J. OPERATOR THEORY
59:1(2008), 179–192

UNBOUNDED OPERATORS ON HILBERT C^*-MODULES OVER C^*-ALGEBRAS OF COMPACT OPERATORS

BORIS GULJAŠ

Communicated by Şerban Strătilă

ABSTRACT. The paper is devoted to closed modular densely defined operators on a Hilbert C^*-module W over the C^*-algebra of compact operators on a Hilbert space. A bijective operation preserving correspondence of the set of all such operators and the set of all closed densely defined operators on a Hilbert space is obtained. The polar decomposition of a closed operator is generalized as well as some results concerning operators with compact resolvent, relative compact operators and the generalized inverse.

KEYWORDS: *C^*-algebra, Hilbert C^*-module, closed operator, closable operator, polar decomposition, relative compact operator, generalized inverse.*

MSC (2000): 46H25, 47C15.

INTRODUCTION

A (left) Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module W equipped with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle : W \times W \to \mathcal{A}$ which is $\mathcal{A}$-linear in the first and conjugate $\mathcal{A}$-linear in the second variable such that W is a Banach space with the norm $\|w\| = \|\langle w|w\rangle\|^{1/2}$.

Throughout this paper we are basically interested in Hilbert modules over the C^*-algebras $\mathcal{A}$ of compact operators on some complex Hilbert space. These modules are characterized by the property that each closed submodule is orthogonally complemented or orthogonally closed (see [5] and [8]). They are generally not self-dual, i.e. a generalization of Riesz representation theorem for bounded modular ($\mathcal{A}$-linear) functionals is not valid. Nevertheless, all bounded modular operators are adjointable.

In the theory of unbounded modular operators densely defined on an arbitrary Hilbert C^*-module it is necessary to suppose the regularity condition (see [4]) on closed operators in order to generalize some basic properties of closed operators on Hilbert spaces. We point out that the regularity condition is fulfilled for all closed modular operators densely defined on Hilbert C^*-modules over C^*-algebras of compact operators.

It is proved in [2] that the mapping $\Psi : \mathbf{B}(W) \to \mathbf{B}(W_e)$, $\Psi(A) = A|_{W_e}$, is a $*$-isomorphism of the C^*-algebra of all bounded modular operators defined on a Hilbert module W over the C^*-algebra of all compact operators on some Hilbert space and the C^*-algebra of bounded operators defined on a Hilbert space W_e contained in W. The main result in the present paper is the extension of this mapping to the set of all modular operators densely defined on W. This extension is an operations preserving bijection from the set of closed modular operators densely defined in W onto the set of closed operators densely defined on W_e. Also, it is a surjective mapping from the set of closable modular operators densely defined on W onto the set of closable operators densely defined on W_e. This mapping enables a natural procedure for lifting the results on densely defined closed and closable modular operators from Hilbert space theory to Hilbert C^*-modules over C^*-algebras of compact operators.

As application of our technique we prove the polar decomposition of a closed modular operator as well as the existence of the unique generalized inverse of a closed modular operator. We also generalize some results on relative compact operators and closed operators with resolvent in the algebra of compact operators.

We also note that the technique used in the paper is applicable in Hilbert H^*-modules. However, a related discussion on closed and closable operators on Hilbert H^*-modules is omitted since the corresponding results are presented in a similar, or even simpler way. For the results on bounded operators on Hilbert H^*-modules we refer to [1] and references therein.

1. CLOSED AND CLOSABLE OPERATORS

Throughout this paper let $\mathbf{K}(H) = \mathbf{K}$ be the fixed elementary C^*-algebra of all compact operators on some complex Hilbert space H of arbitrary dimension. Let W be an arbitrary Hilbert C^*-module over $\mathbf{K}$. We assume that $\mathbf{K}$ acts nontrivially on W. Since $\mathbf{K}$ is simple, this implies that W is full. If W is a Hilbert C^*-module over $\mathbf{K}$ the Cartesian product $W \times W$ is a Hilbert C^*-module over $\mathbf{K}$ with the inner product $\langle (u,v)|(u',v')\rangle_2 = \langle u|u'\rangle + \langle v|v'\rangle$.

In what follows we consider operators defined on a dense submodule of a Hilbert C^*-module W over C^*-algebra $\mathbf{K}$. Let $\mathcal{D}_A \subseteq W$ be the domain of A. An operator is a *modular operator* if it is linear and $A(ax) = aAx$ for all $x \in \mathcal{D}_A$ and all $a \in \mathbf{K}$. We denote the set of all such operators by $\mathcal{L}_{\mathbf{K}}(W)$.

The basic notions are analogous to those in the Hilbert spaces. Operators $A, B \in \mathcal{L}_{\mathbf{K}}(W)$ are adjoint to each other if $\langle Ax|y\rangle = \langle x|By\rangle$ for all $x \in \mathcal{D}_A$ and $y \in \mathcal{D}_B$. In terms of the graph $G(A) = \{(x, Ax) : x \in \mathcal{D}_A\}$ of A and the inverse graph $G'(A) = \{(Ax, x) : x \in \mathcal{D}_A\}$ this is equivalent to $G(A) \perp G'(-B)$

or $G'(-A) \perp G(B)$ in $W \times W$. A modular densely defined operator on a Hilbert C^*-module W over $\mathbf{K}$ possesses the unique maximal operator A^* adjoint to A with the domain $\mathcal{D}_{A^*} = \{x \in W : \exists v \in W,\ \forall y \in \mathcal{D}_A,\ \langle x|Ay\rangle = \langle v|y\rangle\}$ and $G(A^*) = G'(-A)^{\perp}$.

DEFINITION 1.1. An operator $A \in \mathcal{L}_{\mathbf{K}}(W)$ is a *closed operator* if its graph $G(A)$ is a closed submodule of the Hilbert C^*-module $W \times W$. Equivalently, $\mathcal{D}_A$ and the range of A are submodules of W and also, $(x_n)_n$ in $\mathcal{D}_A$, $x_n \to x$ and $Ax_n \to y$ imply $x \in \mathcal{D}_A$ and $y = Ax$. The set of all closed operators in $\mathcal{L}_{\mathbf{K}}(W)$ is denoted by $\mathcal{C}_{\mathbf{K}}(W)$.

$A \in \mathcal{L}_{\mathbf{K}}(W)$ is a *closable operator* if the closure $\overline{G(A)}$ of the graph $G(A)$ is an operator graph itself, i.e. if $(0, y) \in \overline{G(A)}$ imply $y = 0$. Equivalently, $(x_n)_n$ in $\mathcal{D}_A$, $x_n \to 0$ and $Ax_n \to y$ imply $y = 0$. The set of all closable operators in $\mathcal{L}_{\mathbf{K}}(W)$ is denoted by $\mathcal{C}\ell_{\mathbf{K}}(W)$. The operator $B \in \mathcal{C}_{\mathbf{K}}(W)$ such that $G(B) = \overline{G(A)}$ is called the closure of A and denoted by $B = \overline{A}$.

A dense submodule $\mathcal{D} \subset \mathcal{D}_A$ is a *core* of $A \in \mathcal{C}_{\mathbf{K}}(W)$ if $A_0 = A|_{\mathcal{D}}$ satisfies $\overline{A}_0 = A$.

$A \sqsubseteq B$ denotes that B is an extension of A, i.e. $G(A) \subseteq G(B)$.

Let us observe that for $A \in \mathcal{C}\ell_{\mathbf{K}}(W)$ and its closure $\overline{A} \in \mathcal{C}_{\mathbf{K}}(W)$ the domain $\mathcal{D}_A$ is a dense submodule of $\mathcal{D}_{\overline{A}}$. The operator $\overline{A}$ is the smallest closed extension of A, i.e. every closed extension of A is the extension of $\overline{A}$.

For $A \in \mathcal{L}_{\mathbf{K}}(W)$ and $x, y \in \mathcal{D}_A$ we define the inner product $\langle x|y\rangle_A = \langle x|y\rangle + \langle Ax|Ay\rangle$ on $\mathcal{D}_A$. We denote $\|\cdot\|_A = \|\langle\cdot|\cdot\rangle_A\|^{1/2}$ and by $\overline{}^A$ we denote the closure in this norm.

The following characterization of closed modular operators on Hilbert C^*-modules over C^*-algebras is well known from the Hilbert space theory.

