unit is of interest). Often $\mathcal{M}$ is the norm closure of the range of a morphism

$Q : \mathcal{L} \to \mathcal{B}(\mathscr{H})$, where $\mathcal{L}$ is an algebra of complex bounded functions on a set; then we use the notation $Q(\varphi) = \varphi(Q)$.

If $\{J_\alpha\}$ is an approximate unit of $\mathcal{M}$, then the fact that $\mathcal{M}$ is non-degenerate is equivalent to $\lim_\alpha \|J_\alpha u - u\| = 0$ for all $u \in \mathscr{H}$. Due to the Cohen-Hewitt theorem ([9], V-9.2) each $u \in \mathscr{H}$ can be written as $u = Mv$ for some $M \in \mathcal{M}$ and $v \in \mathscr{H}$.

A *reflexive Banach module* is a Banach module such that the Banach space $\mathscr{H}$ is reflexive. Then $\mathscr{H}^*$ is a Banach module with $\mathcal{M}(\mathscr{H}^*) = \{A^* : A \in \mathcal{M}(\mathscr{H})\}$.

Two classes of operators are naturally associated to the Banach module structure: the operators which improve the decay and those which preserve the decay, the notion of decay being defined by the multiplier algebra. Let $\mathscr{H}$ and $\mathscr{K}$ be Banach spaces.

If $\mathscr{K}$ is a Banach module then we denote $\mathcal{B}_0^{\mathrm{l}}(\mathscr{H}, \mathscr{K})$ the norm closed linear subspace generated by the operators MT, with $T \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and $M \in \mathcal{M}(\mathscr{K})$. We say that an operator in $\mathcal{B}_0^{\mathrm{l}}(\mathscr{H}, \mathscr{K})$ *left vanishes at infinity* (or is *decay improving*) with respect to $\mathcal{M}(\mathscr{K})$, if this is not obvious from the context. If J_α is an approximate unit for $\mathcal{M}(\mathscr{K})$, then for an operator $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ we have:

$$\begin{aligned} S \in \mathcal{B}_0^{\mathrm{l}}(\mathscr{H}, \mathscr{K}) \quad &\Leftrightarrow \quad \lim_\alpha \|J_\alpha S - S\| = 0 \\ &\Leftrightarrow \quad S = MT \text{ for some } M \in \mathcal{M}(\mathscr{K}) \text{ and } T \in \mathcal{B}(\mathscr{H}, \mathscr{K}). \end{aligned} \tag{1.1}$$

The second equivalence follows from the Cohen-Hewitt theorem ([9], V-9.2).

If $\mathscr{H}$ is a Banach module then the space $\mathcal{B}_0^{\mathrm{r}}(\mathscr{H}, \mathscr{K})$ of *right vanishing at infinity* is similarly defined. If $\mathscr{H}$ and $\mathscr{K}$ are Banach modules then $\mathcal{B}_0^{\mathrm{l}}(\mathscr{H}, \mathscr{K})$ and $\mathcal{B}_0^{\mathrm{r}}(\mathscr{H}, \mathscr{K})$ are well defined and we set $\mathcal{B}_0(\mathscr{H}, \mathscr{K}) = \mathcal{B}_0^{\mathrm{l}}(\mathscr{H}, \mathscr{K}) \cap \mathcal{B}_0^{\mathrm{r}}(\mathscr{H}, \mathscr{K})$.

The following facts are easy to prove. We have $\mathcal{K}(\mathscr{H}, \mathscr{K}) \subset \mathcal{B}_0^{\mathrm{l}}(\mathscr{H}, \mathscr{K})$. If $\mathscr{K}$ is a reflexive Banach module and $S \in \mathcal{B}_0^{\mathrm{l}}(\mathscr{H}, \mathscr{K})$ then $S^* \in \mathcal{B}_0^{\mathrm{r}}(\mathscr{K}^*, \mathscr{H}^*)$. If $\mathscr{H}$ is a reflexive Banach module, then $\mathcal{K}(\mathscr{H}, \mathscr{K}) \subset \mathcal{B}_0^{\mathrm{r}}(\mathscr{H}, \mathscr{K})$.

If $\mathscr{H}, \mathscr{K}$ are Banach modules then we set

$$\mathcal{B}_q^{\mathrm{l}}(\mathscr{H}, \mathscr{K}) = \{S \in \mathcal{B}(\mathscr{H}, \mathscr{K}) : M \in \mathcal{M}(\mathscr{H}) \Rightarrow SM \in \mathcal{B}_0^{\mathrm{l}}(\mathscr{H}, \mathscr{K})\}.$$

This is the space of *left quasilocal*, or *left decay preserving*, operators. The space $\mathcal{B}_q^{\mathrm{r}}(\mathscr{H}, \mathscr{K})$ of *right quasilocal*, or *right decay preserving*, operators is similarly defined and we set $\mathcal{B}_q(\mathscr{H}, \mathscr{K}) = \mathcal{B}_q^{\mathrm{l}}(\mathscr{H}, \mathscr{K}) \cap \mathcal{B}_q^{\mathrm{r}}(\mathscr{H}, \mathscr{K})$. We clearly have:

PROPOSITION 1.1. *Let $\{J_\alpha\}$ be an approximate unit for $\mathcal{M}(\mathscr{H})$ and let S be an operator in $\mathcal{B}(\mathscr{H}, \mathscr{K})$. Then S is left decay preserving if and only if one of the following conditions is satisfied:*

(i) *$SJ_\alpha \in \mathcal{B}_0^{\mathrm{l}}(\mathscr{H}, \mathscr{K})$ for all α;*

(ii) *for each $M \in \mathcal{M}(\mathscr{H})$ there are $T \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and $N \in \mathcal{M}(\mathscr{K})$ such that $SM = NT$.*

PROPOSITION 1.2. *The spaces $\mathcal{B}_q^{\mathrm{l}}$ have the following properties:*

(i) *if $S \in \mathcal{B}_q^{\mathrm{l}}(\mathscr{H}, \mathscr{K})$ and $T \in \mathcal{B}_q^{\mathrm{l}}(\mathscr{G}, \mathscr{H})$ then $ST \in \mathcal{B}_q^{\mathrm{l}}(\mathscr{G}, \mathscr{K})$;*

(ii) *if* $\mathscr{H}, \mathscr{K}$ *are reflexive and* $S \in \mathcal{B}^{\mathrm{l}}_q(\mathscr{H}, \mathscr{K})$, *then* $S^* \in \mathcal{B}^{\mathrm{r}}_q(\mathscr{K}^*, \mathscr{H}^*)$;
(iii) *if* $\mathscr{H}$ *is a Hilbert module then* $\mathcal{B}_q(\mathscr{H})$ *is a unital* C^**-subalgebra of* $\mathcal{B}(\mathscr{H})$.

2. COMPACT PERTURBATIONS IN BANACH MODULES

A *Friedrichs module* is a couple $(\mathscr{G}, \mathscr{H})$ consisting of a Hilbert module $\mathscr{H}$ and a Hilbert space $\mathscr{G}$ such that $\mathscr{G} \subset \mathscr{H}$ continuously and densely. We say that $(\mathscr{G}, \mathscr{H})$ is a *compact Friedrichs module* if $\mathcal{M}(\mathscr{H}) \subset \mathcal{K}(\mathscr{G}, \mathscr{H})$. In such situations we always identify $\mathscr{H}^* = \mathscr{H}$ and thus get a Gelfand triplet $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$. If $(\mathscr{G}, \mathscr{H})$ is a compact Friedrichs module then each $M \in \mathcal{M}(\mathscr{H})$ extends to a compact operator $M : \mathscr{H} \to \mathscr{G}^*$, hence we have $\mathcal{M}(\mathscr{H}) \subset \mathcal{K}(\mathscr{G}, \mathscr{H}) \cap \mathcal{K}(\mathscr{H}, \mathscr{G}^*)$.

In this section *we fix a compact Friedrichs module* $(\mathscr{G}, \mathscr{H})$ and we set $\|\cdot\| = \|\cdot\|_{\mathscr{H}}$. The following easy to prove observation will be useful:

$$R \in \mathcal{B}^{\mathrm{l}}_0(\mathscr{H}) \quad \text{and} \quad R\mathscr{H} \subset \mathscr{G} \Rightarrow R \in \mathcal{K}(\mathscr{H}). \tag{2.1}$$

We are interested in criteria which ensure that an operator B is a compact perturbation of an operator A, both operators being unbounded operators in $\mathscr{H}$ obtained as restrictions of some bounded operators $\mathscr{G} \to \mathscr{G}^*$. The following is a general assumption (suggested by the statement of Theorem 2.1 in [18]) which will always be fulfilled:

$$\text{(AB)} \quad \begin{cases} A, B \text{ are densely defined operators in } \mathscr{H},\ \rho(A) \cap \rho(B) \neq \emptyset \text{ and:} \\ (1)\ \mathcal{D}(A) \subset \mathscr{G} \text{ densely, } \mathcal{D}(A^*) \subset \mathscr{G},\ \mathcal{D}(B) \subset \mathscr{G}; \\ (2)\ A, B \text{ extend to continuous operators } \widetilde{A}, \widetilde{B} \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*). \end{cases}$$

THEOREM 2.1. *Let* A, B *satisfy assumption* (AB) *and let us assume that there are a Banach module* $\mathscr{K}$ *and operators* $S \in \mathcal{B}(\mathscr{K}, \mathscr{G}^*)$ *and* $T \in \mathcal{B}^{\mathrm{l}}_0(\mathscr{G}, \mathscr{K})$ *such that* $\widetilde{B} - \widetilde{A} = ST$ *and* $(A - z)^{-1}S \in \mathcal{B}^{\mathrm{l}}_q(\mathscr{K}, \mathscr{H})$ *for some* $z \in \rho(A) \cap \rho(B)$. *Then the operator* B *is a compact perturbation of the operator* A *and* $\sigma_{\mathrm{ess}}(B) = \sigma_{\mathrm{ess}}(A)$.

Proof. The rôle of the assumption (AB) is to allow us to give a rigorous meaning to the formal relation, where $z \in \rho(A) \cap \rho(B)$,

$$(A - z)^{-1} - (B - z)^{-1} = (A - z)^{-1}(B - A)(B - z)^{-1}. \tag{2.2}$$

Recall that $z \in \rho(A)$ if and only if $\bar{z} \in \rho(A^*)$ and then $(A^* - \bar{z})^{-1} = (A - z)^{-1*}$. Thus we have $(A - z)^{-1*}\mathscr{H} \subset \mathscr{G}$ by the assumption (AB) and this allows one to deduce that $(A - z)^{-1}$ extends to a unique continuous operator $\mathscr{G}^* \to \mathscr{H}$, that we shall denote for the moment by R_z. From $R_z(A - z)u = u$ for $u \in \mathcal{D}(A)$ we get, by density of $\mathcal{D}(A)$ in $\mathscr{G}$ and continuity, $R_z(\widetilde{A} - z)u = u$ for $u \in \mathscr{G}$, in particular

$$(B - z)^{-1} = R_z(\widetilde{A} - z)(B - z)^{-1}.$$

On the other hand, the identity

$$(A - z)^{-1} = (A - z)^{-1}(B - z)(B - z)^{-1} = R_z(\widetilde{B} - z)(B - z)^{-1}$$

is trivial. Subtracting the last two relations we get

$$(A-z)^{-1}-(B-z)^{-1}=R_z(\widetilde{B}-\widetilde{A})(B-z)^{-1}$$

Since R_z is uniquely determined as extension of $(A-z)^{-1}$ to a continuous map $\mathscr{G}^*\to\mathscr{H}$, we shall keep the notation $(A-z)^{-1}$ for it. With this convention, the rigorous version of (2.2) that we shall use is:

$$(A-z)^{-1}-(B-z)^{-1}=(A-z)^{-1}(\widetilde{B}-\widetilde{A})(B-z)^{-1}. \tag{2.3}$$

We show that $R\equiv(A-z)^{-1}-(B-z)^{-1}\in\mathcal{B}_0^1(\mathscr{H})$. This suffices to prove the theorem because the domains of A and B are included in $\mathscr{G}$, hence $R\mathscr{H}\subset\mathscr{G}$, and we may use (2.1). Now due to (2.3) and to the factorization assumption, we can write R as a product $R=[(A-z)^{-1}S][T(B-z)^{-1}]$ where the first factor is in $\mathcal{B}_q^1(\mathscr{K},\mathscr{H})$ and the second in $\mathcal{B}_0^1(\mathscr{H},\mathscr{K})$, so the product is in $\mathcal{B}_0^1(\mathscr{H})$. ∎

REMARKS 2.2. (i) We could have stated the assumptions of Theorem 2.1 in an apparently more general form, namely $B-A=\sum_{k=1}^{n}S_kT_k$ with operators $S_k\in\mathcal{B}(\mathscr{K}_k,\mathscr{G}^*)$ and $T_k\in\mathcal{B}(\mathscr{G},\mathscr{K}_k)$. But we are reduced to the stated version of the assumption by considering the Hilbert module $\mathscr{K}=\bigoplus\mathscr{K}_k$ and $S=\bigoplus S_k, T=\bigoplus T_k$.

(ii) If $V\in\mathcal{K}(\mathscr{G},\mathscr{G}^*)$ and if $\mathscr{K}$ is an infinite dimensional module, then there are operators $S\in\mathcal{B}(\mathscr{K},\mathscr{G}^*)$ and $T\in\mathcal{K}(\mathscr{G},\mathscr{K})$ such that $V=ST$ (the proof is an easy exercise). This and the preceding remark show that compact contributions to $\widetilde{B}-\widetilde{A}$ are trivially covered by the factorization assumption.

EXAMPLE 2.3. We construct operators with the properties required in (AB) by the following method. Let $\mathscr{G}_a$, $\mathscr{G}_b$ be Hilbert spaces with $\mathscr{G}\subset\mathscr{G}_a\subset\mathscr{H}$ and $\mathscr{G}\subset\mathscr{G}_b\subset\mathscr{H}$ continuously and densely. Thus we have two scales

$$\mathscr{G}\subset\mathscr{G}_a\subset\mathscr{H}\subset\mathscr{G}_a^*\subset\mathscr{G}^* \quad\text{and}\quad \mathscr{G}\subset\mathscr{G}_b\subset\mathscr{H}\subset\mathscr{G}_b^*\subset\mathscr{G}^*.$$

Then let $A_0\in\mathcal{B}(\mathscr{G}_a,\mathscr{G}_a^*)$ and $B_0\in\mathcal{B}(\mathscr{G}_b,\mathscr{G}_b^*)$ such that $A_0-z:\mathscr{G}_a\to\mathscr{G}_a^*$ and $B_0-z:\mathscr{G}_b\to\mathscr{G}_b^*$ are bijective for some number z. By Lemma 2.4 we can associate to A_0, B_0 closed densely defined operators $A=\widehat{A}_0, B=\widehat{B}_0$ in $\mathscr{H}$, such that $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ are dense subspaces of $\mathscr{G}_a$ and $\mathcal{D}(B)$ and $\mathcal{D}(B^*)$ are dense subspaces of $\mathscr{G}_b$. If we also have $\mathcal{D}(A)\subset\mathscr{G}$ densely, $\mathcal{D}(A^*)\subset\mathscr{G}$ and $\mathcal{D}(B)\subset\mathscr{G}$, then all the conditions of the assumption (AB) are fulfilled with $\widetilde{A}=A_0|\mathscr{G}$ and $\widetilde{B}=B_0|\mathscr{G}$.

The next results require some preliminary considerations on operators acting in a Gelfand triplet. To an operator $S\in\mathcal{B}(\mathscr{G},\mathscr{G}^*)$ (which is the same as a continuous sesquilinear form on $\mathscr{G}$) we associate an operator $\widehat{S}$ acting in $\mathscr{H}$ according to the rules: $\mathcal{D}(\widehat{S})=S^{-1}(\mathscr{H})$, $\widehat{S}=S|\mathcal{D}(\widehat{S})$. Due to the identification $\mathscr{G}^{**}=\mathscr{G}$, the operator S^* is an element of $\mathcal{B}(\mathscr{G},\mathscr{G}^*)$, so $\widehat{S^*}$ makes sense. On the other hand, if $\widehat{S}$ is densely defined in $\mathscr{H}$ then the adjoint $\widehat{S}^*$ of $\widehat{S}$ with respect to $\mathscr{H}$ is also well defined and we clearly have $\widehat{S^*}\subset\widehat{S}^*$.

The proof of the next lemma is an easy exercise.

LEMMA 2.4. *If $S - z : \mathscr{G} \to \mathscr{G}^*$ is bijective for some $z \in \mathbb{C}$, then $\widehat{S}$ is a closed densely defined operator, we have $\widehat{S}^* = \widehat{S^*}$ and $z \in \rho(\widehat{S})$. Moreover, the domains $\mathcal{D}(\widehat{S})$ and $\mathcal{D}(\widehat{S}^*)$ are dense subspaces of $\mathscr{G}$.*

A standard example of operator satisfying the condition required in Lemma 2.4 is a *coercive* operator, i.e. such that $\mathrm{Re}\,\langle u, Su\rangle \geqslant \mu\|u\|^2_{\mathscr{G}} - \nu\|u\|^2_{\mathscr{H}}$ for some strictly positive constants μ, ν and all $u \in \mathscr{G}$. Indeed, replacing S by $S + \nu$, we may assume $\mathrm{Re}\,\langle u, Su\rangle \geqslant \mu\|u\|^2_{\mathscr{G}}$. Since S^* verifies the same estimate, this clearly gives $\|Su\|_{\mathscr{G}^*} \geqslant \mu\|u\|_{\mathscr{G}}$ and $\|S^*u\|_{\mathscr{G}^*} \geqslant \mu\|u\|_{\mathscr{G}}$ for all $u \in \mathscr{G}$. Thus S and S^* are injective operators with closed range, hence they are bijective.

If the operators A, B from Theorem 2.1 are self-adjoint then the condition (AB) becomes quite explicit. Indeed, to each self-adjoint A one associates the Gelfand triplet $\mathcal{D}(|A|^{1/2}) \subset \mathscr{H} \subset \mathcal{D}(|A|^{1/2})^*$ and A extends to a continuous operator $A_0 : \mathcal{D}(|A|^{1/2}) \to \mathcal{D}(|A|^{1/2})^*$ which fulfills the conditions of Lemma 2.4, one has $\widehat{A}_0 = A$, and we may ask $\mathcal{D}(A) \subset \mathscr{G} \subset \mathcal{D}(|A|^{1/2})$.

The next result is convenient for applications to differential operators in divergence form. Recall that if $(\mathscr{E}, \mathscr{K})$ is a Friedrichs module then we have a natural embedding $\mathcal{B}(\mathscr{K}) \subset \mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ so the space

$$\mathcal{B}^1_{00}(\mathscr{E}, \mathscr{E}^*) = \text{ norm closure of } \mathcal{B}^1_0(\mathscr{K}) \text{ in } \mathcal{B}(\mathscr{E}, \mathscr{E}^*) \tag{2.4}$$

is well defined. The operators D^*aD and D^*bD considered below belong to $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ and we denote by Δ_a and Δ_b the operators on $\mathscr{H}$ associated to them as explained above. These operators are closed and densely defined (Lemma 2.4).

