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# WEAK PRODUCT DECOMPOSITIONS AND HANKEL OPERATORS ON VECTOR-VALUED BERGMAN SPACES

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ABSTRACT. We obtain some weak product decomposition theorems, which represent the Bergman space analogues to Sarason's theorem for operator-valued Hardy spaces, respectively, to the Ferguson-Lacey theorem for Hardy spaces on product domains. We also characterize the compact little Hankel operators on vector-valued Bergman spaces.

KEYWORDS: *Weak product decomposition, compact Hankel operator, vector-valued Bergman space.*

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## INTRODUCTION

Product decompositions of function spaces play an important role in complex analysis and operator theory, and they are closely related to the study of Hankel operators. For instance, the well-known Hardy space factorization $H^1 = H^2H^2$, with control on the norms of the factors, together with Fefferman's theorem yield the characterization of bounded Hankel operators on $H^2$, i.e. a Hankel operator is bounded on $H^2$ if and only if the analytic part of its symbol is a BMOA function. These problems become considerably more difficult in the case of vector-valued function spaces or in several complex variables. One such example is the characterization of the boundedness in terms of the symbol for the Hankel operator on vector-valued Hardy spaces, which is still not completely understood in the infinite dimensional case. There is however an operator-valued analogue of the decomposition of $H^1$ mentioned above which was found by Sarason [14]. He showed that $H^1(\mathbb{D}, \mathcal{S}^1) = H^2(\mathbb{D}, \mathcal{S}^2)H^2(\mathbb{D}, \mathcal{S}^2)$, where $\mathcal{S}^1, \mathcal{S}^2$ denote the trace class, respectively Hilbert-Schmidt operators on a separable Hibert space $\mathcal{H}$. This implies that the space of bounded Hankel operators acting on vector-valued Hardy spaces can be identified with the dual of $H^1(\mathbb{D}, \mathcal{S}^1)$ (see also [12]).

A similar role is played by weak product decompositions, that is a decomposition in sums of products of elements of two given spaces, with control on the norms. For example, in their recent study of Hankel operators on the Hardy space on the bidisc, $H^2(\mathbb{D}^2)$, Ferguson and Lacey [8] proved that any $f \in H^1(\mathbb{D}^2)$ can be written as

$$f = \sum_{i=1}^{\infty} g_i h_i,$$

where $g_i, h_i \in H^2(\mathbb{D}^2)$ and

$$\sum_{i=1}^{\infty} \|g_i\|_{H^2(\mathbb{D}^2)} \|h_i\|_{H^2(\mathbb{D}^2)} \leqslant k \|f\|_{H^1(\mathbb{D}^2)}.$$

In the terminology introduced above this is a weak product decomposition of $H^1(\mathbb{D}^2)$ with factors in $H^2(\mathbb{D}^2)$. The factorization of scalar-valued Bergman space functions was obtained by Horowitz [11]. Weak product decompositions for scalar Bergman spaces are related to atomic decompositions (see [9]) and they also appear in a very recent description of the preduals of the $Q_p$-spaces (see [1]).

The aim of this paper is to study weak product decompositions and little Hankel operators on vector-valued Bergman spaces. From now on, whenever we use the terminology Hankel operator, we refer to the *little* Hankel operator. Given $\alpha > -1$, $p > 0$, and a Banach space $Y$, the standard weighted Bergman space $L_a^{p,\alpha}(\mathbb{D}, Y)$ is the space of $Y$-valued functions $f$, holomorphic on $\mathbb{D}$, with

$$\|f\|_{L_a^{p,\alpha}(\mathbb{D},Y)} = \Big( \int_{\mathbb{D}} \|f(z)\|_Y^p \, \mathrm{d}A_\alpha(z) \Big)^{1/p},$$

where $\mathrm{d}A$ denotes the normalized area measure on the unit disc $\mathbb{D}$ in the complex plane, and $\mathrm{d}A_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \mathrm{d}A(z)$. We are going to establish the Bergman space analogues of the theorems by Sarason [14] and Ferguson-Lacey [8] mentioned above, and to give a characterization of the compactness of the Hankel operators on vector-valued Bergman spaces. As an application, we give a short proof of Zhu's [15] characterization of compact Hankel operators on Bergman spaces on the polydisc.

Our analogue to Sarason's theorem is proved in Section 2 and states that any function $f \in L_a^{1,\alpha}(\mathbb{D}, \mathcal{S}^1)$ can be written as

$$f = \sum_{n=1}^{\infty} g_n h_n, \tag{0.1}$$

with $g_n \in L_a^{r,\alpha}(\mathbb{D}, \mathcal{S}^2)$, $h_n \in L_a^{s,\alpha}(\mathbb{D}, \mathcal{S}^2)$ and

$$\sum_{n=1}^{\infty} \|g_n\|_{L_a^{r,\alpha}(\mathbb{D},\mathcal{S}^2)} \|h_n\|_{L_a^{s,\alpha}(\mathbb{D},\mathcal{S}^2)} \sim \|f\|_{L_a^{1,\alpha}(\mathbb{D},\mathcal{S}^1)},$$

where $1/r + 1/s = 1$. As in [2], our approach differs from the one used in [14] for Hardy spaces. The key step is an atomic decomposition theorem for the spaces $L_a^{1,\alpha}(\mathbb{D}, X^*)$, where $X$ is a separable Banach space. As in [1], this atomic decomposition is deduced from sampling theorems via the duality $(\mathcal{B}_0(X))^* =$

$L_a^{1,\alpha}(\mathbb{D}, X^*)$. This duality theorem completes results by Arregui and Blasco [3] and is presented in Section 1. As a direct consequence of this result, we deduce that the dual of $L_a^{1,\alpha}(\mathbb{D}, \mathcal{S}^1)$ can be identified with the bounded Hankel operators on $L_a^{2,\alpha}(\mathbb{D}, \mathcal{H})$. This together with a result by Arregui and Blasco [3] that identifies the dual of $L_a^{1,\alpha}(\mathbb{D}, \mathcal{S}^1)$ with $\mathcal{B}(\mathcal{L})$, where $\mathcal{L}$ denotes the space of bounded linear operators on $\mathcal{H}$, yields a characterization of the bounded Hankel operators on $L_a^{2,\alpha}(\mathbb{D}, \mathcal{H})$. An alternative direct proof of this fact can be found in [2]. We should point out here that in the study of Hankel operators on Bergman spaces we encounter a completely different situation to the one in Hardy spaces. For example, given a separable Hilbert space $\mathcal{H}$, it is shown in [2] that for a Foguel-Hankel matrix

$$\begin{pmatrix} M_z^* & \Gamma_f \\ 0 & M_z \end{pmatrix},$$

where $\Gamma_f$ is the Hankel operator with symbol $f$ on $L_a^{2,\alpha}(\mathbb{D}, \mathcal{H})$, and $M_z$ is the operator of multiplication by $z$ on $L_a^{2,\alpha}(\mathbb{D}, \mathcal{H})$, the power boundedness, the polynomial boundedness and the complete polynomial boundedness are equivalent. This is in contrast to Pisier's famous example [13] of such a matrix on vector-valued Hardy spaces, which is polynomially bounded but not completely polynomially bounded.

We continue the investigation of such operators in Section 3. We prove that a Hankel operator $\Gamma_f$ defined on $L_a^{2,\alpha}(\mathbb{D}, \mathcal{H})$, is compact if and only if its symbol $f$ is an analytic function on $\mathbb{D}$ with values in the space of compact operators on $\mathcal{H}$, that satisfies the following little Bloch condition

$$\|f'(z)\|(1 - |z|) \to 0 \quad \text{as } |z| \to 1.$$

Since the values $f'(z)$ can be computed with the formula

$$f'(z) = (\alpha + 2) \int_{\mathbb{D}} f(w) \frac{\overline{w}}{(1 - \overline{w}z)^{\alpha+3}} \, \mathrm{d}A_\alpha(w),$$

we can say that, in the scalar case, the Hankel operator satisfies the normalised reproducing kernel synthesis, i.e. boundedness and compactness can be characterized considering only the action of the operator on the normalised reproducing kernels. With the appropriate interpretation of the reproducing kernels, this result remains true in the vector-valued case as well.

The Bergman space analogue of the Ferguson-Lacey [8] result is proved in Section 4. We show that any function $f \in L_a^{1,\alpha}(\mathbb{D}^n)$ can be written as

$$f = \sum_{i=1}^{\infty} g_i h_i,$$

where $g_i, h_i \in L_a^{2,\alpha}(\mathbb{D}^n)$ and

$$\sum_{i=1}^{\infty} \|g_i\|_{L_a^{2,\alpha}(\mathbb{D}^n)} \|h_i\|_{L_a^{2,\alpha}(\mathbb{D}^n)} \sim \|f\|_{L_a^{1,\alpha}(\mathbb{D}^n)}.$$

This type of results for $\mathbb{D}^n$ are obtained by an inductive process with respect to the number of variables. As a byproduct of this technique, we also obtain an analogue of Sarason's theorem for operator-valued Bergman spaces on the polydisc.

Our analogue of the Ferguson-Lacey result together with the compactness theorem proven in Section 3 imply Zhu's [15] characterization of compactness for Hankel operators on Bergman spaces on the polydisc. Moreover, our approach yields the following very natural formulation of the necessary and sufficient conditions for the Hankel operator to be bounded, respectively compact. The Hankel operator $\Gamma_f$ is bounded on $L_a^{2,\alpha}(\mathbb{D}^n)$ if and only if its holomorphic symbol $f : \mathbb{D}^n \to \mathbb{C}$ belongs to the Bloch-type space $\mathcal{B}^{(n)}$ defined inductively by $\mathcal{B}^{(1)} = \mathcal{B}$, $\mathcal{B}^{(n)} = \mathcal{B}(\mathcal{B}^{(n-1)})$, $n \geqslant 2$.

Similarly, we show that $\Gamma_f$ is compact if and only if $f \in \mathcal{B}_0^{(n)}$, where $\mathcal{B}_0^{(1)} = \mathcal{B}_0$, $\mathcal{B}_0^{(n)} = \mathcal{B}_0(\mathcal{B}_0^{(n-1)})$, $n \geqslant 2$.

## 1. PRELIMINARIES

Throughout our considerations $k$ will stand for a generic positive constant whose dependence on parameters will be indicated whenever it is relevant. Moreover, by "$E_1 \sim E_2$" we denote two functions that are comparable, i.e. there is a constant $k > 0$ independent of the argument such that $kE_2 \geqslant E_1 \geqslant 1/k\, E_2$.

