J. OPERATOR THEORY
59:1(2008), 211–234

LIPSCHITZ AND COMMUTATOR ESTIMATES IN SYMMETRIC OPERATOR SPACES.

DENIS POTAPOV and FYODOR SUKOCHEV

Communicated by Nikolai K. Nikolski

ABSTRACT. This paper studies Lipschitz and commutator estimates in (non-commutative) symmetric operator spaces $\mathbb{E}$ associated with a general semi-finite von Neumann algebra $\mathcal{M}$ taken in its left regular representation. In particular, we show that if f' is of bounded variation and $\mathbb{E}$ is a reflexive (non-commutative) L_p-space on $\mathcal{M}$, then the Lipschitz estimate

(*) $$\|f(a) - f(b)\|_{\mathbb{E}} \leqslant c_f \, \|a - b\|_{\mathbb{E}},$$

holds for arbitrary self-adjoint operators a and b affiliated with $\mathcal{M}$.

KEYWORDS: *Non-commutative function spaces, commutators, perturbations.*

MSC (2000): 46L52, 47B47.

1. INTRODUCTION

In the type I setting, when $\mathcal{M}$ coincides with the algebra $\mathcal{L}(\mathcal{H})$ of all linear bounded operators on a Hilbert space $\mathcal{H}$ (i.e. when symmetric operator spaces (respectively, non-commutative L_p-spaces) are symmetrically normed ideals $\mathcal{C}^E$ of compact operators on $\mathcal{H}$ (respectively Schatten-von Neumann p-classes $\mathcal{C}^p$) [18]), the result described in the Abstract was first obtained in [13] under the extra assumption that a and b are bounded self-adjoint operators from $\mathcal{L}(\mathcal{H})$. For the special case of the absolute value function, this result was later extended to the type II setting in [15], where the assumption that $a, b \in \mathcal{M}$ was replaced with the assumption that these operators are taken from the algebra $\widetilde{\mathcal{M}}$ of all τ-measurable operators affiliated with $\mathcal{M}$ [17] (all the relevant terms and definitions are also given in the next section). The methods from the (abstract) harmonic analysis used in both [13] and [15] also allowed parallel commutator estimates of the type

(1.1) $$\|[f(x), a]\|_{\mathbb{E}} \leqslant c_f \, \|[x, a]\|_{\mathbb{E}}, \quad \|[x, f(a)]\|_{\mathbb{E}} \leqslant c_f \, \|[x, a]\|_{\mathbb{E}}, \quad x = x^* \in \mathbb{E}$$

with similar restrictions on the operator a. Earlier, in the series of papers [2], [3], [4], where the type I technique of double operator integrals had been developed,

M. Birman and M. Solomyak were able to obtain Lipschitz estimates (*) for arbitrary ideals $\mathcal{C}^E$, in particular for the class of C^1-functions f whose first derivative is Hölder of order ε for arbitrary small $\varepsilon > 0$. Later, in [5], the same authors were able to apply their technique also to commutator estimates (1.1) in ideals $\mathcal{C}^E$, and strengthen the results of [13] for reflexive Schatten-von Neumann p-classes, by removing the assumption that a and b are bounded.

The double operator integral technique has been recently extended to the type II setting in [25], [23], [24], where the estimates (*) in Abstract and (1.1) are obtained in the setting of general symmetric operator spaces $\mathbb{E}$ and general semifinite von Neumann algebras $\mathcal{M}$. However, in the case of reflexive non-commutative L_p-spaces on $\mathcal{M}$, the results obtained in those articles are weaker than the corresponding result of [5] due to the restrictive τ-measurability assumption imposed on the operators a and b in those papers. The fact that this assumption is restrictive is clearly seen from the fact that in all interesting applications of the estimates (*) in Abstract and (1.1) in quantum mechanics (see e.g. [6]) and in non-commutative geometry (see e.g. [10]) it is not satisfied. In fact, even in the simplest example of interest (see e.g. Example 7.1 and Remark 7.4 of [24]), when the algebra $\mathcal{M} = L_\infty(\mathbb{R})$ acts on $\mathcal{H} = L_2(\mathbb{R})$ via multiplication and the operator a is given by the differentiation $\frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}$, it is clear that a does not belong to the algebra $\widetilde{\mathcal{M}}$ (furthermore, it is not even affiliated with $\mathcal{M}$). The problem of obtaining the estimates (*) in Abstract and (1.1) for general self-adjoint operators a and b and not just for τ-measurable and for not necessarily continuously differentiable functions f is non-trivial: the difference in the assumptions renders many existing techniques inapplicable. For example, the fact that our functions are not C^1 prevents us from using the approach developed in [24] (based, in turn, on an earlier idea from [1]), which ultimately views the first inequality in (1.1) as a statement that f is an operator differentiable function and thus must be continuously differentiable. It is, perhaps, also instructive to refer to [7] where a problem, arising in the type II quantized calculus similar to the estimates (*) and (1.1), has obtained completely different resolution depending on whether operators in question were τ-measurable or just affiliated (see Theorem 0.3(i) and (ii) of [7] and discussion on p. 144).

