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SPECTRAL PICTURES OF OPERATOR-VALUED WEIGHTED BI-SHIFTS

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Dedicated to Salaheddine Bourhim, our little bundle of joy, who came into our lives on August 31st 2006

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ABSTRACT. In this paper, we introduce operator-valued weighted bi-shifts on the Hilbert space $l^2(\mathbb{N}, \mathcal{H})$, of all square-summable sequences whose elements are in a complex Hilbert space $\mathcal{H}$, and study their spectral and local spectral properties. We determine the spectrum and its parts of such bi-shifts, and compute their local spectrum at most points of $l^2(\mathbb{N}, \mathcal{H})$. Furthermore, we provide necessary and sufficient conditions for an operator-valued weighted bi-shift to enjoy the single-valued extension property.

KEYWORDS: *Weighted shift, weighted bi-shift, spectrum, point spectrum, approximate point spectrum, local spectrum, the single-valued extension property.*

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INTRODUCTION

Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$, and let $\mathbb{K}$ stand either for the set of all integers $\mathbb{Z}$ or the set of all nonnegative integers $\mathbb{N}$. Let $l^2(\mathbb{K}, \mathcal{H})$ be the usual Hilbert space of all square-summable sequences $(x_n)_{n\in\mathbb{K}}$ whose elements are in $\mathcal{H}$. The corresponding inner product is defined by

$$\langle (x_n)_{n\in\mathbb{K}}; (y_n)_{n\in\mathbb{K}} \rangle := \sum_{n\in\mathbb{K}} \langle x_n; y_n \rangle, \quad ((x_n)_{n\in\mathbb{K}},\ (y_n)_{n\in\mathbb{K}} \in l^2(\mathbb{K}, \mathcal{H})).$$

Let $\mathcal{A} := (A_n)_{n\geqslant 0}$ be a sequence of uniformly bounded invertible operators of $\mathcal{L}(\mathcal{H})$. The corresponding *unilateral operator-weighted shift* is defined by

$$S_{\mathcal{A}}\, x := (0, A_0x_0, A_1x_1, A_2x_2, \ldots), \quad x = (x_n)_{n\geqslant 0} \in l^2(\mathbb{N}, \mathcal{H}),$$

and its adjoint is given by

$$S_{\mathcal{A}}{}^*\, x := (A_0^*x_1, A_1^*x_2, A_2^*x_3, \ldots), \quad x = (x_n)_{n\geqslant 0} \in l^2(\mathbb{N}, \mathcal{H}).$$

Throughout this paper, we assume in addition that $\mathcal{A}^{-1} := (A_n^{-1})_{n\geqslant 0}$ is also a uniformly bounded sequence. The *operator-weighted bi-shift*, $B_{\mathcal{A}}$, with the weight sequence $\mathcal{A} = (A_n)_{n\geqslant 0}$ is the operator on $l^2(\mathbb{N}, \mathcal{H})$ defined by

$$B_{\mathcal{A}} := S_{\mathcal{A}} + S^*_{\mathcal{A}^{*-1}}.$$

This operator has the following tridiagonal operator matrix representation

$$\begin{bmatrix} 0 & {A_0}^{-1} & 0 & 0 & \cdots & \cdots \\ A_0 & 0 & {A_1}^{-1} & 0 & \cdots & \cdots \\ 0 & A_1 & 0 & {A_2}^{-1} & 0 & \cdots \\ 0 & 0 & A_2 & 0 & {A_3}^{-1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

Clearly, the adjoint of $B_{\mathcal{A}}$ is nothing but the weighted bi-shift with the weight sequence $\mathcal{A}^{*-1} = ({A_n}^{*-1})_{n\geqslant 0}$, and therefore has a similar tridiagonal operator matrix representation.

Note that when $\mathcal{H} = \mathbb{C}$, since in this case each A_n is a multiplication operator on $\mathbb{C}$ by a nonzero scalar $\alpha_n \in \mathbb{C}$, these bi-shifts are exactly the scalar weighted bi-shifts which have been introduced and studied, in [8], by A. Atzmon and M. Sodin. They gave the complete description of their spectrum and point spectrum, and considered these bi-shifts mainly in the case where each α_n is positive and $\lim_{n\to+\infty} \alpha_n = 1$. They showed that if the weight sequence $(\alpha_n)_{n\geqslant 0}$ satisfies certain additional growth conditions, then the spectrum of every part of either the corresponding bi-shift B_α or its adjoint B^*_α equals the whole interval $[-2, 2]$; a part of an operator is its restriction to a closed invariant subspace. In this case, such bi-shifts are said to be completely indecomposable. The question whether any scalar weighted bi-shift has a proper closed invariant subspace was affirmatively answered later by A. Atzmon in [7].

In this paper, we study the spectral and local spectral pictures of operator-valued weighted bi-shifts. In Section 2, we use elementary methods to determine the spectrum and its parts of an operator-valued weighted bi-shift. In Section 3, we characterize those bi-shifts who enjoy the single-valued extension property. While in Section 4, we aim at determining the local spectra of operator-valued weighted bi-shifts.

1. PREMIMINARIES

In this section we gather together some usual notations and basic facts from spectral and local spectral theory. We shall also collect some basic facts about the spectra of the unilateral operator-weighted shift $S_{\mathcal{A}}$. Our references are the books of Aiena [1] and of Laursen and Neumann [21], and the papers [12] and [20].

1.1. BASIC FACTS FROM SPECTRAL THEORY. For an operator $T \in \mathcal{L}(\mathcal{H})$, we denote as usual by $r(T)$ its spectral radius given by

$$r(T) := \lim_{n\to+\infty} \|T^n\|^{1/n} = \max\{|\lambda| : \lambda \in \sigma(T)\},$$

where $\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not invertible}\}$ is the spectrum of T. We also denote by $m(T) := \inf\{\|Tx\| : \|x\| = 1\}$ the lower bound of T, and recall that the sequence $(m(T^n)^{1/n})_{n\geqslant 1}$ converges and its limit, denoted by $r_1(T)$, coincides with its supremum and satisfies

$$r_1(T) := \lim_{n\to+\infty} m(T^n)^{1/n} = \min\{|\lambda| : \lambda \in \sigma_{\text{ap}}(T)\},$$

where $\sigma_{\text{ap}}(T) := \{\lambda \in \mathbb{C} : m(T - \lambda) = 0\}$ is the approximate point spectrum of T; see [22]. The point spectrum of the operator T is defined by $\sigma_{\text{p}}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not injective}\}$, while the surjectivity spectrum of T is defined by $\sigma_{\text{su}}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not surjective}\}$. Both the approximate point spectrum and the surjectivity spectrum of T are nonempty closed subsets of $\sigma(T)$, and are dual to each other, in the sense that $\sigma_{\text{ap}}(T) = \overline{\sigma_{\text{su}}(T^*)}$ and $\sigma_{\text{su}}(T) = \overline{\sigma_{\text{ap}}(T^*)}$; see Theorem 2.42 of [1]. Here, T^* denotes the adjoint of T.

The following result summarizes some well known facts about the spectrum and its parts.

PROPOSITION 1.1. *For an operator $T \in \mathcal{L}(\mathcal{H})$, the following assertions hold:*

(i) $\sigma(T) = \sigma_{\text{su}}(T) \cup \sigma_{\text{p}}(T) = \sigma_{\text{ap}}(T) \cup \overline{\sigma_{\text{p}}(T^*)}$.

(ii) $\partial\sigma(T) \subset \sigma_{\text{ap}}(T) \cap \sigma_{\text{su}}(T) \subset \sigma(T)$.

(iii) *The set $\sigma(T)\backslash\sigma_{\text{ap}}(T)$ is open, and $\sigma(T)\backslash\sigma_{\text{ap}}(T) \subset \operatorname{int}(\overline{\sigma_{\text{p}}(T^*)})$.*

(iv) *The set $\sigma(T)\backslash\sigma_{\text{su}}(T)$ is also open, and $\sigma(T)\backslash\sigma_{\text{su}}(T) \subset \operatorname{int}(\sigma_{\text{p}}(T))$.*

(v) *If T is invertible, then $m(T) = 1/\|T^{-1}\|$.*

(iv) $m(T_1)m(T_2) \leqslant m(T_1T_2) \leqslant \|T_1\|m(T_2)$ *for all $T_1, T_2 \in \mathcal{L}(\mathcal{H})$.*

1.2. BASIC FACTS FROM LOCAL SPECTRAL THEORY. The single-valued extension property was first introduced by N. Dunford [16], [17] and has, successively, received a more systematic treatment in Dunford-Schwartz [18]. It plays an important role in local spectral theory; see the monograph of Laursen and Neumann [21].

DEFINITION 1.2. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single-valued extension property* provided that for every open set $O \subset \mathbb{C}$ there exists no nonzero analytic $\mathcal{H}$-valued function ϕ on O such that

$$(T - \lambda)\phi(\lambda) = 0, \quad (\lambda \in O). \tag{1.1}$$

Recently, there has been a flurry of activity regarding the localization of the single-valued extension property in the sense of J.K. Finch [19] which has been widely studied in recent papers [2], [4], [3], [6], [5]. The recent monograph by P. Aiena [1] contains further details.