THEOREM 2.5. *Let $(\mathscr{E}, \mathscr{K})$ be an arbitrary Friedrichs module and let us assume that $D \in \mathcal{B}(\mathscr{G}, \mathscr{E})$, $a, b \in \mathcal{B}(\mathscr{E}, \mathscr{E}^*)$, and $z \in \mathbb{C}$ are such that:*
(i) *the operators $D^*aD - z$ and $D^*bD - z$ are bijective maps $\mathscr{G} \to \mathscr{G}^*$;*
(ii) *$a - b \in \mathcal{B}^1_{00}(\mathscr{E}, \mathscr{E}^*)$;*
(iii) *$D(\Delta^*_a - \bar{z})^{-1} \in \mathcal{B}^{\mathrm{r}}_q(\mathscr{H}, \mathscr{K})$.*
Then Δ_b is a compact perturbation of Δ_a.

Proof. Clearly $\Delta_a - z$ and $\Delta_b - z$ extend to bijections $\mathscr{G} \to \mathscr{G}^*$ and

$$R := (\Delta_a - z)^{-1} - (\Delta_b - z)^{-1} = (\Delta_a - z)^{-1} D^*(b - a) D(\Delta_b - z)^{-1}$$

holds in $\mathcal{B}(\mathscr{G}^*, \mathscr{G})$, hence in $\mathcal{B}(\mathscr{H})$. Since the domains of Δ_a and Δ_b are included in $\mathscr{G}$, we have $R\mathscr{H} \subset \mathscr{G}$. Thus, according to (2.1), it suffices to show that $R \in \mathcal{B}^1_0(\mathscr{H})$. Since the space $\mathcal{B}^1_0(\mathscr{H})$ is norm closed and since by hypothesis we can approach $b - a$ in norm in $\mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ by operators in $\mathcal{B}^1_0(\mathscr{K})$, it suffices to show

$$(D(\Delta^*_a - \bar{z})^{-1})^* c D(\Delta_b - z)^{-1} \in \mathcal{B}^1_0(\mathscr{H})$$

if $c \in \mathcal{B}^1_0(\mathscr{K})$. But this is clear because $cD(\Delta_b - z)^{-1}$ belongs to $\mathcal{B}^1_0(\mathscr{H}, \mathscr{K})$ and $(D(\Delta^*_a - \bar{z})^{-1})^*$ belongs to $\mathcal{B}^1_q(\mathscr{K}, \mathscr{H})$ by Proposition 1.2. ∎

The spaces $\mathcal{B}^{\mathrm{r}}_{00}(\mathscr{E},\mathscr{E}^*)$ and $\mathcal{B}_{00}(\mathscr{E},\mathscr{E}^*)$ are defined in an obvious way and we have

$$\mathcal{K}(\mathscr{E},\mathscr{E}^*) \subset \mathcal{B}_{00}(\mathscr{E},\mathscr{E}^*) \tag{2.5}$$

because $\mathcal{K}(\mathscr{K})$ is a dense subset of $\mathcal{K}(\mathscr{E},\mathscr{E}^*)$ and $\mathcal{K}(\mathscr{K}) \subset \mathcal{B}_0(\mathscr{K})$. So we could assume $a - b \in \mathcal{K}(\mathscr{E},\mathscr{E}^*)$, but this case is trivial in the context of this paper. Although the space $\mathcal{B}^{\mathrm{l}}_{00}(\mathscr{E},\mathscr{E}^*)$ is much larger than $\mathcal{K}(\mathscr{E},\mathscr{E}^*)$, it is not satisfactory in some applications, cf. Remark 4.5. However, we can allow still more general perturbations and obtain more explicit results if we impose more structure on the modules. In Section 3 we describe such improvements for a class of Banach modules over abelian groups.

Condition (iii) of Theorem 2.5 is not easy to verify in the situations of interest to us, so we describe now a perturbative method for checking it. For the rest of this section we fix two Friedrichs modules $(\mathscr{G},\mathscr{H})$ and $(\mathscr{E},\mathscr{K})$ and a continuous operator $D : \mathscr{G} \to \mathscr{E}$. Let $a \in \mathcal{B}(\mathscr{E},\mathscr{E}^*)$ such that the operator D^*aD is coercive, i.e.

$$\mathrm{Re}\,\langle Du, aDu\rangle \geqslant \mu\|u\|^2_{\mathscr{G}} - \nu\|u\|^2_{\mathscr{H}} \tag{2.6}$$

for some strictly positive constants μ,ν and all $u \in \mathscr{G}$. Then, as explained before, if $\mathrm{Re}\, z \leqslant -\nu$ the operator $D^*aD - z$ is a bijective map $\mathscr{G} \to \mathscr{G}^*$ and

$$\|(D^*aD - z)^{-1}\|_{\mathcal{B}(\mathscr{G}^*,\mathscr{G})} \leqslant \mu^{-1}. \tag{2.7}$$

Note that a^* has all these properties too so the closed densely defined operators Δ_a and Δ_{a^*} in $\mathscr{H}$ are well defined, their domains are dense subsets of $\mathscr{G}$, and we have $\Delta_a^* = \Delta_{a^*}$. It is easy to check that $\|(\Delta_a - z)^{-1}\|_{\mathcal{B}(\mathscr{H})} \leqslant |\mathrm{Re}\, z + \nu|^{-1}$ if $\mathrm{Re}\,(z+\nu) < 0$. Since a and a^* play a symmetric role, it will suffice to consider $\Delta_a - z$ in place of $\Delta_a^* - \bar{z}$ in condition (iii) of Theorem 2.5.

Now let c be a second operator with the same properties as a. We assume, without loss of generality, that it satisfies an estimate like (2.6) with the same constants μ,ν.

PROPOSITION 2.6. *Assume that*

$$D(\Delta_c - z)^{-1} \in \mathcal{B}^{\mathrm{r}}_q(\mathscr{H},\mathscr{K}) \quad \text{and} \quad D(D^*cD - z)^{-1}D^* \in \mathcal{B}^{\mathrm{r}}_q(\mathscr{K})$$

for some z with $\mathrm{Re}\, z \leqslant -\nu$. *If $a - c \in \mathcal{B}^{\mathrm{r}}_q(\mathscr{K})$ then*

$$D(\Delta_a - z)^{-1} \in \mathcal{B}^{\mathrm{r}}_q(\mathscr{H},\mathscr{K}) \quad \text{and} \quad D(D^*aD - z)^{-1}D^* \in \mathcal{B}^{\mathrm{r}}_q(\mathscr{K}).$$

A similar assertion holds for the spaces $\mathcal{B}^{\mathrm{l}}_q$.

Proof. Let $V = D^*(a-c)D$ and $L_t = (1-t)D^*cD + tD^*aD = D^*cD + tV$. For z as in the statement of the proposition we have $\mathrm{Re}\,\langle u,(L_t - z)u\rangle \geqslant \mu\|u\|^2_{\mathscr{G}}$ if $0 \leqslant t \leqslant 1$. Hence there is $\varepsilon > 0$ such that $\mathrm{Re}\,\langle u,(L_t - z)u\rangle \geqslant \mu/2\|u\|^2_{\mathscr{G}}$ if $-\varepsilon \leqslant t \leqslant 1+\varepsilon$, in particular $\|(L_t - z)^{-1}\|_{\mathcal{B}(\mathscr{G}^*,\mathscr{G})} \leqslant 2/\mu$ for all such t. If $-\varepsilon \leqslant s \leqslant 1+\varepsilon$ and

$|t-s|\|V(L_t-z)^{-1}\|_{\mathcal{B}(\mathscr{G}^*,\mathscr{G})} < 1$ we get a norm convergent expansion in $\mathcal{B}(\mathscr{G}^*,\mathscr{G})$

$$(L_t - z)^{-1} = (L_s - z - (s-t)V)^{-1} = \sum_{k\geqslant 0}(s-t)^k (L_s - z)^{-1}[V(L_s-z)^{-1}]^k$$

so the map $t \mapsto (L_t - z)^{-1} \in \mathcal{B}(\mathscr{G}^*,\mathscr{G})$ is real analytic on the interval $]-\varepsilon, 1+\varepsilon[$. Let us denote Δ_t the operator in $\mathscr{H}$ associated to L_t then we see that the maps $t \mapsto D(\Delta_t - z)^{-1} \in \mathcal{B}(\mathscr{H},\mathscr{K})$ and $t \mapsto D(L_t-z)^{-1}D^* \in \mathcal{B}(\mathscr{K})$ are real analytic on the same interval. The set of decay preserving operators is a closed subspace of the Banach space $\mathcal{B}(\mathscr{H},\mathscr{K})$ and an analytic function which on a nonempty open set takes values in a closed subspace remains in that subspace for ever. Thus it suffices to show that $D(\Delta_t - z)^{-1} \in \mathcal{B}^{\mathrm{r}}_q(\mathscr{H},\mathscr{K})$ for small positive values of t. Similarly, we need to prove $D(L_t-z)^{-1}D^* \in \mathcal{B}(\mathscr{K})$ only for small t. To prove the first assertion for example, we take $s=0$ above and get a norm convergent series in $\mathcal{B}(\mathscr{H},\mathscr{K})$:

$$D(L_t - z)^{-1} = \sum_{k\geqslant 0}(-t)^k D(D^*cD - z)^{-1}[D^*(a-c)D(D^*cD-z)^{-1}]^k.$$

It is clear that each term belongs to $\mathcal{B}^{\mathrm{r}}_q(\mathscr{H},\mathscr{K})$. ∎

3. BANACH MODULES OVER ABELIAN GROUPS

In this section we fix a locally compact non-compact abelian group X with the group operation denoted additively and let X^* be its dual group. For example, X could be $\mathbb{R}^n, \mathbb{Z}^n$, or a finite dimensional vector space over the field of p-adic numbers. A *Banach X-module* over the group X is a Banach space $\mathscr{H}$ equipped with a strongly continuous representation $\{V_k\}$ of X^* on $\mathscr{H}$. If $\mathscr{H}$ is a Hilbert space and the V_k are unitary operators we say that $\mathscr{H}$ is a *Hilbert X-module*. Note that we shall use the same notation V_k for the representations of X^* in different spaces $\mathscr{H}$ whenever this does not lead to ambiguities.

Such *a Banach X-module has a canonical structure of Banach module* that we now define. We choose Haar measures $\mathrm{d}x$ and $\mathrm{d}k$ on X and X^* normalized by the following condition: if the Fourier transform of a function φ on X is given by $(\mathcal{F}\varphi)(k) \equiv \widehat{\varphi}(k) = \int_X \overline{k(x)}\varphi(x)\mathrm{d}x$ then $\varphi(x) = \int_{X^*} k(x)\widehat{\varphi}(k)\mathrm{d}k$. Recall that $X^{**} = X$. Let $C^{(a)}(X) := \mathcal{F}L^1_{\mathrm{c}}(X^*)$ be the set of Fourier transforms of integrable functions with compact support on X^*. It is easy to see that $C^{(a)}(X)$ is a $*$-algebra for the usual algebraic operations; more precisely, it is a dense subalgebra of $C_0(X)$ invariant under conjugation. For $\varphi \in C^{(a)}(X)$ we set

$$\varphi(Q) = \int_{X^*} V_k \widehat{\varphi}(k)\mathrm{d}k. \tag{3.1}$$

This definition is determined by the formal requirement $k(Q) = V_k$. Then

$$\mathcal{M} := \text{ norm closure of } \{\varphi(Q) : \varphi \in C^{(a)}(X)\} \text{ in } \mathcal{B}(\mathscr{H}) \tag{3.2}$$

is a Banach subalgebra of $\mathcal{B}(\mathscr{H})$ which has an approximate unit consisting of elements of the form $e_\alpha(Q)$ with $e_\alpha \in C^{(a)}(X)$ (let α be a compact neighborhood of the identity in X^* and $\widehat{e}_\alpha = \chi_\alpha/|\alpha|$, where $|\alpha|$ is the Haar measure). Thus the couple $(\mathscr{H}, \mathcal{M})$ is a Banach module, which gives us a canonical Banach module structure on $\mathscr{H}$.

The adjoint of a reflexive Banach X-module has a natural structure of Banach X-module. Indeed, a weakly continuous representation is strongly continuous, so we can equip the adjoint space $\mathscr{H}^*$ with the Banach X-module structure defined by $k \mapsto (V_{\overline{k}})^*$, where $\overline{k} = k^{-1}$ is the complex conjugate of k.

We show now that, in the case of Banach X-modules over groups, the decay preserving property is related to regularity in the sense of the next definition. Let $\mathscr{H}$ and $\mathscr{K}$ be Banach X-modules. We say that a continuous operator $S : \mathscr{H} \to \mathscr{K}$ *is of class* $C^{\mathrm{u}}(Q)$, and we write $S \in C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{K})$, if the map $k \mapsto V_k^{-1} S V_k \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ is norm continuous. The class of regular operators is stable under algebraic operations:

PROPOSITION 3.1. *Let $\mathscr{G}, \mathscr{H}, \mathscr{K}$ be Banach X-modules.*
(i) *If $S \in C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{K})$ and $T \in C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{H})$ then $ST \in C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{K})$.*
(ii) *If $S \in C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{K})$ is bijective, then $S^{-1} \in C^{\mathrm{u}}(Q; \mathscr{K}, \mathscr{H})$.*
(iii) *If $S \in C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{K})$ and $\mathscr{H}, \mathscr{G}$ are reflexive, then $S^* \in C^{\mathrm{u}}(Q; \mathscr{K}^*, \mathscr{H}^*)$.*

PROPOSITION 3.2. *If $T \in C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{K})$ then T is decay preserving.*

Proof. We show that $\varphi(Q)T \in \mathcal{B}_0^{\mathrm{r}}(\mathscr{H}, \mathscr{K})$ if $\varphi \in C^{(a)}(X)$. A similar argument gives $T\varphi(Q) \in \mathcal{B}_0^{\mathrm{l}}(\mathscr{H}, \mathscr{K})$. Set $T_k = V_k T V_k^{-1}$, then

$$\varphi(Q)T = \int_{X^*} \widehat{\varphi}(k) V_k T \mathrm{d}k = \int_{X^*} T_k \widehat{\varphi}(k) V_k \mathrm{d}k.$$

Since $k \mapsto T_k$ is norm continuous on the compact support of $\widehat{\varphi}$, for each $\varepsilon > 0$ we can construct, with the help of a partition of unity, functions $\theta_i \in C_{\mathrm{c}}(X^*)$ and operators $S_i \in \mathcal{B}(\mathscr{H}, \mathscr{K})$, such that $\left\| T_k - \sum_{i=1}^n \theta_i(k) S_i \right\| < \varepsilon$ if $\widehat{\varphi}(k) \neq 0$. Thus

$$\left\| \varphi(Q)T - \sum_{i=1}^n \int_{X^*} \theta_i(k) S_i \widehat{\varphi}(k) V_k \mathrm{d}k \right\| \leqslant \varepsilon \sum_{i=1}^n \int_{X^*} |\widehat{\varphi}(k)| \|V_k\|_{\mathcal{B}(\mathscr{H})} \mathrm{d}k.$$

Now, since $\mathcal{B}_0^{\mathrm{r}}(\mathscr{H}, \mathscr{K})$ is a norm closed subspace, it suffices to show that the operator $\int_{X^*} \theta_i(k) S_i \widehat{\varphi}(k) V_k \mathrm{d}k$ belongs to $\mathcal{B}_0^{\mathrm{r}}(\mathscr{H}, \mathscr{K})$ for each i. But if ψ_i is the inverse Fourier transform of $\theta_i \widehat{\varphi}$ then this is $S_i \psi_i(Q)$ and $\psi_i \in C^{(a)}(X)$. ∎

Note that one can develop a pseudo-differential calculus on X and Proposition 3.2 shows that many pseudo-differential operators on $L^p(X)$ are decay preserving.

Our next purpose is to improve the results of Section 2 in the present context. A *Friedrichs X-module* over the group X is a Friedrichs couple $(\mathscr{G},\mathscr{H})$ such that $\mathscr{H}$ is a Hilbert X-module with the following properties: $V_k\mathscr{G}\subset\mathscr{G}$ for all $k\in X^*$ and if $u\in\mathscr{G}$ and $K\subset X^*$ is compact then $\sup_{k\in K}\|V_k u\|_{\mathscr{G}}<\infty$. It is clear that $V_k\mathscr{G}\subset\mathscr{G}$ implies $V_k\in\mathcal{B}(\mathscr{G})$ and that the local boundedness condition implies that the map $k\mapsto V_k\in\mathcal{B}(\mathscr{G})$ is a weakly, hence strongly, continuous representation of X^* on $\mathscr{G}$ (not unitary in general). The local boundedness condition is automatically satisfied if X^* is second countable.

Thus, *if $(\mathscr{G},\mathscr{H})$ is a Friedrichs X-module, then $\mathscr{G}$ is equipped with a canonical Banach X-module structure*. Then, by taking adjoints, we get a natural Banach X-module structure on $\mathscr{G}^*$ too. Our definitions are such that after the identifications $\mathscr{G}\subset\mathscr{H}\subset\mathscr{G}^*$ the restriction to $\mathscr{H}$ of the operator V_k acting in $\mathscr{G}^*$ is just the initial V_k. Indeed, we have $V_k^*=V_k^{-1}=V_{\bar{k}}$ in $\mathscr{H}$. Thus there is no ambiguity in using the same notation V_k for the representation of X^* in the spaces $\mathscr{G},\mathscr{H}$ and $\mathscr{G}^*$.

PROPOSITION 3.3. *If $\mathscr{K}$ is a Banach space then $\mathcal{B}^1_0(\mathscr{K},\mathscr{G})\subset\mathcal{B}^1_0(\mathscr{K},\mathscr{H})$, and if $\mathscr{K}$ is a Banach module then $\mathcal{B}^1_q(\mathscr{K},\mathscr{G})\subset\mathcal{B}^1_q(\mathscr{K},\mathscr{H})$.*

Proof. We start with some general remarks. Assume that $\mathcal{A}$ is a Banach algebra with approximate unit and that a morphism $\Phi:\mathcal{A}\to\mathcal{M}(\mathscr{H})$ with dense image is given. Then, by the Cohen-Hewitt theorem ([9], V-9.2) , each $u\in\mathscr{H}$ can be written as $u=Av$ where $A\in\Phi(\mathcal{A})$ and $v\in\mathscr{H}$. Many such algebras can be constructed in the present context. Indeed, if ω is a sub-multiplicative function on X^*, i.e. a Borel map $X^*\to[1,\infty[$ satisfying $\omega(k'k'')\leqslant\omega(k')\omega(k'')$ (hence ω is locally bounded), let $C^{(\omega)}(X)$ be the set of functions φ whose Fourier transform $\widehat{\varphi}$ satisfies

$$\|\varphi\|_{C^{(\omega)}}:=\int_{X^*}|\widehat{\varphi}(k)|\omega(k)\mathrm{d}k<\infty. \tag{3.3}$$

Then $C^{(\omega)}(X)$ is a subalgebra of $C_0(X)$ and is a Banach algebra for the norm (3.3). Moreover, $C^{(a)}(X)\subset C^{(\omega)}(X)$ densely and the net $\{e_\alpha\}$ defined after (3.2) is an approximate unit of $C^{(\omega)}(X)$. If $\|V_k\|_{\mathcal{B}(\mathscr{H})}\leqslant c\omega(k)$ for some number $c>0$ then $\varphi(Q)$ is well defined for each $\varphi\in C^{(\omega)}(X)$ by the relation (3.1) and $\Phi(\varphi)=\varphi(Q)$ is a continuous morphism with dense range of $C^{(\omega)}(X)$ into $\mathcal{M}(\mathscr{H})$.