PROPOSITION 1.2. *Let $A \in \mathcal{L}_{\mathbf{K}}(W)$. Then A belongs to $\mathcal{C}_{\mathbf{K}}(W)$ if and only if the submodule $\mathcal{D}_A$ is a Hilbert C^*-module over $\mathbf{K}$ with the inner product $\langle\cdot|\cdot\rangle_A$. When this is the case $A : \mathcal{D}_A \to W$ is a bounded operator with respect to the norm $\|\cdot\|_A$ on $\mathcal{D}_A$.*

Proof. The statement follows from the inequality

$$\max\{\|x\|, \|Ax\|\} \leqslant \|x\|_A \leqslant (\|x\|^2 + \|Ax\|^2)^{1/2}, \quad \forall x \in \mathcal{D}_A, \tag{1.1}$$

as in Hilbert spaces. ∎

Clearly, for $A \in \mathcal{C}_{\mathbf{K}}(W)$ the submodule Ker$A$ is closed in $\|\cdot\|_A$ and $\|\cdot\|$.

As defined in [4], an operator $A \in \mathcal{C}_{\mathbf{K}}(W)$ is *regular* if there exists a densely defined adjoint operator A^* such that the range of $I + A^*A$ is dense in W. It is proved by A. Pal [6] that in Hilbert C^*-modules over C^*-algebras of compact operators the regularity condition is equivalent to a generally weaker semi-regularity condition (i.e. that A and A^* are densely defined). The direct consequence of the property that every closed submodule in $\mathbf{K}$-module W is complemented is that any operator in $\mathcal{C}_{\mathbf{K}}(W)$ is regular.

PROPOSITION 1.3. *Let A be in $\mathcal{C}\ell_{\mathbf{K}}(W)$. Then its adjoint A^* is a densely defined closed modular operator, i.e. $A^* \in \mathcal{C}_{\mathbf{K}}(W)$. Moreover, if $A \in \mathcal{C}_{\mathbf{K}}(W)$ then* $\mathrm{Im}\,(I + A^*A) = W$ *holds true.*

Proof. If $A \in \mathcal{C}\ell_{\mathbf{K}}(W)$, as demonstrated above, then there exists the modular adjoint operator A^* with closed graph $G(A^*) = G'(-A)^{\perp}$. Then $G'(-A^*) = G(A)^{\perp}$ and $G(A)^{\perp\perp} = \overline{G(A)}$ imply $\overline{G(A)} \oplus G'(-A^*) = W \times W$.

It remains to prove that $\mathcal{D}_{A^*}$ is dense in W. Let $y \in W$ be such that $y \perp \mathcal{D}_{A^*}$. Then $(0, y) \in G'(-A^*)^{\perp} = \overline{G(A)}$ and, because $\overline{G(A)}$ is a graph, we have $y = 0$. This implies that A^* is densely defined.

The regularity of $A \in \mathcal{C}_{\mathbf{K}}(W)$ follows analogously as in the case of Hilbert spaces (see Teorem V.3.24 of [3]). ∎

We end this introductory section with a remark on Hilbert C^*-modules over arbitrary C^*-algebras of compact operators.

REMARK 1.4. Let W be a full Hilbert C^*-module over an arbitrary C^*-algebra $\mathcal{A}$ of compact operators. An arbitrary C^*-algebra of compact operators is of the form $\mathcal{A} = \bigoplus_{\lambda \in \Lambda} \mathbf{K}_\lambda$, i.e. $\mathcal{A}$ is a direct sum of elementary C^*-algebras $\mathbf{K}_\lambda = \mathbf{K}(H_\lambda)$. For each $\lambda \in \Lambda$ consider the associated ideal submodule $W_\lambda = \overline{\mathrm{span}\{\mathbf{K}_\lambda W\}}$. Notice that W_λ, regarded as a Hilbert $\mathbf{K}_\lambda$-module is full, hence by the Hewitt-Cohen factorization theorem (see Proposition 2.31 of [10]) $W_\lambda = \mathbf{K}_\lambda W_\lambda$. Consequently, we have $W_\lambda = \mathbf{K}_\lambda W$. W admits the decomposition into the orthogonal sum $W = \bigoplus_{\lambda \in \Lambda} W_\lambda$ as well as $W \times W = \bigoplus_{\lambda \in \Lambda} (W_\lambda \times W_\lambda)$. All operators in $\mathcal{L}_{\mathcal{A}}(W)$ are reduced by all W_λ, i.e. $G(A) = \bigoplus_{\lambda} G(A_\lambda)$ with $A_\lambda = A|_{\mathcal{D}_A \cap W_\lambda}$, $G(A_\lambda) \subset W_\lambda \times W_\lambda$, $(\lambda \in \Lambda)$. Moreover, $A \in \mathcal{C}_{\mathcal{A}}(W)$ if and only if $A_\lambda \in \mathcal{C}_{\mathbf{K}_\lambda}(W_\lambda)$ for all $\lambda \in \Lambda$. This enables us to reduce our attention to the case of a Hilbert C^*-module over an elementary C^*-algebra $\mathbf{K} = \mathbf{K}(H)$.

2. DESCRIPTION OF $\mathcal{C}_{\mathbf{K}}(W)$

It is proved in [2] that each Hilbert C^*-module over the C^*-algebra of compact operators possesses an orthonormal basis consisting of basic vectors ($x \in W$ is a basic vector if $\langle x|x\rangle$ is a minimal projection). Orthonormal bases of that form admit the unique Fourier expansion for any vector in W. For any minimal projection e in $\mathbf{K}$ we define a subspace $W_e = eW$ which is also a Hilbert space (see [2]). The scalar product $(\cdot|\cdot) : W_e \times W_e \to \mathbb{C}$ is the restriction of the inner product $\langle \cdot|\cdot\rangle$ modulo projection e, i.e. $(x|y)e = \langle x|y\rangle$ for all $x, y \in W_e$. Any orthonormal basis for the Hilbert space W_e is also an orthonormal basis for the Hilbert C^*-module W and $W = \overline{\mathrm{span}\{\mathbf{K}W_e\}}$. Also, for any submodule $Y \subseteq W$ (possibly non closed) we have $\overline{eY} = e\overline{Y}$. Moreover, if $eY = \{0\}$ then $Y = \{0\}$. Namely, for $y \in Y$ we have $eay = 0$ for all $a \in \mathbf{K}$ and, since $\overline{\mathrm{span}\{\mathbf{K}e\mathbf{K}\}} = \mathbf{K}$, $beay = 0$ for all $a, b \in \mathbf{K}$ imply $y = 0$.

For $A \in \mathcal{C}\ell_{\mathbf{K}}(W)$ we have $e\mathcal{D}_A \subseteq W_e$ and $\overline{e\mathcal{D}_A} = W_e$. In what follows by the restriction $A|_{W_e}$ of an operator $A \in \mathcal{C}_{\mathbf{K}}(W)$ we mean the restriction of A onto the subspace $e\mathcal{D}_A$. It is easy to see that a Hilbert space W_e reduces all operators in $\mathcal{L}_{\mathbf{K}}(W)$. For any $A \in \mathcal{C}_{\mathbf{K}}(W)$ the induced operator $\widehat{A} = A|_{W_e} : \mathcal{D}_{\widehat{A}} \to W_e$, $\mathcal{D}_{\widehat{A}} = e\mathcal{D}_A$, is in the set $\mathcal{C}(W_e)$ of all closed operators densely defined on W_e. Namely, $G(\widehat{A}) = eG(A) = e\overline{G(A)} = \overline{eG(A)}$, hence $G(\widehat{A})$ is closed.

Our main goal is to connect the sets $\mathcal{C}_{\mathbf{K}}(W)$ and $\mathcal{C}(W_e)$. The basic technical tools are contained in the following lemmas.