DEFINITION 1.3. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to enjoy *the single-valued extension property at a point* $\lambda_0 \in \mathbb{C}$ if for every open disc O centered at λ_0, the equation (1.1) has no nontrivial analytic solution ϕ on O.

The set of all points on which T fails to have the single-valued extension property will be denoted by $\Re(T)$. It is an open subset of $\mathbb{C}$ contained in $\mathrm{int}(\sigma_{\mathrm{p}}(T))$, and is empty precisely when T has the single-valued extension property.

In [5], P. Aiena and O. Monsalve showed that operators that do not have the single-valued extension property at a given point in $\mathbb{C}$ may be characterized by means of some typical tools of the local spectral theory. To state their useful characterization, we recall that the *local resolvent set*, $\rho_T(x)$, of an operator $T \in \mathcal{L}(\mathcal{H})$ at point $x \in \mathcal{H}$ is defined to be the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $\phi : U \to \mathcal{H}$ which satisfies $(T - \lambda)\phi(\lambda) = x$, $(\lambda \in U)$. Evidently, it is an open subset of $\mathbb{C}$ which contains $\rho(T)$ the resolvent set of T. The *local spectrum* of T at x is defined by

$$\sigma_T(x) := \mathbb{C} \backslash \rho_T(x),$$

and is a closed subset (possibly empty) of $\sigma(T)$.

We shall make use of this useful characterization quoted from Theorem 1.9 in [5].

THEOREM 1.4. *An operator $T \in \mathcal{L}(\mathcal{H})$ does not have the single-valued extension property at a point $\lambda \in \mathbb{C}$ if and only if there exists a nonzero $x \in \ker(T - \lambda)$ such that $\sigma_T(x) = \emptyset$.*

Finally, we recall that the *local spectral radius* of an operator $T \in \mathcal{L}(\mathcal{H})$ at a vector $x \in \mathcal{H}$ is defined by

$$r_T(x) := \limsup_{n \to +\infty} \|T^n x\|^{1/n},$$

and equals $\max\{|\lambda| : \lambda \in \sigma_T(x)\}$ provided that T has the single-valued extension property; see Theorem 2.21 of [1].

1.3. SPECTRA OF $S_{\mathcal{A}}$. Throughout this paper, we let $(B_n)_{n \geqslant 0}$ be the sequence given by

$$B_n = \begin{cases} A_{n-1}A_{n-2}\cdots A_1 A_0 & \text{if } n > 0, \\ 1 & \text{if } n = 0, \end{cases}$$

and set

$$R(S_{\mathcal{A}}) := \sup\{1/\limsup \|B_n^{*\,-1} x\|^{1/n} : x \in \mathcal{H},\ \|x\| = 1\}.$$

The following known result summarizes the complete description of some parts of the spectrum of a unilateral operator-weighted shift.

LEMMA 1.5. *The following assertions hold:*

(i) $r(S_{\mathcal{A}}) = \lim_{n \to +\infty} [\sup_{k \geqslant 0} \|B_{n+k} B_k^{-1}\|]^{1/n}$.

(ii) $r_1(S_{\mathcal{A}}) = \lim_{n \to +\infty} [\inf_{k \geqslant 0} \{1/\|B_k B_{n+k}^{-1}\|\}]^{1/n}$.

(iii) $\sigma(S_{\mathcal{A}}) = \sigma_{\mathrm{su}}(S_{\mathcal{A}}) = \{\lambda \in \mathbb{C} : |\lambda| \leqslant r(S_{\mathcal{A}})\}$, *and* $\sigma_{\mathrm{p}}(S_{\mathcal{A}}) = \emptyset$.
(iv) $\{\lambda \in \mathbb{C} : |\lambda| < R(S_{\mathcal{A}})\} \subset \sigma_{\mathrm{p}}(S_{\mathcal{A}}{}^*) \subset \{\lambda \in \mathbb{C} : |\lambda| \leqslant R(S_{\mathcal{A}})\}$.
(v) $\Re(S_{\mathcal{A}}{}^*) = \{\lambda \in \mathbb{C} : |\lambda| < R(S_{\mathcal{A}})\}$.

For the proof see [12] and [20].

2. SPECTRA OF $B_{\mathcal{A}}$

In what follows, we denote by φ the function defined on $\mathbb{C}\backslash\{0\}$ by

$$\varphi(\xi) := \xi + \xi^{-1}, \quad \xi \in \mathbb{C}\backslash\{0\}.$$

In the proofs of the main results of this section, we need some lemmas. The first one is quoted from Lemma 3.1 in [8], and is in fact easy to prove.

LEMMA 2.1. *If T_1 and T_2 are operators in $\mathcal{L}(\mathcal{H})$ such that $T_2T_1 = 1$, then*

$$\varphi(\{\rho(T_1) \cap \rho(T_2)\}\backslash\{0\}) \subset \rho(T_1 + T_2).$$

Proof. Since $T_2T_1 = 1$, for all $\xi \in \mathbb{C}\backslash\{0\}$ we have the next relation and the desired statement follows:

$$(\varphi(\xi) - T_1 - T_2) = \xi^{-1}(\xi - T_2)(\xi - T_1). \quad \blacksquare \tag{2.1}$$

The spectrum of $B_{\mathcal{A}}$ is described in term of the following quantity:

$$\delta := \max\{r(S_{\mathcal{A}}), r(S_{\mathcal{A}^{*-1}})\}.$$

LEMMA 2.2. *We always have $\delta \geqslant 1$. Moreover, if $\delta = 1$ then*

$$r(S_{\mathcal{A}}) = r(S_{\mathcal{A}^{*-1}}) = 1.$$

Proof. For every positive integers n, k, we have

$$1 \leqslant \|A_n \cdots A_{n+k}\| \|A_{n+k}^{-1} \cdots A_n^{-1}\| = \|A_n \cdots A_{n+k}\| \|A_n^{*-1} \cdots A_{n+k}^{*-1}\| \leqslant \|S_{\mathcal{A}}{}^n\| \|S_{\mathcal{A}^{*-1}}{}^n\|.$$

From this it follows that $1 \leqslant r(S_{\mathcal{A}})r(S_{\mathcal{A}^{*-1}}) \leqslant \delta^2$, and the proof is therefore complete. $\blacksquare$

2.1. POINT SPECTRUM OF $B_{\mathcal{A}}$.

We describe the point spectrum of the bi-shift $B_{\mathcal{A}}$. For this, we need to fix some notations and make some useful observations.

For every $x \in \mathcal{H}$, we let $x^{(n)} := (\delta_{n,k}x)_{k\in\mathbb{K}}$, $(n \in \mathbb{K})$, where $\delta_{n,k}$ is the usual Kronecker-Delta symbol. Let x be a nonzero fixed element of $\mathcal{H}$, and let B_x be the restriction of $B_{\mathcal{A}}$ on $M_x := \bigvee\{B_{\mathcal{A}}{}^n x^{(0)} : n \geqslant 0\}$. For every $n \geqslant 0$, we put

$$v_n := (B_n x)^{(n)} / \|B_n x\| \quad \text{and} \quad \alpha_n = \|B_{n+1}x\| / \|B_n x\|.$$

We have $B_{\mathcal{A}}x^{(0)} = B_{\mathcal{A}}v_0 = \alpha_0 v_1$, and

$$B_{\mathcal{A}}v_n = \alpha_n v_{n+1} + \alpha_{n-1}{}^{-1} v_{n-1}$$

for all $n \geqslant 1$. From this, we see that $(v_n)_{n\geqslant 0}$ is an orthonormal basis of M_x, and B_x is a scalar weighted bi-shift with the weight sequence $(\alpha_n)_{n\geqslant 0}$ and is cyclic with a cyclic vector $v_0 = x^{(0)}$.

PROPOSITION 2.3. *The point spectrum of* $B_{\mathcal{A}}$ *is given by* $\sigma_{\mathrm{p}}(B_{\mathcal{A}}) = \bigcup_{x\neq 0} \sigma_{\mathrm{p}}(B_x)$. *More precisely, the following assertions hold:*

(i) *If* $R(S_{\mathcal{A}^{*-1}}) \leqslant 1$ *and* $(B_n x)_{n\geqslant 0} \notin l^2(\mathbb{N},\mathcal{H})$ *for all nonzero* $x \in \mathcal{H}$, *then* $\sigma_{\mathrm{p}}(B_{\mathcal{A}}) = \emptyset$.

(ii) *If* $R(S_{\mathcal{A}^{*-1}}) \leqslant 1$ *and* $(B_n x)_{n\geqslant 0} \in l^2(\mathbb{N},\mathcal{H})$ *for some nonzero* $x \in \mathcal{H}$, *then* $(-2,2) \subset \sigma_{\mathrm{p}}(B_{\mathcal{A}}) \subset [-2,2]$.

(iii) *If* $R(S_{\mathcal{A}^{*-1}}) > 1$, *then*

$$\{a + ib \in \mathbb{C} : a^2/c_1^2 + b^2/c_2^2 < 1\} \subset \sigma_{\mathrm{p}}(B_{\mathcal{A}}) \subset \{a + ib \in \mathbb{C} : a^2/c_1^2 + b^2/c_2^2 \leqslant 1\},$$

where $c_1 := R(S_{\mathcal{A}^{*-1}}) + R(S_{\mathcal{A}^{*-1}})^{-1}$ *and* $c_2 := R(S_{\mathcal{A}^{*-1}}) - R(S_{\mathcal{A}^{*-1}})^{-1}$.