Now the proposition follows easily. If $S\in\mathcal{B}^1_0(\mathscr{K},\mathscr{G})$ then $S=\varphi(Q)T$ for some $\varphi\in C^{(\omega)}(X)$ with $\omega(k)=\sup(1,\|V_k\|_{\mathcal{B}(\mathscr{G})})$ and some $T\in\mathcal{B}(\mathscr{K},\mathscr{G})$. But clearly $\varphi(Q)$ belongs to the multiplier algebra of $\mathscr{H}$ and $T\in\mathcal{B}(\mathscr{K},\mathscr{H})$. ∎

REMARK 3.4. Giving a Hilbert X-module structure on a Hilbert space $\mathscr{H}$ is equivalent with giving a morphism with non-degenerate range $\varphi\mapsto\varphi(Q)$ from the set of all bounded Borel functions on X into $\mathcal{B}(\mathscr{H})$. The relation between the two structures is determined by the condition $V_k = k(Q)$. The non-trivial part of this assertion follows from the estimate $\|\varphi(Q)\| \leqslant \sup|\varphi|$ if $\varphi(Q)$ is defined as in (3.1), see [15]. Then it is easy to check that $(\mathscr{G}, \mathscr{H})$ is a compact Friedrichs module if and only if we have $\varphi(Q) \in \mathcal{K}(\mathscr{G}, \mathscr{H})$ for all $\varphi \in C_0(X)$.

The following is a corollary of Theorem 2.1 which covers interesting examples of differential operators of any order. Then we shall treat operators in divergence form.

THEOREM 3.5. *Let $(\mathscr{G}, \mathscr{H})$ be a compact Friedrichs X-module such that condition* (AB) *from page* 121 *is satisfied. Assume also that $\widetilde{A} - z : \mathscr{G} \to \mathscr{G}^*$ is bijective for some $z \in \rho(A) \cap \rho(B)$ and that $\widetilde{A} \in C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{G}^*)$. If $\widetilde{B} - \widetilde{A} \in \mathcal{B}_0^1(\mathscr{G}, \mathscr{G}^*)$, then B is a compact perturbation of A.*

Proof. We apply Theorem 2.1 with $\mathscr{K} = \mathscr{G}^*$, S the identity operator and $T = \widetilde{B} - \widetilde{A}$. Then $(\widetilde{A} - z)^{-1}$ is of class $C^{\mathrm{u}}(Q; \mathscr{G}^*, \mathscr{G})$ by (ii) of Proposition 3.1, hence $(\widetilde{A} - z)^{-1} \in \mathcal{B}_q(\mathscr{G}^*, \mathscr{G})$ by Proposition 3.2. But this is stronger than the relation $(\widetilde{A} - z)^{-1} \in \mathcal{B}_q^1(\mathscr{G}^*, \mathscr{H})$, as follows from Proposition 3.3. ∎

LEMMA 3.6. *Let $(\mathscr{G}, \mathscr{H})$ and $(\mathscr{E}, \mathscr{K})$ be Friedrichs X-modules over the group X. Let $D \in \mathcal{B}(\mathscr{G}, \mathscr{E})$ and $a \in \mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ be of class $C^{\mathrm{u}}(Q)$ and such that the map $D^*aD - z : \mathscr{G} \to \mathscr{G}^*$ is bijective for some number z. If Δ_a is the operator on $\mathscr{H}$ associated to D^*aD, then the operator $D(\Delta_a - z)^{-1} \in \mathcal{B}(\mathscr{H}, \mathscr{E})$ is decay preserving.*

Proof. The lemma is a consequence of Propositions 3.1 and 3.2. Indeed, due to Proposition 3.2, it suffices to show that the operator $D(\Delta_a - z)^{-1}$ is of class $C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{E})$. We shall prove more, namely that $D(D^*aD - z)^{-1}$ is of class $C^{\mathrm{u}}(Q; \mathscr{G}^*, \mathscr{E})$. Since D is of class $C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{E})$, and due to (i) of Proposition 3.1, it suffices to show that $(D^*aD - z)^{-1}$ is of class $C^{\mathrm{u}}(Q; \mathscr{G}^*, \mathscr{G})$. But $D^*aD - z$ is of class $C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{G}^*)$ by (i) and (iii) of Proposition 3.1 and is a bijective map $\mathscr{G} \to \mathscr{G}^*$, so the result follows from (ii) of Proposition 3.1. ∎

THEOREM 3.7. *Let $(\mathscr{G}, \mathscr{H})$ be a compact Friedrichs X-module and $(\mathscr{E}, \mathscr{K})$ a Friedrichs X-module and let $D \in \mathcal{B}(\mathscr{G}, \mathscr{E})$ and $a, b \in \mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ be operators of class $C^{\mathrm{u}}(Q)$ such that $D^*aD - z$ and $D^*bD - z$ are bijective maps $\mathscr{G} \to \mathscr{G}^*$ for some complex number z. If $a - b \in \mathcal{B}_0^1(\mathscr{E}, \mathscr{E}^*)$ then Δ_b is a compact perturbation of Δ_a.*

Proof. The proof is a repetition of that of Theorem 2.5. The only difference is that we write directly

$$R = (D(\Delta_a^* - \overline{z})^{-1})^*(b - a)D(\Delta_b - z)^{-1}$$

and observe that $(b - a)D(\Delta_b - z)^{-1} \in \mathcal{B}_0^1(\mathscr{H}, \mathscr{E}^*)$ and that $(D(\Delta_a^* - \overline{z})^{-1})^*$ as an operator $\mathscr{E}^* \to \mathscr{H}$ is decay preserving by (ii) of Proposition 1.2 and because the operator $D(\Delta_a^* - \overline{z})^{-1} : \mathscr{H} \to \mathscr{E}$ is decay preserving by Lemma 3.6. ∎

Let E be a finite dimensional complex Hilbert space and $\mathscr{H} = L^2(X;E)$ with the Hilbert X-module structure defined by the multiplication operators $V_k = k(Q)$. We assume that X is not discrete, so X^* is not compact. Let $w : X^* \to [1,\infty[$ be a continuous function with $w(k) \to \infty$ as $k \to \infty$ and such that $w(k'k) \leqslant \omega(k')w(k)$ holds for some function ω and all k',k. If ω is the smallest function satisfying such an estimate, then $\omega(k+p) \leqslant \omega(k)\omega(p)$. From now on we assume that ω is a Borel function $X^* \to [1,\infty[$ satisfying this submultiplicativity condition.

Then $w(P) = \mathcal{F}^{-1}M_w\mathcal{F}$ is a self-adjoint operator on $\mathscr{H}$ with $w(P) \geqslant 1$, where M_w is the operator of multiplication by w in $L^2(X^*)$. We denote $\mathscr{H}^w = \mathcal{D}(w(P))$ and equip it with the Banach X-module structure given by the norm $\|u\|_w = \|w(P)u\|$ and the representation $V_k|\mathscr{H}^w$. Obviously, this space is a generalization of the usual notion of Sobolev spaces (that such spaces are natural in the context of hypoelliptic operators is shown in Section 10.1 from [12]).

LEMMA 3.8. *$(\mathscr{H}^w,\mathscr{H})$ is a compact Friedrichs X-module.*

Proof. If $\varphi \in C_0(X)$ then $\varphi(Q)w(P)^{-1}$ is a compact operator because w^{-1} belongs to $C_0(X)$, hence $\varphi(Q) \in \mathcal{K}(\mathscr{H}^w,\mathscr{H})$. Then observe that $V_k^{-1}w(P)V_k = w(kP)$ and $w(kP) \leqslant \omega(k)w(P)$. Thus V_k leaves invariant $\mathscr{H}^w$ and we have the estimate $\|V_k\|_{\mathcal{B}(\mathscr{H}^w)} \leqslant \omega(k)$. ∎

Let us call *uniformly hypoelliptic* an operator A on $\mathscr{H}$ such that there are w as above and an operator $\widetilde{A} \in \mathcal{B}(\mathscr{H}^w,\mathscr{H}^{w*})$ such that $\widetilde{A} - z : \mathscr{H}^w \to \mathscr{H}^{w*}$ is bijective for some complex z and such that A is the operator induced by $\widetilde{A}$ in $\mathscr{H}$ (see the Section 2). For example, the constant coefficients case with $E = \mathbb{C}$ corresponds to the choice $A = h(P)$ with $h : X^* \to \mathbb{C}$ a Borel function such that $c'w^2 \leqslant 1 + |h| \leqslant c''w^2$ and such that the range of h is not dense in $\mathbb{C}$. It is clear that Theorem 3.5 allows one to show the stability of the essential spectrum of such operators under perturbations which are small at infinity. We stress that *the differential operators covered by these results can be of any order* and that in the usual case when the coefficients are complex measurable functions a condition of the type $\widetilde{A} \in C^{\mathrm{u}}(Q;\mathscr{H}^w,\mathscr{H}^{w*})$ is very general. The only condition really relevant in this context is $\widetilde{B} - \widetilde{A} \in \mathcal{B}_0^1(\mathscr{H}^w,\mathscr{H}^{w*})$ and the main point is that it allows perturbations of the higher order coefficients even in the non-smooth case.

EXAMPLE 3.9. We give an application of Theorem 3.5 to the case of Dirac operators. Let $X = \mathbb{R}^n$ and let $\alpha_0 \equiv \beta,\alpha_1,\ldots,\alpha_n$ be symmetric operators on E such that $\alpha_j\alpha_k + \alpha_k\alpha_j = \delta_{jk}$. The free Dirac operator is $D = \sum_{k=1}^{n} \alpha_k P_k + m\beta$ for some real number m. The natural compact Friedrichs X-module in this context is $(\mathscr{H}^{1/2},\mathscr{H})$, where $\mathscr{H}^{1/2}$ is the Sobolev space of order $1/2$, cf. the next section. Let V,W be measurable functions on X with values symmetric operators on E and such that the operators of multiplication by V and W define continuous maps $\mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$. Assume that $D + V + \mathrm{i}$ and $D + W + \mathrm{i}$ are bijective maps $\mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$ and denote A and B the self-adjoint operators in $\mathscr{H}$ induced by $D+V$ and $D+W$. If $V-W \in \mathcal{B}_0(\mathscr{H}^{1/2}, \mathscr{H}^{-1/2})$ then B is a compact perturbation of A, and $\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(A)$.

We keep the notations and assumptions of the preceding example and we denote $U = W - V$, so that formally $B = A + U$. Our result improves the existing results concerning the stability of the essential spectrum of Dirac Hamiltonians in two respects. First, there are no regularity assumptions on the "unperturbed" operator A besides the fact that $A + \mathrm{i}$ extends to a bijective continuous map $\mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$; for example, this allows V to have local singularities of Coulomb type in the optimal class for which self-adjointness results have been proved. Second, the perturbation U can be as singular as V locally and it must decay at infinity in a quite weak sense, namely we ask $\lim_{r\to\infty} \|\theta(Q/r)U\|_{\mathscr{H}^{1/2}\to\mathscr{H}^{-1/2}} = 0$ where θ is as in (4.1). As far as we know, in the previous results there is always an assumption which implies $\mathcal{D}(A) \subset \mathscr{H}^s$ for some $s > 1/2$ (as we explained in the introduction, this drastically simplifies the arguments) and the decay condition is stronger. In fact, usually $s = 1$ because A is just the free Dirac operator, i.e. $V = 0$. We refer to [17] for results which are typical for the later developments on this question.

4. OPERATORS IN DIVERGENCE FORM ON EUCLIDEAN SPACES

The results of this section follow from Theorem 3.7. Here X is the additive group $\mathbb{R}^n$ and we identify X^* with X by setting $k(x) = \exp(\mathrm{i}\langle x, k\rangle)$, where $\langle x, k\rangle$ is the scalar product in X. Let E be a finite dimensional Hilbert space and $\mathscr{H} = L^2(X; E)$ with the Hilbert X-module structure defined by $(V_k u)(x) = \exp(\mathrm{i}\langle x, k\rangle)u(x)$.

For each real number s let $\mathscr{H}^s := \mathscr{H}^s(\mathbb{R}^n; E)$ be the Hilbert space of E-valued distributions u on $\mathbb{R}^n$ such that $\|u\|_s^2 := \int (1+|k|^2)^s |\widehat{u}(k)|^2 \mathrm{d}k < \infty$, where $\widehat{u}$ is the Fourier transform of u. This is the usual Sobolev space of order s on $\mathbb{R}^n$. Note that $V_k\mathscr{H}^s \subset \mathscr{H}^s$ and $\|V_k\| \leqslant C(1+|k|)^s$ if $s > 0$ from which we get a canonical X-Banach module structure on $\mathscr{H}^s$ for any real s.

This Banach module structure can also be defined as follows. The algebra $\mathscr{S}$ of Schwartz test functions on $\mathbb{R}^n$ is naturally embedded in $\mathcal{B}(\mathscr{H}^s)$, a function $\varphi \in \mathscr{S}$ being identified with the operator of multiplication by φ on $\mathscr{H}^s$. If we denote by $\mathcal{M}^s$ the closure of $\mathscr{S}$ in $\mathcal{B}(\mathscr{H}^s)$, then clearly $(\mathscr{H}^s, \mathcal{M}^s)$ is a Banach module and this Banach module is a Hilbert module if and only if $s = 0$. The module adjoint to $(\mathscr{H}^s, \mathcal{M}^s)$ is identified with $(\mathscr{H}^{-s}, \mathcal{M}^{-s})$. Note that $\mathcal{M}^s$ can be realized as a subalgebra of $\mathcal{M}^0 = C_0(\mathbb{R}^n)$, namely $\mathcal{M}^s$ is the completion of $\mathscr{S}$ for the norm $\|\varphi\|_{\mathcal{M}^s} := \sup_{\|u\|_s=1} \|\varphi u\|_s$, and then we have $\mathcal{M}^s = \mathcal{M}^{-s}$ isometrically and $\mathcal{M}^s \subset \mathcal{M}^t$ if $s \geqslant t \geqslant 0$ (by interpolation).

Thus for each $s > 0$ we get a compact Friedrichs X-module $(\mathscr{H}^s, \mathscr{H})$ the associated Gelfand triplet being $\mathscr{H}^s \subset \mathscr{H} \subset \mathscr{H}^{-s}$. In fact, if $\varphi \in C_0(\mathbb{R}^n)$ then the operator of multiplication by φ is a compact operator $\mathscr{H}^s \to \mathscr{H}$.

Let us describe the objects which appear in Theorem 3.7 in the present context. We fix an integer $m \geqslant 1$ and take $\mathscr{G} = \mathscr{H}^m$. Let $\mathscr{K} = \bigoplus_{|\alpha| \leqslant m} \mathscr{H}_\alpha$, where $\mathscr{H}_\alpha \equiv \mathscr{H}$, with the natural direct sum Hilbert X-module structure. Here α are multi-indices $\alpha \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then we define

$$\mathscr{E} = \bigoplus_{|\alpha| \leqslant m} \mathscr{H}^{m-|\alpha|} = \{(u_\alpha)_{|\alpha| \leqslant m} \in \mathscr{K} : u_\alpha \in \mathscr{H}^{m-|\alpha|}\}$$

equipped with the Hilbert direct sum structure. It is obvious that $(\mathscr{E}, \mathscr{K})$ is a Friedrichs X-module (but not compact).

We set $P_k = -\mathrm{i}\partial_k$, where ∂_k is the derivative with respect to the k-th variable, and $P^\alpha = P_1^{\alpha_1} \cdots P_n^{\alpha_n}$ if $\alpha \in \mathbb{N}^n$. Then for $u \in \mathscr{G}$ let $Du = (P^\alpha u)_{|\alpha| \leqslant m} \in \mathscr{K}$. Since

$$\|Du\|^2 = \sum_{|\alpha| \leqslant m} \|P^\alpha u\|^2 = \|u\|^2_{\mathscr{H}^m}$$

we see that $D : \mathscr{G} \to \mathscr{K}$ is a linear isometry. Moreover, we have defined $\mathscr{E}$ so as to have $D\mathscr{G} \subset \mathscr{E}$, hence $D \in \mathcal{B}(\mathscr{G}, \mathscr{E})$. We have $D \in C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{E})$ because the components of the operator $V_k^{-1} D V_k$ are the operators $V_k^{-1} P^\alpha V_k = (P + k)^\alpha$ which are polynomials in k with coefficients in $\mathcal{B}(\mathscr{G}, \mathscr{H})$ (because $|\alpha| \leqslant m$).

We shall identify $\mathscr{H}^* = \mathscr{H}$ and $\mathscr{K}^* = \mathscr{K}$, which implies $\mathscr{G}^* = \mathscr{H}^{-m}$ and

$$\mathscr{E}^* = \bigoplus_{|\alpha| \leqslant m} \mathscr{H}^{|\alpha| - m}.$$

Then $D^* \in \mathcal{B}(\mathscr{E}^*, \mathscr{G}^*)$ acts as follows: $D^*(u_\alpha)_{|\alpha| \leqslant m} = \sum_{|\alpha| \leqslant m} P^\alpha u_\alpha$.

An operator $a \in \mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ can be identified with a matrix $a = (a_{\alpha\beta})$ of operators $a_{\alpha\beta} \in \mathcal{B}(\mathscr{H}^{m-|\beta|}, \mathscr{H}^{|\alpha|-m})$, where $|\alpha| \leqslant m$ and $|\beta| \leqslant m$, such that

$$a(u_\beta)_{|\beta| \leqslant m} = \Big(\sum_{|\beta| \leqslant m} a_{\alpha\beta} u_\beta \Big)_{|\alpha| \leqslant m}.$$

Then $D^* a D = \sum_{|\alpha|, |\beta| \leqslant m} P^\alpha a_{\alpha\beta} P^\beta$ which is a general version of a differential operator in divergence form. We must, however, emphasize that our $a_{\alpha\beta}$ are not necessarily ($B(E)$ valued) functions, they could be pseudo-differential or more general operators.

In view of the statement of the next theorem we note that since the Sobolev spaces are Banach X-modules the class of regularity $C^{\mathrm{u}}(Q; \mathscr{H}^s, \mathscr{H}^t)$ is well defined for all real s, t. A bounded operator $S : \mathscr{H}^s \to \mathscr{H}^t$ belongs to this class if and only if the map $k \mapsto V_{-k} S V_k \in \mathcal{B}(\mathscr{H}^s, \mathscr{H}^t)$ is norm continuous. In particular, this condition is trivially satisfied if S is the operator of multiplication by a function, because then V_k commutes with S. Since the coefficients $a_{\alpha\beta}$ of the differential expression $D^* a D$ are usually assumed to be functions, this is a quite weak restriction in the setting of the next theorem. The condition $S \in \mathcal{B}_0^1(\mathscr{H}^s, \mathscr{H}^t)$ is also well defined and it is easily seen that it is equivalent to

$$\lim_{r\to\infty} \|\theta(Q/r)S\|_{\mathscr{H}^s\to\mathscr{H}^t} = 0 \tag{4.1}$$

where θ is a C^∞ function on X equal to zero on a neighborhood of the origin and equal to one on a neighborhood of infinity. Now we can state the following immediate consequence of Theorem 3.7.