Let $\mathbb{D}^n$ be the polydisc in $\mathbb{C}^n$. We denote by $\mathrm{d}A$ the normalized area measure on $\mathbb{D}$, and for $\alpha > -1$, we let $\mathrm{d}A_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \mathrm{d}A(z)$. Given a Banach space $Y$, and $p > 0$, we consider the standard weighted vector-valued Bergman spaces $L_a^{p,\alpha}(\mathbb{D}^n, Y)$, which consist of holomorphic functions $f : \mathbb{D}^n \to Y$ with

$$\|f\|_{L_a^{p,\alpha}(\mathbb{D}^n,Y)} = \Big( \int\limits_{\mathbb{D}^n} \|f(z_1,\dots,z_n)\|_Y^p \, \mathrm{d}A_\alpha(z_1) \, \cdots \, \mathrm{d}A_\alpha(z_n) \Big)^{1/p} < \infty,$$

and note that, for $n \geqslant 2$, $L_a^{p,\alpha}(\mathbb{D}^n, Y) = L_a^{p,\alpha}(\mathbb{D}, L_a^{p,\alpha}(\mathbb{D}^{n-1}, Y))$ When $Y = \mathbb{C}$ we shall simply denote these spaces by $L_a^{p,\alpha}(\mathbb{D}^n)$, while for $n = 1$, $Y = \mathbb{C}$ we shall denote these spaces by $L_a^{p,\alpha}$. Moreover, we denote by $\mathcal{B}(Y)$, the vector-valued Bloch space which consists of holomorphic functions $f : \mathbb{D} \to Y$ that satisfy

$$\|f\|_{\mathcal{B}(Y)} = \sup_{z\in\mathbb{D}} (1-|z|^2)\|f'(z)\|_Y < \infty.$$

In case $Y = \mathbb{C}$, we shall simply write $\mathcal{B}$ instead of $\mathcal{B}(\mathbb{C})$. For $\mathcal{B}^{(1)} = \mathcal{B}$, $\mathcal{B}^{(2)} = \mathcal{B}(\mathcal{B})$, we define inductively the spaces $\mathcal{B}^{(n)} = \mathcal{B}(\mathcal{B}^{(n-1)})$, $n \geqslant 2$. One can easily show that

$$\mathcal{B}^{(n)} = \Big\{ f : \mathbb{D}^n \to \mathbb{C} \text{ holomorphic } : \max_{\beta\in\mathcal{A}_n} \sup_{z\in\mathbb{D}^n} |(\partial^\beta f(\beta\cdot z)| \, (1-|z|)^\beta < \infty \Big\},$$

where $\mathcal{A}_n$ denotes the set of multi-indices $\beta = (\beta_1, \dots, \beta_n)$ with the entries $\beta_i \in \{0,1\}, 1 \leqslant i \leqslant n$, and

$$z = (z_1, \dots, z_n), \quad \beta \cdot z = (\beta_1 z_1, \dots, \beta_n z_n),$$
$$\partial^\beta = \partial_{z_1}^{\beta_1} \cdots \partial_{z_n}^{\beta_n}, \quad (1 - |z|)^\beta = (1 - |z_1|)^{\beta_1} \cdots (1 - |z_n|)^{\beta_n}.$$

The little Bloch space $\mathcal{B}_0(Y)$, will be the subspace of $\mathcal{B}(Y)$, consisting of $Y$-valued functions $f$, analytic on $\mathbb{D}$, for which the following holds

$$(1 - |z|^2)\|f'(z)\|_Y \to 0 \quad \text{as } |z| \to 1.$$

We shall also consider the spaces $\mathcal{B}_0^{(n)}$, $n \geqslant 1$, defined as $\mathcal{B}_0^{(n)} = \mathcal{B}_0(\mathcal{B}_0^{(n-1)})$ for $n \geqslant 2$, with $\mathcal{B}_0^{(1)} = \mathcal{B}_0$. An alternative characterization of functions in $\mathcal{B}_0^{(n)}$ is given by the following result.

PROPOSITION 1.1. *We have*

$$\mathcal{B}_0^{(n)} = \Big\{ f \text{ analytic on } \mathbb{D}^n : \lim_{(1-|z|)^\beta \to 0} (1 - |z|)^\beta |\partial^\beta f(\beta z)| = 0 \quad \text{for } \beta \in \mathcal{A}_n \Big\},$$

*the convergence being uniform in* $z \in \mathbb{D}^n$.

*Proof.* The fact that functions with the property specified in the statement belong to $\mathcal{B}_0^{(n)}$ follows at once; we only need to notice that the boundedness of point evaluations on $\mathcal{B}^{(n)}$ ensures $f'(z_1)\,(z_2, \dots, z_n) = \partial_{z_1} f(z_1, \dots, z_n)$.

The converse inclusion can be proved by induction. For $n = 1$ the statement is true by definition. Assume the statement is true for $n - 1$ with $n \geqslant 2$, and let us prove that it holds for $\mathcal{B}_0^{(n)}$ as well.

Let $f \in \mathcal{B}_0^{(n)}$, $\beta = (\beta_1, \dots, \beta_n) \in \mathcal{A}_n$, and denote $w = (z_2, \dots, z_n)$. If $\beta_1 = 0$, then, since $\mathbb{D}^{n-1} \ni w \mapsto f(0, w) \in \mathcal{B}_0^{(n-1)}$, we clearly have $\lim_{(1-|z|)^\beta \to 0} (1 - |z|)^\beta |\partial^\beta f(\beta z)| = 0$, uniformly for $z \in \mathbb{D}^n$. Suppose now $\beta_1 = 1$ and denote $\widetilde{\beta} = (0, \beta_2, \dots, \beta_n)$, $\widehat{\beta} = (\beta_2, \dots, \beta_n)$. Let $\varepsilon > 0$. Then, since $f \in \mathcal{B}_0^{(n)}$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$(1 - |z|)^\beta |\partial^\beta f(\beta z)| < \varepsilon, \quad z \in \mathbb{D}^n, 1 - |z_1| < \delta.$$

For $z_1 \in \mathbb{D}$ and $w \in \mathbb{D}^{n-1}$ we define

$$g_w(z_1) = (1 - |z|)^{\widetilde{\beta}} \partial^{\widetilde{\beta}} f(z_1, \widehat{\beta} w), \quad z = (z_1, w).$$

Since the set $K = \{z_1 \in \mathbb{D} : \ |z_1| \leqslant 1 - \delta\}$ is compact we can find $C > 0$ such that

$$|g_w''(z_1)| < C, \quad w \in \mathbb{D}^{n-1}, z_1 \in K.$$

Indeed, the map $\mathbb{D} \ni z_1 \mapsto f(z_1) \in \mathcal{B}_0^{(n-1)}$ is analytic, so that

$$\sup_{z_1 \in K} \|f''(z_1)\|_{\mathcal{B}^{(n-1)}} < C.$$

Thus

$$|g''_w(z_1)| = (1-|z|)^{\widetilde{\beta}}|\partial^2_{z_1}\partial^{\widetilde{\beta}} f(z_1, \widehat{\beta}w)| < C, \quad z_1 \in K,\ w \in \mathbb{D}^{n-1}.$$

The compactness of $K$ also ensures the existence of $N$ disks $D(z_{1,k}, r_k), k=1,\dots,N$, centered at $z_{1,k} \in K$ and of radius $r_k < \varepsilon/C$, such that $K \subset \bigcup_{k=1}^{N} D(z_{1,k}, r_k) \subset \mathbb{D}$. Given $z_1 \in K$, if $|z_{1,k} - z_1| < \varepsilon/C$, then

$$\begin{aligned} |(1-|z|)^{\beta}\partial^{\beta} f(z_1, \widehat{\beta}w) - (1-|z|)^{\beta}\partial^{\beta} f(z_{1,k}, \widehat{\beta}w)| &\leqslant |g'_w(z_1) - g'_w(z_{1,k})| \\ &\leqslant C\,|z_1 - z_{1,k}| < \varepsilon, \quad w \in \mathbb{D}^{n-1}, \end{aligned}$$

so that

$$(1-|z|)^{\beta}|\partial^{\beta} f(z_1, \widehat{\beta}w)| \leqslant (1-|z|)^{\widetilde{\beta}}|\partial^{\beta} f(z_{1,k}, \widehat{\beta}w)| + \varepsilon, \quad z_1 \in K.$$

Combined with the induction hypothesis this yields

$$\lim_{(1-|z|)^{\beta}\to 0} (1-|z|)^{\beta}|\partial^{\beta} f(z_1, \widehat{\beta}w)| < \varepsilon \quad \text{uniformly for } |z_1| \leqslant 1-\delta,\ w \in \mathbb{D}^{n-1}.$$

From the above we deduce that

$$\lim_{(1-|z|)^{\beta}\to 0} (1-|z|)^{\beta}|\partial^{\beta} f(\beta z)| < \varepsilon + \varepsilon = 2\varepsilon$$

uniformly in $z \in \mathbb{D}^n$, which completes the proof. ∎

Let $\mathcal{H}$ be a separable Hilbert space and $T : \mathbb{D} \to \mathcal{L}(\mathcal{H})$ a holomorphic operator-valued function, where $\mathcal{L}(\mathcal{H})$ denotes the bounded linear operators on $\mathcal{H}$. The Hankel operator $\Gamma_T$ is defined by means of the Hankel form

$$\langle \Gamma_T\, x, y\rangle = \lim_{r\to 1} \int_{|z|\leqslant r} \langle T(z)\, x(\overline{z}), y(z)\rangle\, \mathrm{d}A(z),$$

where $x, y$ are $\mathcal{H}$-valued holomorphic functions in a disk of radius strictly larger than 1 (as it is well-known these functions form a dense subset in $L^{2,\alpha}_a(\mathbb{D}, \mathcal{H})$). It turns out (see [2]) that $\Gamma_T$ extends to a bounded linear operator on $L^{2,\alpha}_a(\mathbb{D}, \mathcal{H})$ if and only if $T \in \mathcal{B}(\mathcal{L}(\mathcal{H}))$ and, in addition, $\|\Gamma_T\| \sim \|T\|_{\mathcal{B}(\mathcal{L}(\mathcal{H}))}$.

From now on we let $X$ be a separable Banach space. We shall frequently use in the sequel the following lemmas. The second one is a direct application of the Stokes' lemma (see [2]).