LEMMA 2.1. *Let W be a Hilbert C^*-module over an elementary algebra $\mathbf{K}$ and $\widehat{A} \in \mathcal{C}(W_e)$. A closed submodule $G = \overline{\mathrm{span}\{\mathbf{K}G(\widehat{A})\}} \subset W \times W$ is an operator graph and defines $A \in \mathcal{C}_{\mathbf{K}}(W)$ such that $G(A) = G$, $e\mathcal{D}_A = \mathcal{D}_{\widehat{A}}$. Also $\mathcal{D}_A = \overline{\mathrm{span}\{\mathbf{K}\mathcal{D}_{\widehat{A}}\}}^A$, $\mathrm{Ker}\,A = \overline{\mathrm{span}\{\mathbf{K}\mathrm{Ker}\,\widehat{A}\}}^A = \overline{\mathrm{span}\{\mathbf{K}\mathrm{Ker}\widehat{A}\}}$ and $\overline{\mathrm{Im}\,A} = \overline{\mathrm{span}\{\mathbf{K}\mathrm{Im}\,\widehat{A}\}}$ hold true.*

Moreover, A is a minimal extension of $\widehat{A}$ in $\mathcal{C}_{\mathbf{K}}(W)$, i.e. for any other $B \in \mathcal{C}_{\mathbf{K}}(W)$ with the property $e\mathcal{D}_B \supseteq \mathcal{D}_{\widehat{A}}$ and $B|_{\mathcal{D}_{\widehat{A}}} = \widehat{A}$ we have $A \sqsubseteq B$.

Proof. $\widehat{A} \in \mathcal{C}(W_e)$ implies that $W_e \times W_e = G(\widehat{A}) \oplus G'(-\widehat{A}^*)$ is an orthogonal sum of closed subspaces. From the properties of a minimal projection e we have $eG = G(\widehat{A})$. If $(0, y) \in G$ then $(0, ay) \in G$ for all $a \in \mathbf{K}$. Now $(0, eay) \in G(\widehat{A})$ implies $eay = 0$ for all $a \in \mathbf{K}$, hence $y = 0$. G is closed in $W \times W$ by definition and there is a unique $A \in \mathcal{C}_{\mathbf{K}}(W)$ such that $G(A) = G$. Clearly, $e\mathcal{D}_A = \mathcal{D}_{\widehat{A}}$ and $A|_{\mathcal{D}_{\widehat{A}}} = \widehat{A}$.

The scalar product in $W_e \times W_e$ restricted to the subspace $G(\widehat{A})$ is of the form $\langle (u, \widehat{A}u) | (v, \widehat{A}v) \rangle_2 = (u|v)e + (\widehat{A}u|\widehat{A}v)e = (u|v)_{\widehat{A}}e$. Let $((v_n, \widehat{A}v_n))_{n \in I}$ be an orthonormal basis in $G(\widehat{A})$. Then $(v_n)_{n \in I}$ is an orthonormal basis in the Hilbert space $(\mathcal{D}_{\widehat{A}}, (\cdot|\cdot)_A)$. Because $((v_n, \widehat{A}v_n))_{n \in I}$ is also an orthonormal basis in $G(A)$ (see Remark 4 in [2]) any element $(x, Ax) \in G(A)$ is of the form

$$x = \sum_n (\langle x|v_n \rangle + \langle Ax|\widehat{A}v_n \rangle) v_n = \sum_n \langle x|v_n \rangle_A v_n, \tag{2.1}$$

$$Ax = \sum_n (\langle x|v_n \rangle + \langle Ax|\widehat{A}v_n \rangle) \widehat{A}v_n = \sum_n \langle x|v_n \rangle_A \widehat{A}v_n. \tag{2.2}$$

We note that convergence in (2.1) is with respect to $\|\cdot\|_A$ (and $\|\cdot\|$), and in (2.2) with respect to $\|\cdot\|$. From (2.1) we have $\mathcal{D}_A = \overline{\mathrm{span}\{\mathbf{K}\mathcal{D}_{\widehat{A}}\}}^A$. The norms $\|\cdot\|$ and $\|\cdot\|_A$ coincide on $\mathrm{Ker}A$, hence $\mathrm{Ker}\widehat{A}$ and $\mathrm{Ker}A$ are closed with respect to both norms. Clearly, $\overline{\mathrm{span}\{\mathbf{K}\mathrm{Ker}\widehat{A}\}} \subseteq \mathrm{Ker}A$. But $e((\overline{\mathrm{span}\{\mathbf{K}\mathrm{Ker}\,\widehat{A}\}})^{\perp} \cap \mathrm{Ker}A) = (\mathrm{Ker}\widehat{A})^{\perp} \cap \mathrm{Ker}\widehat{A} = \{0\}$ implies $\mathrm{Ker}A = \overline{\mathrm{span}\{\mathbf{K}\mathrm{Ker}\widehat{A}\}}$. Clearly, $\mathrm{span}\{\mathbf{K}\mathrm{Im}\widehat{A}\} = \mathcal{I} \subseteq \mathrm{Im}A$. By (2.2), we have $\mathrm{Im}A \subseteq \overline{\mathcal{I}}$, hence $\overline{\mathrm{Im}A} = \overline{\mathcal{I}}$.

Let $B \in \mathcal{C}_K(W)$ be such that $B|_{\mathcal{D}_{\widehat{A}}} = \widehat{A}$ and let $x \in \mathcal{D}_A$. After defining a net $(x_J)_{J \subset I}$ over all finite sets $J \subset I$, $x_J = \sum_{n \in J} \langle x | v_n \rangle_A v_n$, we find $Bx_J = \sum_{n \in J} \langle x | v_n \rangle_A \widehat{A} v_n = Ax_J$. Because (2.1) and (2.2) imply $x_J \to x$, $Bx_J \to y = Ax$ (since B is closed) we conclude $x \in \mathcal{D}_B$ and $Bx = Ax$, i.e. $A \sqsubseteq B$. ∎

LEMMA 2.2. *Let W be a Hilbert C^*-module over an elementary algebra $\mathbf{K}$. If $A \in \mathcal{C}_{\mathbf{K}}(W)$ and $B \in \mathcal{C}\ell_{\mathbf{K}}(W)$ are such that $A \sqsubseteq B$ and $e\mathcal{D}_A = e\mathcal{D}_B$ then $A = B$.*

Proof. Clearly, $A \sqsubseteq B$ implies $G(A) \subseteq G(B) \subseteq G(\overline{B})$. Let us suppose $G(A) \neq G(B)$. Since $G(A)$ and $G(\overline{B})$ are closed submodules of the Hilbert C^*-module $W \times W$, there exists $G \neq \{(0,0)\}$ such that $G(\overline{B}) = G(A) \oplus G$. If we denote $G_0 = G \cap G(B)$ then $G(B) = G(A) \oplus G_0$. Clearly, G_0 is a dense submodule in G while $G(\overline{B}) = \overline{G(B)}$. The condition $e\mathcal{D}_A = e\mathcal{D}_B$ and $A \sqsubseteq B$ imply $eG(B) = eG(A)$ and $eG_0 = \{(0,0)\}$. Because G_0 is a submodule, we have $G_0 = \{(0,0)\}$, i.e. $G(A) = G(B)$ and $A = B$. ∎

LEMMA 2.3. *Let W be a Hilbert C^*-module over an elementary algebra $\mathbf{K}$, $A \in \mathcal{L}_{\mathbf{K}}(W)$ and $\widehat{A} = A|_{e\mathcal{D}_A}$. Then $A \in \mathcal{C}\ell_{\mathbf{K}}(W)$ if and only if $\widehat{A} \in \mathcal{C}\ell(W_e)$.*

Proof. Let $\widehat{A} \in \mathcal{C}\ell(W)$ and let (x_n) be a sequence in $\mathcal{D}_A$ such that $x_n \to 0$ and $Ax_n \to y$ for some $y \in W$. Then for any $a \in \mathbf{K}$, we have $eax_n \to 0$ and $A(eax_n) \to eay$. Because $\widehat{A} = A|_{e\mathcal{D}_A}$ is a closable operator we have $eay = 0$ for all $a \in \mathbf{K}$, hence $y = 0$. The opposite assertion is obvious. ∎

Our next theorem gives a description of the set of all densely defined closed modular operators on Hilbert C^*-modules over C^*-algebras of compact operators. It also shows that the natural $*$-isomorphism Ψ between $\mathbf{B}_{\mathbf{K}}(W)$ and $\mathbf{B}(W_e)$, defined in [2], can be extended to the $*$-preserving bijection between $\mathcal{C}_{\mathbf{K}}(W)$ and $\mathcal{C}(W_e)$.