Proof. We trivially have $\sigma_{\mathrm{p}}(B_x) \subset \sigma_{\mathrm{p}}(B_{\mathcal{A}})$ for all nonzero element $x \in \mathcal{H}$, and $\bigcup_{x\neq 0} \sigma_{\mathrm{p}}(B_x) \subset \sigma_{\mathrm{p}}(B_{\mathcal{A}})$.

Conversely, suppose that $\lambda \in \mathbb{C}$ is an eigenvalue for $B_{\mathcal{A}}$ and $(x_n)_{n\geqslant 0} \in l^2(\mathbb{N},\mathcal{H})$ is a corresponding eigenvector. We have $({A_0}^{-1}x_1, A_0x_0 + {A_1}^{-1}x_2, A_1x_1 + {A_2}^{-1}x_3, A_2x_2 + {A_3}^{-1}x_4, \ldots) = (\lambda x_n)_{n\geqslant 0}$. Then

$$x_1 = \lambda B_1 x_0, \quad x_2 = (\lambda^2 - 1)B_2 x_0, \quad x_3 = \lambda(\lambda^2 - 2)B_3 x_0, \quad \cdots .$$

In fact, by induction one shows that there is a sequence of polynomials $(p_n)_{n\geqslant 0}$ such that

$$x_n = p_n(\lambda)B_n x_0$$

for all $n \geqslant 0$. This shows that $x_0 \neq 0$, and $(x_n)_{n\geqslant 0} \in M_x$. Thus, $\lambda \in \sigma_{\mathrm{p}}(B_{x_0})$, and $\sigma_{\mathrm{p}}(B_{\mathcal{A}}) \subset \bigcup_{x\neq 0} \sigma_{\mathrm{p}}(B_x)$, as desired.

Note that, since $S^*_{\mathcal{A}^{*-1}} S_{\mathcal{A}} = 1$, we have

$$(\varphi(\lambda) - B_{\mathcal{A}}) = \lambda^{-1}(1 - \lambda S^*_{\mathcal{A}^{*-1}})(1 - \lambda S_{\mathcal{A}}) = \lambda^{-1}(\lambda - S^*_{\mathcal{A}^{*-1}})(\lambda - S_{\mathcal{A}})$$

for all nonzero $\lambda \in \mathbb{C}$. From these identities, we see that if $\varphi(\lambda) \in \sigma_{\mathrm{p}}(B_{\mathcal{A}})$ for some nonzero $\lambda \in \mathbb{C}$, then both λ and λ^{-1} are in $\sigma_{\mathrm{p}}(S^*_{\mathcal{A}^{*-1}})$. Therefore,

$$\sigma_{\mathrm{p}}(B_{\mathcal{A}}) \subset \varphi(\sigma_{\mathrm{p}}(S^*_{\mathcal{A}^{*-1}}) \cap \sigma_{\mathrm{p}}(S^*_{\mathcal{A}^{*-1}})^{-1}). \tag{2.2}$$

(i) Assume that $R(S_{\mathcal{A}^{*-1}}) \leqslant 1$ and that $(B_n x)_{n\geqslant 0} \notin l^2(\mathbb{N},\mathcal{H})$ for all nonzero $x \in \mathcal{H}$. By Lemma 1.5(iv), we have $\sigma_{\mathrm{p}}(S^*_{\mathcal{A}^{*-1}}) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. In view of (2.2), we see that $\sigma_{\mathrm{p}}(B_{\mathcal{A}}) = \emptyset$, and the first statement is established.

(ii) Assume that $R(S_{\mathcal{A}^{*-1}}) \leqslant 1$ and that there is a nonzero $x \in \mathcal{H}$ such that $\sum_{n\geqslant 0} \|B_n x\|^2 < +\infty$. In this case, we in fact have $R(S_{\mathcal{A}^{*-1}}) = 1$, and by Lemma 1.5(iv), we see that $\sigma_{\mathrm{p}}(S^*_{\mathcal{A}^{*-1}}) = \{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$. In view of (2.2), we see that $\sigma_{\mathrm{p}}(B_{\mathcal{A}})$ is contained in $[-2,2]$.

Since $\sum_{n\geqslant 0} \|B_n x\|^2 < +\infty$, the function $k_x(\theta) := (B_n x \mathrm{e}^{-\mathrm{i}(n+1)\theta})_{n\geqslant 0}$ is well defined for all $\theta \in [0, 2\pi]$. Moreover, for every $\theta \in [0, 2\pi]$, we have

$$(2\cos\theta - B_{\mathcal{A}})k_x(\theta) = (\mathrm{e}^{-\mathrm{i}\theta} - S^*_{\mathcal{A}^{*-1}})k_x(\theta) + (\mathrm{e}^{\mathrm{i}\theta} - S_{\mathcal{A}})k_x(\theta) = 0 + x^{(0)}.$$

Changing θ by $-\theta$, one also gets that $(2\cos\theta - B_{\mathcal{A}})k_x(-\theta) = x^{(0)}$ for all $\theta \in [0, 2\pi]$. Thus, for every $\theta \in (0, 2\pi)$, we have

$$(2\cos\theta - B_{\mathcal{A}})(\sin((n+1)\theta)B_n x)_{n\geqslant 0} = (2\cos\theta - B_{\mathcal{A}})(k_x(-\theta) - k_x(\theta)) = 0.$$

This shows that $(-2, 2) \subset \sigma_{\mathrm{p}}(B_{\mathcal{A}})$, as desired.

(iii) Suppose that $R(S_{\mathcal{A}^{*-1}}) > 1$. By (2.2) and Lemma 1.5(iii), we have $\sigma_{\mathrm{p}}(B_{\mathcal{A}}) \subset \{a + \mathrm{i}b \in \mathbb{C} : a^2/c_1^2 + b^2/c_2^2 \leqslant 1\}$. For a nonzero element $x \in \mathcal{H}$ with $r_{S_{\mathcal{A}}}(x^{(0)}) < 1$, we set $O_x := \{\lambda \in \mathbb{C} : r_{S_{\mathcal{A}}}(x^{(0)}) < |\lambda| < r_{S_{\mathcal{A}}}(x^{(0)})^{-1}\}$, and let $k_x(\lambda) := (B_n x/\lambda^{n+1})_{n\geqslant 0}$, $(\lambda \in O_x)$. Just as before, one can verify that $(\varphi(\lambda) - B_{\mathcal{A}})(k_x(\lambda) - k_x(\lambda^{-1})) = 0$ for all $\lambda \in O_x$, and $\varphi(O_x) \subset \sigma_{\mathrm{p}}(B_{\mathcal{A}})$. Since x is an arbitrary nonzero element in $\mathcal{H}$ for which $r_{S_{\mathcal{A}}}(x^{(0)}) < 1$, we infer that $\{a + \mathrm{i}b \in \mathbb{C} : a^2/c_1^2 + b^2/c_2^2 < 1\} \subset \sigma_{\mathrm{p}}(B_{\mathcal{A}})$. The proof is therefore complete. ∎

2.2. SPECTRUM OF $B_{\mathcal{A}}$. The following result is an extension of Proposition 3.3 in [8] to the general setting of operator-valued weighted bi-shifts. We provide a simple and direct proof.

THEOREM 2.4. *If $\delta = 1$ then the spectrum of $B_{\mathcal{A}}$ is the interval $[-2, 2]$.*

Proof. Since $\delta = 1$, it follows from Lemma 1.5(iii) and Lemma 2.2 that $\sigma(S_{\mathcal{A}}) = \sigma(S_{\mathcal{A}^{*-1}}) = \{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$. Thus,

$$\{\lambda \in \mathbb{C} : |\lambda| = 1\} \subset \sigma_{\mathrm{ap}}(S_{\mathcal{A}}) \cap \sigma_{\mathrm{ap}}(S_{\mathcal{A}^{*-1}}).$$

As $S^*_{\mathcal{A}^{*-1}} S_{\mathcal{A}} = 1$, it follows from Lemma 2.1 that

$$\sigma(B_{\mathcal{A}}) \subset [-2, 2].$$

Now, suppose for the sake of contradiction that $\lambda - B_{\mathcal{A}}$ is invertible for some $\lambda \in [-2, 2]$. Take a real number θ for which $2\cos\theta = \lambda$ and note that

$$(\lambda - B_{\mathcal{A}}) = \mathrm{e}^{-\mathrm{i}\theta}(\mathrm{e}^{\mathrm{i}\theta} - S^*_{\mathcal{A}^{*-1}})(\mathrm{e}^{\mathrm{i}\theta} - S_{\mathcal{A}}).$$

It follows from this equation that $\mathrm{e}^{\mathrm{i}\theta} - S^*_{\mathcal{A}^{*-1}}$ is right invertible and $\mathrm{e}^{\mathrm{i}\theta} - S_{\mathcal{A}}$ is left invertible and therefore $\mathrm{e}^{-\mathrm{i}\theta} \notin \sigma_{\mathrm{ap}}(S_{\mathcal{A}^{*-1}})$ and $\mathrm{e}^{\mathrm{i}\theta} \notin \sigma_{\mathrm{ap}}(S_{\mathcal{A}})$. We have a contradiction and the proof is complete. ∎

With a little more effort, a proof similar to that of Theorem 2.4 yields the following result.