LEMMA 1.2 ([10], Theorem 1.7). *For any $\alpha \in (-1, \infty)$ and any $\beta > 0$, there exists $k > 0$ such that the following estimate holds*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha}}{|1-z\overline{w}|^{2+\alpha+\beta}}\, \mathrm{d}A(w) \leqslant k\frac{1}{(1-|z|^2)^{\beta}}, \quad z \in \mathbb{D}.$$

LEMMA 1.3. *Let $g \in L_a^{1,\alpha}(\mathbb{D}, X^*)$ and $f : \mathbb{D} \to X$ be analytic in a neighborhood of $\overline{\mathbb{D}}$. Then the following equality holds:*

$$\int_{\mathbb{D}} \langle f(z), g(\overline{z})\rangle \mathrm{d}A_\alpha(z) = \frac{1}{\alpha+1} \int_{\mathbb{D}} \Big\langle f'(z), \frac{g(\overline{z})-g(0)}{\overline{z}} \Big\rangle (1-|z|^2)\mathrm{d}A_\alpha(z) + \langle f(0), g(0)\rangle.$$

Let now $\mathcal{B}_{00}(X)$ be the closed subspace of $\mathcal{B}_0(X)$, consisting of functions that vanish at 0. Our next theorem shows that the dual of $\mathcal{B}_{00}(X)$ can be identified with $L_a^{1,\alpha}(\mathbb{D}, X^*)$. This completes a result by Arregui and Blasco [3]. Our method of proof is, however, different from the one in [3] and essentially follows the argument in the scalar-valued case. Recall that the functions analytic in a neighborhood of $\overline{\mathbb{D}}$ form a dense subset of $\mathcal{B}_0(X)$ (see [3]).

THEOREM 1.4. *The dual of $\mathcal{B}_{00}(X)$ can be identified with $L_a^{1,\alpha}(\mathbb{D}, X^*)$, $\alpha > -1$, under the pairing*

$$\langle f, g\rangle = \int_{\mathbb{D}} \langle f(z), g(\overline{z})\rangle \, \mathrm{d}A_\alpha(z), \tag{1.1}$$

*where $g \in L_a^{1,\alpha}(\mathbb{D}, X^*)$ and $f$ is an $X$-valued function, analytic in a neighborhood of $\overline{\mathbb{D}}$, $f(0) = 0$. This identification is an isomorphism from $(\mathcal{B}_{00}(X))^*$ onto $L_a^{1,\alpha}(\mathbb{D}, X^*)$.*

*Proof.* Let $f$ be an $X$-valued function, analytic on a neighborhood of $\overline{\mathbb{D}}$ with $f(0) = 0$. Then, for $g \in L_a^{1,\alpha}(\mathbb{D}, X^*)$, Lemma 1.3 gives

$$\Big|\int_{\mathbb{D}} \langle f(z), g(\overline{z})\rangle \mathrm{d}A_\alpha(z)\Big| = \Big|\int_{\mathbb{D}} \Big\langle f'(z), \frac{g(\overline{z})-g(0)}{\overline{z}}\Big\rangle (1-|z|^2)\frac{\mathrm{d}A_\alpha(z)}{\alpha+1}\Big| \leqslant k\|f\|_{\mathcal{B}(X)}\|g\|_{L_a^{1,\alpha}(\mathbb{D},X^*)}.$$

From this we immediately deduce that the mapping $f \mapsto \langle f, g\rangle$ extends to a bounded linear functional on $\mathcal{B}_{00}(X)$ for any $g \in L_a^{1,\alpha}(\mathbb{D}, X^*)$. Moreover, distinct functions in $L_a^{1,\alpha}(\mathbb{D}, X^*)$ induce distinct linear functionals on $\mathcal{B}_{00}(X)$. Indeed, if $\langle z^n x, g_1\rangle = \langle z^n x, g_2\rangle$, for all $x \in X$ and $n \in \mathbb{N}$, then the Taylor coefficients of $g_1$ equal the Taylor coefficients of $g_2$.

Now we shall show that every bounded linear functional on $\mathcal{B}_{00}(X)$ is induced by a function in $L_a^{1,\alpha}(\mathbb{D}, X^*)$. Let $\phi$ be a bounded linear functional on $\mathcal{B}_{00}(X)$ and put

$$g_\phi(\lambda)(x) = \phi\Big(\frac{1}{(1-\zeta\,\lambda)^{\alpha+2}} x\Big),$$

where $\zeta$ denotes the identity function of $\mathbb{D}$. We want to show that $g_\phi \in L_a^{1,\alpha}(\mathbb{D}, X^*)$ and $\langle f, \phi\rangle = \langle f, g_\phi\rangle$, for all $f \in \mathcal{B}_{00}(X)$. To this end consider the embedding $J : \mathcal{B}_{00}(X) \to C(\overline{\mathbb{D}}, X)$

$$J(f)(z) = f'(z)(1-|z|^2), \quad z \in \mathbb{D}.$$

Clearly $J$ is an isometry, and hence $\psi = \phi J^{-1}$ defines a bounded linear functional on a closed subspace of $C(\overline{\mathbb{D}}, X)$ (i.e. the range of $J$), and by the Hahn-Banach theorem it extends to a bounded linear functional on $C(\overline{\mathbb{D}}, X)$ preserving its norm.

Then (see [6]) there exists an $X^*$-valued countably additive measure $\mu$ on $\overline{\mathbb{D}}$ such that

$$g_\phi(\lambda)(x) = \phi\Big(\frac{1}{(1-z\lambda)^{\alpha+2}}x\Big) = \psi\Big(\frac{(\alpha+2)\lambda(1-|z|^2)}{(1-z\lambda)^{\alpha+3}}x\Big)$$
$$= (\alpha+2)\int_{\overline{\mathbb{D}}} \frac{\lambda(1-|z|^2)}{(1-z\lambda)^{\alpha+3}}x\,\mathrm{d}\mu(z) \quad \lambda\in\mathbb{D}.$$

The measure $\mu$ induces (see [6] for details) a finite nonnegative regular Borel measure $|\mu|$ on $\overline{\mathbb{D}}$, which is called the variation of $\mu$, and satisfies

$$\Big|\int_{\overline{\mathbb{D}}} h(z)\,\mathrm{d}\mu(z)\Big| \leqslant \int_{\overline{\mathbb{D}}} \|h(z)\|\,\mathrm{d}|\mu|(z),$$

for any $h\in C(\overline{\mathbb{D}},X)$. In particular, we have $|\mu|(\overline{\mathbb{D}})\leqslant\|\phi\|$. Then

$$|g_\phi(\lambda)(x)| \leqslant (\alpha+2)\int_{\overline{\mathbb{D}}} \frac{1-|z|^2}{|1-z\lambda|^{\alpha+3}}\,\mathrm{d}|\mu|(z)\,\|x\|, \quad \lambda\in\mathbb{D},$$

and hence

$$\|g_\phi(\lambda)\| \leqslant (\alpha+2)\int_{\overline{\mathbb{D}}} \frac{1-|z|^2}{|1-z\lambda|^{\alpha+3}}\,\mathrm{d}|\mu|(z).$$

Now, by Fubini's theorem and Lemma 1.2, we deduce

$$\int_{\mathbb{D}} \|g_\phi(\lambda)\|\,\mathrm{d}A_\alpha(\lambda) \leqslant (\alpha+2)\int_{\mathbb{D}}\int_{\overline{\mathbb{D}}} \frac{1-|z|^2}{|1-z\lambda|^{\alpha+3}}\,\mathrm{d}|\mu|(z)\,\mathrm{d}A_\alpha(\lambda)$$
$$= (\alpha+2)\int_{\overline{\mathbb{D}}}\int_{\mathbb{D}} \frac{1-|z|^2}{|1-z\lambda|^{\alpha+3}}\,\mathrm{d}A_\alpha(\lambda)\,\mathrm{d}|\mu|(z)$$
$$\leqslant k\int_{\overline{\mathbb{D}}} \mathrm{d}|\mu|(z) \leqslant k|\mu|(\overline{\mathbb{D}}) \leqslant k\|\phi\|.$$

This shows that $g_\phi\in L^{1,\alpha}_a(\mathbb{D},X^*)$.

For an $X$-valued function $f$, analytic in a neighborhood of $\overline{\mathbb{D}}$ and with $f(0)=0$, Lemma 1.3 gives

$$(\alpha+1)\langle f,g_\phi\rangle = \int_{\mathbb{D}} \langle f'(z),\, g_\phi(\overline{z})-g_\phi(0)\rangle\frac{1-|z|^2}{\overline{z}}\,\mathrm{d}A_\alpha(z)$$
$$= \int_{\mathbb{D}} \phi\Big(\frac{f'(z)}{(1-\overline{z}\lambda)^{\alpha+2}} - f'(z)\Big)\frac{1-|z|^2}{\overline{z}}\,\mathrm{d}A_\alpha(z) \tag{1.2}$$
$$= \phi\Big(\int_{\mathbb{D}} \Big(\frac{f'(z)}{(1-\overline{z}\lambda)^{\alpha+2}} - f'(z)\Big)\frac{1-|z|^2}{\overline{z}}\,\mathrm{d}A_\alpha(z)\Big).$$

The last step above is obtained as follows. For $r < 1$ we split the inner integral as below

$$\phi\Big(\int_{|z|<r}\Big(\frac{f'(z)}{(1-\bar{z}\lambda)^{\alpha+2}}-f'(z)\Big)\frac{1-|z|^2}{\bar{z}}\mathrm{d}A_\alpha(z)\Big)+\phi\Big(\int_{|z|>r}\Big(\frac{f'(z)}{(1-\bar{z}\lambda)^{\alpha+2}}-f'(z)\Big)\frac{1-|z|^2}{\bar{z}}\mathrm{d}A_\alpha(z)\Big).$$

Since $f \in \mathcal{B}_{00}(X)$ that the second term above tends to 0 as $r \to 1$, while in the first term we clearly can interchange the action of $\phi$ with the integration. By a standard formula in Bergman spaces (see [16]), we now get from (1.2)

$$\langle f, g_\phi \rangle = \phi(f(\lambda) - f(0)) = \phi(f),$$

which completes the proof. ∎

The next theorem will be the main tool in our further investigations.