PROPOSITION 4.1. *Let $a_{\alpha\beta}$ and $b_{\alpha\beta}$ be operators of class $C^{\mathrm{u}}(\mathscr{H}^{m-|\beta|}, \mathscr{H}^{|\alpha|-m})$ and such that the operators $D^*aD - z$ and $D^*bD - z$ are bijective maps $\mathscr{H}^m \to \mathscr{H}^{-m}$ for some complex z. Let Δ_a and Δ_b be the operators in $\mathscr{H}$ associated to D^*aD and D^*bD respectively. Assume that*

$$\lim_{r\to\infty} \|\theta(Q/r)(a_{\alpha\beta} - b_{\alpha\beta})\|_{\mathscr{H}^{m-|\beta|}\to\mathscr{H}^{|\alpha|-m}} = 0 \tag{4.2}$$

for each α, β, where θ is a function as above. Then Δ_b is a compact perturbation of Δ_a and the operators Δ_a and Δ_b have the same essential spectrum.

EXAMPLE 4.2. In the simplest case the coefficients $a_{\alpha\beta}$ and $b_{\alpha\beta}$ of the principal parts (i.e. $|\alpha| = |\beta| = m$) are functions. Then the conditions become: $a_{\alpha\beta}$ and $b_{\alpha\beta}$ belong to $L^\infty(X)$ and $|a_{\alpha\beta}(x) - b_{\alpha\beta}(x)| \to 0$ as $|x| \to \infty$. Of course, the assumptions on the lowest order coefficients are much more general.

EXAMPLE 4.3. We show here that "highly oscillating potentials" leave invariant the essential spectrum. If $m = 1$ then the terms of order one of D^*aD are of the form $S = \sum_{k=1}^{n} (P_k v'_k + v''_k P_k)$, where $v'_k \in \mathcal{B}(\mathscr{H}^1, \mathscr{H})$ and $v''_k \in \mathcal{B}(\mathscr{H}, \mathscr{H}^{-1})$. Choose $v_k \in \mathcal{B}(\mathscr{H}^1, \mathscr{H})$ symmetric in $\mathscr{H}$ and let $v'_k = \mathrm{i}v_k, v''_k = -\mathrm{i}v_k$. Then $S = [\mathrm{i}P, v] \equiv \operatorname{div} v$, with natural notations, can also be thought as a term of order zero. Now assume that v_k are bounded Borel functions and consider a similar term $T = [\mathrm{i}P, w]$ for D^*bD. Then the condition $|v_k(x) - w_k(x)| \to 0$ as $|x| \to \infty$ suffices to ensure the stability of the essential spectrum. However, the difference $S - T$ could be a function which does not tend to zero at infinity in a simple sense, being only "highly oscillating". An explicit example in the case $n = 1$ is the following: a perturbation of the form $\exp(x)(1+|x|)^{-1}\cos(\exp(x))$ is allowed because it is the derivative of $(1+|x|)^{-1}\sin(\exp(x))$ plus a function which tends to zero at infinity.

In order to apply Proposition 4.1 we need that $D^*aD - z : \mathscr{H}^m \to \mathscr{H}^{-m}$ be bijective for some $z \in \mathbb{C}$, and similarly for b. A standard way of checking this is to require the following coercivity condition:

$$\text{(C)} \quad \begin{cases} \text{there are } \mu, \nu > 0 \text{ such that for all } u \in \mathscr{H}^m : \\ \sum_{|\alpha|,|\beta|\leqslant m} \operatorname{Re}\langle P^\alpha u, a_{\alpha\beta} P^\beta u\rangle \geqslant \mu \|u\|^2_{\mathscr{H}^m} - \nu\|u\|^2_{\mathscr{H}}. \end{cases}$$

EXAMPLE 4.4. One often imposes a stronger ellipticity condition that we describe below. Observe that the coefficients of the highest order part of D^*aD defined by $A_0 = \sum_{|\alpha|=|\beta|=m} P^\alpha a_{\alpha\beta} P^\beta$ are operators $a_{\alpha\beta} \in \mathcal{B}(\mathscr{H})$. Then ellipticity means:

$$\text{(Ell)} \quad \begin{cases} \text{there is } \mu > 0 \text{ such that if } u_\alpha \in \mathscr{H} \text{ for } |\alpha| = m \text{ then} \\ \sum_{|\alpha|=|\beta|=m} \operatorname{Re} \langle u_\alpha, a_{\alpha\beta} u_\beta \rangle \geqslant \mu \sum_{|\alpha|=m} \|u_\alpha\|^2_{\mathscr{H}}. \end{cases}$$

Our conditions on the lower order terms being quite general (e.g. the $a_{\alpha\beta}$ could be differential operators, so the terms of formally lower order could be of order $2m$ in fact) we have to supplement the ellipticity condition (Ell) with a condition saying that the rest of the terms $A_1 = \sum_{|\alpha|+|\beta|<2m} P^\alpha a_{\alpha\beta} P^\beta$ is small with respect to A_0. For example, we may require the existence of some $\delta < \mu$ and $\gamma > 0$ such that

$$\Big| \sum_{|\alpha|+|\beta|<2m} \operatorname{Re} \langle P^\alpha u, a_{\alpha\beta} P^\beta u \rangle \Big| \leqslant \delta \|u\|^2_{\mathscr{H}^m} + \gamma \|u\|^2_{\mathscr{H}}. \tag{4.3}$$

This is satisfied if $A_1 \mathscr{H}^m \subset \mathscr{H}^{-m+\theta}$ for some $\theta > 0$, because for each $\varepsilon > 0$ there is $c(\varepsilon) < \infty$ such that $\|u\|_{\mathscr{H}^{m-\theta}} \leqslant \varepsilon \|u\|_{\mathscr{H}^m} + c(\varepsilon)\|u\|_{\mathscr{H}}$.

REMARK 4.5. If we use Theorem 2.5 in the context of this section then we get the same conditions on the coefficients $a_{\alpha\beta} - b_{\alpha\beta}$ of the principal part (i.e. $|\alpha| = |\beta| = m$) but those on the lower order coefficients are less general. Indeed, if $s + t > 0$ the space $\mathcal{B}^1_{00}(\mathscr{H}^s, \mathscr{H}^{-t})$ defined as the closure of $\mathcal{B}^1_0(\mathscr{H})$ in $\mathcal{B}(\mathscr{H}^s, \mathscr{H}^{-t})$ does not contain operators of order $s + t$, while $\mathcal{B}^1_0(\mathscr{H}^s, \mathscr{H}^{-t})$ contains such operators.

5. ON MAUREY'S FACTORIZATION THEOREM

The next is due to Bernard Maurey ([16], Theorems 2 and 8). The L^p spaces refer to an arbitrary positive measure space (X, μ) and $\mathscr{H}$ is a Hilbert space.

THEOREM 5.1. *Let $1 < p < 2$ and $T \in \mathcal{B}(\mathscr{H}, L^p)$. Then there is $R \in \mathcal{B}(\mathscr{H}, L^2)$ and there is a function $g \in L^q$, where $1/p = 1/2 + 1/q$, such that $T = g(Q)R$.*

Before going on to our main purpose, we shall state an easy consequence of this theorem which is needed in Sections 7 and 8. Let $\{\mathscr{K}(x)\}_{x\in X}$ be a measurable family of Hilbert spaces (see Chapter II of [7]) such that the dimension of $\mathscr{K}(x)$ is $\leqslant N$ for some finite N. Let $\mathscr{K} = \int_X^\oplus \mathscr{K}(x)\mathrm{d}\mu(x)$ be the corresponding direct integral and for each $p \geqslant 1$ let $\mathscr{K}_p$ be the space of (μ-equivalence classes) of measurable vector fields v such that $\int_X \|v(x)\|^p_{\mathscr{K}(x)}\mathrm{d}\mu(x) < \infty$. Thus $\mathscr{K}_p$ is naturally a Banach space and $\mathscr{K}_2 = \mathscr{K}$.

COROLLARY 5.2. *Let $T \in \mathcal{B}(\mathscr{H}, \mathscr{K}_p)$ where p satisfies $1 < p < 2$. Then there are $R \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and $g \in L^q$, with $q = 2p/(2-p)$, such that $T = g(Q)R$.*

Proof. For each $n = 1, \dots, N$ let X_n be the set of x such that the dimension of $\mathscr{K}(x)$ is equal to n. Then X is the disjoint union of the measurable sets X_n. For each x there is n such that $x \in X_n$ and we can choose a unitary map $j(x) : \mathscr{K}(x) \to \mathbb{C}^n$ such that $\{j_x\}$ be a measurable family of operators. Let J be the operator acting on vector fields according to the rule $(Jv)(x) = j(x)v(x)$, let Π_n be the operator of multiplication by χ_{X_n}, and let $T_n \equiv \Pi_n J T \in \mathcal{B}(\mathscr{H}, L^p(X_n; \mathbb{C}^n))$. We can write $T_n = (T_n^k)_{1 \leqslant k \leqslant n}$ with $T_n^k \in \mathcal{B}(\mathscr{H}, L^p(X_n))$ and Maurey's theorem gives us a factorization $T_n^k = g_n^k(Q) S_n^k$ with $S_n^k \in \mathcal{B}(\mathscr{H}, L^2(X_n))$ and $g_n^k \in L^q(X_n)$, and clearly we may assume $g_n^k \geqslant 0$. Let $g_n = \sup_k g_n^k \in L^q(X_n)$ and $S_n \in \mathcal{B}(\mathscr{H}, L^2(X_n; \mathbb{C}^n))$ be the operator with components $(g_n^k g_n^{-1})(Q) R_n^k$. Then $T_n = g_n(Q) S_n$ and if we define $R_n = J^{-1} S_n$ we get

$$g_n(Q) R_n = J^{-1} g_n(Q) S_n = J^{-1} T_n = \Pi_n T.$$

Thus, if we define $g = \sum_n \chi_{X_n} g_n$ and $R = \sum \Pi_n R_n$, we get $T = g(Q)R$. ∎

Our purpose in the rest of this section is to extend Theorem 5.1 to spaces of measurable functions which are more general then L^p and do not seem to be covered by the results existing in the literature [13]. Our proof follows closely that of Maurey. The following general fact will be needed. Let (X, μ) be a σ-finite positive measure space and let $L^0(X)$ be the space of equivalence classes of complex valued measurable functions on X with the topology of convergence in measure. Let $\mathscr{L}$ be a Banach space with $\mathscr{L} \subset L^0(X)$ linearly and continuously and such that if $f \in L^0(X)$, $g \in \mathscr{L}$ and $|f| \leqslant |g|$ (μ-a.e.) then $f \in \mathscr{L}$ and $\|f\|_{\mathscr{L}} \leqslant \|g\|_{\mathscr{L}}$. The next result is a rather straightforward consequence of Khinchin's inequality 1.10 in [8] (see also Section 8 of [19]).

PROPOSITION 5.3. *There is a number C, independent of $\mathscr{L}$, such that for any Hilbert space $\mathscr{H}$ and any $T \in \mathcal{B}(\mathscr{H}, \mathscr{L})$ the following inequality holds*

$$\Big\| \Big(\sum_j |T u_j|^2 \Big)^{1/2} \Big\|_{\mathscr{L}} \leqslant C \|T\|_{\mathcal{B}(\mathscr{H}, \mathscr{L})} \Big(\sum_j \|u_j\|^2 \Big)^{1/2} \tag{5.1}$$

for all finite families $\{u_j\}$ of vectors in $\mathscr{H}$.

From now on we work in a setting adapted to our needs in Section 7, although it is clear that we could treat by the same methods a general abstract situation. Let $X = \mathbb{R}^n$ equipped with the Lebesgue measure, denote $Z = \mathbb{Z}^n$, and for each $a \in Z$ let $K_a = a + K$, where $K =]-1/2, 1/2]^n$, so that K_a is a unit cube centered at a and we have $X = \bigcup_{a \in Z} K_a$ disjoint union. Let χ_a be the characteristic function of K_a and if $f : X \to \mathbb{C}$ let $f_a = f|K_a$. We fix a number $1 < p < 2$ and a family $\{\lambda_a\}_{a \in Z}$ of strictly positive numbers $\lambda_a > 0$ and we define $\mathscr{L} \equiv \ell^2_\lambda(L^p)$ as the

Banach space of all (equivalence classes) of complex functions f on X such that

$$\|f\|_{\mathscr{L}} := \Big(\sum_{a\in Z} \|\lambda_a \chi_a f\|^2_{L^p} \Big)^{1/2} < \infty. \tag{5.2}$$

Here $L^p = L^p(X)$ but note that, by identifying $\chi_a f \equiv f_a$, we can also interpret $\mathscr{L}$ as a conveniently normed direct sum of the spaces $L^p(K_a)$, see page XIV in [8]. If $\lambda_a = 1$ for all a we set $\ell^2_\lambda(L^p) = \ell^2(L^p)$. Observe that $\ell^2(L^2) = L^2(X)$.

Let q be given by $1/p = 1/2 + 1/q$, so that $1 < p < 2 < q < \infty$. We also need the space $\mathscr{M} \equiv \ell^\infty_\lambda(L^q)$ defined by the condition

$$\|g\|_{\mathscr{M}} := \sup_{a\in Z} \|\lambda_a \chi_a g\|_{L^q} < \infty. \tag{5.3}$$

The definitions are chosen such that $\|gu\|_{\mathscr{L}} \leqslant \|g\|_{\mathscr{M}} \|u\|_{L^2}$ where $L^2 = L^2(X)$. As explained in page XV of [8], the space $\mathscr{M}$ is naturally identified with the dual space of the Banach space $\mathscr{M}_* \equiv \ell^1_{\lambda^{-1}}(L^{q'})$, where $1/q + 1/q' = 1$, defined by the norm

$$\|h\|_{\mathscr{M}_*} := \sum_{a\in Z} \|\lambda_a^{-1} \chi_a h\|_{L^{q'}}.$$

The $\sigma(\mathscr{M}, \mathscr{M}_*)$-topology on $\mathscr{M}$ will be called w^*-topology. Clearly

$$\mathscr{M}_1^+ = \{g \in \mathscr{M} : g \geqslant 0, \|g\|_{\mathscr{M}} \leqslant 1\}$$

is a convex compact subset of $\mathscr{M}$ for the w^*-topology.

LEMMA 5.4. *For each $f \in \mathscr{L}$ there is $g \in \mathscr{M}_1^+$ such that $\|f\|_{\mathscr{L}} = \|g^{-1} f\|_{L^2}$.*

Proof. We can assume $f \geqslant 0$. Since $1 = p/2 + p/q$, we have:

$$\|f_a\|_{L^p} = \|f_a\|^{p/2}_{L^p} \|f_a\|^{p/q}_{L^p} = \|f_a^{p/2}\|_{L^2} \|f_a^{p/q}\|_{L^q} = \|f_a^{-p/q} f\|_{L^2} \|f_a^{p/q}\|_{L^q}$$

with the usual convention $0/0 = 0$. Now we define g_a on K_a as follows. If $f_a = 0$ then we take any $g_a \geqslant 0$ satisfying $\lambda_a \|g_a\|_{L^q} = 1$. If $f_a \neq 0$ let

$$g_a = \lambda_a^{-1} \big(f_a / \|f_a\|_{L^p}\big)^{p/q} = \lambda_a^{-1} \|f_a^{p/q}\|^{-1}_{L^q} f_a^{p/q}.$$

Thus we have $\lambda_a \|g_a\|_{L^q} = 1$ for all a, in particular $\|g\|_{\mathscr{M}} = 1$. By the preceding computations we also have $\|f_a\|_{L^p} = \|g_a^{-1} f_a\|_{L^2} \|g_a\|_{L^q}$ and so

$$\|f\|^2_{\mathscr{L}} = \sum \lambda_a^2 \|f_a\|^2_{L^p} = \sum \lambda_a^2 \|g_a\|^2_{L^q} \|g_a^{-1} f_a\|^2_{L^2} = \sum \|g_a^{-1} f_a\|^2_{L^2}$$

which is just $\|g^{-1} f\|^2_{L^2}$. ∎

The main technical result follows.

PROPOSITION 5.5. *Let $(f^u)_{u\in U}$ be a family of functions in $\mathscr{L}$ such that, for each $\alpha = (\alpha_u)_{u\in U}$ with $\alpha_u \in \mathbb{R}$, $\alpha_u \geqslant 0$ and $\alpha_u \neq 0$ for at most a finite number of u, the function $f^\alpha := \Big(\sum_u |\alpha_u f^u|^2\Big)^{1/2}$ satisfies $\|f^\alpha\|_{\mathscr{L}} \leqslant \|\alpha\|_{\ell^2(U)}$. Then there is $g \in \mathscr{M}_1^+$ such that $\|g^{-1} f^u\|_{L^2} \leqslant 1$ for all $u \in U$.*

Proof. We shall use the Ky Fan Lemma ([8], 9.10): *Let $\mathcal{K}$ be a compact convex subset of a Hausdorff topological vector space and let $\mathscr{F}$ be a convex set of functions $F : \mathcal{K} \to]-\infty, +\infty]$ such that each $F \in \mathscr{F}$ is convex and lower semicontinuous. If for each $F \in \mathscr{F}$ there is $g \in \mathcal{K}$ such that $F(g) \leqslant 0$, then there is $g \in \mathcal{K}$ such that $F(g) \leqslant 0$ for all $F \in \mathscr{F}$.* For each α as above we define $F_\alpha : \mathscr{M}_1^+ \to]-\infty, +\infty]$ by:

$$F_\alpha(g) = \|g^{-1} f^\alpha\|_{L^2}^2 - \|\alpha\|_{\ell^2(U)}^2 = \sum_u \alpha_u^2 \big(\|g^{-1} f^u\|_{L^2}^2 - 1\big).$$

Then $\mathscr{F}$ will be the set of all functions F_α and $\mathscr{K} = \mathscr{M}_1^+$ (with the w^*-topology) which is a convex compact set as we saw before. From the second representation of F_α given above it follows that $\mathscr{F}$ is a convex set. Each F_α is a convex function because $\|g^{-1} f^\alpha\|_{L^2}^2 = \int g^{-2} (f^\alpha)^2 \mathrm{d}x$ and the map $t \mapsto t^{-2}$ is convex on $[0, \infty[$. We shall prove in a moment that F_α is lower semicontinuous. From Lemma 5.4 it follows that there is $g_\alpha \in \mathscr{K}$ such that $\|f^\alpha\|_{\mathscr{L}} = \|g_\alpha^{-1} f^\alpha\|_{L^2}$. Our assumptions imply $F_\alpha(g_\alpha) = \|f^\alpha\|_{\mathscr{L}}^2 - \|\alpha\|_{\ell^2(U)}^2 \leqslant 0$. Then Ky Fan's Lemma shows that one can choose $g \in \mathscr{K}$ such that $F_\alpha(g) \leqslant 0$ for all α, which finishes the proof of the proposition.