THEOREM 2.4. *Let W be a Hilbert C^*-module over $\mathbf{K}$ and let $e \in \mathbf{K}$ be any minimal projection. Then $\Psi : \mathcal{C}_{\mathbf{K}}(W) \to \mathcal{C}(W_e)$, $\Psi(A) = A|_{W_e}$ is a bijective operation preserving mapping of $\mathcal{C}_{\mathbf{K}}(W)$ onto $\mathcal{C}(W_e)$.*

Proof. Because $G(\Psi(A)) = eG(A)$ is a closed subspace of $eW \times eW \subset W \times W$, we conclude $\Psi(A) = A|_{W_e} \in \mathcal{C}(W_e)$. From Lemma 2.1 it follows that Ψ is surjective.

Let us suppose that $A, B \in \mathcal{C}_{\mathbf{K}}(W)$ are such that $\Psi(A) = \Psi(B) = \widehat{C} \in \mathcal{C}(W_e)$ with $\mathcal{D}_{\widehat{C}} = e\mathcal{D}_A = e\mathcal{D}_B$. By Lemma 2.1, there exists a minimal operator $C \in \mathcal{C}_{\mathbf{K}}(W)$, $e\mathcal{D}_C = \mathcal{D}_{\widehat{C}}$, such that $C \sqsubseteq A$ and $C \sqsubseteq B$. The injectivity follows from Lemma 2.2.

Let $B = \Psi^{-1}(\Psi(A)^*)$ be a closed operator. From Lemma 2.1 we have $G(B) = \overline{\operatorname{span}\{\mathbf{K}G(\Psi(A)^*)\}}$. $W_e \times W_e = G'(-\Psi(A)) \oplus G(\Psi(A)^*)$ implies $G'(-A) \perp G(B)$, hence $\langle Ax | y \rangle = \langle x | By \rangle$ for all $x \in \mathcal{D}_A$ and all $y \in \mathcal{D}_B$. This implies

$B \subseteq A^*$ because A^* is the maximal closed operator with this property. But $W \times W = G'(-A) \oplus G(A^*)$ (and hence $W_e \times W_e = eG'(-A) \oplus eG(A^*)$) implies $eG(A^*) = G'(-\Psi(A))^{\perp} = G(\Psi(A)^*)$, i.e. $e\mathcal{D}_{A^*} = \mathcal{D}_{\Psi(A)^*} = \mathcal{D}_{\Psi(B)} = e\mathcal{D}_B$. By Lemma 2.2, this gives $B = A^*$. The equality $\Psi(A^*) = \Psi(A)^*$ follows from the construction of B.

Let $\mathrm{Im}B \subseteq \mathcal{D}_A$ and $A, B, AB \in \mathcal{C}_{\mathbf{K}}(W)$ (then $\mathcal{D}_{AB} = \mathcal{D}_B$). Then $e\mathrm{Im}B \subseteq e\mathcal{D}_A$ and $\Psi(A)\Psi(B)$ is well defined on $e\mathcal{D}_B = e\mathcal{D}_{AB}$. Obviously, $AB|_{e\mathcal{D}_{AB}} = \Psi(A)\Psi(B)$, hence the uniqueness from Lemma 2.2 implies $\Psi(AB) = \Psi(A)\Psi(B)$.

If $A, B \in \mathcal{C}_{\mathbf{K}}(W)$, $\mathcal{D}_A = \mathcal{D}_B$ and $\alpha A + \beta B \in \mathcal{C}_{\mathbf{K}}(W)$ then $\Psi(\alpha A + \beta B) = \alpha\Psi(A) + \beta\Psi(B)$. Namely, $\alpha\Psi(A) + \beta\Psi(B)$ is a restriction of a closed operator and $\mathcal{D}_{\alpha A+\beta B} = \mathcal{D}_A = \mathcal{D}_B$, so by injectivity we have the equality above. ∎

The next result extends the assertion from the above theorem.

COROLLARY 2.5. *If $A, B \in \mathcal{C}_{\mathbf{K}}(W)$ such that Im $B \subseteq \mathcal{D}_A$ and $AB \in \mathcal{C}\ell_{\mathbf{K}}(W)$, then $\Psi(\overline{AB}) = \overline{\Psi(A)\Psi(B)}$.*

Proof. $G(\overline{AB}) = \overline{G(AB)}$ implies $G(\Psi(\overline{AB})) = e\overline{G(AB)} = \overline{eG(AB)} = \overline{\{(ex, ABex) : x \in \mathcal{D}_B\}} = \overline{\{(y, \Psi(A)\Psi(B)y) : y \in e\mathcal{D}_B\}} = G(\overline{\Psi(A)\Psi(B)})$. ∎

The function Ψ from Theorem 2.4 can be extended to the function $\widetilde{\Psi}$ from the set of all densely defined operators on Hilbert C^*-module over $\mathbf{K}$, i.e. $\widetilde{\Psi} : \mathcal{L}_{\mathbf{K}}(W) \to \mathcal{L}(W_e)$, $\widetilde{\Psi}(A) = A|_{e\mathcal{D}_A}$. It follows from Lemma 2.3 that the restriction $\widehat{\Psi} = \widetilde{\Psi}|_{\mathcal{C}\ell_{\mathbf{K}}(W)}$ is a function $\widehat{\Psi} : \mathcal{C}\ell_{\mathbf{K}}(W) \to \mathcal{C}\ell(W_e)$.

$\widehat{\Psi}$ is not injective. For any $A \in \mathcal{C}_{\mathbf{K}}(W)$ there exists a nonclosed $B \in \mathcal{C}\ell_{\mathbf{K}}(W)$ such that $\overline{B} = A$ and $eG(B) = eG(A)$. Let $\widehat{A} = A|_{e\mathcal{D}_A}$ and $eG(A) = G(\widehat{A})$. We define a (nonclosed) submodule $G = \mathrm{span}\{\mathbf{K}eG(A)\}$. The set $G \subset G(A)$ is a graph and $\overline{G} = G(A)$.

That $\widehat{\Psi}$ is surjective, follows also from Lemma 2.3. Let $A_0 \in \mathcal{C}\ell(W_e)$ and define G as above. $eG = G(A_0)$ and $G = G(A)$ is an operator graph for some $A \in \mathcal{L}_{\mathbf{K}}(W)$. Namely, because $\mathcal{D}_{A_0}$ is dense in W_e, $\mathcal{D}_A$ is dense in W. Now, $e\mathcal{D}_A = \mathcal{D}_{A_0}$ implies by Lemma 2.3 that $A \in \mathcal{C}\ell_{\mathbf{K}}(W)$.

In what follows we assume the same meaning for the symbols $\widetilde{\Psi}$, $\widehat{\Psi}$ and Ψ.

COROLLARY 2.6. *If $A \in \mathcal{C}\ell_{\mathbf{K}}(W)$ then $\Psi(\overline{A}) = \overline{\Psi(A)}$.*

Proof. The subspace $\mathcal{D}_{\Psi(A)} = e\mathcal{D}_A \subseteq e\mathcal{D}_{\overline{A}} = \mathcal{D}_{\Psi(\overline{A})}$ is dense in $\mathcal{D}_{\Psi(\overline{A})}$ and $\overline{G(\Psi(A))} = \overline{eG(A)} = e\overline{G(A)} = eG(\overline{A}) = G(\Psi(\overline{A}))$. ∎

The next result gives a connection between closed and closable operators if their restrictions coincide.

COROLLARY 2.7. *If $A \in \mathcal{C}_{\mathbf{K}}(W)$ and $B \in \mathcal{C}\ell_{\mathbf{K}}(W)$ are such that $\Psi(A) = \Psi(B)$ then $A = \overline{B}$.*

Proof. Clearly, $e\mathcal{D}_A = e\mathcal{D}_B \subseteq e\mathcal{D}_{\overline{B}}$. If $e\mathcal{D}_B \neq e\mathcal{D}_{\overline{B}}$ then $G(\Psi(B)) \subset G(\Psi(\overline{B}))$ and, because $\Psi(B) = \Psi(A)$ is closed, we conclude $G(\overline{\Psi(B)}) \subset G(\Psi(\overline{B}))$, hence $\overline{\Psi(B)} \neq \Psi(\overline{B})$ which is impossible by Corollary 2.6. Consequently, we have $e\mathcal{D}_A = e\mathcal{D}_{\overline{B}}$. By the injectivity of Ψ on closed operators, this implies $A = \overline{B}$. ∎

REMARK 2.8. In Corollary 2.7 we see the only, but crucial, disadvantage of the fact that Ψ is not injection on the set of closable operators. One cannot deduce that an operator in $\mathcal{L}_{\mathbf{K}}(W)$ is closed if its restriction on the fixed Hilbert space W_e is in $\mathcal{C}(W_e)$. Using Lemma 2.2, we can only deduce $A \in \mathcal{C}\ell_{\mathbf{K}}(W)$. Moreover, one cannot deduce that an operator is closed if its restriction on W_e is closed for each minimal projection $e \in \mathbf{K}$. We show this by the following example.