THEOREM 1.5. *There exists a sequence $(\lambda_i)_{i\geqslant 1} \subseteq \mathbb{D}$ such that any $g \in L_a^{1,\alpha}(\mathbb{D}, X^*)$ can be written as*

$$g(\lambda) = g(0) + \sum_{i=1}^{\infty} C_i \frac{\lambda(1-|\lambda_i|^2)}{(1-\lambda_i\lambda)^{3+\alpha}}, \quad \lambda \in \mathbb{D},$$

*with $(C_i)_i = (C_i(g))_i \in \ell^1(X^*)$ satisfying*

$$\|(C_i)_{i\geqslant 1}\|_{\ell^1(X^*)} \sim \|g\|_{L_a^{1,\alpha}(\mathbb{D},X^*)}.$$

*Proof.* There exists a sequence $(\lambda_i)_{i\geqslant 1} \subseteq \mathbb{D}$, with no accumulation points in $\mathbb{D}$, such that the following relation holds

$$\|f\|_{\mathcal{B}(X)} \sim \sup_{i\geqslant 1} \|f'(\lambda_i)\|(1-|\lambda_i|^2), \quad f \in \mathcal{B}_{00}(X), \tag{1.3}$$

with constants independent of $f$. Indeed, consider for example a sequence $(\lambda_i)_{i\geqslant 1} \subseteq \mathbb{D}$ that satisfies

$$\inf_{i\geqslant 1} |z-\lambda_i| < c(1-|z|), \quad z \in \mathbb{D},$$

for some constant $0 < c < 1$. It is easy to see that if $c$ is sufficiently small, relation (1.3) holds for such a sequence $(\lambda_i)_{i\geqslant 1}$ (see [1] for more details). Let $c_0(X)$ denote the space of sequences in $X$ that tend to 0. Consider the embedding $\mathcal{I} : \mathcal{B}_{00}(X) \to c_0(X)$ given by

$$\mathcal{I}(f) = (f'(\lambda_i)(1-|\lambda_i|^2))_{i\in\mathbb{N}},$$

where the $\lambda_i$'s satisfy (1.3). Note that $\mathcal{I}$ is bounded below and above.

Now let $g \in L_a^{1,\alpha}(\mathbb{D}, X^*)$. By Theorem 1.4, $g$ defines via (1.1) a bounded linear functional $\phi_g \in (\mathcal{B}_{00}(X))^*$ with $\|\phi_g\| \sim \|g\|_{L_a^{1,\alpha}(\mathbb{D},X^*)}$. Hence $\psi = \phi_g\mathcal{I}^{-1}$ defines a bounded linear functional on the range of $\mathcal{I}$ which is a closed subspace of $c_0(X)$, and, by the Hahn-Banach theorem, it extends to a bounded linear functional on $c_0(X)$ with the same norm. But the dual of $c_0(X)$ can be identified with

$\ell^1(X^*)$ with the natural pairing. Hence there exists $(c_i)_i \in \ell^1(X^*)$ such that for any $f \in \mathcal{B}_{00}(X)$ we have

$$\phi_g(f) = \psi((f'(\lambda_i)(1-|\lambda_i|^2))_i) = \sum_{i=1}^{\infty} \langle f'(\lambda_i)(1-|\lambda_i|^2), c_i \rangle.$$

Clearly $\|(c_i)_i\|_{\ell^1(X^*)} \sim \|\phi_g\| \sim \|g\|_{L_a^{1,\alpha}(\mathbb{D},X^*)}$. In particular, for $x \in X, \lambda \in \mathbb{D}$ and $f_\lambda(z) = \frac{1}{(\alpha+2)\overline{\lambda}}\left(\frac{1}{(1-\overline{\lambda}z)^{\alpha+2}} - 1\right) x$, we get

$$\phi_g(f_\lambda) = \sum_{i=1}^{\infty} \langle f'_\lambda(\lambda_i)(1-|\lambda_i|^2), c_i \rangle = \sum_{i=1}^{\infty} \langle x, c_i \rangle \frac{1-|\lambda_i|^2}{(1-\lambda_i\overline{\lambda})^{3+\alpha}}. \tag{1.4}$$

On the other hand, by Lemma 1.3 it follows that

$$\phi_g(f_\lambda) = \int_{\mathbb{D}} \left\langle f'_\lambda(z), \frac{g(\overline{z})-g(0)}{\overline{z}} \right\rangle (1-|z|^2)\,\mathrm{d}A_\alpha(z) = \int_{\mathbb{D}} \left\langle x, \frac{g(\overline{z})-g(0)}{\overline{z}} \right\rangle \frac{1-|z|^2}{(1-z\overline{\lambda})^{3+\alpha}}\,\mathrm{d}A_\alpha(z)$$

$$= \frac{\alpha+1}{\alpha+2} \left\langle x, \frac{g(\overline{\lambda})-g(0)}{\overline{\lambda}} \right\rangle, \quad \lambda \in \mathbb{D}. \tag{1.5}$$

From (1.4), (1.5) we now get

$$g(\lambda) = g(0) + \sum_{i=1}^{\infty} C_i \frac{\lambda(1-|\lambda_i|^2)}{(1-\lambda_i\lambda)^{3+\alpha}}, \quad \lambda \in \mathbb{D},$$

where $C_i = (\alpha+2)/(\alpha+1)\, c_i$. ∎

## 2. A THEOREM OF SARASON TYPE ON BERGMAN SPACES

Let $\mathcal{H}$ be a separable Hilbert space. For $t > 0$ we let $\mathcal{S}^t$ denote the corresponding Schatten class, which consists of compact linear operators $T$ on $\mathcal{H}$, such that the eigenvalues of the modulus $|T|$ form an $\ell^t$ sequence. In particular, $\mathcal{S}^1 = \mathcal{S}^1(\mathcal{H})$, $\mathcal{S}^2 = \mathcal{S}^2(\mathcal{H})$ denote the set of trace class operators on $\mathcal{H}$, respectively the set of Hilbert-Schmidt operators on $\mathcal{H}$.

The main result of this section gives the Bergman space analogue of Sarason's theorem mentioned in the introduction.

THEOREM 2.1. *Let $t \geqslant 1, t_1, t_2, r, s > 0$ be such that $1/t_1 + 1/t_2 = 1/t$, $1/r + 1/s = 1$. Then, any function $f \in L_a^{1,\alpha}(\mathbb{D}, \mathcal{S}^t), \alpha > -1$, can be written as*

$$f = \sum_{n=1}^{\infty} g_n h_n,$$

*with $g_n \in L_a^{r,\alpha}(\mathbb{D}, \mathcal{S}^{t_1})$, $h_n \in L_a^{s,\alpha}(\mathbb{D}, \mathcal{S}^{t_2})$ and*

$$\sum_{n=1}^{\infty} \|g_n\|_{L_a^{r,\alpha}(\mathbb{D},\mathcal{S}^{t_1})} \|h_n\|_{L_a^{s,\alpha}(\mathbb{D},\mathcal{S}^{t_2})} \sim \|f\|_{L_a^{1,\alpha}(\mathbb{D},\mathcal{S}^t)}.$$

*Proof.* In view of the fact that $\mathcal{S}^1 = (\mathcal{K}(\mathcal{H}))^*$ and $(\mathcal{S}^t)^* = \mathcal{S}^{t'}$, for $t > 1, 1/t + 1/t' = 1$, we can apply Theorem 1.5 to deduce that, for any $f \in L_a^{1,\alpha}(\mathbb{D}, \mathcal{S}^t)$, there are $C_i \in \mathcal{S}^t$ and $\lambda_i \in \mathbb{D}$, $i \geqslant 1$, such that

$$f(\lambda) = f(0) + \sum_{i=1}^{\infty} C_i \frac{\lambda(1-|\lambda_i|^2)}{(1-\lambda_i\lambda)^{3+\alpha}}, \quad \lambda \in \mathbb{D},$$

with $(C_i)_i = (C_i(f))_i \in \ell^1(\mathcal{S}^t)$ satisfying

$$\|(C_i)_{i\geqslant 1}\|_{\ell^1(\mathcal{S}^t)} \sim \|f\|_{L_a^{1,\alpha}(\mathbb{D},\mathcal{S}^t)}.$$

Denote $C_0 = f(0)$ and let $C_i = W_i|C_i|$ be the polar decomposition of $C_i$, $i \geqslant 0$, (see [5]). For $\lambda \in \mathbb{D}$, $i \geqslant 1$, put

$$g_i(\lambda) = W_i|C_i|^{t/t_1}\lambda\frac{(1-|\lambda_i|^2)^{1/r}}{(1-\lambda_i\lambda)^{(3+\alpha)/r}}, \qquad g_0(\lambda) = W_0|C_0|^{t/t_1},$$

$$h_i(\lambda) = |C_i|^{t/t_2}\frac{(1-|\lambda_i|^2)^{1/s}}{(1-\lambda_i\lambda)^{(3+\alpha)/s}}, \qquad h_0(\lambda) = |C_0|^{t/t_2}.$$

Note that $g_i \in L_a^{r,\alpha}(\mathbb{D}, \mathcal{S}^{t_1})$, $h_i \in L_a^{s,\alpha}(\mathbb{D}, \mathcal{S}^{t_2})$ and

$$f = \sum_{i=0}^{\infty} g_i h_i. \tag{2.1}$$

By Lemma 1.2 we obtain

$$\|g_i\|_{L_a^{r,\alpha}(\mathbb{D},\mathcal{S}^{t_1})} \sim \|W_i|C_i|^{t/t_1}\|_{\mathcal{S}^{t_1}} \sim \||C_i|^{t/t_1}\|_{\mathcal{S}^{t_1}} = k\|C_i\|_{\mathcal{S}_t}^{t/t_1},$$

while

$$\|h_i\|_{L_a^{s,\alpha}(\mathbb{D},\mathcal{S}^{t_2})} \sim \||C_i|^{t/t_2}\|_{\mathcal{S}^{t_2}} = k\|C_i\|_{\mathcal{S}_t}^{t/t_2}, \quad i \geqslant 0.$$

Thus

$$\sum_{i=0}^{\infty} \|g_i\|_{L_a^{r,\alpha}(\mathbb{D},\mathcal{S}^{t_1})}\|h_i\|_{L_a^{s,\alpha}(\mathbb{D},\mathcal{S}^{t_2})} \sim \sum_{i=1}^{\infty} \|C_i\|_{\mathcal{S}^t} \sim \|f\|_{L_a^{1,\alpha}(\mathbb{D},\mathcal{S}^t)}.$$

With this, the proof is complete. ∎

For the sake of completeness, we shall apply this theorem to identify the dual of $L_a^{1,\alpha}(\mathbb{D}, \mathcal{S}^1)$ with the bounded Hankel operators on $L_a^{2,\alpha}(\mathbb{D}, \mathcal{H})$ and with the vector-valued Bloch-space $\mathcal{B}(\mathcal{L}(\mathcal{H}))$. As mentioned in the introduction, one could derive this also from [2] together with a theorem by Arregui and Blasco [3], that identifies the dual of $L_a^{1,\alpha}(\mathbb{D}, \mathcal{S}^1)$ with $\mathcal{B}(\mathcal{L}(\mathcal{H}))$ with the natural pairing given in (2.2) below.