It remains to show the lower semicontinuity of F_α. For this it suffices to prove that $g \mapsto \|g^{-1} f\|_{L^2}^2 \in [0, \infty]$ is lower semicontinuous on $\mathscr{K}$ if $f \in \mathscr{L}, f \geqslant 0$. But

$$\|g^{-1} f\|_{L^2}^2 = \sum_a \int_{K_a} g_a^{-2} f_a^2 \mathrm{d}x$$

and the set of lower semicontinuous functions $\mathscr{K} \to [0, \infty]$ is stable under sums and upper bounds of arbitrary families. So it suffices to show that each map $g \mapsto \int_{K_a} g_a^{-2} f_a^2 \mathrm{d}x$ is lower semicontinuous. This map can be written as a composition $\phi \circ J_a$ where $J_a : \mathscr{M} \to L^q(K_a)$ is the restriction map $J_a g = g_a$ and $\phi : L^q(K_a) \to [0, \infty]$ is defined by $\phi(\theta) = \int_{K_a} \theta^{-2} f_a^2 \mathrm{d}x$. The map J_a is continuous if we equip $L^q(K_a)$ with the weak topology and $\mathscr{M}$ with the w^*-topology because it is the adjoint of the norm continuous map $L^{q'}(K_a) \to \mathscr{M}_*$ which sends u into the function equal to u on K_a and 0 elsewhere. Thus it suffices to show that ϕ is lower semicontinuous on the positive part of $L^q(K_a)$ equipped with the weak topology and for this we can use exactly the same argument as Maurey. We must prove that the set $\{\theta \in L^q(K_a) : \theta \geqslant 0, \phi(\theta) \leqslant r\}$ is weakly closed for each real r. Since ϕ is convex, this set is convex, so it suffices to show that it is norm closed. But this is clear by the Fatou Lemma. ∎

THEOREM 5.6. *If $\mathscr{H}$ is a Hilbert space and $T \in \mathcal{B}(\mathscr{H}, \mathscr{L})$ then one can find an operator $R \in \mathcal{B}(\mathscr{H}, L^2)$ and a positive function $g \in \mathscr{M}$ such that $T = g(Q)R$.*

Proof. Let U be the unit ball of $\mathscr{H}$ and for each $u \in U$ let $f^u = Tu$. From Proposition 5.3 we get

$$\|f^\alpha\|_{\mathscr{L}} = \Big\| \Big(\sum_u |T(\alpha_u u)|^2\Big)^{1/2} \Big\|_{\mathscr{L}} \leqslant A \Big(\sum_u \|\alpha_u u\|^2\Big)^{1/2} \leqslant A \Big(\sum_u |\alpha_u|^2\Big)^{1/2}$$

where $A = C\|T\|_{\mathcal{B}(\mathscr{H},\mathscr{L})}$. Since there is no loss of generality in assuming $A \leqslant 1$, we see that the assumptions of Proposition 5.5 are satisfied. So there is $g \in \mathscr{M}_1^+$ such that $\|g^{-1}Tu\|_{L^2(X)} \leqslant 1$ for all $u \in U$. Thus it suffices to define R by the rule $Ru = g^{-1}Tu$ for all $u \in \mathscr{H}$. ∎

6. WEAK DECAY PRESERVING OPERATORS

The purpose of the next two sections is to reconsider the examples from Section 4 and to prove stability results for perturbations which decay in a weaker sense. This section contains some preparatory material concerning several classes of operators which preserve decay in some weak senses.

We first consider a measure space (X, μ) with $\mu(X) = \infty$ and define the class of functions which "vanish at infinity" as follows. Let us say that a set $F \subset X$ is of *cofinite measure* if its complement F^c is of finite (exterior) measure. The family $\mathcal{F}_\mu$ of sets of cofinite measure is clearly a filter. If φ is a function on X then $\lim_{\mathcal{F}_\mu} \varphi = 0$ means that for each $\varepsilon > 0$ the set where $|\varphi(x)| \geqslant \varepsilon$ is of finite measure. We denote $B_\mu(X)$ the C^*-subalgebra of $L^\infty(X)$ consisting of functions such that $\lim_{\mathcal{F}_\mu} \varphi = 0$. Let $\mathcal{N}_\mu$ be the set of (equivalence classes of) Borel subsets of finite measure of X. Then $\{\chi_N\}_{N\in\mathcal{N}_\mu}$ is an approximate unit of $B_\mu(X)$ because for each $\varphi \in B_\mu(X)$ and each $\varepsilon > 0$ we have $N = \{x : |\varphi(x)| \geqslant \varepsilon\} \in \mathcal{N}_\mu$ and ess-sup $|\varphi - \chi_N\varphi| \leqslant \varepsilon$. Now it is clear that $L^2(X)$ and, more generally, any direct integral of Hilbert spaces over X, has a natural Hilbert module structure with $B_\mu(X)$ as multiplier algebra. We shall speak of $\mathcal{F}_\mu$-decay preserving operators when we refer to this algebra.

Let $\{\mathscr{H}(x)\}_{x\in X}$ and $\{\mathscr{K}(x)\}_{x\in X}$ be measurable families of Hilbert spaces with dimensions $\leqslant N$ for some finite N. We shall use the notations introduced before Corollary 5.2.

THEOREM 6.1. *Let $S \in \mathcal{B}(\mathscr{H},\mathscr{K}) \cap \mathcal{B}(\mathscr{H}_p,\mathscr{K}_p)$ for some $p \neq 2$. If $p < 2$ then S is left $\mathcal{F}_\mu$-decay preserving and if $p > 2$ then S is right $\mathcal{F}_\mu$-decay preserving.*

Proof. We shall consider only the case $p < 2$, the assertion for $p > 2$ follows by observing that $S^* \in \mathcal{B}(\mathscr{K},\mathscr{H}) \cap \mathcal{B}(\mathscr{K}_{p'},\mathscr{H}_{p'})$ and then using Proposition 1.2. We prove that for each measurable set N of finite measure the operator $T = S\chi_N(Q)$ has the property: if $\varepsilon > 0$ then there is a Borel set $F \in \mathcal{F}_\mu$ such that $\|\chi_F(Q)T\| \leqslant \varepsilon$ (then Proposition 1.1 implies that S is left $\mathcal{F}_\mu$-decay preserving). Since N is of finite measure, $\chi_N(Q)$ is a bounded operator $\mathscr{H} \to \mathscr{H}_p$, hence $T \in \mathcal{B}(\mathscr{H},\mathscr{K}_p)$. The rest of the proof is a straightforward application of Corollary 5.2. Let $a > 0$ real and let F be the set of points x such that $|g(x)| \leqslant a$. Since $g \in L^q$ with $q < \infty$, we have $F \in \mathcal{F}_\mu$ and

$$\|\chi_F(Q)T\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} = \|\chi_F(Q)g(Q)R\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \leqslant a\|R\|_{\mathscr{H},\mathscr{K})}.$$

Thus it suffices to choose a such that $a\|R\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} = \varepsilon$. ∎

Now let X be a locally compact non-compact abelian group and let $\mathscr{H}$ be a Hilbert X-module. Then, due to Remark 3.4, the operator $\varphi(Q) \in \mathcal{B}(\mathscr{H})$ is well defined for all $\varphi \in B(X)$. If $\mathcal{F}$ is a filter finer than the Fréchet filter on X then $B_{\mathcal{F}}(X) = \{\varphi \in B(X) : \lim_{\mathcal{F}} \varphi = 0\}$ is a C^*-algebra and we can consider on $\mathscr{H}$ the Hilbert module structure defined by the multiplier algebra $\mathcal{M}_{\mathcal{F}} = \{\varphi(Q) : \varphi \in B_{\mathcal{F}}(X)\}$. The corresponding classes of decay improving or decay preserving operators will be called (left or right) *$\mathcal{F}$-vanishing at infinity* or of (left or right) *$\mathcal{F}$-decay preserving* operators.

LEMMA 6.2. *If $\mathscr{H}, \mathscr{K}$ are Hilbert X-modules then an operator $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ is left $\mathcal{F}$-decay preserving if and only if for each Borel set N with $N^{\mathrm{c}} = X \setminus N \in \mathcal{F}$ and for each $\varepsilon > 0$ there is a Borel set $F \in \mathcal{F}$ such that $\|\chi_F(Q)S\chi_N(Q)\| \leqslant \varepsilon$.*

Proof. We note first that the family of operators $\chi_N(Q)$, where N runs over the family of Borel sets with complement in $\mathcal{F}$, is an approximate unit for $B_{\mathcal{F}}(X)$. Indeed, if $\varepsilon > 0$ and $\varphi \in B_{\mathcal{F}}(X)$ then the set $N = \{x : |\varphi(x)| > \varepsilon\}$ is Borel, its complement is in $\mathcal{F}$, and $\sup_x |\varphi(x)(1 - \chi_N(x))| \leqslant \varepsilon$. Thus, according to Proposition 1.1, S is left $\mathcal{F}$-decay preserving if and only if $S\chi_N(Q)$ is left $\mathcal{F}$-vanishing at infinity for each N. Now the result follows from (1.1). ∎

As a first example we may take $\mathcal{F} = \mathcal{F}_\mu$ the filter of sets of cofinite Haar measure. A much more interesting filter $\mathcal{F}_{\mathrm{w}}$ is defined as follows. Let $|K|$ be the exterior (Haar) measure of a set $K \subset X$ and let $K_a = a + K$ if $a \in X$. A subset N is called w-*small (at infinity)* if $\lim_{a\to\infty} |N \cap K_a| = 0$ for some compact neighborhood K of the origin. The complement of a w-small set will be called w-*large (at infinity)*. The family $\mathcal{F}_{\mathrm{w}}$ of all w-large sets is clearly a filter on X finer than the Fréchet filter.

We give descriptions of the functions $\varphi \in B_{\mathcal{F}_{\mathrm{w}}}(X)$ which explain the importance of the filter $\mathcal{F}_{\mathrm{w}}$. We say that $\varphi \in B(X)$ is *weakly vanishing (at infinity)* if

$$\lim_{a\to\infty} \int_{a+K} |\varphi(x)|\mathrm{d}x = 0 \text{ for each compact set } K. \tag{6.1}$$

We shall denote by $B_{\mathrm{w}}(X)$ the set of functions φ satisfying (6.1). This is clearly a C^*-algebra. Note that it suffices that the convergence condition in (6.1) be satisfied for only one compact set K with non-empty interior.

Observe that a Borel set is w-small if and only if its characteristic function weakly vanishes at infinity. Denote $f * g$ the convolution of two functions on X.

PROPOSITION 6.3. *For a $\varphi \in B(X)$ the following conditions are equivalent:*

(i) *φ is weakly vanishing;*

(ii) *$\theta * |\varphi| \in C_0(X)$ if $\theta \in C_{\mathrm{c}}(X)$;*

(iii) *$\lim_{\mathcal{F}_{\mathrm{w}}} \varphi = 0$;*

(iv) *$\varphi(Q)\psi(P)$ is a compact operator on $L^2(X)$ for all $\psi \in C_0(X)$.*

Proof. The equivalence of (i) and (ii) is clear because $\int_{K_a} |\varphi| \mathrm{d}x = (\chi_K * |\varphi|)(a)$. Then (iii) means that for each $\varepsilon > 0$ the Borel set N where $|\varphi(x)| > \varepsilon$ is w-small. Since $\chi_N \leqslant \varphi/\varepsilon$, the implication (ii) $\Rightarrow$ (iii) is clear, while the reciprocal implication follows from $\chi_K * |\varphi| \leqslant \sup|\varphi|\chi_K * \chi_N + \varepsilon|K|$. If (iv) holds, let us choose ψ such that its Fourier transform $\widehat{\psi}$ be a positive function in $C_c(X)$ and let $f \in C_c(X)$ be positive and not zero. Since $\psi(P)f$ is essentially the convolution of $\widehat{\psi}$ with f, there is a compact set K with non-empty interior such that $\psi(P)f \geqslant c\chi_K$ with a number $c > 0$. Let U_a be the unitary operator of translation by a in $L^2(X)$, then $U_a f \to 0$ weakly when $a \to \infty$, hence $\|\varphi(Q)U_a\psi(P)f\| = \|\varphi(Q)\psi(P)U_a f\| \to 0$. Since $U_a^* \varphi(Q) U_a = \varphi(Q-a)$ we get $\|\varphi(Q-a)\chi_K\| \to 0$, hence (i) holds.

Finally, let us prove that (i) $\Rightarrow$ (iv). It suffices to prove that $\varphi(Q)\psi(P)$ is compact if $\widehat{\psi} \in C_c(X)$ and for this it suffices that $\overline{\psi}(P)|\varphi|^2(Q)\psi(P)$ be compact. Since $\xi := |\varphi|^2 \in B_w(X)$ and since $\psi(P)$ is the operator of convolution by a function $\theta \in C_c(X)$, we are reduced to proving that the integral operator S with kernel $S(x,y) = \int \overline{\theta}(z-x)\xi(z)\theta(z-y)\mathrm{d}z$ is compact. If $K = \operatorname{supp}\theta$ and Λ is the compact set $K - K$, then clearly there is a number C such that

$$|S(x,y)| \leqslant C \int_{K_x} \xi(z)\mathrm{d}z \chi_\Lambda(x-y) \equiv \phi(x)\chi_\Lambda(x-y)$$

where $\phi \in C_0(X)$. The last term here is a kernel which defines a compact operator T. Thus $\eta(Q)S$ is a Hilbert-Schmidt operator for each $\eta \in C_c(X)$ and from the preceding estimate we get $\|(S - \eta(Q)S)u\| \leqslant \|(1-\eta(Q))T|u|\|$ for each $u \in L^2(X)$. Thus $\|S - \eta(Q)S\| \leqslant \|(1-\eta(Q))T\|$ and the right hand side tends to zero if $\eta \equiv \eta_\alpha$ is an approximate unit for $C_0(X)$. ∎

The main restriction we have to impose on $\mathcal{F}$ comes from the fact that the Friedrichs couple $(\mathscr{G}, \mathscr{H})$ which is involved in our abstract compactness criteria must be such that $\varphi(Q) \in \mathcal{K}(\mathscr{G}, \mathscr{H})$ if $\varphi \in B_{\mathcal{F}}(X)$. The preceding proposition shows that $\mathcal{F}_w$ is the finest filter which satisfies this condition in the case of interest for differential operators:

COROLLARY 6.4. *Let X be an Euclidean space, $\mathscr{H} = L^2(X)$, and let $\mathscr{G} = \mathscr{H}^s$ be a Sobolev space of order $s > 0$. If $\varphi \in B(X)$ then $\varphi(Q) \in \mathcal{K}(\mathscr{G}, \mathscr{H})$ if and only if*

$$\lim_{a\to\infty} \int_{|x-a|\leqslant 1} |\varphi(x)|\mathrm{d}x = 0. \tag{6.2}$$

We mention that the importance of such a condition in questions of stability of the essential spectrum has been noticed in [11], [14], [18],[21].

For technical reasons it is convenient to consider the following class of filters defined in terms of the metric and measure space structure of X. We shall assume from now on in this section that X is an Euclidean space, although most of what we do extends to more general locally compact groups or metric spaces (in the context of Section 8). We set $B_a(r) = \{x \in X : |x-a| < r\}$, $B_a = B_a(1)$ and $B(r) = B_0(r)$. To each function $\nu : X \to]0,\infty[$ such that $\liminf_{a\to\infty} \nu(a) = 0$ we associate a set of subsets of X as follows:

$$\mathscr{N}_\nu = \Big\{ N \subset X : \limsup_{a\to\infty} \nu(a)^{-1} |N \cap B_a| < \infty \Big\}. \tag{6.3}$$

Clearly $\mathscr{F}_\nu = \{F \subset X : F^{\mathrm{c}} \in \mathscr{N}_\nu\}$ is a filter on X finer than the Fréchet filter.

THEOREM 6.5. *Let $X = \mathbb{R}^n$ and let $\nu : X \to]0,\infty[$ such that $\liminf_{a\to\infty} \nu(a) = 0$ and $\sup_{|b-a|\leqslant r} \nu(b)/\nu(a) < \infty$ for each real r. If $S \in \mathcal{B}(L^2(X))$ is of class $C^{\mathrm{u}}(Q)$ and if $S \in \mathcal{B}(L^p(X))$ for some $p < 2$, then S is left $\mathscr{F}_\nu$-decay preserving.*

Proof. We shall use the following terminology: an operator $S : \mathscr{H} \to \mathscr{K}$ acting between two Hilbert X-modules is called of finite range if there is a compact neighborhood Λ of the origin such that for any compact sets $H, K \subset X$ with $(H-K) \cap \Lambda = \varnothing$ we have $\chi_H(Q) S \chi_K(Q) = 0$. For $\theta \in L^1(X^*)$ we define

$$S_\theta = \int_{X^*} V_k^* S V_k \theta(k) \mathrm{d}k. \tag{6.4}$$

In order to explain the main idea of the proof we note that a formal computation involving the spectral measure $E(A) = \chi_A(Q)$ of the representation V_k gives

$$\varphi(Q) S_\theta \psi(Q) = \int_X \int_X \widehat{\theta}(x-y) \varphi(x) \psi(y) E(\mathrm{d}x) S E(\mathrm{d}y) \tag{6.5}$$

for all $\varphi, \psi \in B(X)$. This clearly implies the following:

$$(*) \quad \begin{cases} \text{If the support of } \widehat{\theta} \text{ is a compact set } \Lambda \text{ and if } \operatorname{supp} \varphi \cap (\Lambda + \operatorname{supp} \psi) \\ = \varnothing \ \text{ then } \varphi(Q) S_\theta \psi(Q) = 0. \end{cases}$$

We shall not give a rigorous justification of (6.5) but we shall prove the preceding assertion, which suffices for our purposes. Observe that if $(*)$ holds for a certain set of operators S then it also holds for the strongly closed linear subspace of $\mathcal{B}(\mathscr{H}, \mathscr{K})$ generated by it. So it suffices to prove $(*)$ for S an operator of rank one $Sf = v\langle u, f\rangle$ with some fixed $u \in \mathscr{H}$ and $v \in \mathscr{K}$. Now the computation giving (6.5) obviously makes sense in the weak topology and gives for $f \in \mathscr{H}$ and $g \in \mathscr{K}$:

$$\langle g, \varphi(Q) S_\theta \psi(Q) f\rangle = \int_X \int_X \widehat{\theta}(x-y) \varphi(x) \psi(y) \langle g, E(\mathrm{d}x) u\rangle \langle u, E(\mathrm{d}y) f\rangle,$$

hence $(*)$ holds for such S.

Observe that if $S \in C^{\mathrm{u}}(Q)$ then S is norm limit of operators of the form S_θ. For this it suffices to take $\theta = |K|^{-1} \chi_K$ where K runs over the set of open relatively compact neighbourhoods of the neutral element of X^*, $|K|$ being the

Haar measure of K. Then, by approximating conveniently θ in L^1 norm, one shows that S is norm limit of operators S_θ such that $\widehat{\theta}$ has compact support.