THEOREM 2.5. *If $\delta \neq 1$, then the spectrum of $B_{\mathcal{A}}$ is the solid ellipse*

$$\{a + \mathrm{i}b \in \mathbb{C} : a^2/{\delta_1}^2 + b^2/{\delta_2}^2 \leqslant 1\},$$

where $\delta_1 = \delta + \delta^{-1}$ and $\delta_2 = \delta - \delta^{-1}$.

Proof. By Lemma 1.5(ii) and Lemma 2.1, we have

$$\sigma(B_{\mathcal{A}}) \subset \{a + ib \in \mathbb{C} : a^2/\delta_1{}^2 + b^2/\delta_2{}^2 \leqslant 1\}.$$

Note that, since $B^*_{\mathcal{A}} = B_{\mathcal{A}^{*-1}}$, we may and shall assume without loss of generality that $\delta = r(S_{\mathcal{A}^{*-1}})$. So, to prove the reverse inclusion, we only need to show that $\varphi(\lambda) \in \sigma(B_{\mathcal{A}})$ for all $\max(1, R(S_{\mathcal{A}^{*-1}})) \leqslant |\lambda| \leqslant \delta = r(S_{\mathcal{A}^{*-1}})$; see Proposition 2.3. Assume that there is λ_0 such that $\max(1, R(S_{\mathcal{A}^{*-1}})) \leqslant |\lambda_0| \leqslant \delta = r(S_{\mathcal{A}^{*-1}})$ and $\varphi(\lambda_0) - B_{\mathcal{A}}$ is invertible. Since

$$(\varphi(\lambda_0) - B_{\mathcal{A}}) = \lambda_0^{-1}(\lambda_0 - S^*_{\mathcal{A}^{*-1}})(\lambda_0 - S_{\mathcal{A}}),$$

we note that $(\lambda_0 - S^*_{\mathcal{A}^{*-1}})$ is right invertible, and $\lambda_0 \notin \sigma_{\mathrm{su}}(S^*_{\mathcal{A}^{*-1}})$. From this, we have

$$\lambda_0 \in \sigma(S^*_{\mathcal{A}^{*-1}}) \backslash \sigma_{\mathrm{su}}(S^*_{\mathcal{A}^{*-1}}) \subset \mathrm{int}(\sigma_{\mathrm{p}}(S^*_{\mathcal{A}^{*-1}}));$$

see Proposition 1.1(iv). And so, $|\lambda_0| < R(S_{\mathcal{A}^{*-1}})$; see Lemma 1.5(iv). This contradicts the hypothesis and finishes the proof. ∎

2.3. APPROXIMATE POINT SPECTRUM AND SURJECTIVITY SPECTRUM OF $B_{\mathcal{A}}$. The description of the approximate point spectrum of $B_{\mathcal{A}}$ depends on the one of the approximate point spectrum of $S_{\mathcal{A}}$ which is not yet settled, and is still an open problem.

PROPOSITION 2.6. *The following assertions hold:*
(i) *If* $R(S_{\mathcal{A}}) \leqslant 1$, *then* $\sigma_{\mathrm{ap}}(B_{\mathcal{A}}) = \sigma(B_{\mathcal{A}})$.
(ii) *If* $r(S_{\mathcal{A}^{*-1}}) \leqslant 1$, *then* $\sigma_{\mathrm{ap}}(B_{\mathcal{A}}) = \varphi(\sigma_{\mathrm{ap}}(S_{\mathcal{A}}))$.
(iii) *If* $r(S_{\mathcal{A}^{*-1}}) > 1$ *and* $R(S_{\mathcal{A}}) > 1$, *then*

$$\sigma_{\mathrm{ap}}(B_{\mathcal{A}}) \cup \varphi(\{\lambda \notin \sigma_{\mathrm{ap}}(S_{\mathcal{A}}) : a < |\lambda| < b\}) = \varphi(\sigma_{\mathrm{ap}}(S_{\mathcal{A}}) \cup \sigma(S_{\mathcal{A}^{*-1}})),$$

where $a = \max(1, R(S_{\mathcal{A}^{*-1}}))$, *and* $b = \min(r(S_{\mathcal{A}^{*-1}}), R(S_{\mathcal{A}}))$.

Proof. (i) If $R(S_{\mathcal{A}}) \leqslant 1$, then $\mathrm{int}(\sigma_{\mathrm{p}}(B^*_{\mathcal{A}})) = \emptyset$; see Proposition 2.3. By Proposition 1.1(iii), the desired identity holds.

(ii) Assume that $r(S_{\mathcal{A}^{*-1}}) \leqslant 1$. We always have

$$(\varphi(\lambda) - B_{\mathcal{A}}) = \lambda^{-1}(\lambda - S^*_{\mathcal{A}^{*-1}})(\lambda - S_{\mathcal{A}}) \tag{2.3}$$

for all $\lambda \neq 0$. In view of Proposition 1.1(vi), we see that

$$\varphi(\sigma_{\mathrm{ap}}(S_{\mathcal{A}})) \subset \sigma_{\mathrm{ap}}(B_{\mathcal{A}}). \tag{2.4}$$

Assume that there is $\mu \in \sigma_{\mathrm{ap}}(B_{\mathcal{A}}) \backslash \varphi(\sigma_{\mathrm{ap}}(S_{\mathcal{A}}))$, and let $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$ and $\mu = \varphi(\lambda)$. We claim that $\lambda - S^*_{\mathcal{A}^{*-1}}$ is invertible. Of course, the claim is trivial provided that $r(S_{\mathcal{A}^{*-1}}) < 1$. So, we may assume that $r(S_{\mathcal{A}^{*-1}}) = 1$. In this case, we have $\{\xi \in \mathbb{C} : |\xi| = 1\} \subset \sigma_{\mathrm{ap}}(S_{\mathcal{A}})$ as $r_1(S_{\mathcal{A}}) = 1/r(S_{\mathcal{A}^{*-1}}) = 1$; see Lemma 1.5(i)–(ii). This implies that $|\lambda| > 1 = r(S_{\mathcal{A}^{*-1}})$, and the claim is proved in both cases.

From this claim, (2.3), Proposition 1.1(v)–(vi), we have

$$0=m(\mu-B_{\mathcal{A}})=m(\varphi(\lambda)-B_{\mathcal{A}})\geqslant|\lambda^{-1}|m(\lambda-{S^*}_{\mathcal{A}^{*-1}})m(\lambda-S_{\mathcal{A}})=\frac{|\lambda^{-1}|m(\lambda-S_{\mathcal{A}})}{\|(\lambda-{S^*}_{\mathcal{A}^{*-1}})^{-1}\|}.$$

This shows that $m(\lambda - S_{\mathcal{A}}) = 0$, and contradicts the fact that $\lambda \notin \sigma_{\rm ap}(S_{\mathcal{A}})$. The reverse inclusion of (2.4) is therefore proved.

(iii) Assume that $r(S_{\mathcal{A}^{*-1}}) > 1$ and $R(S_{\mathcal{A}}) > 1$. We trivially have

$$\sigma_{\rm p}(B_{\mathcal{A}})) \subset \sigma_{\rm ap}(B_{\mathcal{A}}). \tag{2.5}$$

From (2.3) and Proposition 1.1(vi), we get that

$$\varphi(\sigma_{\rm ap}(S_{\mathcal{A}})) \subset \sigma_{\rm ap}(B_{\mathcal{A}}) \subset \varphi(\sigma_{\rm ap}(S_{\mathcal{A}}) \cup \sigma_{\rm ap}(S^*_{\mathcal{A}^{*-1}})). \tag{2.6}$$

Note that, since $\sigma_{\rm ap}({S_{\mathcal{A}^{*-1}}}^*)=\overline{\sigma_{\rm su}(S_{\mathcal{A}^{*-1}})}=\sigma_{\rm su}(S_{\mathcal{A}^{*-1}})=\sigma(S_{\mathcal{A}^{*-1}})$, (2.6) becomes

$$\varphi(\sigma_{\rm ap}(S_{\mathcal{A}})) \subset \sigma_{\rm ap}(B_{\mathcal{A}}) \subset \varphi(\sigma_{\rm ap}(S_{\mathcal{A}}) \cup \sigma(S_{\mathcal{A}^{*-1}})). \tag{2.7}$$

In view of Proposition 1.1(i), we have $\sigma(B_{\mathcal{A}})\backslash\sigma_{\rm p}({B_{\mathcal{A}}}^*) \subset \sigma_{\rm ap}(B_{\mathcal{A}})$. By Proposition 2.3 and Theorem 2.5, we get

$$\varphi(\{\lambda \in \mathbb{C} : R(S_{\mathcal{A}})) \leqslant |\lambda| \leqslant \delta\}) \subset \sigma_{\rm ap}(B_{\mathcal{A}}). \tag{2.8}$$

Moreover, just as in the proof of (ii), we see that if $\lambda \in \mathbb{C}$ such that $r(S_{\mathcal{A}^{*-1}}) < |\lambda| < \delta$, then $\varphi(\lambda) \in \sigma_{\rm ap}(B_{\mathcal{A}})$ if and only if $\lambda \in \sigma_{\rm ap}(S_{\mathcal{A}})$. From this fact, (2.5), Proposition 2.3, (2.7) and (2.8), one derives the desired identity. ∎

The following is a dual formulation of what has been shown above.