Given $B \in (L_a^{1,\alpha}(\mathbb{D}, \mathcal{S}^1))^*$, where $(L_a^{1,\alpha}(\mathbb{D}, \mathcal{S}^1))^*$ denotes the dual of $L_a^{1,\alpha}(\mathbb{D}, \mathcal{S}^1)$, we can associate to it the analytic operator-valued function $b : \mathbb{D} \to \mathcal{L}(\mathcal{H})$,

$$\langle b(z)\, x, y\rangle = \overline{B(K_z(\cdot)\, y \otimes x)}, \quad x, y \in \mathcal{H},$$

where

$$K_z(w) = \frac{1}{(1-w\bar{z})^{\alpha+2}}, \quad w,z \in \mathbb{D} \text{ and } y \otimes x : \mathcal{H} \to \mathcal{H}, \quad y \otimes x(f) = \langle f, x \rangle y, \quad f \in \mathcal{H}.$$

Some of the considerations below are the Bergman space analogues of those made in [12].

COROLLARY 2.2. *Let $b$, $B$ be as above. Then :*

(1) *If $F \in L_a^{1,\alpha}(\mathbb{D}, \mathcal{S}^1)$ is an $\mathcal{S}^1$-valued analytic function in a neighbourhood of $\overline{\mathbb{D}}$, we have, with $b^*(z)$ is the adjoint of $b(z)$ for $z \in \mathbb{D}$,*

$$BF = \int_{\mathbb{D}} \operatorname{tr}\,(b^*(z)F(z))\,\mathrm{d}A_\alpha(z). \tag{2.2}$$

(2) *If $\Gamma_b$ is the Hankel operator defined on $L_a^{2,\alpha}(\mathbb{D}, \mathcal{H})$ with symbol $b$, then the following quantities are comparable:*

(i) $\|B\|_*$;

(ii) $\|\Gamma_b\|$;

(iii) $\|b\|_{\mathcal{B}(\mathcal{L}(\mathcal{H}))} = \sup_{z \in \mathbb{D}} (1-|z|^2)\,\|b'(z)\|$ .

The result in Corollary 2.2 is in contrast to the situation in the Hardy space with the Bloch condition replaced by BMO, where the above inequalities are true only for finite dimensional subspaces. Investigations of this problem were made in [12] where, roughly speaking, it is shown that the norm of a Hankel operator cannot exceed a constant multiple of the logarithm of the dimension times the BMO-norm of the symbol, and that this estimate is sharp.

The proof of Corollary 2.2 relies on a further identification of Hankel operators that act on different spaces. More precisely, given $b$ as above, there is a natural way to associate to it a Hankel operator $\Gamma_{T(b)}$ acting on $L_a^{2,\alpha}(\mathbb{D}, \mathcal{S}^2)$. Its symbol $T(b) : \mathbb{D} \mapsto \mathcal{L}(\mathcal{S}^2)$ is defined by

$$T(b)(z)S = b(z)S, \quad S \in \mathcal{S}^2. \tag{2.3}$$

The following lemma expresses a general fact which holds in all weighted Bergman spaces or even Hardy spaces.

LEMMA 2.3. *Let $b \in L_a^1(\mathbb{D}, \mathcal{L}(\mathcal{H}))$. Then $\Gamma_{T(b)}$ is bounded on $L_a^{2,\alpha}(\mathbb{D}, \mathcal{S}^2)$ if and only if $\Gamma_b$ is bounded on $L_a^{2,\alpha}(\mathbb{D}, \mathcal{H})$, and*

$$\|\Gamma_{T(b)}\| = \|\Gamma_b\|.$$

*Proof.* One inequality is almost obvious. Suppose that $\Gamma_{T(b)}$ extends to a bounded operator on $L_a^{2,\alpha}(\mathbb{D}, \mathcal{S}^2)$. Denote by $\{e_n\}_n$ the standard orthonormal basis of $\mathcal{H}$ and let $x, y$ be $\mathcal{H}$-valued functions, analytic in a neighborhood of $\overline{\mathbb{D}}$. Then for a fixed $n \in \mathbb{N}$

$$\langle b(z)x(\bar{z}), y(z) \rangle = \langle b(z)x(\bar{z}) \otimes e_n, y(z) \otimes e_n \rangle_{\mathcal{S}^2} = \langle T(b)(z)\, x(\bar{z}) \otimes e_n, y(z) \otimes e_n \rangle_{\mathcal{S}^2},$$

which implies

$$\langle \Gamma_b x, y\rangle_{L^{2,\alpha}_a(\mathbb{D},\mathcal{H})} = \langle T(b)(z)x(\bar{z})\otimes e_n, y(z)\otimes e_n\rangle_{L^{2,\alpha}_a(\mathbb{D},\mathcal{S}^2)} = \langle \Gamma_{T(b)} x\otimes e_n, y\otimes e_n\rangle_{L^{2,\alpha}_a(\mathbb{D},\mathcal{S}^2)}.$$

From this we immediately deduce that

$$\|\Gamma_b\| \leqslant \|\Gamma_{T(b)}\|.$$

To prove the reverse inequality let $F, G \in L^{2,\alpha}_a(\mathbb{D},\mathcal{S}^2)$ be $\mathcal{S}^2$-valued analytic functions in a neighbourhood of $\overline{\mathbb{D}}$ (this class of functions is dense in $L^{2,\alpha}_a(\mathbb{D},\mathcal{S}^2)$) and note that

$$\begin{aligned}\langle \Gamma_{T(b)} F, G\rangle_{L^{2,\alpha}_a(\mathbb{D},\mathcal{S}^2)} &= \int_{\mathbb{D}} \langle T(b)(z)F(\bar{z}),\, G(z)\rangle_{\mathcal{S}^2}\, \mathrm{d}A_\alpha(z)\\ &= \int_{\mathbb{D}} \sum_{n=1}^{\infty} \langle T(b)(z)F(\bar{z})e_n,\, G(z)e_n\rangle\, \mathrm{d}A_\alpha(z)\\ &= \int_{\mathbb{D}} \sum_{n=1}^{\infty} \langle b(z)F(\bar{z})e_n,\, G(z)e_n\rangle\, \mathrm{d}A_\alpha(z),\end{aligned}$$

by relation (2.3). Hence,

$$\begin{aligned}|\langle \Gamma_{T(b)} F, G\rangle_{L^{2,\alpha}_a(\mathbb{D},\mathcal{S}^2)}| &\leqslant \sum_{n=1}^{\infty}\int_{\mathbb{D}} |\langle b(z)F(\bar{z})e_n, G(z)e_n\rangle| \mathrm{d}A_\alpha(z) = \sum_{n=1}^{\infty} |\langle \Gamma_b F e_n, G e_n\rangle|\\ &\leqslant \sum_{n=1}^{\infty} \|\Gamma_b\|\, \|Fe_n\|_{L^{2,\alpha}_a(\mathbb{D},\mathcal{H})} \|Ge_n\|_{L^{2,\alpha}_a(\mathbb{D},\mathcal{H})} \leqslant \|\Gamma_b\| \|F\|_{L^{2,\alpha}_a(\mathbb{D},\mathcal{S}^2)}\, \|G\|_{L^{2,\alpha}_a(\mathbb{D},\mathcal{S}^2)},\end{aligned}$$

by the Cauchy-Schwarz inequality. Thus

$$\|\Gamma_{T(b)}\| \leqslant \|\Gamma_b\|,$$

and with this the proof is complete. ∎

*Poof of Corollary* 2.2. (1) Taking into account the fact that

$$\int_{\mathbb{D}} |\partial_{\bar{z}} K_z(\xi)|\, \mathrm{d}A_\alpha(\xi) \leqslant \frac{k}{1-|z|^2},$$

one can easily see that $b$ satisfies

$$\sup_{z\in\mathbb{D}} (1-|z|^2)\, \|b'(z)\| < k\|B\|_*,$$

and hence the right-hand-side of (2.2) is well-defined. Now, to prove (2.2), let $\{e_n\}_n$ be an orthonormal basis for $\mathcal{H}$ and let $P_M$ denote the orthogonal projection on span$\{e_1,\dots,e_M\}$, $M \geqslant 1$. Then

$$
\begin{aligned}
&\int_{\mathbb{D}} \operatorname{tr}\,(b^*(z)F(z)P_M)\,\mathrm{d}A_\alpha(z)\\
&\quad = \int_{\mathbb{D}} \sum_{n=1}^{\infty} \langle F(z)P_M e_n, b(z)e_n\rangle\,\mathrm{d}A_\alpha(z) = \int_{\mathbb{D}} \sum_{n=1}^{M} B(K_z(\cdot)\; F(z)e_n \otimes e_n)\,\mathrm{d}A_\alpha(z)\\
&\quad = \int_{\mathbb{D}} B\Big(\sum_{n=1}^{M} K_z(\cdot)\; F(z)e_n \otimes e_n\Big)\,\mathrm{d}A_\alpha(z) = B\Big(\int_{\mathbb{D}} K_z(\cdot) \sum_{n=1}^{M} F(z)e_n \otimes e_n\,\mathrm{d}A_\alpha(z)\Big)\\
&\quad = B\Big(\sum_{n=1}^{\infty} F(\cdot)P_M e_n \otimes e_n\Big) = B(FP_M).
\end{aligned}
$$

But $FP_M \to F$ in $L_a^{1,\alpha}(\mathbb{D},\mathcal{S}^1)$, so, if we let $M \to \infty$ in the above relation we obtain (2.2).