We now briefly explain the technical difficulties (and our strategy) arising in the setting of commutator estimates. Suppose that the operator a is not τ-measurable and that $x \in \mathcal{M}$. Among various definitions of the symbol $[a, x]$ in the literature (allowing the treatment of the situation when all three operators, a, x and $[a, x]$ may be unbounded), we have chosen the least restrictive approach articulated in [6], allowing us to consider a wider class of operators than those in [24] and [7]. We say that $[a, x] \in \mathbb{E}$ if and only if the subspace $x^{-1}(\mathrm{Dom}\, a) \cap \mathrm{Dom}\, a$ contains a core of the operator a which is invariant under the unitary group $\{\mathrm{e}^{\mathrm{i}ta}\}_{t\in\mathbb{R}}$ and the operator $xa - ax$, initially defined on that subspace, is closable with closure $[a, x]$ belonging to $\mathbb{E}$. Assume (for brevity) that the core above coincides with Dom a (it is of interest to observe that the latter assumption is *automatically* satisfied in the type I setting and more generally, when $\mathbb{E} \subseteq \mathcal{M}$, see Lemma 5.1 below). Then for a (τ-measurable) operator $y := [a, x] \in \mathbb{E}$ with a τ-dense domain (see next section), we have $\mathrm{Dom}\, a \subseteq \mathrm{Dom}\, y$. Now, our general strategy in proving the estimate (1.1) consists in linking with a given function f a linear operator T_{ψ_f}, which is bounded on $\mathbb{E}$ (indeed, T_{ψ_f} is just the double operator integral defined via the divided difference function ψ_f, see Definition 2.3 below) and for which the relation

$$[a, f(x)] = T_{\psi_f}([a, x]), \tag{1.2}$$

holds. The double operator integral T_{ψ_f} is a bounded linear operator on $\mathbb{E}$ defined via a complicated process of vector-valued integration with respect to a finitely additive measure and the relationship between the domain of the image $z := T_{\psi_f}([x, a])$ and that of a is not clear. On the other hand, if (1.2) were to hold, we should have (at the very least) that $\mathrm{Dom}\, a \subseteq \mathrm{Dom}\, z$ and $(f(x))^{-1}(\mathrm{Dom}\, a) \cap \mathrm{Dom}\, a \neq \emptyset$. This is a serious obstacle, which is specific to the type II setting. Indeed, if $\mathcal{M}$ is a type I factor, then the operator z is necessarily bounded (due to the obvious embedding $\mathcal{C}^E \subseteq \mathcal{M}$) and so, the embedding $\mathrm{Dom}\, a \subseteq \mathrm{Dom}\, z = \mathcal{H}$ is trivial.

We solve this problem and achieve a complete extension of the type I result of [5] to a general semifinite von Neumann algebra $\mathcal{M}$ under the additional assumption that the latter algebra is acting on $\mathcal{H}$ in standard form. In many circumstances the latter assumption is automatically satisfied and in many cases our results may be transferred to general von Neumann algebras. We illustrate this in the final section of this paper suggesting a simple and straightforward variant of the proofs of corresponding type I results, yielding an additional insight into methods used in [5]. In the following section, we present necessary preliminaries from the theory of non-commutative integration and a revised version of double operator integration theory from [25], [23] and [24]. We derive Lipschitz estimates and the second commutator estimates from (1.1) in Section 3; the first commutator estimates from (1.1) are obtained in Section 4.

2. PRELIMINARIES

Throughout the text $\mathcal{M}$ is a semi-finite von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$ equipped with a semi-finite faithful normal trace τ. The unit element of $\mathcal{M}$ is denoted by $\mathbf{1}$. A densely defined operator a is called *affiliated with the algebra* $\mathcal{M}$ if and only if $u^*au = a$, for every $u \in \mathcal{M}'$. We refer the reader to [27] for the general theory of von Neumann algebras.