PROPOSITION 2.7. *We have the following indentities:*

(i) *If* $R(S_{\mathcal{A}^{*-1}}) \leqslant 1$, *then* $\sigma_{\rm su}(B_{\mathcal{A}}) = \sigma(B_{\mathcal{A}})$.

(ii) *If* $r(S_{\mathcal{A}}) \leqslant 1$, *then* $\sigma_{\rm su}(B_{\mathcal{A}}) = \varphi(\sigma_{\rm ap}(S_{\mathcal{A}^{*-1}}))$.

(iii) *If* $r(S_{\mathcal{A}}) > 1$ *and* $R(S_{\mathcal{A}^{*-1}}) > 1$, *then*

$$\sigma_{\rm su}(B_{\mathcal{A}}) \cup \varphi(\{\lambda \notin \sigma_{\rm ap}(S_{\mathcal{A}^{*-1}}) : a^* < |\lambda| < b^*\}) = \varphi(\sigma_{\rm ap}(S_{\mathcal{A}^{*-1}}) \cup \sigma(S_{\mathcal{A}})),$$

where $a^* = \max(1, R(S_{\mathcal{A}}))$, *and* $b^* = \min(r(S_{\mathcal{A}}), R(S_{\mathcal{A}^{*-1}}))$.

2.4. ESSENTIAL SPECTRUM OF $B_{\mathcal{A}}$. In the rest of this section, we assume that $\mathcal{H} = \mathbb{C}^r$ and shall describe the essential spectrum of $B_{\mathcal{A}}$. Here, r is a fixed natural number.

For a nonzero subspace V of $\mathbb{C}^r$, we associate two quantities:

$$\kappa_+(V) := \lim_{n\to+\infty}\{\sup\{[(\|B_{n+k}x\|/\|B_kx\|)]^{1/n} : k \geqslant 0,\ x \in V,\ x \neq 0\}\},$$

$$\kappa_-(V) := \lim_{n\to+\infty}\{\inf\{[(\|B_{n+k}x\|/\|B_kx\|)]^{1/n} : k \geqslant 0,\ x \in V,\ x \neq 0\}\}.$$

When $V = \{0\}$, we set by convention

$$\kappa_+(\{0\}) := 0 \quad \text{and} \quad \kappa_-(\{0\}) := +\infty.$$

These are called *the discrete upper and lower Bohl exponents* of $S_{\mathcal{A}}$ corresponding to the subspace V, respectively.

DEFINITION 2.8. (i) A projection P of $\mathbb{C}^r$ is said to be *splitting projection* for $S_{\mathcal{A}}$ if

$$\kappa_+(\operatorname{ran}P) < \kappa_-(\ker P).$$

(ii) A *canonical set* of splitting projections for $S_{\mathcal{A}}$ is a maximal set $(P_i)_{0\leqslant i\leqslant l}$ of splitting projections for $S_{\mathcal{A}}$ such that

$$\operatorname{rank} P_i < \operatorname{rank} P_{i+1} \quad (0 \leqslant i \leqslant l-1).$$

In [9], A. Ben-Artzi and I. Gohberg proved that a canonical set $(P_i)_{0\leqslant i\leqslant l}$ of splitting projections for $S_{\mathcal{A}}$ is unique, and the following inequalities always hold:

$$\begin{aligned} 0 = \kappa_+(\operatorname{ran}P_0) < \kappa_-(\ker P_0) \leqslant \kappa_+(\operatorname{ran}P_1) < \cdots \\ < \kappa_-(\ker P_{l-1}) \leqslant \kappa_+(\operatorname{ran}P_l) < \kappa_-(\ker P_l) = +\infty, \quad l \leqslant r. \end{aligned}$$

The following result gives the complete description of the essential spectrum of $B_{\mathcal{A}}$. Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *Fredholm* if $\operatorname{ran}T$ is closed and both $\ker T$ and $\ker T^*$ are finite dimensional. The *essential spectrum*, $\sigma_{\mathrm{e}}(T)$, is the set of all $\lambda \in \mathbb{C}$ for which $T-\lambda$ is not Fredholm. It is a closed subset of $\sigma(T)$, and is, in fact, the spectrum $\sigma(\pi(T))$ in the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ of $\pi(T)$, where $\mathcal{K}(\mathcal{H})$ is the closed ideal of all compact operators on $\mathcal{H}$ and π is the natural quotient map from $\mathcal{L}(\mathcal{H})$ onto $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

PROPOSITION 2.9. *If $(P_i)_{0\leqslant i\leqslant l}$ is a canonical set of splitting projections for $S_{\mathcal{A}}$ then*

$$\sigma_{\mathrm{e}}(B_{\mathcal{A}}) = \bigcup_{i=0}^{l-1} \varphi(\{\lambda \in \mathbb{C} : \kappa_-(\ker P_i) \leqslant |\lambda| \leqslant \kappa_+(\operatorname{ran}P_{i+1})\}).$$

In particular, if $\mathcal{H} = \mathbb{C}$, then

$$\sigma_{\mathrm{e}}(B_{\mathcal{A}}) = \varphi(\{\lambda \in \mathbb{C} : r_1(S_{\mathcal{A}}) \leqslant |\lambda| \leqslant r(S_{\mathcal{A}})\}).$$

Proof. Let $P \in \mathcal{L}(l^2(\mathbb{N},\mathcal{H}))$ be the finite rank projection defined by

$$P((x_n)_{n\geqslant 0}) = (x_0, 0, 0, \ldots), \quad ((x_n)_{n\geqslant 0} \in l^2(\mathbb{N},\mathcal{H})).$$

Note that, since $S^*_{\mathcal{A}^{*-1}} S_{\mathcal{A}} = 1$ and $S_{\mathcal{A}} S^*_{\mathcal{A}^{*-1}} = 1 - P$, the element $\pi(S_{\mathcal{A}})$ is invertible in the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, and its inverse is $\pi(S^*_{\mathcal{A}^{*-1}})$. By the spectral mapping theorem, we have

$$\sigma_{\mathrm{e}}(B_{\mathcal{A}}) = \sigma(\pi(B_{\mathcal{A}})) = \sigma(\pi(S_{\mathcal{A}}) + \pi(S_{\mathcal{A}})^{-1}) = \varphi(\sigma_{\mathrm{e}}(S_{\mathcal{A}})).$$

The result now follows from Chapter 2, Theorem 1.6 in [9]. ∎

3. THE SINGLE-VALUED EXTENSION PROPERTY FOR $B_{\mathcal{A}}$

In this section, we give a necessary and sufficient condition for $B_{\mathcal{A}}$ to enjoy the single-valued extension property. The arguments of the proof are influenced by ideas from [14].

THEOREM 3.1. *The following statements are equivalent:*

(i) *The bi-shift $B_{\mathcal{A}}$ has the single-valued extension property.*

(ii) *Each B_x has the single-valued extension property.*

(iii) $r_{S_{\mathcal{A}}}(x^{(0)}) \geqslant 1$ *for all nonzero $x \in \mathcal{H}$.*

(iv) $R(S_{\mathcal{A}^{*-1}}) \leqslant 1$.

Moreover, if $B_{\mathcal{A}}$ does not have the single-valued extension property, then

$$\Re(B_{\mathcal{A}}) = \{a + ib \in \mathbb{C} : a^2/c_1^2 + b^2/c_2^2 < 1\},$$

where $c_1 := R(S_{\mathcal{A}^{*-1}}) + R(S_{\mathcal{A}^{*-1}})^{-1}$ and $c_2 := R(S_{\mathcal{A}^{*-1}}) - R(S_{\mathcal{A}^{*-1}})^{-1}$.