(2) The fact that (ii)$\sim$(iii) is a direct application of Theorem 3.1 in [2]. We shall only show that (i)$\sim$(ii). Note that

$$\overline{B(GH)} = \langle \Gamma_{T(b)}\widetilde{H}, G\rangle_{L_a^{2,\alpha}(\mathbb{D},\mathcal{S}^2)}, \tag{2.4}$$

where $G, H$ are $\mathcal{S}^2$-valued functions, analytic in a neighborhood of $\overline{\mathbb{D}}$, and $\widetilde{H}(z) = H^*(\overline{z})$. This follows immediately from

$$
\begin{aligned}
\overline{\operatorname{tr}(b^*(z)G(z)H(z))} &= \operatorname{tr}(H^*(z)G^*(z)b(z)) = \operatorname{tr}(G^*(z)b(z)H^*(z))\\
&= \langle T(b)(z)H^*(z), G(z)\rangle_{\mathcal{S}^2} = \langle T(b)(z)\widetilde{H}(\overline{z}), G(z)\rangle_{\mathcal{S}^2}, \quad z \in \mathbb{D}.
\end{aligned}
$$

From relation (2.4) we obtain the inequality

$$\|B\|_* \geqslant \|\Gamma_{T(b)}\|.$$

On the other hand, given $F \in L_a^{1,\alpha}(\mathbb{D},\mathcal{S}^1)$, we deduce by Theorem 2.1 that there are $G_n, H_n \in L_a^{2,\alpha}(\mathbb{D},\mathcal{S}^2)$, $n = 1,2,\dots$ such that $F = \sum_{n=1}^{\infty} G_n H_n$, where the series converges in $L_a^{1,\alpha}(\mathbb{D},\mathcal{S}^1)$, and

$$\sum_{n=1}^{\infty} \|G_n\|_{L_a^{2,\alpha}(\mathbb{D},\mathcal{S}^2)} \|H_n\|_{L_a^{2,\alpha}(\mathbb{D},\mathcal{S}^2)} \leqslant k\,\|F\|_{L_a^{1,\alpha}(\mathbb{D},\mathcal{S}^1)}.$$

Hence

$$
\begin{aligned}
|B(F)| &= \Big|\sum_{n=1}^{\infty} B(G_n H_n)\Big| = \Big|\sum_{n=1}^{\infty} \overline{\langle \Gamma_{T(b)}\widetilde{H}_n, G_n\rangle}_{L_a^{2,\alpha}(\mathbb{D},\mathcal{S}^2)}\Big|\\
&\leqslant \sum_{n=1}^{\infty} \|\Gamma_{T(b)}\| \cdot \|G_n\|_{L_a^{2,\alpha}(\mathbb{D},\mathcal{S}^2)} \cdot \|H_n\|_{L_a^{2,\alpha}(\mathbb{D},\mathcal{S}^2)} \leqslant k\|\Gamma_{T(b)}\|\,\|F\|_{L_a^{1,\alpha}(\mathbb{D},\mathcal{S}^1)}.
\end{aligned}
$$

Therefore $\|B\|_* \sim \|\Gamma_{T(b)}\|$. Combine this with Lemma 2.3 to deduce that the quantity in (i) is comparable to the quantity in (ii). ∎

## 3. COMPACT HANKEL OPERATORS ON VECTOR-VALUED BERGMAN SPACES

Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{K}(\mathcal{H})$ the space of compact operators on $\mathcal{H}$. Recall the result in [2] mentioned in the previous section which asserts that the Hankel operator $\Gamma_T$ is bounded on $L_a^{2,\alpha}(\mathbb{D},\mathcal{H})$ if and only if $T\in\mathcal{B}(\mathcal{L}(\mathcal{H}))$ and

$$\|\Gamma_T\| \sim \|T\|_{\mathcal{B}(\mathcal{L}(\mathcal{H}))}. \tag{3.1}$$

The following theorem provides a compactness criterion for $\Gamma_T$.

THEOREM 3.1. *$\Gamma_T$ is compact on $L_a^{2,\alpha}(\mathbb{D},\mathcal{H})$ if and only if $T \in \mathcal{B}_0(\mathcal{K}(\mathcal{H}))$.*

*Proof.* Assume first that $T \in \mathcal{B}_0(\mathcal{K}(\mathcal{H}))$ and denote by $T_r(z) = T(rz)$, $z \in \mathbb{D}$, $r < 1$. By a standard argument, analogous to the one used in the scalar case, it follows that $\|T_r(z) - T(z)\|_{\mathcal{B}(\mathcal{H})} \to 0$ as $r \to \infty$. Then by (3.1) we get $\|\Gamma_{T_r} - \Gamma_T\| \to 0$ as $r \to \infty$, and hence it is enough to show that $\Gamma_{T_r}$ is compact. But since $T_r$ is analytic on a neighborhood of $\overline{\mathbb{D}}$, it can be approximated by its Taylor polymonials in the $\mathcal{B}(\mathcal{H})$-norm, i.e.

$$\Big\|T_r(z) - \sum_{k=0}^{n} \widehat{T}(k)r^k z^k\Big\|_{\mathcal{B}(\mathcal{H})} \to 0, \quad \text{as } n \to \infty,$$

where $\widehat{T}(k) \in \mathcal{K}(\mathcal{H})$ are the Taylor coefficients of $T$. Again, by (3.1), it now follows that it is enough to show that $\Gamma_{\widehat{T}(k)z^k}$ is a compact operator for all $n, k \in \mathbb{N}$. But this is easily proven since one can approximate $\widehat{T}(k)$ with finite rank operators $\widehat{T}_p(k)$, and then one approximates $\Gamma_{\widehat{T}(k)z^k}$ with the operators $\Gamma_{\widehat{T}_p(k)z^k}$ which are also finite rank operators. With this, the "if part" is proven.

Let us now prove the "only if" part. If $\Gamma_T$ is compact, then it is bounded and by (3.1) we get $T \in \mathcal{B}(\mathcal{L}(\mathcal{H}))$. We shall first prove that the Taylor coefficients $\widehat{T}(k), k \geqslant 0$ of $T$ are compact operators on $\mathcal{H}$. For this let $f_k \in \mathcal{H}$, $f_k \xrightarrow{w} 0$ as $k \to \infty$ (here $\xrightarrow{w}$ denotes weak convergence). For a fixed $n_0 \in \mathbb{N}$, set

$$x_k(z) = z^{n_0} f_k, \quad y_k(z) = \widehat{T}(n_0) f_k, \quad z \in \mathbb{D}, k \in \mathbb{N}.$$

It is straightforward to check that for $k \to \infty$ both sequences $(x_k)$, $(f_k)$ converge weakly to zero in $L_a^{2,\alpha}(\mathbb{D},\mathcal{H})$.

Then, since $\Gamma_T$ is compact we get $\langle \Gamma_T x_k, y_k\rangle \to 0$ as $k \to \infty$. But $\langle \Gamma_T x_k, y_k\rangle = \frac{n_0!\Gamma(\alpha+2)}{\Gamma(n_0+\alpha+2)}\|\widehat{T}(n_0)f_k\|^2 \underset{k\to\infty}{\longrightarrow} 0$, which shows that $\widehat{T}(n_0)$ is compact on $\mathcal{H}$, and, as $n_0$ is arbitrary, this holds for all $n_0 \in \mathbb{N}$. By the above and the pointwise convergence of the Taylor series of $T$, we deduce that $T(z)$ is compact for any $z \in \mathbb{D}$.

It remains to show that $T$ verifies the "little Bloch" condition. For any $a, b \in \mathcal{H}$, the function $z \mapsto \langle T(z)a, b\rangle$ belongs to $L_a^{1,\alpha}$ since $T \in \mathcal{B}(\mathcal{L}(\mathcal{H}))$, and then by a standard formula for Bergman spaces

$$\langle T''(z)a,b\rangle = (\alpha+2)(\alpha+3)\int_{\mathbb{D}} \langle T(w)a,b\rangle \frac{\overline{w}^2}{(1-\overline{w}z)^{\alpha+4}}\,\mathrm{d}A_\alpha(w). \tag{3.2}$$

For each fixed $z \in \mathbb{D}$, the operator $T''(z)$ is compact and hence there exists $x_0 = x_0(z) \in \mathcal{H}$ with $\|x_0(z)\| = 1$ such that $\|T''(z)\| = \|T''(z)x_0(z)\|$. Put now $y_0(z) = T''(z)x_0(z)/\|T''(z)\|$. Note that $\|y_0(z)\| = 1$ and

$$\|T''(z)\| = \|T''(z)x_0(z)\| = \langle T''(z)x_0(z), y_0(z)\rangle.$$

By (3.2) and the above relation we get

$$\begin{aligned}(1-|z|^2)^2\|T''(z)\| &= (\alpha+2)(\alpha+3)\int_{\mathbb{D}} \langle T(w)x_0(z), y_0(z)\rangle \frac{\overline{w}^2(1-|z|^2)^2}{(1-\overline{w}z)^{\alpha+4}}\,\mathrm{d}A_\alpha(w)\\ &= \langle \Gamma_T x_z, y_z\rangle, \qquad (3.3)\end{aligned}$$

where

$$x_z(w) = \frac{x_0(z)(1-|z|^2)}{(1-wz)^{2+\alpha/2}}, \quad y_z(w) = (\alpha+2)(\alpha+3)\frac{w^2 y_0(z)(1-|z|^2)}{(1-w\overline{z})^{2+\alpha/2}}, \quad w,z \in \mathbb{D}.$$

Let us now show that

$$x_z, y_z \xrightarrow{w} 0 \quad \text{as } |z| \to 1. \tag{3.4}$$

In view of (3.3) and the compactness of $\Gamma_T$, this implies $(1-|z|^2)^2 T''(z) \to 0$ as $|z| \to 1$, which is equivalent to $T'(z)(1-|z|^2) \to 0$ as $|z| \to 1$ (see [4] for the proof in the scalar case; the proof in the vector-valued case is analogous).

Thus it remains to prove (3.4). To this end, denote for each $w \in \mathbb{D}$, $a \in \mathcal{H}$

$$e_{w,a}(\zeta) = \frac{1}{(1-\zeta\overline{w})^{\alpha+2}}\,a, \quad \zeta \in \mathbb{D}.$$

The functions $e_{z,a} \in L_a^{2,\alpha}(\mathbb{D},\mathcal{H})$ induce bounded linear functionals on $L_a^{2,\alpha}(\mathbb{D},\mathcal{H})$ with the natural pairing, and their linear span is dense in $L_a^{2,\alpha}(\mathbb{D},\mathcal{H})$. Note that, for any $w \in \mathbb{D}$,

$$\langle x_z, e_{w,a}\rangle = \langle x_z(w), a\rangle \to 0, \quad \langle y_z, e_{w,a}\rangle = \langle y_z(w), a\rangle \to 0 \quad \text{as } |z| \to 1,$$

and hence, in order to prove (3.4), it is enough to have

$$\sup_{z\in\mathbb{D}} \|x_z\| < \infty, \quad \sup_{z\in\mathbb{D}} \|y_z\| < \infty,$$

but these follow immediately from Lemma 1.2. ∎

## 4. PRODUCT DECOMPOSITIONS FOR BERGMAN SPACES ON THE POLYDISC

As mentioned in the Introduction, Ferguson and Lacey [8] proved a weak product decomposition theorem for the Hardy space $H^1(\mathbb{D}^2)$. Our aim is to give an analogue of this result for the Bergman space.