A closed and densely defined linear operator $a : \mathrm{Dom}\, a \to \mathcal{H}$ is called τ*-measurable* if a is affiliated with $\mathcal{M}$ and the space $\mathrm{Dom}\, a$ is τ-dense, i.e. for every $\varepsilon > 0$ there is an orthogonal projection $p \in \mathcal{M}$ such that $p(\mathcal{H}) \subseteq \mathrm{Dom}\, a$

and $\tau(\mathbf{1} - p) < \varepsilon$. The collection of all τ-measurable operators will be denoted by $\widetilde{\mathcal{M}}$. We denote the spectral measure of a self-adjoint operator a by E^a, so $E^a : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$, where $\mathcal{B}(\mathbb{R})$ is the σ-algebra of all Borel subsets of $\mathbb{R}$. Observe that $E^{|a|}(B) \in \mathcal{M}$ whenever a is affiliated with $\mathcal{M}$ for all $B \in \mathcal{B}(\mathbb{R})$. For every $a \in \widetilde{\mathcal{M}}$ the generalised singular value function $\mu(a) : [0,\infty) \to [0,\infty]$ is defined by

$$\mu_t(a) = \inf\{s \geqslant 0 : \tau(E^{|a|}(s,\infty)) \leqslant t\}, \quad t \geqslant 0.$$

The set of all $a \in \widetilde{\mathcal{M}}$ such that $\mu_t(a) \to 0$ as $t \to \infty$ is denoted as $\widetilde{\mathcal{M}}_0$. We will say that the net of τ-measurable operators $\{x_\alpha\}$ converge to an operator x with respect to the measure topology if and only if $\mu_t(x_\alpha) \to \mu_t(x)$ for every $t \geqslant 0$. $\widetilde{\mathcal{M}}$ is a complete topological algebra with respect to the measure topology. We refer the reader to [22], [28], [17] for the theory of τ-measurable operators.

Throughout the text let $E = E(0,\infty)$ be a symmetric Banach function space, i.e. $E = E(0,\infty)$ is a rearrangement invariant Banach function space on $[0,\infty)$ with the additional property that $f, g \in E$ and $g \prec\prec f$ imply that $\|g\|_E \leqslant \|f\|_E$. Here $g \prec\prec f$ denotes submajorization in the sense of Hardy, Littlewood and Polya, i.e.

$$\int_0^t \mu_s(g)\,\mathrm{d}s \leqslant \int_0^t \mu_s(f)\,\mathrm{d}s, \quad t > 0.$$

The non-commutative symmetric space $\mathbb{E} = E(\mathcal{M},\tau)$ is defined by

$$\mathbb{E} = \{a \in \widetilde{\mathcal{M}} : \mu(a) \in E\} \quad \text{with } \|a\|_{\mathbb{E}} = \|\mu(a)\|_E.$$

If $E = L^p$, $1 \leqslant p \leqslant \infty$, then $\mathbb{L}^p$ is the classical non-commutative L^p-space. We note that the space $\mathbb{L}^\infty$ coincides with $\mathcal{M}$ and $\mathbb{L}^1$ coincides with the predual $\mathcal{M}_*$ of the von Neumann algebra $\mathcal{M}$, via duality given by $\langle x, y\rangle := \tau(xy)$, $x \in \mathcal{M}, y \in \mathbb{L}^1$. We shall need the following generalised Hölder inequality for non-commutative L^p-spaces, cf. Theorem 4.2.(i) of [17],

$$\|\xi\eta\|_{\mathbb{L}^s} \leqslant \|\xi\|_{\mathbb{L}^p}\,\|\eta\|_{\mathbb{L}^q}, \quad \frac{1}{s} = \frac{1}{p} + \frac{1}{q},\ 1 \leqslant s, p, q \leqslant \infty. \tag{2.1}$$

The Köthe dual $\mathbb{E}^\times$ of a symmetric space $\mathbb{E}$ is the symmetric space given by

$$\mathbb{E}^\times = \{a \in \widetilde{\mathcal{M}} : ab \in \mathbb{L}^1, \text{whenever } b \in \mathbb{E} \text{ and } \|a\|_{\mathbb{E}^\times} := \sup_{b \in \mathbb{L}^1 \cap \mathbb{L}^\infty, \|b\|_{\mathbb{E}} \leqslant 1} \tau(ab) < \infty\}, \tag{2.2}$$

see, for example, [14]. It is a subspace of the dual space $\mathbb{E}^*$ (the norms $\|\cdot\|_{\mathbb{E}^\times}$ and $\|\cdot\|_{\mathbb{E}^*}$ coincide on $\mathbb{E}^\times$) and $\mathbb{E}^\times = \mathbb{E}^*$ if and only if the space E is separable. We say that $\mathbb{E}$ has the Fatou norm (respectively, the Fatou property) if and only if the natural embedding $\mathbb{E} \subseteq \mathbb{E}^{\times\times}$ is isometrical (respectively, isometrical bijection). We have $(\mathbb{L}^1 \cap \mathbb{L}^\infty)^\times = \mathbb{L}^1 + \mathbb{L}^\infty$, $(\mathbb{L}^1 + \mathbb{L}^\infty)^\times = \mathbb{L}^1 \cap \mathbb{L}^\infty$ and the continuous embeddings

$$\mathbb{L}^1 \cap \mathbb{L}^\infty \subseteq \mathbb{E}, \quad \mathbb{E}^\times \subseteq \mathbb{L}^1 + \mathbb{L}^\infty$$

hold for every symmetric operator space $\mathbb{E}$. It follows from (2.2) that $\mathbb{L}^1 \cap \mathbb{L}^\infty$ separates points in $\mathbb{L}^1 + \mathbb{L}^\infty$, i.e. if, for $a \in \mathbb{L}^1 + \mathbb{L}^\infty$, $\tau(ab) = 0$ whenever $b \in \mathbb{L}^1 \cap \mathbb{L}^\infty$, then $a = 0$. The following lemma relates the operator and the Banach topological structures of the algebra $\mathcal{M}$.