Proof. First of all, let us prove that if x is a nonzero element of $\mathcal{H}$, then

$$\sigma_{B_x}((x_n)_{n\geqslant 0}) = \sigma_{B_{\mathcal{A}}}((x_n)_{n\geqslant 0}) \tag{3.1}$$

for all $(x_n)_{n\geqslant 0} \in M_x$. Fix an element $(x_n)_{n\geqslant 0} \in M_x$, and note that this inclusion $\sigma_{B_{\mathcal{A}}}((x_n)_{n\geqslant 0}) \subset \sigma_{B_x}((x_n)_{n\geqslant 0})$ always holds. Now, let $\phi = (\phi_n)_{n\geqslant 0}$ be an $l^2(\mathbb{N}, \mathcal{H})$-valued analytic function on some open set $U \subset \rho_{B_{\mathcal{A}}}((x_n)_{n\geqslant 0})$ such that $(B_{\mathcal{A}} - \lambda)\phi(\lambda) = (x_n)_{n\geqslant 0}$ for all $\lambda \in U$. We have

$$A_0{}^{-1}\phi_1 - \lambda\phi_0 = x_0; \quad A_{n-1}\phi_{n-1} + A_n{}^{-1}\phi_{n+1} - \lambda\phi_n = x_n \quad \text{if } n \geqslant 1.$$

For every $n \geqslant 0$, let $F_n(\lambda) := P_n\phi_n(\lambda)$, $(\lambda \in U)$, where P_n is the canonical projection from $\mathcal{H}$ onto $M_n := \mathbb{C}B_n x$. As $A_n M_n = M_{n+1}$ and $\|F_n(\lambda)\| \leqslant \|\phi_n(\lambda)\|$ for all $n \geqslant 0$ and all $\lambda \in U$, we see that the function $F(\lambda) := (F_n(\lambda))_{n\in\mathbb{Z}}$, $(\lambda \in U)$, is an M_x-valued analytic function on U, and that

$$A_0{}^{-1}F_1 - \lambda F_0 = x_0; \quad A_{n-1}F_{n-1} + A_n{}^{-1}F_{n+1} - \lambda F_n = x_n \quad \text{if } n \geqslant 1.$$

From this it follows that $(B_x - \lambda)F(\lambda) = (x_n)_{n\geqslant 0}$, $(\lambda \in U)$. This shows that $U \subset \rho_{B_{\mathcal{A}}}((x_n)_{n\geqslant 0})$, and $\sigma_{B_x}((x_n)_{n\geqslant 0}) \subset \sigma_{B_{\mathcal{A}}}((x_n)_{n\geqslant 0})$. Thus the desired identity is established.

Obviously, the equivalence (iii)$\Longleftrightarrow$(iv) always holds. To prove the first part of the theorem, we shall establish the following equivalence (i)$\Longleftrightarrow$(ii) and (ii)$\Longleftrightarrow$(iii).

The implication (i)$\Longrightarrow$(ii) is obvious. Conversely, assume that $B_{\mathcal{A}}$ does not have the single-valued extension property. By Theorem 1.9 of [5], there are $\lambda_0 \in \mathbb{C}$ and a nonzero element $(x_n)_{n\geqslant 0} \in l^2(\mathbb{N}, \mathcal{H})$ such that $(B_{\mathcal{A}} - \lambda_0)(x_n)_{n\geqslant 0} = 0$ and $\sigma_{B_{\mathcal{A}}}((x_n)_{n\geqslant 0}) = \emptyset$. Just as in the proof of Proposition 2.3, we see that $x_0 \neq 0$ and $(x_n)_{n\geqslant 0} \in M_{x_0}$. Hence, $(x_n)_{n\geqslant 0}$ is, in fact, an eigenvector of B_{x_0}. By (3.1), we have

$$\sigma_{B_{x_0}}((x_n)_{n\geqslant 0}) = \sigma_{B_{\mathcal{A}}}((x_n)_{n\geqslant 0}) = \emptyset.$$

By Theorem 1.9 of [5], we note that B_{x_0} does not have the single-valued extension property. This shows that the implication (ii)$\Longrightarrow$(i) always holds.

Let x be a nonzero element of $\mathcal{H}$ such that $r_{S_{\mathcal{A}}}(x^{(0)}) \geqslant 1$. By Proposition 2.3, we see that $\operatorname{int}(\sigma_{\mathrm{p}}(B_x)) = \emptyset$, and B_x has the single-valued extension property. Thus the implication (iii)$\Longrightarrow$(ii) holds true.

Conversely, assume that there is a nonzero element $x\in\mathcal{H}$ such that $r_{S_{\mathcal{A}}}(x^{(0)}) < 1$. Set $O_x := \{\lambda \in \mathbb{C} : r_{S_{\mathcal{A}}}(x^{(0)}) < |\lambda| < r_{S_{\mathcal{A}}}(x^{(0)})^{-1}\}$, and let $k_x(\lambda) := (B_n x/\lambda^{n+1})_{n\geqslant 0}$, $(\lambda \in O_x)$. For every $\lambda \in O_x$, we have

$$(\varphi(\lambda) - B_{\mathcal{A}})(k_x(\lambda) - k_x(\lambda^{-1})) = 0 \tag{3.2}$$

for all $\lambda \in O_x$. Set $U_x := \varphi(O_x)$, and pick $\mu_0 = \varphi(\lambda_0) \in U_x$ such that $\lambda_0 \neq \pm 1$ and $|\lambda_0| < r_{S_{\mathcal{A}}}(x^{(0)})^{-1}$. As $\varphi'(\lambda_0) = 1 - 1/{\lambda_0}^2 \neq 0$, there is small open neighbourhood O of λ_0 such that $\varphi_0 := \varphi_{|O}$ is a bijective bianalytic function from O onto $U := \varphi(O)$ which is an open neighbourhood of μ_0. Set

$$\Phi_x(\mu) = k_x({\varphi_0}^{-1}(\mu)) - k_x(1{\varphi_0}^{-1}(\mu)), \quad (\mu \in U).$$

It is an M_x-valued analytic function, and satisfies trivially the equation

$$(\mu - B_{\mathcal{A}})\Phi_x(\mu) = (\mu - B_x)\Phi_x(\mu) = 0, \quad (\mu \in U);$$

see (3.2). This shows that $\mu_0 \in \Re(B_x)$, and $U_x\backslash\{0\} \subset \Re(B_x) \subset \Re(B_{\mathcal{A}})$. As $B^*_{x_0}$ is a cyclic operator, the set $\Re(B_x)$ is simply connected; combine Theorem 2.2 of [10] and Proposition II.7.12 of [15]. Thus, 0 is also in $\Re(B_x)$, and

$$U_x = \varphi(O_x) \subset \Re(B_x) \subset \Re(B_{\mathcal{A}}).$$

This shows that $B_{\mathcal{A}}$ does not have the single-valued extension property, and establishes the implication (ii)$\Longrightarrow$(iii).

Assume that $B_{\mathcal{A}}$ does not have the single-valued extension property. By what has gone before, we have

$$\begin{aligned}\Re(B_{\mathcal{A}}) &= \bigcup\{\Re(B_x) : x \in \mathcal{H}\backslash\{0\},\ r_{S_{\mathcal{A}}}(x^{(0)}) < 1\}\\ &= \bigcup\{\varphi(O_x) : x \in \mathcal{H}\backslash\{0\},\ r_{S_{\mathcal{A}}}(x^{(0)}) < 1\} = \{a+ib \in \mathbb{C} : a^2/c_1^2 + b^2/c_2^2 < 1\}.\end{aligned}$$

This finishes the proof. ∎

4. LOCAL SPECTRA OF $B_{\mathcal{A}}$

In this section, we aim at giving a complete description of the local spectrum of $B_{\mathcal{A}}$ at most point of $l^2(\mathbb{N}, \mathcal{H})$. For this, we need to fix some notations, and provide a useful model representation of a weighted bi-shift.

Throughout the rest of this paper, let S be the bilateral operator-weighted shift defined on $l^2(\mathbb{Z}, \mathcal{H})$ by

$$Sx := (\ldots, A_1^{-1}x_{-3}, A_0^{-1}x_{-2}, [x_{-1}], x_0, A_0x_1, A_1x_2, \ldots),$$

where for an element $x = (\ldots, x_{-2}, x_{-1}, [x_0], x_1, x_2, \ldots) \in l^2(\mathbb{Z}, \mathcal{H})$, we denote by $[x_0]$ the $0th$ coordinate of x. This bilateral operator-weighted shift is invertible, and its inverse is defined on $l^2(\mathbb{Z}, \mathcal{H})$ by

$$S^{-1}(x_n)_{n\in\mathbb{Z}} = (\ldots, A_1x_{-2}, A_0x_{-1}, x_0, [x_1], A_0^{-1}x_2, A_1^{-1}x_3, \ldots).$$

We shall now see that $B_{\mathcal{A}}$ is similar to $S + S^{-1}$ when restricted to one of its proper closed invariant subspaces. Indeed, let

$$\mathcal{K} := \{(x_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z}, \mathcal{H}) : x_{-n} = -x_n \text{ for all } n \geqslant 0\},$$

and note that $\mathcal{K}$ is a nontrivial closed invariant subspace of $S + S^{-1}$. Now, let $V : \mathcal{K} \to l^2(\mathcal{H}, \mathbb{N})$ be the invertible operator given by

$$V(x_n)_{n\in\mathbb{Z}} := (x_{n+1})_{n\geqslant 0}, \quad (x_n)_{n\in\mathbb{Z}} \in \mathcal{K}.$$

A simple computation shows that

$$B_{\mathcal{A}}V = V(S + S^{-1})_{|\mathcal{K}}, \tag{4.1}$$

and the claim is proved.

This representation of the bi-shift $B_{\mathcal{A}}$ as a part of the operator $S + S^{-1}$ together with some lemmas will be used to determine the local spectrum of $B_{\mathcal{A}}$ at most points of $l^2(\mathbb{N}, \mathcal{H})$.