EXAMPLE 2.9. Let $\mathbf{K}$ be the elementary C^*-algebra of compact operators on an infinite dimensional Hilbert space and let W be any infinite $\mathbf{K}$-dimensional Hilbert module over $\mathbf{K}$. Let us take a bounded operator $A \in \mathbf{B}_{\mathbf{K}}(W) \subset \mathcal{C}_{\mathbf{K}}(W)$ ($\mathcal{D}_A = W$). Let $\mathcal{F}$ denotes the (nonclosed) ideal of operators of finite rank in $\mathbf{K}$. Any minimal projection e in $\mathbf{K}$ has rank one, i.e. $e \in \mathcal{F}$, hence $e\mathbf{K} \subseteq \mathcal{F}$. This gives $e\mathbf{K} = e(e\mathbf{K}) \subseteq e\mathcal{F}$. The opposite inclusion is obvious, so we have $e\mathbf{K} = e\mathcal{F}$. We denote by $\mathcal{D}_0 = \operatorname{span}\{\mathcal{F}W\}$. This is a submodule of W and $\overline{\mathcal{D}}_0 = W$ ($\mathcal{F}$ contains an approximate unit for $\mathbf{K}$). Clearly, $A_0 = A|_{\mathcal{D}_0}$ is not closed because a bounded operator is closed if and only if its domain is closed. Further, $G(A_0) = \operatorname{span}\{\mathcal{F}G(A)\}$, hence $\overline{A}_0 = A$. Obviously, for any minimal projection e we have $e\mathcal{D}_0 = eW$.

In the light of the preceding example, the next observation will be useful.

COROLLARY 2.10. *Let W be a Hilbert C^*-module over an elementary algebra $\mathbf{K}$, $A \in \mathcal{L}_{\mathbf{K}}(W)$ and $\widehat{A} = A|_{e\mathcal{D}_A}$. Then A is bounded if and only if $\widehat{A}$ is bounded.*

Proof. Clearly, if A is bounded then its restriction $\widehat{A}$ is bounded. Let us suppose that $\widehat{A}$ is bounded. Then $\widehat{B} = \overline{\widehat{A}} \in \mathbf{B}(W_e)$ and we have $B \in \mathbf{B}_{\mathbf{K}}(W)$ such that $\Psi(B) = \widehat{B}$. By Lemma 2.3, $A \in \mathcal{C}\ell_{\mathbf{K}}(W)$ and Corollary 2.6 implies $\Psi(\overline{A}) = \overline{\Psi(A)} = \widehat{B} = \Psi(B)$. Then, by Theorem 2.4, we have $B = \overline{A}$. ∎

3. LIFTING RESULTS FROM HILBERT SPACE THEORY

In this section we generalize some results on closed operators on Hilbert spaces.

3.1. THE POLAR DECOMPOSITION IN $\mathcal{C}_{\mathbf{K}}(W)$.

If A is a closed densely defined operator on a Hilbert space then there exists a unique decomposition of the form $A = US$ where $S = (A^*A)^{1/2}$, $\mathcal{D}_S = \mathcal{D}_A$, is a selfadjoint operator and U is a partial isometry (see VI. Section 2.7 of [3]). As the first application of previous results we show that the same assertion is true for operators in Hilbert C^*-modules over the C^*-algebra $\mathbf{K}$.

THEOREM 3.1. *Let $A \in \mathcal{C}_{\mathbf{K}}(W)$. Then there exists a unique selfadjoint operator $S = (A^*A)^{1/2}$, $\mathcal{D}_S = \mathcal{D}_A$, and a partial isometry U with the initial set $\overline{\mathrm{Im} S}$ and the final set $\overline{\mathrm{Im} A}$ such that $A = US$.*

Proof. We apply the corresponding Hilbert space result to $\Psi(A) \in \mathcal{C}(W_e)$, thus $\Psi(A) = \widehat{U}\widehat{S}$ with $\widehat{S} = (\Psi(A)^*\Psi(A))^{1/2}$, $\mathcal{D}_{\widehat{S}} = \mathcal{D}_{\Psi(A)}$, where $\widehat{U}$ is a partial isometry with the initial set $\overline{\mathrm{Im}\widehat{S}}$ and the final set $\overline{\mathrm{Im}\Psi(A)}$. We have $\widehat{S} = \widehat{U}^*\Psi(A)$, hence $\widehat{S} \in \mathcal{C}(W_e)$. Let $S = \Psi^{-1}(\widehat{S}) \in \mathcal{C}_{\mathbf{K}}(W)$ and $U = \Psi^{-1}(\widehat{U}) \in \mathbf{B}_{\mathbf{K}}(W)$. U is a partial isometry because Ψ is a $*$-isomorphism of $\mathbf{B}_{\mathbf{K}}(W)$ and $\mathbf{B}(W_e)$. By Lemma 2.3 and Corollary 2.7, $\Psi(U^*A) = \widehat{U}^*\Psi(A) = \widehat{S}$ and $\Psi(US) = \widehat{U}\widehat{S} = \Psi(A)$ imply $\overline{U^*A} = S$ and $\overline{US} = A$ respectively. Now, $\mathcal{D}_A = \mathcal{D}_{U^*A} \subseteq \mathcal{D}_S = \mathcal{D}_{US} \subseteq \mathcal{D}_A$ gives $\mathcal{D}_A = \mathcal{D}_S$, hence $U^*A, US \in \mathcal{C}_{\mathbf{K}}(W)$ and $S = U^*A$, $A = US$. Analogously, $\Psi(UU^*A) = \widehat{U}\widehat{U}^*\Psi(A) = \Psi(A)$ and $\Psi(U^*US) = \widehat{U}^*\widehat{U}\widehat{S} = \widehat{S}$ imply $UU^*A = A$ and $U^*US = S$ respectively, hence U is a partial isometry with the initial set $\overline{\mathrm{Im} S}$ and the final set $\overline{\mathrm{Im} A}$. Obviously, $A^*A = SU^*US = S^2$. ∎

3.2. GENERALIZED INVERSES IN $\mathcal{C}_{\mathbf{K}}(W)$. Generalized inverses of closed densely defined operators in Hilbert spaces are systematically investigated in [7]. We recall that any operator $A \in \mathcal{C}(H)$, where H is a Hilbert space, possesses a unique generalized inverse $A^- \in \mathcal{C}(H)$ such that $AA^-A = A$, $A^-AA^- = A^-$, $(A^-A)^* = \overline{A^-A}$, $(AA^-)^* = \overline{AA^-}$, $\mathcal{D}_{A^-} = \mathrm{Im} A \oplus \mathrm{Ker} A^*$. Operator $\overline{AA^-}$ is the orthogonal projection onto the subspace $\overline{\mathrm{Im} A} = (\mathrm{Ker} A^*)^\perp$, $\mathcal{D}_{AA^-} = \mathcal{D}_{A^-}$, $\mathrm{Im} A = \{x \in \mathcal{D}_{A^-} : AA^-x = x\}$, and $\overline{A^-A}$ is the orthogonal projection onto the subspace $\overline{\mathrm{Im} A^-} = (\mathrm{Ker} A)^\perp = \overline{\mathrm{Im} A^*}$, $\mathcal{D}_{A^-A} = \mathcal{D}_A$, $\mathrm{Im} A^- = \{y \in \mathcal{D}_A : A^-Ay = y\}$. Also, we have $\mathrm{Ker} A^- = \mathrm{Ker} A^*$ and $\mathcal{D}_A = \mathrm{Im} A^- \oplus \mathrm{Ker} A$.

From the topological point of view a Hilbert $\mathbf{K}$-module W is a Banach space with the norm which is not strictly convex in general. Nevertheless, we show that the notion of generalized inverse for operator in $\mathcal{C}_{\mathbf{K}}(W)$ is well founded.