LEMMA 2.1 ([27], Lemma 1.2, Theorem 1.10). *The weak operator* (wo)*-topology,* $\sigma(\mathbb{L}^\infty, \mathbb{L}^1 \cap \mathbb{L}^\infty)$*-topology and* $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$*-topology coincide on the unit ball* $\mathcal{M}_1$ *of the von Neumann algebra* $\mathcal{M}$.

LEMMA 2.2. *If* $z \in \mathbb{L}^1 + \mathbb{L}^\infty$, $D \subseteq \mathcal{H}$ *is a dense subspace affiliated with* $\mathcal{M}$ *and* $z' = z|_D$, *then* $\overline{z}' = z$.

Proof. Since D is affiliated with $\mathcal{M}$, it follows that z' is also affiliated with $\mathcal{M}$. Since D is dense and $z' \subseteq z$, we have $z^* \subseteq z'^*$. The operator z^* is τ-measurable, therefore, z'^* is τ-measurable also and $z'^* = z^*$, cf. Lemma 2.1 of [19]. Passing to the second adjoints, we obtain $\overline{z}' = z'^{**} = z^{**} = z$. ∎

Let us next recall several basic results of the theory of double operator integrals developed recently in [25]. By $\mathcal{L}(\mathbb{E})$ we denote the algebra of all bounded linear operators on $\mathbb{E}$. Suppose a, b are self-adjoint operators affiliated with $\mathcal{M}$. Let $E^a, E^b : \mathcal{B}(\mathbb{R}) \to \mathcal{M}$ be the corresponding spectral measures. For every $B \in \mathcal{B}(\mathbb{R})$, we define the projections $P^a_{\mathbb{E}}(B), Q^b_{\mathbb{E}}(B) : \mathbb{E} \to \mathbb{E}$ by

$$P^a_{\mathbb{E}}(B)x = E^a(B)x, \quad Q^b_{\mathbb{E}}(B)x = xE^b(B), \quad x \in \mathbb{E}.$$

Evidently, $P^a_{\mathbb{E}}, Q^b_{\mathbb{E}} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathbb{E})$ are two commuting (countably additive) spectral measures (in the sense of Chapter X in [16]) satisfying

$$\|P^a_{\mathbb{E}}(B)\|_{\mathcal{L}(\mathbb{E})} \leqslant 1 \quad \text{and} \quad \|Q^b_{\mathbb{E}}(B)\|_{\mathcal{L}(\mathbb{E})} \leqslant 1, \quad B \in \mathcal{B}(\mathbb{R}).$$

We denote by $\mathcal{A}$ the algebra of subsets of $\mathbb{R}^2$ generated by all Borel rectangles $A \times B$ with $A, B \in \mathcal{B}(\mathbb{R})$. Let $P^a_{\mathbb{E}} \otimes Q^b_{\mathbb{E}} : \mathcal{A} \to \mathcal{L}(\mathbb{E})$ be the product measure, that is,

$$P^a_{\mathbb{E}} \otimes Q^b_{\mathbb{E}}(A \times B) = P^a_{\mathbb{E}}(A)Q^b_{\mathbb{E}}(B)$$

for all $A, B \in \mathcal{B}(\mathbb{R})$. It is easily verified that $P^a_{\mathbb{E}} \otimes Q^b_{\mathbb{E}}$ is a (finitely additive) spectral measure on $\mathcal{A}$.

If $\mathbb{E} = \mathbb{L}^2$, then $P^a_{\mathbb{L}^2}$ and $Q^b_{\mathbb{L}^2}$ take their values in orthogonal projections of the Hilbert space $\mathbb{L}^2$. As is well known the product measure $P^a_{\mathbb{L}^2} \otimes Q^b_{\mathbb{L}^2}$ extends uniquely from $\mathcal{A}$ to a countably additive spectral measure on the Borel σ-algebra of $\mathbb{R}^2$, $\mathcal{B}(\mathbb{R}^2)$, taking its values in orthogonal projections of $\mathbb{L}^2$. This extension is denoted by $P^a_{\mathbb{L}^2} \otimes Q^b_{\mathbb{L}^2}$ as well, so

$$P^a_{\mathbb{L}^2} \otimes Q^b_{\mathbb{L}^2} : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathbb{L}^2).$$

In particular, $\langle P^a_{\mathbb{L}^2} \otimes Q^b_{\mathbb{L}^2} x, y \rangle$ is a σ-additive complex-valued measure on $\mathbb{R}^2$ for every $x, y \in \mathbb{L}^2$, where $\langle x, y \rangle := \tau(xy)$ (see also Remark 3.1 of [25]).