LEMMA 4.1. *If $(x_n)_{n\in\mathbb{Z}} \in \mathcal{K}$, then $\sigma_{S+S^{-1}}((x_n)_{n\in\mathbb{Z}}) = \sigma_{B_{\mathcal{A}}}((x_{n+1})_{n\geqslant 0})$.*

Proof. We always have

$$\begin{aligned}\sigma_{S+S^{-1}}((x_n)_{n\in\mathbb{Z}}) \subset \sigma_{(S+S^{-1})_{|\mathcal{K}}}((x_n)_{n\in\mathbb{Z}}) &= \sigma_{V^{-1}B_{\mathcal{A}}V}((x_n)_{n\in\mathbb{Z}}) \\ &= \sigma_{B_{\mathcal{A}}}(V(x_n)_{n\in\mathbb{Z}}) = \sigma_{B_{\mathcal{A}}}((x_{n+1})_{n\geqslant 0}).\end{aligned}$$

Conversely, let $\phi = (\phi_n)_{n\in\mathbb{Z}}$ be a $l^2(\mathbb{Z}, \mathcal{H})$-valued analytic function on some open set $U \subset \rho_{S+S^{-1}}((x_n)_{n\in\mathbb{Z}})$ such that

$$(S + S^{-1} - \lambda)\phi(\lambda) = (x_n)_{n\in\mathbb{Z}}, \quad (\lambda \in U).$$

We have

$$\begin{cases} A_n\phi_{-n-1} + A_{n+1}^{-1}\phi_{-n-3} - \lambda\phi_{-n-2} = -x_{n+2} & \text{if } n \geqslant 0, \\ A_0^{-1}\phi_{-2} + \phi_0 - \lambda\phi_{-1} = -x_1, \\ \phi_{-1} + \phi_1 - \lambda\phi_0 = 0,\ \phi_0 + A_0^{-1}\phi_2 - \lambda\phi_1 = x_1, \\ A_n\phi_{n+1} + A_{n+1}^{-1}\phi_{n+3} - \lambda\phi_{n+2} = x_{n+2} & \text{if } n \geqslant 0. \end{cases} \tag{4.2}$$

Now, we set

$$\widetilde{\phi}_n = \frac{\phi_n - \phi_{-n}}{2}, \quad (n \geqslant 1),$$

and note that $\widetilde{\phi} = (\widetilde{\phi}_{n+1})_{n\geqslant 0}$ is a $l^2(\mathbb{N}, \mathcal{H})$-valued analytic function on U. In view of (4.2), we have

$$A_0^{-1}\widetilde{\phi}_2 - \lambda\widetilde{\phi}_1 = x_1; \quad A_n\widetilde{\phi}_{n+1} + A_{n+1}^{-1}\widetilde{\phi}_{n+3} - \lambda\widetilde{\phi}_{n+2} = x_{n+2} \quad \text{if } n \geqslant 0.$$

From this it follows that $(B_{\mathcal{A}} - \lambda)(\widetilde{\phi}_{n+1}(\lambda))_{n\geqslant 0} = (x_{n+1})_{n\geqslant 0}$, $(\lambda \in U)$, and $U \subset \rho_{B_{\mathcal{A}}}((x_{n+1})_{n\geqslant 0})$. Thus, as desired,

$$\sigma_{B_{\mathcal{A}}}((x_{n+1})_{n\geqslant 0}) \subset \sigma_{S+S^{-1}}((x_n)_{n\in\mathbb{Z}}). \quad ∎$$

Before stating the next lemma, we shall fix some notations. For the sake of completeness of the reader, we shall sketch its proof.

Assume that T is a bilateral operator-weighted shift on $l^2(\mathbb{Z},\mathcal{H})$ with a weight sequence $(T_n)_{n\in\mathbb{Z}}$ of uniformly bounded invertible operators of $\mathcal{L}(\mathcal{H})$. For every $n\in\mathbb{Z}$, we set

$$\widehat{T}_n := \begin{cases} T_{n-1}T_{n-2}\cdots T_1T_0 & \text{if } n>0,\\ 1 & \text{if } n=0,\\ T_n^{-1}T_{n+1}^{-1}\cdots T_{-2}^{-1}T_{-1}^{-1} & \text{if } n<0.\end{cases}$$

For $x\in\mathcal{H}$, we set

$$\widehat{\mathcal{H}}_T(x) := \bigvee\{(\widehat{T}_n x)^{(n)} : n\in\mathbb{Z}\},$$

and define

$$n(T,x) := \liminf_{n\to+\infty}\|\widehat{T}_{-n}x\|^{-1/n}, \quad p(T,x) := \limsup_{n\to+\infty}\|\widehat{T}_n x\|^{1/n}.$$

For every element $y:=(y_n)_{n\in\mathbb{Z}}\in l^2(\mathbb{Z},\mathcal{H})$, we define

$$R^-(T,y) := \limsup_{n\to+\infty}\|\widehat{T}_{-n}^{-1}y_{-n}\|^{1/n}, \quad R^+(T,y) := 1/\limsup_{n\to+\infty}\|\widehat{T}_n^{-1}y_n\|^{1/n}.$$

LEMMA 4.2. *Let x be a nonzero element of $\mathcal{H}$, and let $y=(y_n)_{n\in\mathbb{Z}}\in\widehat{\mathcal{H}}_T(x)$. If $R^-(T,y)<n(T,x)\leqslant p(T,x)<R^+(T,y)$, then*

$$\sigma_T(y)=\{\lambda\in\mathbb{C} : n(T,x)\leqslant|\lambda|\leqslant p(T,x)\}.$$

Proof. The proof is an immediate consequence of Theorem 4.7(a) in [13], once one observes that $\sigma_T(z)=\sigma_{T_{|\widehat{\mathcal{H}}_T(x)}}(z)$ for all $z\in\widehat{\mathcal{H}}_T(x)$; see Lemma 2.1 of [11]. ∎

LEMMA 4.3. *If $\delta=1$, then $r_{S_\mathcal{A}}((x_n)_{n\geqslant0})=1$ for all nonzero element $(x_n)_{n\geqslant0}\in l^2(\mathbb{N},\mathcal{H})$.*

Proof. First, we fix a nonzero element $x\in\mathcal{H}$, and prove that $r_{S_\mathcal{A}}(x^{(n)})=1$ for all $n\geqslant0$. Since ${S_\mathcal{A}}^n({B_n}^{-1}x)^{(0)}=x^{(n)}$, it suffices to prove this identity for $n=0$; see [23].

Indeed, for every integer k, we have $\|x\|\leqslant\|B_k^{-1}\|\|B_kx\|\leqslant\|S^k_{\mathcal{A}^{*-1}}\|\|B_kx\|$. And so, $\|x\|^{1/k}\leqslant\|S^k_{\mathcal{A}^{*-1}}\|^{1/k}\|B_kx\|^{1/k}$ for all $k\geqslant1$. From this it follows that

$$1\leqslant r(S_{\mathcal{A}^{*-1}})r_{S_\mathcal{A}}(x^{(0)})\leqslant r(S_{\mathcal{A}^{*-1}})r(S_\mathcal{A}).$$

As $\delta=1$, it follows from Lemma 2.2 that $r(S_\mathcal{A})=r(S_{\mathcal{A}^{*-1}})=1$, and $r_{S_\mathcal{A}}(x^{(0)})=1$.

Now, assume that $x:=(x_n)_{n\geqslant0}$ is a nonzero element of $l^2(\mathbb{N},\mathcal{H})$. So, there is an integer $k\geqslant0$ such that $x_k\neq0$. Since, $\|{S_\mathcal{A}}^n x\|^2=\sum_{i=0}^{+\infty}\|B_{n+i}B_i^{-1}x_i\|^2$, $(n\geqslant0)$, we have

$$\|B_{n+k}B_k^{-1}x_k\|^{1/n}\leqslant\|S^n_\mathcal{A}x\|^{1/n}, \quad (n\geqslant0).$$

By taking lim sup as $n \to +\infty$, we get $r_{S_{\mathcal{A}}}(x_k^{(k)}) \leqslant r_{S_{\mathcal{A}}}(x)$. From this and what has shown above, we infer that $r_{S_{\mathcal{A}}}(x) = 1$. The proof is therefore complete. ∎

For every $y = (y_n)_{n\geqslant 0} \in l^2(\mathbb{N}, \mathcal{H})$, we define

$$R(y) := 1/ \limsup_{n\to+\infty} \|B_n^{-1}y_n\|^{1/n}.$$

Now, we are able to state and prove one of the main results of this section.

THEOREM 4.4. *Assume that $\delta = 1$. Let x be a nonzero element of $\mathcal{H}$, and let $y = (y_n)_{n\geqslant 0} \in \bigvee\{B_{\mathcal{A}}{}^n x^{(0)} : n \geqslant 0\}$. If $1 < R(y)$, then*

$$\sigma_{B_{\mathcal{A}}}(y) = [-2, 2].$$

In particular, if y is a finite combination of elements of $\{B_{\mathcal{A}}{}^n x^{(0)} : n \geqslant 0\}$, then $\sigma_{B_{\mathcal{A}}}(y) = [-2, 2]$.