DEFINITION 3.2. Let W be a Hilbert C^*-module over the C^*-algebra $\mathbf{K}$ and $A \in \mathcal{C}_{\mathbf{K}}(W)$. An operator $A^- \in \mathcal{C}_{\mathbf{K}}(W)$ is called a *generalized inverse* of A if $\mathcal{D}_{A^-} = \mathrm{Im} A \oplus \mathrm{Ker} A^*$, $AA^-A = A$, $A^-AA^- = A^-$, $(A^-A)^* = \overline{A^-A}$ and $(AA^-)^* = \overline{AA^-}$.

In Hilbert H^*-modules, which are Hilbert spaces as well, the existence of a generalized inverse of a closed densely defined modular operator is not in question. It is also easy to verify its modularity. The same conclusion in a Hilbert C^*-module over the C^*-algebra $\mathbf{K}$ is not obvious.

THEOREM 3.3. *If $A \in \mathcal{C}_{\mathbf{K}}(W)$ then there exist the unique generalized inverse $A^- \in \mathcal{C}_{\mathbf{K}}(W)$ and $\Psi(A^-) = \Psi(A)^-$ with Ψ as in Theorem* 2.4. *Further,* $\mathrm{Ker} A^- = \mathrm{Ker} A^*$ *and* $\mathcal{D}_A = \mathrm{Im} A^- \oplus \mathrm{Ker} A$ *hold true.*

Proof. If $A \in \mathcal{C}_{\mathbf{K}}(W)$ then, by Theorem 2.4, we have a uniquely defined $\Psi(A) \in \mathcal{C}(W_e)$. By the Hilbert space theory there exists a generalized inverse $\Psi(A)^- \in \mathcal{C}(W_e)$ and $\mathcal{D}_{\Psi(A)^-} = \mathrm{Im}\Psi(A) \oplus \mathrm{Ker}\Psi(A)^*$. Let $B \in \mathcal{C}_{\mathbf{K}}(W)$ be the unique operator such that $B = \Psi^{-1}(\Psi(A)^-)$. We claim that B is the unique operator in $\mathcal{C}_{\mathbf{K}}(W)$ which satisfies the conditions in Definition 3.2.

First we prove $\mathcal{D}_B = \mathrm{Im}\, A \oplus \mathrm{Ker}A^*$ and $\mathcal{D}_A = \mathrm{Im}\, B \oplus \mathrm{Ker}A$. By Lemma 2.1, we have $\mathcal{D}_B = \overline{\mathrm{span}\{\mathbf{K}\mathcal{D}_{\Psi(A)}\}}^B$ and $\mathrm{Ker}B = \overline{\mathrm{span}\{\mathbf{K}\mathrm{Ker}\Psi(A)^-\}}^B = \overline{\mathrm{span}\{\mathbf{K}\mathrm{Ker}\Psi(A)^*\}}$ $= \mathrm{Ker}A^*$. Subspaces $\mathrm{Im}\Psi(A)$ and $\mathrm{Ker}\Psi(A)^* = \mathrm{Ker}\Psi(A)^-$ are orthogonal subspaces in the scalar product $(\cdot|\cdot)$ as well as in $(\cdot|\cdot)_{\Psi(A)^-}$. Hence $\mathrm{span}\{\mathbf{K}\mathrm{Im}\Psi(A)\}$ and $\mathrm{Ker}A^* = \mathrm{Ker}B$ are orthogonal submodules in the inner products $\langle\cdot|\cdot\rangle$ and $\langle\cdot|\cdot\rangle_B$. Then $\mathcal{D}_{\Psi(A)^-} = \mathrm{Im}\Psi(A) \oplus \mathrm{Ker}\Psi(A)^-$ by Proposition 1.2 and Lemma 2.1 implies $\mathcal{D}_B = \overline{\mathrm{span}\{\mathbf{K}\mathrm{Im}\Psi(A)\}}^B \oplus \mathrm{Ker}B$. We have to prove $\mathrm{Im}A = \overline{\mathrm{span}\{\mathbf{K}\mathrm{Im}\Psi(A)\}}^B$. Let $(u_n)_{n\in I}$ be an orthonormal basis in the Hilbert space $(\mathrm{Im}\,\Psi(A), (\cdot|\cdot)_{\Psi(A)^-})$ (it is also an orthonormal basis for the Hilbert C^*-module $(\overline{\mathrm{span}\{\mathbf{K}\mathrm{Im}\,\Psi(A)\}}^B$, $\langle\cdot|\cdot\rangle_B)$). For $x \in \overline{\mathrm{span}\{\mathbf{K}\mathrm{Im}\Psi(A)\}}^B \subseteq \mathcal{D}_B$ we have $x = \sum_{n\in I} \langle x|u_n\rangle_B u_n$ and $Bx = \sum_{n\in I} \langle x|u_n\rangle_B \Psi(A)^- u_n$. We define nets $(x_J)_{J\subset I}$, $x_J = \sum_{j\in J} \langle x|u_j\rangle_B u_j$, and $(y_J)_{J\subset I}, y_J = Bx_J = \sum_{j\in J} \langle x|u_j\rangle_B \Psi(A)^- u_j$, where J is a finite subset of I. We have $Ay_J = \sum_{j\in J} \langle x|u_j\rangle_B$ $\Psi(A)\Psi(A)^- u_j = \sum_{j\in J} \langle x|u_j\rangle_B u_j = x_j$ because $\Psi(A)\Psi(A)^- u_j = u_j$, $\forall j \in I$. Hence $y_j \to y = Bx$, $Ay_j \to x$ and $A \in \mathcal{C}_{\mathbf{K}}(W)$ imply $Bx \in \mathcal{D}_A$ and $x = Ay \in \mathrm{Im}A$, i.e. $\mathrm{Im}B \subseteq \mathcal{D}_A$ and $\overline{\mathrm{span}\{\mathbf{K}\mathrm{Im}\Psi(A)\}}^B \subseteq \mathrm{Im}A$. Analogously, exchanging the roles of A and B, we have $\mathrm{Im}A \subseteq \mathcal{D}_B$ and $\overline{\mathrm{span}\{\mathbf{K}\mathrm{Im}\Psi(A)^-\}}^A \subseteq \mathrm{Im}B$. Now, $\mathrm{Im}A \subseteq (\,\mathrm{Ker}\, A^*)^{\perp_B} = \overline{\mathrm{span}\{\mathbf{K}\mathrm{Im}\Psi(A)\}}^B$ and $\mathrm{Im}B \subseteq (\mathrm{Ker}A)^{\perp_A} = \overline{\mathrm{span}\{\mathbf{K}\mathrm{Im}\Psi(A)^-\}}^A$ give the equalities.

Previously proved facts on domains ensure that BA, ABA and AB, BAB are well defined operators on $\mathcal{D}_A$ and $\mathcal{D}_B$, respectively. The equalities $ABA = A$ and $BAB = B$ follow by a direct calculation as above. Namely, the equation $BAB = B$ is trivial on $\mathrm{Ker}B$ and for $x \in \mathrm{Im}A$ we have $x = ABx$ as above. $ABA = A$ is proved analogously. Since by Corollary 2.5 $\Psi(\overline{BA}) = \overline{\Psi(A)^-\Psi(A)} = (\Psi(A)^-\Psi(A))^*$ $= (\overline{\Psi(A)^-\Psi(A)})^* = \Psi(\overline{BA})^* = \Psi((\overline{BA})^*) = \Psi((BA)^*)$ and $(BA)^* \in \mathcal{C}_{\mathbf{K}}(W)$, we have $\overline{BA} = (BA)^*$. $\overline{AB} = (AB)^*$ can be seen in the same way.