We denote by $B_{\mathbb{C}}(\mathbb{R}^2)$ the algebra of all complex-valued bounded Borel functions on $\mathbb{R}^2$. For every $\phi \in B_{\mathbb{C}}(\mathbb{R}^2)$, the spectral integral

$$T^{a,b}_{\phi,\mathbb{L}^2} = \int_{\mathbb{R}^2} \phi \,\mathrm{d}(P^a_{\mathbb{L}^2} \otimes Q^b_{\mathbb{L}^2})$$

is a bounded linear operator on $\mathbb{L}^2$ and the mapping $\phi \to T^{a,b}_{\phi,2}$ is an algebra homomorphism from the algebra $B_{\mathbb{C}}(\mathbb{R}^2)$ into $\mathcal{L}(\mathbb{L}^2)$.

The following definition is in fact a special case of Definition 2.9 in [25]. See also Proposition 2.12 in [25] and the discussion there on pages 81–82.

DEFINITION 2.3. A function $\phi \in B_{\mathbb{C}}(\mathbb{R}^2)$ is *integrable with respect to the measure* $P^a_{\mathbb{E}} \otimes Q^b_{\mathbb{E}}$ if and only if there is a bounded linear operator

$$T^{a,b}_{\phi,\mathbb{E}} : \mathbb{E} \to \mathbb{E}$$

satisfying the following conditions:

(i) $T^{a,b}_{\phi,\mathbb{E}}(\mathbb{L}^2 \cap \mathbb{E}) \subseteq \mathbb{L}^2 \cap \mathbb{E}$ and $(T^{a,b}_{\phi,\mathbb{E}})^*(\mathbb{L}^2 \cap \mathbb{E}^\times) \subseteq \mathbb{L}^2 \cap \mathbb{E}^\times$;

(ii) for every $x \in \mathbb{L}^2 \cap \mathbb{E}$ and $y \in \mathbb{L}^2 \cap \mathbb{E}^\times$

$$\langle T^{a,b}_{\phi,\mathbb{E}} x, y\rangle = \int_{\mathbb{R}^2} \phi(\lambda,\mu)\,\mathrm{d}\langle P^a_{\mathbb{E}} \otimes Q^b_{\mathbb{E}} x, y\rangle. \tag{2.3}$$

If such an operator exists, then it is unique, see Definition 2.9 of [25].

The class of all functions $\phi \in B_{\mathbb{C}}(\mathbb{R}^2)$ integrable with respect to $P^a_{\mathbb{E}} \otimes Q^b_{\mathbb{E}}$ for every a, b affiliated with $\mathcal{M}$ will be denoted by $\Phi(\mathbb{E})$. We also set

$$\Phi_s(\mathbb{E}) = \{\phi \in \Phi(\mathbb{E}) : \phi(\lambda,\mu) = \phi(\mu,\lambda),\ \lambda,\mu \in \mathbb{R}\}.$$

It is known that $\Phi_s(\mathbb{L}^q) \subsetneq \Phi_s(\mathbb{L}^p)$ when $2 \leqslant p < q \leqslant \infty$, cf. [20]. It is also known that $\Phi_s(\mathbb{L}^1) = \Phi_s(\mathbb{L}^\infty)$. The next result extends the latter observation.

LEMMA 2.4. *If $\mathbb{E}$ is a symmetric operator space with order continuous norm and has Fatou property, then $\Phi_s(\mathbb{E}) = \Phi_s(\mathbb{E}^*)$. Moreover, $T^{a,b}_{\phi,\mathbb{E}^*} = (T^{b,a}_{\phi,\mathbb{E}})^*$ and $T^{a,b}_{\phi,\mathbb{E}} = (T^{b,a}_{\phi,\mathbb{E}^*})^*|_{\mathbb{E}}$, provided $\phi \in \Phi_s(\mathbb{E}) = \Phi_s(\mathbb{E}^*)$.*