Now we start the proof of the theorem. We can approximate in norm in $\mathcal{B}(L^2(X))$ the operator S by operators which are in $\mathcal{B}(L^2(X)) \cap \mathcal{B}(L^p(X))$ and have finite range. Indeed, the approximation procedure (6.4) used above is such that it leaves $\mathcal{B}(L^2(X)) \cap \mathcal{B}(L^p(X))$ invariant (because V_k are isometries in L^p too). Since the set of left $\mathscr{F}_\nu$-decay preserving operators is norm closed in $\mathcal{B}(L^2(X))$, we may assume in the rest of the proof that S is of finite range. According to Lemma 6.2, it suffices to show that, for a given Borel set $N \in \mathscr{N}_\nu$ and for any number $\varepsilon > 0$, there is a Borel set $M \in \mathscr{N}_\nu$ such that $\|\chi_{M^c}(Q) S \chi_N(Q)\| < \varepsilon$.

In the rest of the proof we shall freely use the notations introduced in Section 5. In particular, q is defined by $1/p = 1/2 + 1/q$. If $f \in L^2(X)$ we have

$$\|\chi_N f\|_{L^p(K_a)} \leqslant \|\chi_N\|_{L^q(K_a)} \|f\|_{L^2(K_a)} \leqslant |N \cap K_a|^{1/q} \|f\|_{L^2(K_a)}.$$

Since $N \in \mathscr{N}_\nu$ we can find a constant c such that $|N \cap K_a| \leqslant c\nu(a)$ (note that the definition (6.3) does not involve the restriction of ν to bounded sets). Thus, if we take $\lambda_a = \nu(a)^{-1/q}$ for $a \in Z \equiv \mathbb{Z}^n$, we get $\chi_N f \in \mathscr{L}$ with the notations of Section 5. In other terms, we see that we have $\chi_N(Q) \in \mathcal{B}(L^2(X), \mathscr{L})$. Let $T = S\chi_N(Q)$ and let us assume that we also have $S \in \mathcal{B}(\mathscr{L})$. Then $T \in \mathcal{B}(L^2(X), \mathscr{L})$ and we can apply the Maurey type factorization theorem Theorem 5.6, where $\mathscr{H} = L^2(X)$. Thus we can write $T = g(Q)R$ for some $R \in \mathcal{B}(L^2(X))$ and some function $g \in \mathscr{M}$, which means that $G := \sup_{a \in Z} \nu(a)^{-1/q} \|g\|_{L^q(K_a)}$ is a finite number. If $t > 0$ and $M = \{x : g(x) > t\}$ then we get for all $a \in Z$:

$$|M \cap K_a| = \|\chi_M\|^q_{L^q(K_a)} \leqslant \|g/t\|^q_{L^q(K_a)} \leqslant (G/t)^q \nu(a).$$

Note that the second condition imposed on ν in Theorem 6.5 can be stated as follows: there is an increasing strictly positive function δ on $[0, \infty[$ such that $\nu(b) \leqslant \delta(|b-a|)\nu(a)$ for all a, b; for we may take $\delta(r) = \sup_{|b-a| \leqslant r} \nu(b)/\nu(a)$. Now let $a \in X$ and let $D(a)$ be the set of $b \in Z$ such that K_b intersects B_a. Clearly $D(a)$ contains at most 2^n points b all of them satisfying $|b-a| \leqslant \sqrt{n}+1$. Hence:

$$|M \cap K_a| \leqslant \sum_{b \in D(a)} |M \cap K_b| \leqslant 2^n \sup_{b \in D(a)} (G/t)^q \nu(b) \leqslant 2^n (G/t)^q \delta(\sqrt{n}+1)\nu(a),$$

which proves that M belongs to $\mathscr{N}_\nu$. On the other hand, we have:

$$\|\chi_{M^c}(Q)T\| = \|\chi_{M^c}(Q)g(Q)R\| \leqslant \|\chi_{M^c} g\|_{L^\infty} \|R\| \leqslant t\|R\|.$$

To finish the proof of the theorem it suffices to take $t = \varepsilon/\|R\|$.

We still have to prove that $S \in \mathcal{B}(\mathscr{L})$. Since S is of finite range, there is a number r such that $\chi_a(Q)\chi_b(Q) = 0$ if $|a-b| \geqslant r$. Then for any $f \in \mathscr{L}$:

$$\sum_a \lambda_a^2 \|\chi_a S f\|^2_{L^p} = \sum_a \lambda_a^2 \Big\| \sum_{|b-a|<r} \chi_a S \chi_b f \Big\|^2_{L^p} \leqslant C \sum_{|b-a|<r} \lambda_a^2 \|\chi_a S \chi_b f\|^2_{L^p}$$

where C is a number depending only on r and n. Since S is bounded in L^p the last term is less than $CC' \sum_{|b-a|<r} \lambda_a^2 \|\chi_b f\|_{L^p}^2$ for some constant C'. Finally, from $\nu(b) \leqslant \delta(|b-a|)\nu(a) \leqslant \delta(r)\nu(a)$ we get

$$\sum_{|a-b|<r} \lambda_a^2 = \sum_{|a-b|<r} \nu(a)^{-2/q} \leqslant L(r)\delta(r)^{2/q}\lambda_b^2$$

where $L(r)$ is the maximum number of points from Z inside a ball of radius r. Thus we have $\|S\|^2_{\mathcal{B}(\mathscr{L})} \leqslant CC'L(r)\delta(r)^{2/q}$. ∎

THEOREM 6.6. *Let $X = \mathbb{R}^n$ and let S be a pseudo-differential operator of class S^0. Then S is $\mathcal{F}_{\mathrm{w}}$-decay preserving in $L^2(X)$, i.e. if $\varphi \in B_{\mathrm{w}}(X)$ then $\varphi(Q)S = T_1\psi_1(Q)$ and $S\varphi(Q) = \psi_2(Q)T_2$ for some $\psi_1, \psi_2 \in B_{\mathrm{w}}(X)$ and $T_1, T_2 \in \mathcal{B}(L^2(X))$.*

Proof. Since the adjoint of S is also a pseudo-differential operator of class S^0, it suffices to show that S is left $\mathcal{F}_{\mathrm{w}}$-decay preserving. We have $S \in \mathcal{B}(L^p(X))$ for all $1 < p < \infty$ and S is of class $C^{\mathrm{u}}(Q)$ because the commutators $[Q_j, S]$ are bounded operators for all $1 \leqslant j \leqslant n$. Thus we can apply Theorem 6.5 and deduce that for any function ν as in the statement of the theorem, for any $\varepsilon > 0$, and for any $N \in \mathscr{N}_\nu$ there is $M \in \mathscr{N}_\nu$ such that $\|\chi_{M^c}(Q)S\chi_N(Q)\| \leqslant \varepsilon$. Now let N be a Borel w-small set, i.e. such that $|N \cap B_a| \to 0$ if $a \to \infty$. We shall prove that there is a function ν with the properties required in Theorem 6.5 and with $\lim_{a\to\infty} \nu(a) = 0$ such that $N \in \mathscr{N}_\nu$. This finishes the proof of the corollary because the relation $M \in \mathscr{N}_\nu$ implies now that M is w-small.

We construct ν as follows. The relation $\theta(r) = \sup_{|a|\geqslant r} |N \cap B_a|$ defines a positive decreasing function on $[0,\infty[$ which tends to zero at infinity and such that $|N \cap B_a| \leqslant \theta(|a|)$ for all $a \in X$. We set $\xi(t) = \theta(0)$ if $0 \leqslant t < 1$ and for $k \geqslant 0$ integer and $2^k \leqslant t < 2^{k+1}$ we define $\xi(t) = \max\{\xi(2^{k-1})/2, \theta(2^k)\}$. So ξ is a strictly positive decreasing function on $[0,\infty[$ which tends to zero at infinity and such that $\theta \leqslant \xi$. Moreover, if $2^k \leqslant s < 2^{k+1}$ and $2^{k+p} \leqslant t < 2^{k+p+1}$ then

$$\xi(t) = \xi(2^{k+p}) \geqslant \xi(2^{k+p-1})/2 \geqslant \cdots \geqslant 2^{-p}\xi(2^k) = 2^{-p}\xi(s)$$

hence $\xi(s) \geqslant \xi(t) \geqslant (s/2t)\xi(s)$ if $1 \leqslant s \leqslant t$. We take $\nu(a) = \xi(|a|)$, so ν is a bounded strictly positive function on X with $\lim_{a\to\infty} \nu(a) = 0$ and $|N \cap B_a| \leqslant \nu(a)$ for all a. If a, b are points with $|a|, |b| \geqslant 1$ and $|a-b| \leqslant r$ then $\nu(b)/\nu(a) \leqslant 1$ if $|a| \leqslant |b|$ and if $|a| > |b|$ then

$$\nu(b)/\nu(a) = \xi(|b|)/\xi(|a|) \leqslant (2|a|)/|b| \leqslant 2(1+r).$$

Thus the second condition imposed on ν in Theorem 6.5 is also satisfied. ∎

REMARK 6.7. The theorem remains true if $\mathcal{F}_{\mathrm{w}}$ is replaced by $\mathcal{F}_\mu$. To prove this it suffices to use Theorem 6.1 and to take into account the fact that a pseudo-differential operator of class S^0 belongs to $\mathcal{B}(L^p(X))$ for all $1 < p < \infty$ and that the adjoint of such an operator is also pseudo-differential of class S^0. In [10] we introduce a third class of filters (of a more topological nature) $\mathcal{F}_L$, which thus define new classes of weakly vanishing at infinity functions, for which Theorem 6.6 is still true.

7. WEAKLY VANISHING PERTURBATIONS

In this section we reconsider the framework of Section 4 and improve, but with a stronger assumption $a \in \mathcal{B}(\mathscr{K})$, the decay condition (4.2). We shall consider on $\mathscr{H}$ the class of "vanishing at infinity" functions corresponding to the algebra $B_{\mathrm{w}}(X)$, in other terms we equip $\mathscr{H}$ with the Hilbert module structure associated to the multiplier algebra $\{\varphi(Q) | \varphi \in B_{\mathrm{w}}(X)\}$. By Corollary 6.4, $(\mathscr{G}, \mathscr{H})$ remains a compact Friedrichs module. The space $\mathscr{K}$ inherits a natural direct sum Hilbert module structure.

We keep the notations and terminology of Sections 4 and 6. We recall that an operator $D^*aD : \mathscr{H}^m \to \mathscr{H}^{-m}$ is coercive if there are numbers $\mu, \nu > 0$ such that

$$\text{(7.1)} \qquad \operatorname{Re} \langle Du, aDu \rangle \geqslant \mu \|u\|^2_{\mathscr{H}^m} - \nu \|u\|^2_{\mathscr{H}} \quad \forall u \in \mathscr{H}^m.$$

Clearly the next lemma remains true if the filter $\mathcal{F}_{\mathrm{w}}$ is replaced by $\mathcal{F}_\mu$.

LEMMA 7.1. *Assume that $a \in \mathcal{B}(\mathscr{K})$ is $\mathcal{F}_{\mathrm{w}}$-decay preserving and that the operator $D^*aD : \mathscr{H}^m \to \mathscr{H}^{-m}$ is coercive. Then $D(\Delta_a - z)^{-1}$ is $\mathcal{F}_{\mathrm{w}}$-decay preserving if* $\operatorname{Re} z \leqslant -\nu$, *where ν is as in* (7.1).

Proof. We shall use Proposition 2.6 with c the identity operator in $\mathscr{K}$, so $\Delta \equiv \Delta_c$ is the operator in $\mathscr{H}$ associated to $D^*D = \sum_{|\alpha| \leqslant m} P^{2\alpha}$, which is the canonical positive isomorphism of $\mathscr{G}$ onto $\mathscr{G}^*$ and (7.1) means $\operatorname{Re} D^*aD \geqslant \mu D^*D - \nu$. We have $D(\Delta - z)^{-1} \in \mathcal{B}_q(\mathscr{H}, \mathscr{K})$ and $D(D^*D - z)^{-1}D^* \in \mathcal{B}_q(\mathscr{K})$ if $\operatorname{Re} z < 0$ because these operators consist of matrices of pseudo-differential operators with constant coefficients of class S^0, so we can use Theorem 6.6. ∎

We now consider two operators $\mathscr{H}^m \to \mathscr{H}^{-m}$ of the form

$$D^*aD = \sum_{|\alpha|, |\beta| \leqslant m} P^\alpha a_{\alpha\beta} P^\beta \quad \text{and} \quad D^*bD = \sum_{|\alpha|, |\beta| \leqslant m} P^\alpha b_{\alpha\beta} P^\beta$$

where the coefficients are continuous operators $a_{\alpha\beta}, b_{\alpha\beta} : \mathscr{H}^{m-|\beta|} \to \mathscr{H}^{|\alpha|-m}$ satisfying some other conditions stated below and denote as usual Δ_a and Δ_b the operators in $\mathscr{H}$ associated to them.

THEOREM 7.2. *Assume that the operators D^*aD and D^*bD are coercive and that their coefficients satisfy the following conditions:*

(i) *$a_{\alpha\beta} \in \mathcal{B}(\mathscr{H})$ and are $\mathcal{F}_{\mathrm{w}}$-decay preserving operators;*

(ii) *if $|\alpha| + |\beta| = 2m$ then $a_{\alpha\beta} - b_{\alpha\beta}$ is left $\mathcal{F}_{\mathrm{w}}$-vanishing at infinity;*

(iii) *if $|\alpha| + |\beta| < 2m$ then $a_{\alpha\beta} - b_{\alpha\beta} \in \mathcal{K}(\mathscr{H}^{m-|\beta|}, \mathscr{H}^{|\alpha|-m})$.*

Then the operator Δ_b is a compact perturbation of Δ_a, in particular $\sigma_{\mathrm{ess}}(\Delta_a) = \sigma_{\mathrm{ess}}(\Delta_b)$.

Proof. We check the conditions of Theorem 2.5. Because of the coercivity assumptions, condition (i) is fulfilled, and (iii) is satisfied by Lemma 7.1. The part of condition (ii) involving the coefficients such that $|\alpha| + |\beta| = 2m$ is satisfied by definition, for the lower order coefficients it suffices to use (2.5). ∎

REMARK 7.3. If $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are bounded Borel functions and $a_{\alpha\beta} - b_{\alpha\beta} \in B_{\mathrm{w}}(X)$ for all α, β, then the conditions (i)–(iii) of the theorem are satisfied. Indeed, in order to check the compactness conditions on the lower order coefficients note that, by Corollary 6.4, if $\varphi \in B_{\mathrm{w}}(X)$ then the operator $\varphi(Q) : \mathscr{H}^s \to \mathscr{H}^{-t}$ is compact if $s, t \geqslant 0$ and one of them is not zero.

The next result is a more general but less explicit version of Theorem 7.2. This is an improvement of Theorem 2.1 in [18], thus it covers some subelliptic operators.

THEOREM 7.4. *Assume that D^*aD satisfies* (7.1) *and that Δ_b is a closed densely defined operator such that there is $z \in \rho(\Delta_b)$ with* $\operatorname{Re} z \leqslant -\nu$. *Moreover, assume that a, b satisfy the conditions* (i)–(iii) *of Theorem* 7.2. *Then the operator Δ_b is a compact perturbation of Δ_a.*

Proof. We shall apply Theorem 2.1 with $A = \Delta_a$ and $B = \Delta_b$. The assumption (AB) is clearly satisfied and we take $\widetilde{A} = D^*aD$ and $\widetilde{B} = D^*bD$, hence $\widetilde{B} - \widetilde{A} = D^*(b-a)D$. Then let $S = D^*$ and $T = (b-a)D$. ∎

Finally, let us note that one should be able to use Theorem 2.1 to treat situations when the coefficients $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are unbounded operators even if $|\alpha| = \beta| = m$ (as in Theorem 3.1 of [18] and [4], [5]), see the framework of Example 2.3, but we shall not pursue this idea here.

8. RIEMANNIAN MANIFOLDS

8.1. In this section X will be at least a locally compact non-compact topological space and we shall consider Hilbert modules associated to it according to the following definition. A Hilbert space $\mathscr{H}$ is a *Hilbert X-module* if a $*$-morphism $Q : C_0(X) \to \mathcal{B}(\mathscr{H})$ with non-degenerate range is given. We shall use the notation $\varphi(Q) \equiv Q(\varphi)$. The Hilbert module structure on $\mathscr{H}$ is defined by the C^*-algebra of operators on $\mathscr{H}$ given by $\mathcal{M} = \{\varphi(Q) : \varphi \in C_0(X)\}$. Hilbert X-modules appear naturally in differential geometry as spaces of sections of vector fiber bundles over X.

If $\mathscr{H}$ is a Hilbert X-module then the morphism Q canonically extends to a $*$-morphism $\varphi \mapsto \varphi(Q)$ of $B(X)$ into $\mathcal{B}(\mathscr{H})$ such that: if $\{\varphi_n\}$ is a bounded sequence in $B(X)$ and $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$ for all $x \in X$, then $\text{s-}\lim_n \varphi_n(Q) = \varphi(Q)$. This follows from standard integration theory argument, see [15]. In particular, a separable Hilbert X-module is essentially a direct integral of Hilbert spaces over X, see II.6.2 of [7].

The Cohen-Hewitt theorem ([9], V-9.2) shows that the classes of decay improving and decay preserving operators associated to $\mathcal{M}$ and to the algebra of operators $\varphi(Q)$ with $\varphi \in B(X)$ and $\lim_{x\to\infty} \varphi(x) = 0$ are identical.

Remark 3.4 implies that if X is a locally compact group then the notion of Hilbert X-module introduced in Section 3 coincides with that considered here.

A Banach module structure defined by an algebra $\mathcal{N}$ on $\mathscr{H}$ is *finer* than that defined by $\mathcal{M}$ if $\mathcal{M} \subset \mathcal{N}$. If $\mathcal{F}$ is a filter on X finer than the Fréchet filter then we can associate to it a Hilbert structure on $\mathscr{H}$ finer than its initial X-module structure by taking $\left\{\varphi(Q) : \lim_{\mathcal{F}} \varphi = 0\right\}$ as multiplier algebra. The decay preserving operators with respect to this new structure will be called $\mathcal{F}$-decay preserving. The decay preserving operators (with respect to the initial X-module structure) are not, in general, $\mathcal{F}$-decay preserving, and one of the main problems that we shall consider later on will be to find $\mathcal{F}$-decay preserving operators for some nontrivial $\mathcal{F}$.

The *support* $\operatorname{supp} u \subset X$ of an element $u \in \mathscr{H}$ is the smallest closed set such that its complement U has the property $\varphi(Q)u = 0$ if $\varphi \in C_c(U)$. Clearly, the set $\mathscr{H}_c$ of elements $u \in \mathscr{H}$ such that $\operatorname{supp} u$ is compact is a dense subspace of $\mathscr{H}$.

Let $\mathscr{H}, \mathscr{K}$ be Hilbert X-modules, let $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$, and let $\varphi, \psi \in C(X)$, not necessarily bounded. We say that $\varphi(Q)S\psi(Q)$ is a bounded operator if there is a constant C such that $\|\xi(Q)\varphi(Q)S\psi(Q)\eta(Q)\| \leqslant C \sup|\xi| \sup|\eta|$ for all $\xi, \eta \in C_c(X)$. The greatest lower bound of the admissible constants C in this estimate is denoted $\|\varphi(Q)S\psi(Q)\|$. The product $\varphi(Q)S\psi(Q)$ is well defined as sesquilinear form on the dense subspace $\mathscr{K}_c \times \mathscr{H}_c$ of $\mathscr{K} \times \mathscr{H}$ and the preceding boundedness notion is equivalent to the continuity of this form for the topology induced by $\mathscr{K} \times \mathscr{H}$. We similarly define the boundedness of the commutator $[S, \varphi(Q)]$.