THEOREM 4.1. *For $\alpha > -1$, any function $f \in L_a^{1,\alpha}(\mathbb{D}^n)$ can be written as*

$$f = \sum_{i=1}^{\infty} g_i h_i, \tag{4.1}$$

*where $g_i, h_i \in L_a^{2,\alpha}(\mathbb{D}^n)$ and*

$$\sum_{i=1}^{\infty} \|g_i\|_{L_a^{2,\alpha}(\mathbb{D}^n)} \|h_i\|_{L_a^{2,\alpha}(\mathbb{D}^n)} \sim \|f\|_{L_a^{1,\alpha}(\mathbb{D}^n)}. \tag{4.2}$$

*Moreover, the functions $g_i, h_i$ above are analytic in a neighborhood of $\overline{\mathbb{D}}^n$.*

*Proof.* We shall prove the theorem by induction over $n$. Let $n = 1$ and apply Theorem 1.5 with $X = \mathbb{C}$ to deduce that any $f \in L_a^{1,\alpha}$ can be written as

$$f(\lambda) = f(0) + \sum_{i=1}^{\infty} c_i \frac{\lambda(1-|\lambda_i|^2)}{(1-\overline{\lambda_i}\lambda)^{3+\alpha}}, \quad \lambda \in \mathbb{D},$$

where $\lambda_i \in \mathbb{D}$ and $(c_i)_i = (c_i(f))_i \in \ell^1$ satisfy

$$\|(c_i)\|_{\ell^1} \sim \|f\|_{L_a^{1,\alpha}}. \tag{4.3}$$

Put $g_0 \equiv 1, h_0 \equiv f(0)$ and

$$g_i(\lambda) = c_i \frac{\lambda(1-|\lambda_i|^2)^{1/2}}{(1-\overline{\lambda_i}\lambda)^{(3+\alpha)/2}}, \quad h_i(\lambda) = \frac{(1-|\lambda_i|^2)^{1/2}}{(1-\overline{\lambda_i}\lambda)^{(3+\alpha)/2}}, \quad \lambda \in \mathbb{D}, i \geqslant 1.$$

By Lemma 1.2 we deduce $\|g_i\|_{L_a^{2,\alpha}} \sim |c_i|$ and $\|h_i\|_{L_a^{2,\alpha}} \sim 1$, and hence, by (4.3), we have

$$\sum_{i=0}^{\infty} \|g_i\|_{L_a^{2,\alpha}} \|h_i\|_{L_a^{2,\alpha}} \sim \|f\|_{L_a^{1,\alpha}}.$$

So the statement is true for $n = 1$. Suppose now it is true for $n \in \mathbb{N}$. By Theorem 1.5 applied for $X^* = L_a^{1,s}(\mathbb{D}^n) = (\mathcal{B}_0^{(n)})^*$, we can write any $f \in L_a^{1,s}(\mathbb{D}^{n+1}) = L_a^{1,s}(\mathbb{D}, L_a^{1,s}(\mathbb{D}^n))$ as

$$f(\lambda) = f(0) + \sum_{i=1}^{\infty} c_i \frac{\lambda(1-|\lambda_i|^2)}{(1-\overline{\lambda_i}\lambda)^{3+\alpha}}, \quad \lambda \in \mathbb{D}, \tag{4.4}$$

where $\lambda_i \in \mathbb{D}$ and $(c_i)_i = (c_i(f))_i \in \ell^1(L_a^{1,\alpha}(\mathbb{D}^n))$ satisfy

$$\|(c_i)\|_{\ell^1(L_a^{1,\alpha}(\mathbb{D}^n))} \sim \|f\|_{L_a^{1,\alpha}(\mathbb{D}^{n+1})}. \tag{4.5}$$

To simplify the notation set $c_0 = f(0)$. By the induction hypothesis we can now write each $c_i \in L_a^{1,\alpha}(\mathbb{D}^n), i \geqslant 0$, as

$$c_i = \sum_{j=1}^{\infty} g_{ij} h_{ij}, \tag{4.6}$$

where $g_{ij}, h_{ij} \in L_a^{2,\alpha}(\mathbb{D}^n)$ and

$$\sum_{j=1}^{\infty} \|g_{ij}\|_{L_a^{2,\alpha}(\mathbb{D}^n)} \|h_{ij}\|_{L_a^{2,\alpha}(\mathbb{D}^n)} \sim \|c_i\|_{L_a^{1,\alpha}(\mathbb{D}^n)}. \tag{4.7}$$

By (4.4)–(4.6) we now get

$$\begin{aligned} f(\lambda, z') &= f(\lambda)(z') \\ &= \sum_{j=1}^{\infty} g_{0j}(z') h_{0j}(z') + \sum_{i,j=1}^{\infty} g_{ij}(z') h_{ij}(z') \frac{\lambda(1-|\lambda_i|^2)}{(1-\lambda_i\lambda)^{3+\alpha}}, \quad z' \in \mathbb{D}^n, \lambda \in \mathbb{D}. \end{aligned}$$

For $i, j \geqslant 1$ and $\lambda \in \mathbb{D}$, $z' \in \mathbb{D}^n$, we put

$$G_{ij}(\lambda, z') = g_{ij}(z') \frac{\lambda(1-|\lambda_i|^2)^{1/2}}{(1-\lambda_i\lambda)^{(3+\alpha)/2}}, \quad \text{and } \; G_{0j}(\lambda, z') = g_{0j}(z');$$

$$H_{ij}(\lambda, z') = h_{ij}(z') \frac{(1-|\lambda_i|^2)^{1/2}}{(1-\lambda_i\lambda)^{(3+\alpha)/2}}, \quad \text{and } \; H_{0j}(\lambda, z') = h_{0j}(z').$$

Clearly

$$f = \sum_{i=0, j=1}^{\infty} G_{ij} H_{ij},$$

and by Lemma 1.2 we deduce that

$$\|G_{ij}\|_{L_a^{2,\alpha}(\mathbb{D}^{n+1})} \sim \|g_{ij}\|_{L_a^{2,\alpha}(\mathbb{D}^n)}, \quad \|H_{ij}\|_{L_a^{2,\alpha}(\mathbb{D}^{n+1})} \sim \|h_{ij}\|_{L_a^{2,\alpha}(\mathbb{D}^n)}, \quad i \geqslant 0, j \geqslant 1.$$

In view of (4.5)–(4.7) we then get

$$\sum_{i=0, j=1}^{\infty} \|G_{ij}\|_{L_a^{2,\alpha}(\mathbb{D}^{n+1})} \|H_{ij}\|_{L_a^{2,\alpha}(\mathbb{D}^{n+1})} \sim \|f\|_{L_a^{1,\alpha}(\mathbb{D}^{n+1})},$$

and the proof is complete. ∎

As mentioned in the introduction, we can combine the approach used in the above proof with our analogue of Sarason's theorem proven in Section 2 to obtain the following

THEOREM 4.2. *Let $t \geqslant 1, t_1, t_2, r, s > 0$ be such that $1/t_1 + 1/t_2 = 1/t$, $1/r + 1/s = 1$. Then, any function $f \in L_a^{1,\alpha}(\mathbb{D}^n, \mathcal{S}^t), \alpha > -1$, can be written as*

$$f = \sum_{n=1}^{\infty} g_n h_n,$$

*with* $g_n \in L_a^{r,\alpha}(\mathbb{D}^n, \mathcal{S}^{t_1})$, $h_n \in L_a^{s,\alpha}(\mathbb{D}^n, \mathcal{S}^{t_2})$ *and*

$$\sum_{n=1}^{\infty} \|g_n\|_{L_a^{r,\alpha}(\mathbb{D}^n,\mathcal{S}^{t_1})} \|h_n\|_{L_a^{s,\alpha}(\mathbb{D}^n,\mathcal{S}^{t_2})} \sim \|f\|_{L_a^{1,\alpha}(\mathbb{D}^n,\mathcal{S}^{t})}.$$

*Sketch of the proof.* For $n = 1$ the result is given in Theorem 2.1, while the inductive step from $n$ to $n+1$ is obtained as in the proof of Theorem 4.1, taking into account $L_a^{1,\alpha}(\mathbb{D}^n, \mathcal{S}^t) = (\mathcal{B}_0^{(n)}(\mathcal{S}^{t'}))^*$ for $t > 1, 1/t + 1/t' = 1$, and $L_a^{1,\alpha}(\mathbb{D}^n, \mathcal{S}^1) = (\mathcal{B}_0^{(n)}(\mathcal{K}(\mathcal{H})))^*$. ∎

Given $f \in L_a^{1,\alpha}(\mathbb{D}^n)$, the Hankel operator $\Gamma_f$ on $L_a^{2,\alpha}(\mathbb{D}^n)$ is defined as usual by

$$\langle \Gamma_f \phi, \psi \rangle = \int_{\mathbb{D}^n} f(z)\phi(\overline{z})\overline{\psi(z)}\, \mathrm{d}A_\alpha(z_1) \cdots \mathrm{d}A_\alpha(z_n),$$

where $\phi, \psi$ are holomorphic in a neighborhood of $\overline{\mathbb{D}}^n$, and $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$.

Our next step is a direct application of the results by Arregui and Blasco [3].

PROPOSITION 4.3. *Let* $\mathcal{B}^{(n)}$ *be the space of analytic functions defined in Section* 2. *The dual of* $L_a^{1,\alpha}(\mathbb{D}^n)$ *can be identified with* $\mathcal{B}^{(n)}$ *with the pairing*

$$\langle f, g \rangle = \int_{\mathbb{D}^n} f(z)\,\overline{g(z)}\, \mathrm{d}A_\alpha(z_1) \cdots \mathrm{d}A_\alpha(z_n), \tag{4.8}$$

*where* $f$ *is analytic in a neighborhood of* $\overline{\mathbb{D}}^n$ *and* $g \in \mathcal{B}^{(n)}$.