Proof. By the assumption, we have $\mathbb{E}^\times = \mathbb{E}^*$ and $\mathbb{E}^{\times\times} = \mathbb{E}$. Fix $\phi \in \Phi_s(\mathbb{E})$ and set $T := T^{b,a}_{\phi,\mathbb{E}}$, for brevity. Let us first show that $\Phi_s(\mathbb{E}) \subseteq \Phi_s(\mathbb{E}^*)$, to this end it is sufficient to show that

$$T^{a,b}_{\phi,\mathbb{E}^*} = T^*. \tag{2.4}$$

Let us also fix $x \in \mathbb{L}^2 \cap \mathbb{E} = \mathbb{L}^2 \cap \mathbb{E}^{\times\times}$, $y \in \mathbb{L}^2 \cap \mathbb{E}^* = \mathbb{L}^2 \cap \mathbb{E}^\times$. It follows from Definition 2.3 that $T(\mathbb{L}^2 \cap \mathbb{E}) \subseteq \mathbb{L}^2 \cap \mathbb{E}$, $T^*(\mathbb{L}^2 \cap \mathbb{E}^\times) \subseteq \mathbb{L}^2 \cap \mathbb{E}^\times$ and $\langle T(x), y\rangle = \int_{\mathbb{R}^2} \phi(\lambda,\mu)\,\mathrm{d}\langle P^b_{\mathbb{E}} \otimes Q^a_{\mathbb{E}}(x), y\rangle$. Consequently, passing to the adjoint operator T^*

in the latter identity, we obtain

$$\langle x, T^*(y)\rangle = \int\limits_{\mathbb{R}^2} \phi(\lambda,\mu)\,\mathrm{d}\langle x, P^a_{\mathbb{E}^\times}\otimes Q^b_{\mathbb{E}^\times}(y)\rangle.$$

Thus, to finish the proof of (2.4), we only need to show that

$$T^{**}(\mathbb{L}^2\cap\mathbb{E}) \subseteq \mathbb{L}^2\cap\mathbb{E}.$$

The latter is apparent, since $T\in\mathcal{L}(\mathbb{E})$ and therefore $T^{**}(z)=T(z)$, $z\in\mathbb{E}$. Thus, we have established that $\Phi_s(\mathbb{E})\subseteq\Phi_s(\mathbb{E}^*)$.

We now fix $\phi\in\Phi_s(\mathbb{E}^*)$ and set $T := T^{b,a}_{\phi,\mathbb{E}^*}\in\mathcal{L}(\mathbb{E}^*)$. To prove that $\Phi_s(\mathbb{E}^*)\subseteq\Phi_s(\mathbb{E})$ it is sufficient to show that

$$T^{a,b}_{\phi,\mathbb{E}} = T^*|_{\mathbb{E}}.$$

Let us again fix $x\in\mathbb{L}^2\cap\mathbb{E}^* = \mathbb{L}^2\cap\mathbb{E}^\times$, $y\in\mathbb{L}^2\cap\mathbb{E}=\mathbb{L}^2\cap\mathbb{E}^{\times\times}$. According to Definition 2.3,

$$T(\mathbb{L}^2\cap\mathbb{E}^\times)\subseteq\mathbb{L}^2\cap\mathbb{E}^\times,\quad T^*(\mathbb{L}^2\cap\mathbb{E})\subseteq\mathbb{L}^2\cap\mathbb{E} \tag{2.5}$$

and

$$\langle T(x), y\rangle = \int\limits_{\mathbb{R}^2}\phi(\lambda,\mu)\,\mathrm{d}\langle P^b_{\mathbb{E}}\otimes Q^a_{\mathbb{E}}(x), y\rangle.$$

Taking the adjoint T^*, we obtain

$$\langle x, T^*(y)\rangle = \int\limits_{\mathbb{R}^2}\phi(\lambda,\mu)\,\mathrm{d}\langle x, P^a_{\mathbb{E}}\otimes Q^b_{\mathbb{E}}(y)\rangle.$$

Thus, we need only to show that $T^*\in\mathcal{L}(\mathbb{E})$ and $T^{**}(\mathbb{L}^2\cap\mathbb{E}^\times)\subseteq\mathbb{L}^2\cap\mathbb{E}^\times$. For the latter, it is sufficient to note that $T\in\mathcal{L}(\mathbb{E}^*)$ and therefore $T^{**}(x)=T(x)$, $x\in\mathbb{E}^*=\mathbb{E}^\times$. For the former, we first show that $T^*(\mathbb{E})\subseteq\mathbb{E}$. Indeed, suppose that $z\in\mathbb{E}$. Since E is separable, there exists a sequence $\{z_k\}_{k=1}^\infty\subseteq\mathbb{L}^2\cap\mathbb{E}$, such that $\lim\limits_{k\to\infty} z_k = z$, where the limit converges with respect to the norm topology in $\mathbb{E}$. Since $T^*\in\mathcal{L}(\mathbb{E}^{**})$ and $\mathbb{E}\subseteq\mathbb{E}^{**}$ isometrically, we obtain that

$$\lim_{k\to 0} T^*(z_k) = T^*(z), \tag{2.6}$$

where the limit converges with respect to the norm topology in $\mathbb{E}^{**}$. In particular, $\{T^*(z_k)\}_{k\geqslant 1}$ is a Cauchy sequence in $\mathbb{E}^{**}$. On the other hand, it follows from (2.5)