PROPOSITION 8.1. *Assume that $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and let $\theta : X \to [1, \infty[$ be a continuous function such that $\lim_{x\to\infty} \theta(x) = \infty$. If $\theta(Q)S\theta^{-1}(Q)$ is a bounded operator then S is left decay preserving. If $\theta^{-1}(Q)S\theta(Q)$ is a bounded operator then S is right decay preserving.*

Proof. Let $K \subset X$ be compact, let $U \subset X$ be a neighborhood of infinity in X, and let $\varphi, \psi \in C(X)$ such that $\operatorname{supp} \varphi \subset K$, $\operatorname{supp} \psi \subset U$ and $|\varphi| \leqslant 1, |\psi| \leqslant 1$. Then $\theta\varphi$ is a bounded function and $\psi\theta^{-1}$ is bounded and can be made as small as we wish by choosing U conveniently. If $\varepsilon > 0$ and U is sufficiently small we have

$$\|\psi(Q)S\varphi(Q)\| \leqslant \|\psi\theta^{-1}\| \cdot \|\theta(Q)S\theta^{-1}(Q)\| \cdot \|\theta\varphi\| \leqslant \varepsilon.$$

Then the result follows from Proposition 1.1(i) and relation (1.1). ∎

The boundedness of $\theta(Q)S\theta^{-1}(Q)$ can be checked by estimating the commutator $[S, \theta(Q)]$; we give an example for the case of metric spaces. Note that on metric spaces one has a natural class of regular functions, namely the Lipschitz functions, for example the functions which give the distance to subsets: $\rho_K(x) = \inf_{y \in K} \rho(x,y)$ for $K \subset X$.

We say that a locally compact metric space (X,ρ) is *proper* if the metric ρ has the property $\lim_{y\to\infty} \rho(x,y) = \infty$ for some (hence for all) points $x \in X$. Equivalently, if X is not compact but the closed balls are compact.

COROLLARY 8.2. *Let (X,ρ) be a proper locally compact metric space. If S belongs to $\mathcal{B}(\mathscr{H},\mathscr{K})$ and if $[S,\theta(Q)]$ is bounded for each positive Lipschitz function θ, then S is decay preserving.*

Proof. Indeed, by taking $\theta = 1+\rho_K$ and by using the notations of the proof of Proposition 8.1, we easily get the following estimate: there is $C<\infty$ depending only on K such that

$$\|\varphi(Q)S\psi(Q)\| \leqslant C(1+\rho(K,U))^{-1}$$

where $\rho(K,U)$ is the distance from K to U. Since S^* has the same properties as S, this proves that S is decay preserving. Note that the boundedness of $[S,\rho_x(Q)]$ for some $x \in X$ suffices in this argument. ∎

8.2. Let $\mathscr{H},\mathscr{K}$ be two Hilbert spaces identified with their adjoints and d a closed densely defined operator mapping $\mathscr{H}$ into $\mathscr{K}$. Let $\mathscr{G} = \mathcal{D}(\mathrm{d})$ equipped with the graph norm, so $\mathscr{G} \subset \mathscr{H}$ continuously and densely and $\mathrm{d} \in \mathcal{B}(\mathscr{G},\mathscr{K})$.

Then the quadratic form $\|\mathrm{d}u\|^2_{\mathscr{K}}$ on $\mathscr{H}$ with domain $\mathscr{G}$ is positive densely defined and closed. Let Δ be the positive self-adjoint operator on $\mathscr{H}$ associated to it. In fact $\Delta = \mathrm{d}^*\mathrm{d}$, where the adjoint d^* of d is a closed densely defined operator mapping $\mathscr{K}$ into $\mathscr{H}$.

Now let $\lambda \in \mathcal{B}(\mathscr{H})$ and $\Lambda \in \mathcal{B}(\mathscr{K})$ be self-adjoint and such that $\lambda \geqslant c$ and $\Lambda \geqslant c$ for some real $c > 0$. Then we can define new Hilbert spaces $\widetilde{\mathscr{H}}$ and $\widetilde{\mathscr{K}}$ as follows:

$$(*)\quad \begin{cases} \widetilde{\mathscr{H}} = \mathscr{H} \text{ as vector space and } \langle u : v\rangle_{\widetilde{\mathscr{H}}} = \langle u : \lambda v\rangle_{\mathscr{H}}, \\ \widetilde{\mathscr{K}} = \mathscr{K} \text{ as vector space and } \langle u : v\rangle_{\widetilde{\mathscr{K}}} = \langle u : \Lambda v\rangle_{\mathscr{K}}. \end{cases}$$

Since $\mathscr{H} = \widetilde{\mathscr{H}}$ and $\mathscr{K} = \widetilde{\mathscr{K}}$ as topological vector spaces, the operator $\mathrm{d} : \mathscr{G} \subset \widetilde{\mathscr{H}} \to \widetilde{\mathscr{K}}$ is still a closed densely defined operator, hence the quadratic form $\|\mathrm{d}u\|^2_{\widetilde{\mathscr{K}}}$ on $\widetilde{\mathscr{H}}$ with domain $\mathscr{G}$ is positive, densely defined and closed. We shall denote by $\widetilde{\Delta}$ the positive self-adjoint operator on $\widetilde{\mathscr{H}}$ associated to it.

We can express $\widetilde{\Delta}$ in more explicit terms as follows. Denote by $\widetilde{\mathrm{d}}$ the operator d when viewed as acting from $\widetilde{\mathscr{H}}$ to $\widetilde{\mathscr{K}}$. Then $\widetilde{\Delta} = \widetilde{\mathrm{d}}^*\widetilde{\mathrm{d}}$, where $\widetilde{\mathrm{d}}^* : \mathcal{D}(\widetilde{\mathrm{d}}^*) \subset \widetilde{\mathscr{K}} \to \widetilde{\mathscr{H}}$ is the adjoint of $\widetilde{\mathrm{d}} = \mathrm{d}$ with respect to the new Hilbert space structures (the spaces $\widetilde{\mathscr{H}},\widetilde{\mathscr{K}}$ being also identified with their adjoints). It is easy to check that $\widetilde{\mathrm{d}}^* = \lambda^{-1}\mathrm{d}^*\Lambda$. Thus $\widetilde{\Delta} = \lambda^{-1}\mathrm{d}^*\Lambda\mathrm{d}$.

From now on (X,ρ) is a proper locally compact metric space and $\mathscr{H}$ and $\mathscr{K}$ are Hilbert X-modules. We set $\operatorname{Lip}\varphi = \sup_{x\neq y} |\varphi(x)-\varphi(y)|\rho(x,y)^{-1}$ for each function φ on X. A closed densely defined map $\mathrm{d} : \mathcal{D}(\mathrm{d}) \subset \mathscr{H} \to \mathscr{K}$ *is a first order operator* if $\exists C \in \mathbb{R}$ such that for each bounded Lipschitz function φ on X

$$|\langle \mathrm{d}^* u, \varphi(Q)v\rangle_{\mathscr{H}} - \langle u, \varphi(Q)\mathrm{d}v\rangle_{\mathscr{K}}| \leqslant C\operatorname{Lip}\varphi \,\|u\|_{\mathscr{K}}\|v\|_{\mathscr{H}}$$

for all $(u,v) \in \mathcal{D}(\mathrm{d}^*) \times \mathcal{D}(\mathrm{d})$. Thus $\langle \mathrm{d}^* u, \varphi(Q)v\rangle - \langle u, \varphi(Q)\mathrm{d}v\rangle$ is a sesquilinear form on the dense subspace $\mathcal{D}(\mathrm{d}^*) \times \mathcal{D}(\mathrm{d})$ of $\mathscr{K}\times\mathscr{H}$ continuous for the topology induced by $\mathscr{K}\times\mathscr{H}$, so there is a continuous operator $[\mathrm{d},\varphi(Q)] : \mathscr{H}\to\mathscr{K}$ such that

$$\langle \mathrm{d}^* u, \varphi(Q)v\rangle_{\mathscr{H}} - \langle u, \varphi(Q)\mathrm{d}v\rangle_{\mathscr{K}} = \langle u, [\mathrm{d},\varphi(Q)]v\rangle_{\mathscr{K}}$$

for all $u \in \mathcal{D}(\mathrm{d}^*)$, $v \in \mathcal{D}(\mathrm{d})$. Moreover, we have $\|[\mathrm{d},\varphi(Q)]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \leqslant C\operatorname{Lip}\varphi$.

LEMMA 8.3. *The operator* $\mathrm{d}(\Delta+1)^{-1}$ *is decay preserving.*

Proof. We shall prove that $S := \mathrm{d}(\Delta+1)^{-1}$ is a decay preserving operator with the help of Corollary 8.2, more precisely we show that $[S,\varphi(Q)]$ is a bounded operator if φ is a positive Lipschitz function. Let $\varepsilon > 0$ and $\varphi_\varepsilon = \varphi(1+\varepsilon\varphi)^{-1}$. Then φ_ε is bounded with $|\varphi_\varepsilon| \leqslant \varepsilon^{-1}$ and $|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)| \leqslant |\varphi(x)-\varphi(y)|$ hence $\operatorname{Lip}\varphi_\varepsilon \leqslant \operatorname{Lip}\varphi$. If $v \in \mathcal{D}(\mathrm{d})$ we have for all $u \in \mathcal{D}(\mathrm{d}^*)$:

$$\begin{aligned}|\langle \mathrm{d}^* u, \varphi_\varepsilon(Q)v\rangle_{\mathscr{H}}| &= |\langle u, \varphi_\varepsilon(Q)\mathrm{d}v\rangle_{\mathscr{K}} + \langle u, [\mathrm{d},\varphi_\varepsilon(Q)]v\rangle_{\mathscr{K}}| \\ &\leqslant \|u\|_{\mathscr{K}}(\varepsilon^{-1}\|\mathrm{d}v\|_{\mathscr{K}} + C\operatorname{Lip}\varphi_\varepsilon\,\|u\|_{\mathscr{H}}).\end{aligned}$$

Hence $\varphi_\varepsilon(Q)v \in \mathcal{D}(\mathrm{d}^{**}) = \mathcal{D}(\mathrm{d})$ because d is closed. Thus $\varphi_\varepsilon(Q)\mathcal{D}(\mathrm{d}) \subset \mathcal{D}(\mathrm{d})$ and by the closed graph theorem we get $\varphi_\varepsilon(Q) \in \mathcal{B}(\mathscr{G})$, where $\mathscr{G}$ is the domain of d equipped with the graph topology. This also implies that $\varphi_\varepsilon(Q)$ extends to an operator in $\mathcal{B}(\mathscr{G}^*)$ (note that $\varphi_\varepsilon(Q)$ is symmetric in $\mathscr{H}$).

Now, if we think of d as a continuous operator $\mathscr{G}\to\mathscr{K}$, then it has an adjoint $\mathrm{d}^* : \mathscr{K}\to\mathscr{G}^*$ which is the unique continuous extension of the operator $\mathrm{d}^* : \mathcal{D}(\mathrm{d}^*) \subset \mathscr{K} \to \mathscr{H} \subset \mathscr{G}^*$. Thus the canonical extension of Δ to an element of $\mathcal{B}(\mathscr{G},\mathscr{G}^*)$ is the product of $\mathrm{d} : \mathscr{G}\to\mathscr{K}$ with $\mathrm{d}^* : \mathscr{K}\to\mathscr{G}^*$ (note $\mathcal{D}(\mathrm{d})$ is the form domain of Δ). Then it is trivial to justify that we have in $\mathcal{B}(\mathscr{G},\mathscr{G}^*)$:

$$[\Delta,\varphi_\varepsilon(Q)] = [\mathrm{d}^*,\varphi_\varepsilon(Q)]\mathrm{d} + \mathrm{d}^*[\mathrm{d},\varphi_\varepsilon(Q)].$$

Here $[\mathrm{d}^*,\varphi_\varepsilon(Q)] = [\varphi_\varepsilon(Q),\mathrm{d}]^* \in \mathcal{B}(\mathscr{K},\mathscr{H})$. Since $\Delta+1 : \mathscr{G}\to\mathscr{G}^*$ is a linear homeomorphism, we then have in $\mathcal{B}(\mathscr{G}^*,\mathscr{G})$:

$$\begin{aligned}&[\varphi_\varepsilon(Q),(\Delta+1)^{-1}] \\ &\quad= (\Delta+1)^{-1}[\Delta,\varphi_\varepsilon(Q)](\Delta+1)^{-1} \\ &\quad= (\Delta+1)^{-1}[\varphi_\varepsilon(Q),\mathrm{d}]^*\mathrm{d}(\Delta+1)^{-1} + (\Delta+1)^{-1}\mathrm{d}^*[\mathrm{d},\varphi_\varepsilon(Q)](\Delta+1)^{-1}.\end{aligned}$$

Finally, taking again into account the fact that $\varphi_\varepsilon(Q)$ leaves $\mathscr{G}$ invariant, we have:

$$[\varphi_\varepsilon(Q), \mathrm{d}(\Delta+1)^{-1}] = [\varphi_\varepsilon(Q), \mathrm{d}](\Delta+1)^{-1} + \mathrm{d}(\Delta+1)^{-1}[\varphi_\varepsilon(Q), \mathrm{d}]^*\mathrm{d}(\Delta+1)^{-1} + \mathrm{d}(\Delta+1)^{-1}\mathrm{d}^*[\mathrm{d}, \varphi_\varepsilon(Q)](\Delta+1)^{-1}.$$

Hence:

$$\begin{aligned}\|[\varphi_\varepsilon(Q),&\mathrm{d}(\Delta+1)^{-1}]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \\ &\leqslant \|[\varphi_\varepsilon(Q), \mathrm{d}]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})}\|(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H})} \\ &\quad + \|\mathrm{d}(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H},\mathscr{K})}\|[\varphi_\varepsilon(Q), \mathrm{d}]^*\|_{\mathcal{B}(\mathscr{K},\mathscr{H})}\|\mathrm{d}(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \\ &\quad + \|\mathrm{d}(\Delta+1)^{-1}\mathrm{d}^*\|_{\mathcal{B}(\mathscr{K},\mathscr{K})}\|[\mathrm{d}, \varphi_\varepsilon(Q)]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})}\|(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H})}.\end{aligned}$$

The most singular factor here is

$$\|\mathrm{d}(\Delta+1)^{-1}\mathrm{d}^*\|_{\mathcal{B}(\mathscr{K},\mathscr{K})} \leqslant \|\mathrm{d}\|_{\mathcal{B}(\mathscr{G},\mathscr{K})}\|(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{G}^*,\mathscr{G})}\|\mathrm{d}^*\|_{\mathcal{B}(\mathscr{K},\mathscr{G}^*)}$$

and this is finite. Thus we get for a finite constant C_1:

$$\|[\varphi_\varepsilon(Q), \mathrm{d}(\Delta+1)^{-1}]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \leqslant C_1\|[\mathrm{d}, \varphi_\varepsilon(Q)]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \leqslant C_1 C \mathrm{Lip}\, \varphi_\varepsilon$$

which is less than $C_1 C \mathrm{Lip}\, \varphi$. Now let $u \in \mathscr{K}_\mathrm{c}$ and $v \in \mathscr{H}_\mathrm{c}$. We get:

$$\begin{aligned}|\langle\varphi(Q)u,&\mathrm{d}(\Delta+1)^{-1}v\rangle - \langle u, \mathrm{d}(\Delta+1)^{-1}\varphi(Q)v\rangle| \\ &= \lim_{\varepsilon\to 0}|\langle\varphi_\varepsilon(Q)u, \mathrm{d}(\Delta+1)^{-1}v\rangle - \langle u, \mathrm{d}(\Delta+1)^{-1}\varphi_\varepsilon(Q)v\rangle| \leqslant C_1\, C\, \mathrm{Lip}\, \varphi.\end{aligned}$$

Thus $[\varphi(Q), \mathrm{d}(\Delta+1)^{-1}]$ is a bounded operator. ∎

THEOREM 8.4. *Let (X, ρ) be a proper locally compact metric space. Assume that $(\mathscr{G}, \mathscr{H})$ is a compact Friedrichs X-module and that $\mathscr{K}$ is a Hilbert X-module. Let* $\mathrm{d}, \lambda, \Lambda$ *be operators satisfying the following conditions:*

(i) d *is a closed first order operator from $\mathscr{H}$ to $\mathscr{K}$ with $\mathcal{D}(\mathrm{d}) = \mathscr{G}$;*

(ii) *λ is a bounded self-adjoint operator on $\mathscr{H}$ with* $\inf \lambda > 0$ *and* $\lambda - 1 \in \mathcal{K}(\mathscr{G}, \mathscr{H})$;

(iii) *Λ is a bounded self-adjoint operator on $\mathscr{K}$ with* $\inf \Lambda > 0$ *and* $\Lambda - 1 \in \mathcal{B}_0(\mathscr{K})$.

Then the self-adjoint operators Δ and $\widetilde{\Delta}$ have the same essential spectrum.

Proof. In this proof, we shall consider $\widetilde{\Delta}$ as an operator acting on $\mathscr{H}$. Since $\widetilde{\mathscr{H}} = \mathscr{H}$ as topological vector spaces and the notion of spectrum is purely topological, $\widetilde{\Delta}$ is a closed densely defined operator on $\mathscr{H}$ and it has the same spectrum as the self-adjoint operator $\widetilde{\Delta}$ on $\widetilde{\mathscr{H}}$. Moreover, if we define the essential spectrum $\sigma_{\mathrm{ess}}(A)$ as the set of $z \in \mathbb{C}$ such that either $\ker(A-z)$ is infinite dimensional or the range of $A - z$ is not closed, we see that the essential spectrum is a topological notion, so $\sigma_{\mathrm{ess}}(\widetilde{\Delta})$ is the same, whether we think of $\widetilde{\Delta}$ as operator on $\mathscr{H}$ or on $\widetilde{\mathscr{H}}$. Finally, with this definition of σ_{ess} we have $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B)$ if $(A-z)^{-1} - (B-z)^{-1}$ is a compact operator for some $z \in \rho(A) \cap \rho(B)$.