The uniqueness of a generalized inverse follows from the bijectivity of Ψ and the fact that any closed operator C with the domain $\mathrm{Im}A \oplus \mathrm{Ker}A^*$ satisfying $ACA = A$, $CAC = C$, $(CA)^* = \overline{CA}$ and $(AC)^* = \overline{AC}$ must satisfy $\Psi(C) = \Psi(A)^-$ (a generalized inverse of a closed operator on a Hilbert space is unique), hence $B = A^-$. Finally, $\Psi(A^-) = \Psi(A)^-$ follows from the construction of B. ∎

Let us briefly discuss the meaning of the existence of the generalized inverse for the solution of the linear equation $Ax = b$ with $A \in \mathcal{C}_{\mathbf{K}}(W)$ and $b \in W$. In

Hilbert H^*-modules the norm is strictly convex and A^-b is the unique solution of the problem $\inf\{\|Ax-b\| : x \in \mathcal{D}_A\}$ with the minimal norm. In Hilbert C^*-modules over C^*-algebras of compact operators the norm is generally not strictly convex. Let $\mathcal{P}_{\rm m}$ denotes the set of all minimal projections in $\mathbf{K}$. From the property of generalized inverses in Hilbert spaces we have $\|AA^-eb - eb\| \leqslant \|Aex - eb\|$ for all $x \in W$ and $e \in \mathcal{P}_{\rm m}$. By the formula $\|x\| = \sup\{\|ex\| : e \in \mathcal{P}_{\rm m}\}$ (see Proposition 2 in [2]) we conclude $\|AA^-b - b\| \leqslant \|Ax - b\|$ for all $x \in W$. Also, for any other solution y we have $\|A^-eb\| \leqslant \|ey\|$ for all $e \in \mathcal{P}_{\rm m}$, hence $\|A^-b\| \leqslant \|y\|$. The uniqueness of such solution fails.

3.3. CLOSED OPERATORS WITH RESOLVENT IN $\mathbf{K}_{\mathbf{K}}(W)$. In the Hilbert space theory the operators with compact resolvent are interesting because of their simple spectral structure. This heavily depends on topological properties of their compact resolvents. Although operators in $\mathbf{K}_{\mathbf{K}}(W)$ are not topologically compact, we are able to prove analogous results for modular operators.

For an operator $A \in \mathcal{C}_{\mathbf{K}}(W)$ the resolvent set $\rho(A)$ and the spectrum $\sigma(A)$ are defined assuming that $A \in \mathcal{C}(W)$ with W regarded as a Banach space.

LEMMA 3.4. *For $A \in \mathcal{C}_{\mathbf{K}}(W)$ we have $\sigma(A) = \sigma(\Psi(A))$. Moreover, the same equality holds for the point, the continuous and the residual spectrum, respectively.*

Proof. For any injective modular operator $A : \mathcal{D}_A \to \operatorname{Im} A$ and its inverse $A^{-1} : \operatorname{Im}A \to \mathcal{D}_A$ (with possibly non dense domains) we have $G(A^{-1}) = G'(A)$. Assuming $\Psi(A) = A|_{e\mathcal{D}_A}$, we have $G(\Psi(A^{-1})) = eG(A^{-1}) = eG'(A) = G'(\Psi(A)) = G(\Psi(A)^{-1})$.

If $A \in \mathcal{C}_{\mathbf{K}}(W)$ then $B = \xi I - A \in \mathcal{C}_{\mathbf{K}}(W)$ for all $\xi \in \mathbb{C}$. If $\xi \in \rho(A)$ then B is a bijection onto W and $B^{-1} \in \mathbf{B}_{\mathbf{K}}(W)$. Clearly, $\Psi(B)$ is a bijection onto W_e and $(\xi I - \Psi(A))^{-1} = \Psi(B^{-1}) = \Psi(B)^{-1} \in \mathbf{B}(W_e)$, hence $\xi \in \rho(\Psi(A))$. Suppose that $\xi \in \rho(\Psi(A))$, i.e. $\Psi(B)$ is a bijection onto W_e and $\Psi(B)^{-1} \in \mathbf{B}(W_e)$. From [2] we know that $\Psi : \mathbf{B}_{\mathbf{K}}(W) \to \mathbf{B}(W_e)$ is a $*$-isomorphism, so we have the unique $C = \Psi^{-1}(\Psi(B)^{-1}) \in \mathbf{B}_{\mathbf{K}}(W) \subseteq \mathcal{C}_{\mathbf{K}}(W)$. By Theorem 3.3, $C = B^-$ and, by Lemma 2.1, $\operatorname{Ker}B = \overline{\operatorname{span}\{\mathbf{K}\operatorname{Ker}\Psi(B)\}} = \{0\}$, hence $C = B^{-1}$. This implies $\xi \in \rho(A)$.

The above connection between kernels implies that B is injective if and only if $\Psi(B)$ is injective, hence $\sigma_p(A) = \sigma_p(\Psi(A))$ follows immediately.

The analog connection between kernels of B^* and $\Psi(B)^*$ implies that $\operatorname{Im}B$ is dense if and only if $\operatorname{Im}\Psi(B)$ is dense. ∎

For $A \in \mathcal{C}_{\mathbf{K}}(W)$, by the multiplicity of an eigenvalue $\alpha \in \sigma_p(A)$, we understand the $\mathbf{K}$-dimension of the closed submodule spanned by all eigenvectors $x \in W$ such that $Ax = \alpha x$.

LEMMA 3.5. *Let $A \in \mathcal{C}_{\mathbf{K}}(W)$ and $\alpha \in \sigma_p(A)$. Then the multiplicities of $\alpha \in \sigma_p(A)$ and $\alpha \in \sigma_p(\Psi(A))$ are equal.*

Proof. Let us denote by W_α the submodule in W spanned by all x such that $Ax=\alpha x$ and by W'_α the subspace in $\overline{\mathcal{D}_{\Psi(A)}}$ spanned by all u such that $\Psi(A)u=\alpha u$.

For any minimal projection $e\in\mathbf{K}$ we have $eW_\alpha = W'_\alpha$. Namely, $W'_\alpha\subseteq W_\alpha$ and $W'_\alpha\subseteq W_e$ imply $W'_\alpha = eW'_\alpha\subseteq eW_\alpha$. On the other hand, $eW_\alpha\subseteq W_e$ implies $eW_\alpha\subseteq W'_\alpha$ by the definition of W'_α. Now, using any basis in W'_α, we have $W_\alpha = \overline{\operatorname{span}\{\mathbf{K}W'_\alpha\}}$ i.e. $\mathbf{K}\text{-}\dim W_\alpha = \dim W'_\alpha$. ∎

For the set of generalized compact operators $\mathbf{K}_{\mathbf{K}}(W)\subseteq\mathbf{B}_{\mathbf{K}}(W)$ we have $\Psi(\mathbf{K}_{\mathbf{K}}(W))=\mathbf{K}(W_e)$ (see Theorem 6 in [2]). The next result is a generalization of Theorem III.6.29. in [3].

THEOREM 3.6. *Let $A\in\mathcal{C}_{\mathbf{K}}(W)$ and assume that there exists $\xi_0\in\rho(A)$ such that $R(\xi_0)=(\xi_0 I-A)^{-1}\in\mathbf{K}_{\mathbf{K}}(W)$. Then the spectrum of A consists entirely of isolated eigenvalues with finite multiplicity, and $R(\xi)=(\xi I-A)^{-1}\in\mathbf{K}_{\mathbf{K}}(W)$ for all $\xi\in\rho(A)$.*

Proof. By the properties of function Ψ, the operator $\Psi(A)$ is a closed densely defined operator on a Hilbert space W_e with compact resolvent $\Psi(R(\xi_0))=(\xi_0 I-\Psi(A))^{-1}$ for some $\xi_0\in\mathbb{C}$. From the Hilbert space theory the statement of the theorem holds true for $\Psi(A)$. From $\sigma(A)=\sigma(\Psi(A))=\sigma_p(\Psi(A))=\sigma_p(A)$ we conclude that $\sigma(A)$ consists only of isolated eigenvalues and the associated subspaces for $\Psi(A)$ are finite dimensional. The statement on multiplicities follows from Lemma 3.5. ∎

3.4. GENERALIZED RELATIVE COMPACTNESS IN $\mathcal{C}_{\mathbf{K}}(W)$. Relative bounded and relative compact operators are of interest in the theory of differential operators on Hilbert spaces (see Chapter IV of [3]). We give some related results for operators on Hilbert C^*-modules over $\mathbf{K}$.

DEFINITION 3.7. Let $T\in\mathcal{C}_{\mathbf{K}}(W)$. An operator $A\in\mathcal{L}_{\mathbf{K}}(W)$ with $\mathcal{D}_A\supseteq\mathcal{D}_T$ is said to be *T-bounded* if $\widetilde{A}=A|_{\mathcal{D}_T}$ is a bounded operator from the Hilbert C^*-module $(\mathcal{D}_T,\langle\cdot|\cdot\rangle_T)$ into W ($\|Ax\|\leqslant\|\widetilde{A}\|_T\|x\|_T$ for all $x\in\mathcal{D}_T$).