$$\{T^*(z_k)\}_{k=1}^\infty\subseteq\mathbb{E}.$$

Since, $\mathbb{E}\subseteq\mathbb{E}^{**}$ isometrically, the latter sequence is also Cauchy in $\mathbb{E}$. Consequently, from (2.6), $T^*(z)\in\mathbb{E}$. Thus, we showed that $T^*(\mathbb{E})\subseteq\mathbb{E}$. Let us recall that $T^*\in\mathcal{L}(\mathbb{E}^{**})$. Consequently, referring to the isometric embedding $\mathbb{E}\subseteq\mathbb{E}^{**}$ again, we obtain that $T^*\in\mathcal{L}(\mathbb{E})$. The lemma is completely proved. ∎

The main interest is attached to the class $\mathfrak{F}(\mathbb{E})$ which contains all Borel measurable function $f : \mathbb{R} \mapsto \mathbb{C}$ such that $\psi_f \in \Phi_s(\mathbb{E})$, where

$$\psi_f(\lambda, \mu) := \frac{f(\lambda) - f(\mu)}{\lambda - \mu}, \quad (\lambda, \mu) \in \mathbb{R},\ \lambda \neq \mu,\ \psi_f(\lambda, \lambda) = 0. \tag{2.7}$$

It is known that the class of all absolutely continuous functions f on $\mathbb{R}$ whose (weak) derivative has a bounded variation is a proper subclass of $\mathfrak{F}(\mathbb{L}^p)$, $1 < p < \infty$, [25]. In particular, $f(\lambda) = |\lambda|$, $\lambda \in \mathbb{R}$ belongs to $\mathfrak{F}(\mathbb{L}^p)$, $1 < p < \infty$. On the other hand, the latter function does not belong to $\mathfrak{F}(\mathbb{L}^\infty) = \mathfrak{F}(\mathbb{L}^1)$, [13]. It is unknown whether $\mathfrak{F}(\mathbb{L}^p) \neq \mathfrak{F}(\mathbb{L}^2)$, when $1 < p < 2$.

We start with the following lemma.

LEMMA 2.5 ([25], Lemma 7.1). *Let a, b be self-adjoint affiliated with $\mathcal{M}$ and let*

$$E_n^a = E^a([-n, n]), \quad E_n^b = E^b([-n, n]), \quad n \geqslant 1$$

be the corresponding spectral projections. If $f \in \mathfrak{F}(\mathbb{E})$, then for every $x \in \mathbb{E}$,

$$T^{a,b}_{\psi_f, \mathbb{E}}(aE_n^a x E_n^b - E_n^a x b E_n^b) = f(a) E_n^a x E_n^b - E_n^a x f(b) E_n^b, \quad n \geqslant 1. \tag{2.8}$$

The following proposition complements the result of Lemma 2.5. We replace the assumption $x \in \mathbb{E}$ with the assumption $ax - xb \in \mathbb{E}$. Recall that the fundamental function of a rearrangement invariant space E is given by $\phi_E(t) := \|\chi_{[0,t)}\|_E$, $t > 0$.

PROPOSITION 2.6. *Let $\mathbb{E} = (\mathbb{E}_*)^\times$, where $\mathbb{E}_*$ is a symmetric operator space with an order continuous norm and the Fatou property. If the fundamental function ϕ_E satisfies*

$$\lim_{t \to \infty} \frac{\phi_E(t)}{t} = 0, \tag{2.9}$$

then, for every complex-valued function f on $\mathbb{R}$ such that $f \in \mathfrak{F}(\mathbb{E})$, we have

$$T^{a,b}_{\psi_f, \mathbb{E}}(ax - xb) = f(a)x - xf(b), \tag{2.10}$$

for all self-adjoint operators $a, b \in \mathcal{M}$ and all operators $x \in \mathcal{M}$ such that $ax - xb \in \mathbb{E}$.