Thus it suffices to prove that $(\Delta+1)^{-1} - (\widetilde{\Delta}+1)^{-1} \in \mathcal{K}(\mathscr{H})$. Now we observe that $\widetilde{\Delta}+1 = \lambda^{-1}\mathrm{d}^*\Lambda\mathrm{d} + 1 = \lambda^{-1}(\mathrm{d}^*\Lambda\mathrm{d} + \lambda)$ and $\Delta_\Lambda = \mathrm{d}^*\Lambda\mathrm{d}$ is the positive self-adjoint operator on $\mathscr{H}$ associated to the closed quadratic form $\|\mathrm{d}u\|^2_{\widetilde{\mathscr{K}}}$ on $\mathscr{H}$ with domain $\mathscr{G}$. Thus $(\widetilde{\Delta}+1)^{-1} = (\Delta_\Lambda + \lambda)^{-1}\lambda$ and

$$(\widetilde{\Delta}+1)^{-1} - (\Delta_\Lambda + \lambda)^{-1} = (\Delta_\Lambda + \lambda)^{-1}(\lambda - 1) = [(\lambda-1)(\Delta_\Lambda+\lambda)^{-1}]^*.$$

The range of $(\Delta_\Lambda + \lambda)^{-1}$ is included in the form domain of $\Delta_\Lambda + \lambda$, which is $\mathscr{G}$. The map $(\Delta_\Lambda + \lambda)^{-1} : \mathscr{H} \to \mathscr{G}$ is continuous, by the closed graph theorem, and $\lambda - 1 : \mathscr{G} \to \mathscr{H}$ is compact. Hence $(\widetilde{\Delta}+1)^{-1} - (\Delta_\Lambda + \lambda)^{-1}$ is compact. Similarly:

$$(\Delta+1)^{-1} - (\Delta_\Lambda + \lambda)^{-1} = (\mathrm{d}^*\mathrm{d}+1)^{-1} - (\mathrm{d}^*\Lambda\mathrm{d}+1)^{-1} \in \mathcal{K}(\mathscr{H}).$$

To prove this we use Theorem 2.5 with: $\mathscr{E} = \mathscr{K}$, $D = \mathrm{d}$, $a = 1$, $b = \Lambda$ and $z = -1$. Since $\mathrm{d}^*\mathrm{d}$ and $\mathrm{d}^*\Lambda\mathrm{d}$ are positive self-adjoint operators on $\mathscr{H}$ with the same form domain $\mathscr{G}$, the first condition of Theorem 2.5 is satisfied. Then the second condition holds because $\Lambda - 1 \in \mathcal{B}^1_0(\mathscr{K})$. Thus it remains to observe that the operator $\mathrm{d}(\Delta+1)^{-1}$ is decay preserving by Lemma 8.3. ∎

8.3. We shall consider now an application of Theorem 8.4 to concrete Riemannian manifolds. It will be clear from what follows that we could treat Lipschitz manifolds with measurable metrics (see [6], [20] for example), but the case of C^1 manifolds with locally bounded metrics suffices as an example. Note also that the proof of Theorem 8.5 covers without any modification the case when X is not C^1 but is a Lipschitz manifold and a countable atlas has been specified (then the tangent space is well defined almost everywhere and the absolute continuity notions we use make sense).

From now on in this section *X is a non-compact differentiable manifold of class C^1*. Then its cotangent manifold T^*X is a topological vector fiber bundle over X whose fiber over x will be denoted T^*_xX. If $u : X \to \mathbb{R}$ is differentiable then $\mathrm{d}u(x) \in T^*_xX$ is its differential at the point x and its differential $\mathrm{d}u$ is a section of T^*X. Thus for the moment d is a linear map defined on the space of real $C^1(X)$ functions to the space of sections of T^*X.

A measurable locally bounded Riemannian structure on X will be called an *R-structure* on X. To be precise, an R-structure is given on X if each T^*_xX is equipped with a quadratic (i.e. generated by scalar product) norm $\|\cdot\|_x$ such that:

$$\text{(R)}\quad \left\{ \begin{array}{l} \text{if } v \text{ is a continuous section of } T^*X \text{ over a compact set } K \text{ such that} \\ v(x) \neq 0 \text{ for } x \in K, \text{ then } x \mapsto \|v(x)\|_x \text{ is a bounded Borel map on} \\ K \text{ and } \|v(x)\|_x \geqslant c \text{ for some number } c > 0 \text{ and all } x \in K. \end{array} \right.$$

Such a structure allows one to construct a metric compatible with the topology on X, the distance between two points being the infimum of the length of the Lipschitz curves connecting the points (see the references above). Since X was assumed to be non-compact, the metric space X is proper if and only if it is complete. If this is the case, we say that *the R-structure is complete*.

It will also be convenient to complexify these structures (i.e. replace T_x^*X by $T_x^*X \otimes \mathbb{C}$ and extend the scalar product as usual) and to keep the same notations for the complexified objects.

We shall consider positive measures μ on X such that:

$$\text{(M)} \quad \begin{cases} \mu \text{ is absolutely continuous and its density is locally bounded} \\ \text{and locally bounded from below by strictly positive constants.} \end{cases}$$

A couple consisting of an R-structure and a measure verifying (M) on X will be called an *RM-structure on X*. To an R-structure we may canonically associate an RM-structure by taking μ equal to the Riemannian volume element.

If an RM-structure is given on X then we may consider the two Hilbert spaces $\mathscr{H} = L^2(X,\mu)$ and $\mathscr{K}$ defined as the completion of the space of continuous sections with compact support of T^*X equipped with the norm

$$\|v\|^2_{\mathscr{K}} = \int_X \|v(x)\|_x^2 \mathrm{d}\mu(x).$$

In fact, $\mathscr{K}$ is the space of (suitably defined) square integrable sections of T^*X.

The operator of exterior differentiation d induces a linear map $C^1_{\mathrm{c}}(X) \to \mathscr{K}$ which is easily seen to be closable as operator from $\mathscr{H}$ to $\mathscr{K}$ (this is a purely local problem and the hypotheses we put on the metric and the measure allow us to reduce ourselves to the Euclidean case). We shall keep the notation d for its closure and we note that its domain $\mathscr{G}$ is the first order Sobolev space $\mathscr{H}^1$ defined in this context as the closure of $C^1_{\mathrm{c}}(X)$ under the norm

$$\|u\|^2_{\mathscr{H}^1} = \int_X (|u(x)|^2 + \|\mathrm{d}u(x)\|_x^2)\mathrm{d}\mu(x).$$

The self-adjoint operator $\Delta = \mathrm{d}^*\mathrm{d}$ in $\mathscr{H}$ associated to the quadratic form $\|\cdot\|^2_{\mathscr{H}^1}$ is the *Laplace operator* associated to the given RM-structure.

Two RM-structures $(\{\|\cdot\|_x\}_{x\in X},\mu)$ and $(\{\|\cdot\|'_x\}_{x\in X},\mu')$ on X are called *equivalent* if there are bounded Borel functions α,β,λ on X with $\alpha \geqslant c$ and $\lambda \geqslant c$ for some number $c > 0$ such that $\alpha(x)\|\cdot\|_x \leqslant \|\cdot\|'_x \leqslant \beta(x)\|\cdot\|_x$ for all x and $\mu' = \lambda\mu$. The distances ρ,ρ' on X associated to these structures satisfy $a\rho \leqslant \rho' \leqslant b\rho$ for some numbers $b \geqslant a > 0$, hence if one of the RM-structures is complete, the second one is also complete. Notice that the spaces $\mathscr{H},\mathscr{K}$ associated to equivalent RM-structures are identical as topological vector spaces.

Two equivalent RM-structures are *strongly equivalent* if the functions α,β,λ can be chosen such that $\lambda(x) \to 1$, $\alpha(x) \to 1$ and $\beta(x) \to 1$ as $x \to \infty$.

THEOREM 8.5. *The Laplace operators associated to strongly equivalent complete* RM-*structures on X have the same essential spectrum.*

Proof. We check that the assumptions of Theorem 8.4 are satisfied. We noted above that X is a proper metric space for the metric associated to the initial Riemann structure. The spaces $\mathscr{H},\mathscr{K}$ have obvious X-module structures and for each $\varphi \in C_c(X)$ the operator $\varphi(Q) : \mathscr{H}^1 \to \mathscr{H}$ is compact. Indeed, by using partitions of unity, we may assume that the support of φ is contained in the domain of a local chart and then we are reduced to a known fact in the Euclidean case. Thus $(\mathscr{G}, \mathscr{H})$ is a compact Friedrichs X-module. To see that d is a first order operator we observe that if φ is Lipschitz then $[\mathrm{d}, \varphi]$ is the operator of multiplication by the differential $\mathrm{d}\varphi$ of φ and the estimate ess-sup $\|\mathrm{d}\varphi(x)\|_x \leqslant \mathrm{Lip}\,\varphi$ is easy to obtain. The conditions on λ in Theorem 8.4 are trivially verified. So it remains to consider the operator Λ. For each $x \in X$ there is a unique operator $\Lambda_0(x)$ on T_x^*X such that $\langle u|v\rangle_x' = \langle u|\Lambda_0(x)v\rangle_x$ for all $u, v \in T_x^*X$ and we have $\alpha(x)^2 \leqslant \Lambda_0(x) \leqslant \beta(x)^2$ by hypothesis. Here the inequalities must be interpreted with respect to the initial scalar product on T_x^*X. Thus the operator Λ on $\mathscr{K}$ is just the operator of multiplication by the function $\Lambda(x) = \lambda(x)\Lambda_0(x)$ and the condition (iii) of Theorem 8.4 is clearly satisfied. ∎

The (strong) equivalence of two R-structures is defined in an obvious way. If μ, μ' are the Riemannian measures associated to strongly equivalent R-structures then the unique function λ such that $\mu' = \lambda\mu$ satisfies $\lambda(x) \to 1$ as $x \to \infty$.

COROLLARY 8.6. *The Laplace operators associated to strongly equivalent complete* R*-structures on X have the same essential spectrum.*

We stress that if one of the Riemannian structures is locally Lipschitz then this result is easy to prove by using local regularity estimates for elliptic equations.

An assumption of the form $\alpha(x) \to 1$ as $x \to \infty$ imposed in the definition of strong equivalence means that the set where $|\alpha(x) - 1| > \varepsilon$ is relatively compact for any $\varepsilon > 0$. We shall consider now a weaker notion of equivalence associated to the filter $\mathcal{F}_\mu$ introduced at the beginning of Section 6.

We first introduce two notions which clearly depend only on the equivalence class of an RM-structure. We say that an RM-structure is of *infinite volume* if $\mu(X) = \infty$. We say that it has the *F-embedding property* if for each Borel set $F \subset X$ of finite measure the operator $\chi_F(Q) : \mathscr{H}^1 \to \mathscr{H}$ is compact.

REMARK 8.7. The F-embedding property is satisfied under quite general conditions. Indeed, the compactness of $\chi_F(Q) : \mathscr{H}^1 \to \mathscr{H}$ is equivalent to the compactness of the operator $\chi_F(Q)(\Delta + 1)^{-1/2}$ in $\mathscr{H}$. Or the set of functions $\varphi \in C([0, \infty[)$ such that $\chi_F(Q)\varphi(\Delta)$ is compact is a closed C^*-subalgebra of $C([0, \infty[)$ so it suffices to find one function φ which generates this algebra such that $\chi_F(Q)\varphi(\Delta)$ be compact. But $\chi_F(Q)\varphi(\Delta)$ is compact if and only if the operator $\chi_F(Q)|\varphi(\Delta)|^2\chi_F(Q)$ is compact, so we see that it suffices to show that for each Borel set F of finite measure there is $t > 0$ such that $\chi_F(Q)\mathrm{e}^{-t\Delta}\chi_F(Q)$ be compact. For example, it suffices that this operator be Hilbert-Schmidt, i.e. that the integral kernel P_t of $\mathrm{e}^{-t\Delta}$ be such that $\int_{F\times F} |P_t(x,y)|^2 \mathrm{d}\mu(x)\mathrm{d}\mu(y) < \infty$, which is true if P_t satisfies a Gaussian upper estimate and the measure of a ball of radius $t^{1/2}$ (for a fixed t) is bounded below by a strictly positive constant (see [2], [3] and references there).

Two infinite volume RM-structures will be called *μ-strongly equivalent* if they are equivalent and if the functions α, β, λ can be chosen such that for each $\varepsilon > 0$ the set where one of the inequalities $|\alpha(x) - 1| > \varepsilon$, $|\beta(x) - 1| > \varepsilon$ or $|\lambda(x) - 1| > \varepsilon$ holds is of finite measure.

We say that an RM-structure is *regular* if there is $p > 2$ such that $\mathrm{d}(\Delta + 1)^{-1}$ induces a bounded operator in L^p. More precisely, this means that there is a constant C such that if $u \in L^2(X) \cap L^p(X)$ then $\mathrm{d}(\Delta + 1)^{-1}u$, which is a section of T^*X of finite L^2 norm, has an L^p norm bounded by $C\|u\|_{L^p}$. If the operator $\mathrm{d}(\Delta + 1)^{-1}\mathrm{d}^*$ also induces a bounded operators in L^p (in an obvious sense), we say that the RM-structure is *strongly regular*. From the relation $\mathrm{d}(\Delta + 1)^{-1}\mathrm{d}^* = [\mathrm{d}(\Delta + 1)^{-1/2}][\mathrm{d}(\Delta + 1)^{-1/2}]^*$ we see that strong regularity follows from: $\exists\varepsilon > 0$ such that $\mathrm{d}(\Delta + 1)^{-1/2}$ induces a bounded operator in L^p for $2 - \varepsilon < p < 2 + \varepsilon$.

THEOREM 8.8. *Let Δ be the Laplace operator associated to an infinite volume complete* RM-*structure on X which has the F-embedding property and is regular. Then the Laplace operator associated to an* RM-*structure μ-strongly equivalent to the given structure has the same essential spectrum as Δ.*

Proof. Let $\Lambda(x)$ be as in the proof of Theorem 8.5. Clearly there is a number $C > 0$ such that $C^{-1} \leqslant \Lambda(x) \leqslant C$ for all x and such that for each $\varepsilon > 0$ the set where $\|\Lambda(x) - 1\| > \varepsilon$ is of finite measure (the inequalities and the norm are computed on T_x^*X, which is equipped with the initial scalar product).

Now we proceed as in the proof of Theorem 8.4 but this time we equip $\mathscr{H}$ and $\mathscr{K}$ with the Hilbert module structures described at the beginning of Section 6. To avoid confusions, we denote $\mathcal{B}_\mu(\mathscr{H})$ and $\mathcal{B}_\mu(\mathscr{K})$ the space of decay improving operators relatively to these new module structures. The F-embedding property implies that $(\mathscr{H}^1, \mathscr{H})$ is a compact Friedrichs module. Moreover, the operator $\lambda(Q) - 1 : \mathscr{H}^1 \to \mathscr{H}$ is compact. Then, as in the proof of Theorem 8.4, we see that it suffices to prove that

$$(\mathrm{d}^*\mathrm{d} + 1)^{-1} - (\mathrm{d}^*\Lambda(Q)\mathrm{d} + 1)^{-1} \in \mathcal{K}(\mathscr{H}).$$

Clearly $\Lambda(Q) - 1 \in \mathcal{B}_\mu(\mathscr{K})$. Now we use Theorem 2.5 exactly as in the proof of Theorem 8.4 and we see that the only condition which remains to be checked is (iii) of Theorem 2.5, i.e. in our case $\mathrm{d}(\Delta + 1)^{-1} \in \mathcal{B}_q^{\mathrm{r}}(\mathscr{H}, \mathscr{K})$, where the decay preserving property is relatively to the algebra $B_\mu(X)$. But this follows from Theorem 6.1. ∎

One may check the regularity property needed in Theorem 8.8 by using the results from [2], [3] concerning the boundedness in L^p of the operator $\mathrm{d}\Delta^{-1/2}$. For example, *it suffices that X be complete, with the doubling volume property, and such that the Poincaré inequality holds in L^2 sense*. Note, however, that these results are much stronger than necessary in our context and that the boundedness of $\mathrm{d}(\Delta+1)^{-1/2}$ should hold under less restrictive assumptions.

The next result does not require regularity assumptions on any of the RM-structures that we want to compare but only on a third one in their equivalence class. Observe that each equivalence class of RM-structures contains one of the same degree of local smoothness as the manifold X (make local regularizations and use a partition of unity).

THEOREM 8.9. *Let Δ_a, Δ_b be the Laplace operators associated to μ-strongly equivalent complete* RM-*structures of infinite volume and having the* F-*embedding property. If these structures are equivalent to a strongly regular* RM-*structure, then Δ_a and Δ_b have the same essential spectrum.*

Proof. Let Δ_c be the Laplace operator associated to the third structure. From Theorem 6.1 it follows that $\mathrm{d}(\Delta_c+1)^{-1}$ and $\mathrm{d}(\Delta_c+1)^{-1}D^*$ are right $\mathcal{F}_\mu$-decay preserving. Then from Proposition 2.6 we see that $\mathrm{d}(\Delta_a+1)^{-1}$ is right $\mathcal{F}_\mu$-decay preserving and we may conclude as in the proof of Theorem 8.8. ∎

It is natural to consider an analogue of the filter $\mathcal{F}_{\mathrm{w}}$ introduced in Section 6 to get an optimal weak decay condition for the stability of the essential spectrum in the present context. The techniques of Section 6 should be relevant for this question.

REMARK 8.10. We shall describe here, without going into details, an abstract framework for the study of the Laplace operator acting on forms. Let $\mathscr{H}$ be a Hilbert space and d a closed densely defined operator in $\mathscr{H}$ such that $\mathrm{d}^2 = 0$. For example, $\mathscr{H}$ could be the space of square integrable differential forms over a C^1 manifold equipped with a measurable locally bounded metric and d the operator of exterior differentiation. We denote $\delta = \mathrm{d}^*$ and we assume that $\mathscr{G} := \mathcal{D}(\mathrm{d}) \cap \mathcal{D}(\delta)$ is dense in $\mathscr{H}$ (which is a rather strong condition in the context of this paper, e.g. in the preceding example it is a differentiability condition on the metric). Then let $D = \mathrm{d} + \delta$ with domain $\mathscr{G}$, observe that $\|Du\|^2 = \|\mathrm{d}u\|^2 + \|\delta u\|^2$ so D is a closed symmetric operator, assume that D is self-adjoint, and define $\Delta = D^2 = \mathrm{d}\delta + \delta\mathrm{d}$ (form sum). Then

$$(\Delta+1)^{-1} = (D+\mathrm{i})^{-1}(D-\mathrm{i})^{-1}. \tag{8.1}$$

Now let $a \in \mathcal{B}(\mathscr{H})$ with $a \geqslant \varepsilon > 0$ and such that $a^{\pm 1}\mathscr{G} \subset \mathscr{G}$ and let $\mathscr{H}_a$ be the Hilbert space which is equal to $\mathscr{H}$ as vector space but is equipped with the new scalar product $\langle u, v\rangle_a = \langle u, av\rangle$. Denote d_a the operator d viewed as operator acting in $\mathscr{H}_a$ with adjoint $\delta_a = a^{-1}\delta a$. We can define as above operators D_a (with domain $\mathscr{G}_a = \mathscr{G}$) and $\Delta_a = D_a^2$ which are self-adjoint in $\mathscr{H}_a$ and satisfy a relation similar to (8.1). Then Δ_a is a compact perturbation of Δ if the operators $(D_a \pm \mathrm{i})^{-1} - (D \pm \mathrm{i})^{-1}$ are compact and this last condition is equivalent to the compactness of the operator $D_a - D : \mathscr{G} \to \mathscr{G}^*$. And this holds if $(\mathscr{G}, \mathscr{H})$ is a compact Hilbert X-module over a metric space X and $a - 1 \in \mathcal{B}_0(\mathscr{H})$.

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