A is said to be a generalized *T-compact* operator if $\widetilde{A}\in\mathbf{K}_{\mathbf{K}}(\mathcal{D}_T,W)$.

If $T\in\mathcal{C}_{\mathbf{K}}(W)$ and $A\in\mathcal{C}\ell_{\mathbf{K}}(W)$ with $\mathcal{D}_A\supseteq\mathcal{D}_T$ than A is T-bounded (see Remark IV.1.5 of [3]).

LEMMA 3.8. *Let $A\in\mathcal{L}_{\mathbf{K}}(W)$ and $T\in\mathcal{C}_{\mathbf{K}}(W)$ with $\mathcal{D}_A\supseteq\mathcal{D}_T$. Then A is a generalized T-compact operator if and only if $\Psi(A)$ is $\Psi(T)$-compact.*

Proof. If $A\in\mathcal{L}_{\mathbf{K}}(W)$ is T-compact, by Corollary 2.6, $\Psi(T)$ is a closed operator and $\mathcal{D}_{\Psi(A)}\supseteq\mathcal{D}_{\Psi(T)}$. Let $\Psi_T:\mathbf{B}_{\mathbf{K}}(\mathcal{D}_T,W)\to\mathbf{B}(e\mathcal{D}_T,W_e)$ be a bijection assuming that $(\mathcal{D}_T,\langle\cdot|\cdot\rangle_T)$ is a Hilbert C^*-module over $\mathbf{K}$ and $(e\mathcal{D}_T,(\cdot|\cdot)_{\Psi(T)})$ is the associated Hilbert space. Applying $\Psi_T(\mathbf{K}_{\mathbf{K}}(\mathcal{D}_T,W))=\mathbf{K}(e\mathcal{D}_T,W_e)$ (see [2]), we conclude that $\Psi_T(\widetilde{A})$ is compact. Since $\Psi_T(\widetilde{A})=\Psi(A)|_{\mathcal{D}_{\Psi(T)}}$, we have that $\Psi(A)$ is $\Psi(T)$-compact.

Conversely, let us suppose that $\Psi(A)$ is $\Psi(T)$-compact. Then $\Psi(\widetilde{A}) = \Psi(A)\widetilde{}$ is a compact operator on a Hilbert space $(\mathcal{D}_{\Psi(T)}, (\cdot|\cdot)_{\Psi(T)})$. Thus $\widetilde{A} = \Psi_T^{-1}(\Psi(A)\widetilde{})$ is a generalized compact operator on a Hilbert C^*-module $(\mathcal{D}_T, \langle\cdot|\cdot\rangle_T)$ and hence A is generalized T-compact. ∎

LEMMA 3.9. *Let $A \in \mathcal{L}_{\mathbf{K}}(W)$ and $T \in \mathcal{C}_{\mathbf{K}}(W)$ with $\mathcal{D}_A \supseteq \mathcal{D}_T$. If A is generalized T-compact operator then A is T-bounded.*

Proof. If A is generalized T-compact then $\widetilde{A} \in \mathbf{K}_{\mathbf{K}}(\mathcal{D}_T, W)$. By Lemma 3.8, this is true if and only if $\Psi(A)$ is $\Psi(T)$-compact, i.e. $\Psi_T(A)$ is compact on a Hilbert space $(\mathcal{D}_{\Psi(T)}, \langle\cdot|\cdot\rangle_{\Psi(T)})$. Hence, $\Psi_T(A)$ is bounded. Therefore, $A|_{\mathcal{D}_T}$ is bounded in $(\mathcal{D}_T, \langle\cdot|\cdot\rangle_T)$ because Ψ_T^{-1} preserves boundedness. Hence, by definition, A is T-bounded. ∎

THEOREM 3.10. *Let $A \in \mathcal{L}_{\mathbf{K}}(W)$ and $T \in \mathcal{C}_{\mathbf{K}}(W)$ with $\mathcal{D}_A \supseteq \mathcal{D}_T$. If A is a generalized T-compact operator then $S = T + \widetilde{A} \in \mathcal{C}_{\mathbf{K}}(W)$, $\mathcal{D}_S = \mathcal{D}_T$, and A is a generalized S-compact operator.*

Proof. In the case $\mathbf{K} = \mathbb{C}$ the statement follows from Theorem IV.1.11 in [3]. By Corollary 2.6, we have $\Psi(A) \in \mathcal{C}\ell(W_e)$ and $\Psi(T) \in \mathcal{C}(W_e)$ with $\mathcal{D}_{\Psi(A)} \supseteq \mathcal{D}_{\Psi(T)}$ and, by Lemma 3.8, $\Psi(A)$ is $\Psi(T)$-compact. Then $\widehat{S} = \Psi(T) + \Psi(\widetilde{A}) \in \mathcal{C}(W_e)$, $\mathcal{D}_{\widehat{S}} = \mathcal{D}_{\Psi(T)}$ and $\Psi(A)$ is generalized $\widehat{S}$-compact. We know that $\Psi(S) = \widehat{S}$ and by Lemma 2.3 this ensures $S \in \mathcal{C}\ell_{\mathbf{K}}(W)$. If we can prove that S is closed then, using Lemma 3.8, we can conclude that A is generalized S-compact. For this purpose we prove that norms $\|\cdot\|_T$ and $\|\cdot\|_S$ are equivalent on $\mathcal{D}_T = \mathcal{D}_S$, hence $(\mathcal{D}_S, \langle\cdot|\cdot\rangle_S)$ is a Hilbert C^*-module and $S \in \mathcal{C}_{\mathbf{K}}(W)$.

For any $x \in \mathcal{D}_T$, by the left hand side of (1.1) in Lemma 2.1, we have $\|x\|_S^2 = \|\langle x|x\rangle_S\| \leqslant \|\langle x|x\rangle_T\| + 2\|Tx\|\|\widetilde{A}x\| + \|\widetilde{A}x\|^2 \leqslant \|x\|_T^2 + 2\|x\|_T\|\widetilde{A}x\| + \|\widetilde{A}x\|^2 \leqslant (1 + \|\widetilde{A}\|_T)^2\|x\|_T^2$, i.e. $\|x\|_S \leqslant K_1\|x\|_T$ with $K_1 = 1 + \|\widetilde{A}\|_T$.

For the opposite inequality let us observe that $\Psi(\widetilde{A})$ is $\Psi(\overline{S})$-bounded, hence by Corollary 2.10 $\widetilde{A}$ is $\overline{S}$-bounded. Analogously as above, for $T = S - \widetilde{A}$ we have $\|x\|_T \leqslant K_2\|x\|_S$ with $K_2 = 1 + \|\widetilde{A}\|_{\overline{S}}$. ∎

Apparently, it is possible to transfer in a similar way many other concepts and results from the Hilbert space theory.

REFERENCES

[1] D. BAKIĆ, B. GULJAŠ, Operators on Hilbert H^*-modules, *J. Operator Theory* **46**(2001), 123–141.

[2] D. BAKIĆ, B. GULJAŠ, Hilbert C^*-modules over C^*-algebras of compact operators, *Acta Sci. Math. (Szeged)* **68**(2002), 249–269.

[3] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin-Heilderberg-New York 1966.

[4] E.C. LANCE, *Hilbert C^*-Modules. A Toolkit for Operator Algebraists*, London Math. Soc. Lecture Notes Ser., vol. 210, Cambridge Univ. Press, Cambridge 1995.

[5] B. MAGAJNA, Hilbert C^*-modules in which all closed submodules are complemented, *Proc. Amer. Math. Soc.* **125**(1997), 849–852.

[6] A. PAL, Regular operators on Hilbert C^*-modules, *J. Operator Theory* **42**(1999), 331–350.

[7] Y.P. PYT′EV, A pseudoinverse operator. Properties and applications [Russian], *Mat. Sb. (NS)* **118**(1982), 19–49.

[8] J. SCHWEIZER, A description of Hilbert C^*-modules in which all closed submodules are orthogonally closed, *Proc. Amer. Math. Soc.* **127**(1999), 2123–2125.

BORIS GULJAŠ, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ZAGREB, BIJENIČKA C. 30, 10000 ZAGREB, CROATIA
E-mail address: guljas@math.hr

Received November 21, 2005.