*Proof.* In [3] it is shown that given a separable Banach space $X$, the dual of $L_a^{1,\alpha}(\mathbb{D}, X)$ can be identified with $\mathcal{B}(X^*)$ under the pairing

$$\langle f, g \rangle = \int_{\mathbb{D}} \langle f(z), g(z) \rangle\, \mathrm{d}A_\alpha(z),$$

where $g \in \mathcal{B}(X^*)$ and $f$ is an $X$-valued function holomorphic on a neighborhood of $\overline{\mathbb{D}}^n$. Note that the $X$-valued functions holomorphic on a neighborhood of $\overline{\mathbb{D}}^n$ are dense in $L_a^{1,\alpha}(\mathbb{D}, X)$. We apply this result for $L_a^{1,\alpha}(\mathbb{D}^n) = L_a^{1,\alpha}(\mathbb{D}, L_a^{1,\alpha}(\mathbb{D}^{n-1}))$, i.e. $X = L_a^{1,\alpha}(\mathbb{D}^{n-1})$, to deduce that $(L_a^{1,\alpha}(\mathbb{D}^n))^*$ can be identified with $\mathcal{B}((L_a^{1,\alpha}(\mathbb{D}^{n-1}))^*)$. We can now continue with the same reasoning for $(L_a^{1,\alpha}(\mathbb{D}^{n-1}))^*$, and, inductively, we obtain that $(L_a^{1,\alpha}(\mathbb{D}^n))^*$ can be identified with $\mathcal{B}^{(n)}$ with the pairing (4.8). ∎

The next two corollaries provide a direct proof for the characterizations of the boundedness and of the compactness for the Hankel operator on $L_a^{2,\alpha}(\mathbb{D}^n)$. Although these characterizations were previously obtained in [16], we give a different formulation of the necessary and sufficient conditions.

COROLLARY 4.4. $\Gamma_f$ *is bounded on* $L_a^{2,\alpha}(\mathbb{D}^n)$ *if and only if* $f \in \mathcal{B}^{(n)}$. *Moreover,*

$$\|\Gamma_f\| \sim \|f\|_{\mathcal{B}^{(n)}}. \tag{4.9}$$

*Proof.* The "if" part follows immediately from $(L_a^{1,\alpha}(\mathbb{D}^n))^* = \mathcal{B}^{(n)}$ and the Cauchy-Schwarz inequality.

Let us now prove the "only if" part. Assume that $\Gamma_f$ is bounded and let $\phi : \mathbb{D}^n \to \mathbb{C}$ be analytic on a neighborhood of $\overline{\mathbb{D}}^n$. We use Theorem 4.1 to deduce that $\phi$ can be written as $\phi = \sum\limits_{i \geqslant 1} g_i h_i$, where $g_i, h_i$ are analytic on a neighborhood of $\overline{\mathbb{D}}^n$ and satisfy

$$\|\phi\|_{L_a^{1,\alpha}(\mathbb{D}^n)} \sim \sum_{i \geqslant 1} \|g_i\|_{L_a^{2,\alpha}(\mathbb{D}^n)} \|h_i\|_{L_a^{2,\alpha}(\mathbb{D}^n)}.$$

Note that the series $\sum\limits_{i \geqslant 1} g_i h_i$ converges also uniformly on the compact subsets of $\overline{\mathbb{D}}^n$ to the function $\phi$.

Then

$$\begin{aligned}\int_{\mathbb{D}^n} \overline{f(z)} \phi(z) \mathrm{d}A_\alpha(z_1) \cdots \mathrm{d}A_\alpha(z_n) &= \lim_{r \to 1} \int_{\mathbb{D}^n} \overline{f(z)} \phi(rz) \mathrm{d}A_\alpha(z_1) \cdots \mathrm{d}A_\alpha(z_n) \\ &= \lim_{r \to 1} \int_{\mathbb{D}^n} \overline{f(z)} \sum_{i \geqslant 1} g_i(rz) h_i(rz) \mathrm{d}A_\alpha(z_1) \cdots \mathrm{d}A_\alpha(z_n) \\ &= \lim_{r \to 1} \sum_{i \geqslant 1} \int_{\mathbb{D}^n} \overline{f(z)} g_i(rz) h_i(rz) \mathrm{d}A_\alpha(z_1) \cdots \mathrm{d}A_\alpha(z_n) \\ &= \lim_{r \to 1} \sum_{i \geqslant 1} \overline{\langle \Gamma_f g_{i_r}^*, h_{i_r} \rangle},\end{aligned}$$

where $h_{i_r}(z) = h_i(rz)$, $g_{i_r}^*(z) = \overline{g_i(r\bar{z})}$, $z \in \mathbb{D}^n$. Hence

$$\begin{aligned}\Big| \int_{\mathbb{D}^n} \overline{f(z)} \phi(z) \mathrm{d}A_\alpha(z_1) \cdots \mathrm{d}A_\alpha(z_n) \Big| &\leqslant \lim_{r \to 1} \sum_{i \geqslant 1} \|\Gamma_f\| \|g_{i_r}^*\|_{L_a^{2,\alpha}(\mathbb{D}^n)} \|h_{i_r}\|_{L_a^{2,\alpha}(\mathbb{D}^n)} \\ &\leqslant \|\Gamma_f\| \sum_{i \geqslant 1} \|g_i\|_{L_a^{2,\alpha}(\mathbb{D}^n)} \|h_i\|_{L_a^{2,\alpha}(\mathbb{D}^n)} \leqslant k \|\phi\|_{L_a^{1,\alpha}(\mathbb{D}^n)}.\end{aligned}$$

Since the functions analytic in a neighborhood of $\overline{\mathbb{D}}^n$ are dense in $L_a^{1,\alpha}(\mathbb{D}^n)$, we deduce that $f$ induces a bounded linear functional on $L_a^{1,\alpha}(\mathbb{D}^n)$, and, by Proposition 4.3, we obtain $f \in \mathcal{B}^{(n)}$. ∎

COROLLARY 4.5. *Consider the space $\mathcal{B}_0^{(n)}$ defined in Section 2 and let $f \in L_a^{1,\alpha}(\mathbb{D}^n)$. Then $\Gamma_f$ extends to a compact operator on $L_a^{2,\alpha}(\mathbb{D}^n)$ if and only if $f \in \mathcal{B}_0^{(n)}$.*

*Proof.* We prove the theorem by induction. For $n = 1$ the statement reduces to a well-known result (see [16]). Now suppose the statement is true for $n - 1$, i.e. the operator $\Gamma_g$ is compact on $L_a^{2,\alpha}(\mathbb{D}^{n-1})$ if and only if $g \in \mathcal{B}_0^{(n-1)}$. Let $f \in \mathcal{B}^{(n)}$. Since the space $L_a^{2,\alpha}(\mathbb{D}^n)$ can be identified with $L_a^{2,\alpha}(\mathbb{D}, L_a^{2,\alpha}(\mathbb{D}^{n-1}))$, then the operator $\Gamma_f$ can be regarded as the Hankel operator $\Gamma_F$ acting on $L_a^{2,\alpha}(\mathbb{D}, L_a^{2,\alpha}(\mathbb{D}^{n-1}))$, where the symbol $F : \mathbb{D} \to \mathcal{L}(L_a^{2,\alpha}(\mathbb{D}^{n-1}))$ is given by

$$F(z_1) = \Gamma_{f(z_1,\zeta)}, \quad z_1 \in \mathbb{D},$$

where $\zeta$ denotes the identity function on $\mathbb{D}^{n-1}$, and $\Gamma_{f(z_1,\zeta)}$ denotes the Hankel operator on $L_a^{2,\alpha}(\mathbb{D}^{n-1})$ with symbol $z' \mapsto f(z_1, z')$, $z' \in \mathbb{D}^{n-1}$. A straightforward argument, based on (4.9) and the fact that $f \in \mathcal{B}^{(n)}$, gives $F \in \mathcal{B}(\mathcal{L}(L_a^{2,\alpha}(\mathbb{D}^{n-1})))$ and

$$F'(z_1) = \Gamma_{\partial_{z_1} f(z_1,\zeta)}, \quad z_1 \in \mathbb{D}. \tag{4.10}$$

Note that for any $\phi, \psi \in L_a^{2,\alpha}(\mathbb{D}^n) = L_a^{2,\alpha}(\mathbb{D}, L_a^{2,\alpha}(\mathbb{D}^{n-1}))$, we have

$$\langle \Gamma_f \phi, \psi \rangle = \langle \Gamma_F \phi, \psi \rangle.$$

Clearly $\Gamma_f$ is compact on $L_a^{2,\alpha}(\mathbb{D}^n)$ if and only if $\Gamma_F$ is compact on $L_a^{2,\alpha}(\mathbb{D}, L_a^{2,\alpha}(\mathbb{D}^{n-1}))$. By Theorem 3.1 we deduce that $\Gamma_F$ is compact if and only if $F \in \mathcal{B}_0(\mathcal{K}(L_a^{2,\alpha}(\mathbb{D}^{n-1})))$, where $\mathcal{K}(L_a^{2,\alpha}(\mathbb{D}^{n-1}))$ denotes the set of compact operators on $L_a^{2,\alpha}(\mathbb{D}^{n-1})$. By the induction hypothesis we deduce that the condition $F(z_1) = \Gamma_{f(z_1,\zeta)} \in \mathcal{K}(L_a^{2,\alpha}(\mathbb{D}^{n-1}))$, for any (fixed) $z_1 \in \mathbb{D}$, is equivalent to $f(z_1, \zeta) \in \mathcal{B}_0^{(n-1)}$, for any (fixed) $z_1 \in \mathbb{D}$. Moreover, in view of (4.9)–(4.10), the condition

$$\|F'(z_1)\|(1 - |z_1|) \to 0 \quad \text{as } |z_1| \to 1,$$

is equivalent to

$$\|\partial_{z_1} f(z_1, \zeta)\|_{\mathcal{B}^{(n-1)}}(1 - |z_1|) = \|\partial_{z_1} f(z_1, \zeta)\|_{\mathcal{B}_0^{(n-1)}}(1 - |z_1|) \to 0 \quad \text{as } |z_1| \to 1,$$

that is $f \in \mathcal{B}_0^{(n)}$. The above considerations show that $\Gamma_f$ is compact on $L_a^{2,\alpha}(\mathbb{D}^n)$ if and only if $f \in \mathcal{B}_0^{(n)}$, which completes the proof. ∎

A similar characterization of bounded and compact Hankel operators has been obtained by Zhu [15]. His conditions are formulated in case $n = 2$ in terms of the derivatives involved in the definition of $\mathcal{B}_0(\mathcal{B}_0)$.

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