Proof. One easily verifies that

$$\bigvee\{B_{\mathcal{A}}{}^n x^{(0)} : n \geqslant 0\} = \bigvee\{(B_n x)^{(n)} : n \geqslant 0\},$$

and that $V^{-1}(\bigvee\{B_{\mathcal{A}}{}^n x^{(0)} : n \geqslant 0\})$ is a subspace of $\mathcal{K}$ which is, of course, contained in $\widehat{\mathcal{H}}_S(x)$. Now, let $z := V^{-1}y$, and keep in mind that z is in $\widehat{\mathcal{H}}_S(x)$. Since $\delta = 1$, we have

$$p(S, x) = \limsup_{n\to+\infty} \|B_{n-1}x\|^n = r_{S_{\mathcal{A}}}(x^0) = 1;$$

see Lemma 4.3. We also have

$$n(S,x)=1/p(S,x)=1, \quad R^-(S,z))=1/R(y)<n(S,x)=p(S,x)=1<R^+(S,z)=R(y).$$

By Lemma 4.2, we have $\sigma_S(z) = \sigma_S(V^{-1}y) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Therefore, it follows from this ([21], Theorem 3.3.8) and Lemma 4.1 that

$$[-2, 2] = \varphi(\sigma_S(V^{-1}y)) = \sigma_{S+S^{-1}}(V^{-1}y) = \sigma_{B_{\mathcal{A}}}(y).$$

To prove the last part of the theorem, it suffices to check that if y is a finite combination of elements of $\{B_{\mathcal{A}}{}^n x^{(0)} : n \geqslant 0\}$, then it is a finite combination of elements of $\{(B_n x)^{(n)} : n \geqslant 0\}$ as well, and $1 < R(y) = +\infty$. ∎

COROLLARY 4.5. *If $\delta = 1$, then we have for all nonzero $x \in \mathcal{H}$, and all $n \geqslant 0$*

$$\sigma_{B_{\mathcal{A}}}(x^{(n)}) = [-2, 2].$$

Proof. For every $x \in \mathcal{H}$ and $n \geqslant 0$, the vector $x^{(n)}$ is a combination of $(B_n^{-1}x)^{(0)}, B_{\mathcal{A}}(B_n^{-1}x)^{(0)}, B_{\mathcal{A}}{}^2(B_n^{-1}x)^{(0)}, \ldots, B_{\mathcal{A}}{}^n(B_n^{-1}x)^{(0)}$. ∎

Next theorem shows that the conclusion of the second part of Theorem 4.4 remains valid for all nonzero finitely supported elements in $l^2(\mathbb{N}, \mathcal{H})$ not necessary in some subspace of the form $\bigvee\{B_{\mathcal{A}}{}^n x^{(0)} : n \geqslant 0\}$, $x \in \mathcal{H}$. To prove that, we need the following elementary lemma which is easy to verify.

LEMMA 4.6. *If p is a nonconstant complex polynomial, then the next function has no continuous extension across the unit circle $\mathbb{T}$:*

$$\phi(\lambda) := \begin{cases} p(\lambda) & \text{if } |\lambda| < 1, \\ p(1/\lambda) & \text{if } |\lambda| > 1. \end{cases}$$

THEOREM 4.7. *If $\delta = 1$, then $\sigma_{B_{\mathcal{A}}}(y) = [-2, 2]$ for all nonzero finitely supported elements $y \in l^2(\mathbb{N}, \mathcal{H})$.*

Proof. Assume that $y = (y_0, y_1, \ldots, y_n, 0, \ldots)$ with $y_n \neq 0$. We only have to show that the function $\lambda \longmapsto (\lambda - B_{\mathcal{A}})^{-1}y$, $\lambda \in \mathbb{C}\backslash[-2, 2]$ has no analytic extension across the interval $[-2, 2]$. To do that, it suffices to show that

$$\lambda \longmapsto (\varphi(\lambda) - B_{\mathcal{A}})^{-1}y, \quad \lambda \in \mathbb{C}\backslash\mathbb{T}$$

has no analytic extension across the unit circle $\mathbb{T}$ since φ is analytic in $\mathbb{C}\backslash\{0\}$ and $\varphi(\mathbb{T}) = [-2, 2]$.

Without loss of generality, we may and shall assume that $y_0 \neq 0$. Since $S^*_{\mathcal{A}^{*-1}} S_{\mathcal{A}} = 1$ and $\sigma(S_{\mathcal{A}}) = \sigma(S^*_{\mathcal{A}^{*-1}}) = \{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$, we have

$$(\varphi(\lambda) - B_{\mathcal{A}})^{-1} = \lambda(1 - \lambda S_{\mathcal{A}})^{-1}(1 - \lambda S^*_{\mathcal{A}^{*-1}})^{-1} \tag{4.3}$$

for all $|\lambda| < 1$. Therefore, for every $|\lambda| < 1$, we have

$$\begin{aligned} &\langle (\varphi(\lambda) - B_{\mathcal{A}})^{-1}y; (y_0)^{(0)} \rangle \\ &\quad = \langle y; (\varphi(\lambda) - B_{\mathcal{A}})^{*-1}(y_0)^{(0)} \rangle = \langle y; (\varphi(\overline{\lambda}) - B^*_{\mathcal{A}})^{-1}(y_0)^{(0)} \rangle \\ &\quad = \langle y; (\varphi(\overline{\lambda}) - B^*_{\mathcal{A}})^{-1}(y_0)^{(0)} \rangle = \langle y; \overline{\lambda}(1 - \overline{\lambda} S_{\mathcal{A}^{*-1}})^{-1}(1 - \overline{\lambda} S^*_{\mathcal{A}})^{-1}(y_0)^{(0)} \rangle \\ &\quad = \langle y; \overline{\lambda}(1 - \overline{\lambda} S_{\mathcal{A}^{*-1}})^{-1}(y_0)^{(0)} \rangle = \Big\langle y; \sum_{i=0}^{+\infty} \overline{\lambda}^{i+1} S^i_{\mathcal{A}^{*-1}}(y_0)^{(0)} \Big\rangle \\ &\quad = \sum_{i=0}^{+\infty} \lambda^{i+1} \langle y; (B_i^{*-1}y_0)^{(i)} \rangle = \sum_{i=0}^{n} \lambda^{i+1} \langle y_i; B_i^{*-1}y_0 \rangle = \sum_{i=0}^{n} \lambda^{i+1} \langle B_i^{-1}y_i; y_0 \rangle. \end{aligned}$$

As $\varphi(\lambda) = \varphi(\lambda^{-1})$ for all $\lambda \neq 0$, we, in fact, have

$$\langle (\varphi(\lambda) - B_{\mathcal{A}})^{-1}y; (y_0)^{(0)} \rangle = \begin{cases} \sum_{i=0}^{n} \lambda^{i+1} \langle B_i^{-1}y_i; y_0 \rangle & \text{if } |\lambda| < 1, \\ \sum_{i=0}^{n} \lambda^{-(i+1)} \langle B_i^{-1}y_i; y_0 \rangle & \text{if } |\lambda| > 1. \end{cases}$$

Since $\langle B_0^{-1}y_0; y_0 \rangle = \|y_0\|^2 \neq 0$, the polynomial $p(z) := \sum_{i=0}^{n} z^{i+1} \langle B_i^{-1}y_i; y_0 \rangle$ is nonconstant, and the function $\langle (\varphi(\lambda) - B_{\mathcal{A}})^{-1}y; (y_0)^{(0)} \rangle$ has no analytic extension across the unit circle $\mathbb{T}$; see Lemma 4.6. Therefore we have as desired:

$$\sigma_{B_{\mathcal{A}}}(y) = [-2, 2]. \quad ∎$$

We close this section with a similar result of Theorem 4.4 but for the case when $\delta \neq 1$. The proof is essentially the same as the one of Theorem 4.4, and is therefore omitted here.

THEOREM 4.8. *Let x be a nonzero element of $\mathcal{H}$. The following statements hold:*

(i) *If $r_{S_{\mathcal{A}}}(x^{(0)}) < 1$, and if y is a finite combinations of elements of $\{B_{\mathcal{A}}{}^n x^{(0)} : n \geqslant 0\}$, then $\sigma_{B_{\mathcal{A}}}(y) = \emptyset$.*

(ii) *If $r_{S_{\mathcal{A}}}(x^{(0)}) = 1$, and if $y \in \bigvee\{B_{\mathcal{A}}{}^n x^{(0)} : n \geqslant 0\}$ such that $1 < R(y)$, then $\sigma_{B_{\mathcal{A}}}(y) = [-2, 2]$.*

(iii) *If $1 < r_{S_{\mathcal{A}}}(x^{(0)})$, and if $y \in \bigvee\{B_{\mathcal{A}}{}^n x^{(0)} : n \geqslant 0\}$ such that $r_{S_{\mathcal{A}}}(x^{(0)}) < R(y)$, then*

$$\sigma_{B_{\mathcal{A}}}(y) = \{a + \mathrm{i}b \in \mathbb{C} : a^2/\delta_{x,1}^2 + b^2/\delta_{x,2}^2 \leqslant 1\},$$

where $\delta_{x,1} = r_{S_{\mathcal{A}}}(x^{(0)}) + 1/r_{S_{\mathcal{A}}}(x^{(0)})$ and $\delta_{x,2} = r_{S_{\mathcal{A}}}(x^{(0)}) - 1/r_{S_{\mathcal{A}}}(x^{(0)})$.

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