Proof. It readily follows from Proposition 6.6 in [25] that there is a net of projections $\{p_\beta\} \subseteq \mathcal{M}$ such that $p_\beta \uparrow \mathbf{1}$, $\tau(p_\beta) < \infty$ and $\|bp_\beta - p_\beta b\|_{\mathbb{E}} \leqslant 1$. Since $p_\beta \uparrow \mathbf{1}$ and $b \in \mathcal{M}$, we have $bp_\beta - p_\beta b \to 0$ in the wo-topology. Moreover, since the unit ball of $\mathbb{E}$ is compact in the $\sigma(\mathbb{E}, \mathbb{E}_*)$-topology, passing to a subnet, if necessary, we may assume that $bp_\beta - p_\beta b$ converges in the $\sigma(\mathbb{E}, \mathbb{E}_*)$-topology. Let us show that this limit is 0. We note that the $\sigma(\mathbb{E}, \mathbb{E}_*)$-topology is stronger than the $\sigma(\mathbb{E}, \mathbb{L}^1 \cap \mathbb{L}^\infty)$-topology, the operators $bp_\beta - p_\beta b$ are uniformly bounded in $\mathbb{L}^\infty$ and the $\sigma(\mathbb{L}^\infty, \mathbb{L}^1 \cap \mathbb{L}^\infty)$-topology coincides with the wo-topology on the unit ball of $\mathcal{M} = \mathbb{L}^\infty$, see Lemma 2.1. Hence, the operators $bp_\beta - p_\beta b$ tend to 0 in the $\sigma(\mathbb{E}, \mathbb{E}_*)$-topology. For every operator $y \in \mathbb{E}_*$, $y(ax - xb) \in \mathbb{L}^1$. Since $p_\beta \to \mathbf{1}$ in the $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$-topology, cf. Lemma 2.1, we obtain that $(ax - xb)p_\beta \to (ax - xb)$ in the $\sigma(\mathbb{E}, \mathbb{E}_*)$-topology.

Since $\tau(p_\beta) < \infty$, $xp_\beta \in \mathbb{E}$, applying Lemma 2.5 to the operator xp_β yields

$$T^{a,b}_{\psi_f,\mathbb{E}}(axp_\beta - xp_\beta b) = f(a)xp_\beta - xp_\beta f(b). \tag{2.11}$$

If follows from Lemma 2.4 that $f \in \mathfrak{F}(\mathbb{E}_*)$ and $T^{a,b}_{\psi_f,\mathbb{E}} = (T^{b,a}_{\psi_f,\mathbb{E}_*})^*$, in particular, the operator $T^{a,b}_{\psi_f,\mathbb{E}}$ is $\sigma(\mathbb{E},\mathbb{E}_*)$-continuous. Now it follows that the left hand side of (2.11) converges to $T^{a,b}_{\psi_f,\mathbb{E}}(ax - xb)$ in the $\sigma(\mathbb{E},\mathbb{E}_*)$-topology. Since $p_\beta \uparrow \mathbf{1}$, the right hand side converges to $f(a)x - xf(b)$ in the wo-topology and since the right hand side is uniformly bounded in $\mathcal{M}$, these limits coincide. ∎

COROLLARY 2.7. *Let a, b be self-adjoint operators affiliated with $\mathcal{M}$, E^a_n and E^b_n, $n \geqslant 1$ be spectral projections as in Lemma* 2.5 *and let $\mathbb{E}$ satisfy the assumptions of Proposition* 2.6. *If $x \in \mathcal{M}$, $aE^a_n xE^b_n - E^a_n xbE^b_n \in \mathbb{E}$, $n \geqslant 1$ and $f \in \mathfrak{F}(\mathbb{E})$, then*

$$T^{a,b}_{\psi_f,\mathbb{E}}(aE^a_n xE^b_n - E^a_n xbE^b_n) = f(a)E^a_n xE^b_n - E^a_n xf(b)E^b_n, \quad n \geqslant 1.$$

Proof. Setting $a_n = aE^a_n$, $b_n = bE^b_n$ and $x_n = E^a_n xE^b_n$, $n \geqslant 1$, we have (by assumption)

$$a_n x_n - x_n b_n = aE^a_n xE^b_n - E^a_n xbE^b_n \in \mathbb{E}.$$

Applying Proposition 2.6 to the operators a_n, b_n and x_n we obtain $T^{a_n,b_n}_{\psi_f,\mathbb{E}}(a_n x_n - x_n b_n) = f(a_n)x_n - x_n f(b_n)$. To finish the proof, we note that, according to Definition 2.3,

$$T^{a_n,b_n}_{\psi_f,\mathbb{E}}(y) = T^{a,b}_{\psi_f,\mathbb{E}}(E^a_n yE^b_n), \quad y \in \mathbb{E},\ n \geqslant 1. \quad ∎$$

The lemma below generalizes a fairly standard argument for groups of operators on symmetric operator spaces and will be used in the sequel.

LEMMA 2.8. *Let $\mathbb{E}$ and $\mathbb{F}$ be symmetric operator spaces, B be a closed ball in $\mathbb{F}$ and $\gamma := \{\gamma_t\}_{t\in\mathbb{R}}$ be a group of contractions in both $\mathbb{E}$ and $\mathbb{F}$. If γ is a strongly (respectively, weakly*) continuous group in $\mathbb{E}$,* Dom δ *is the domain of the strong (respectively, weak*) generator of γ in $\mathbb{E}$, and the function $t \to \|\gamma_t(\xi)\|_{\mathbb{F}}$ is Lebesgue measurable, for every $\xi \in B$, then the set* Dom $\delta \cap B$ *is invariant with respect to γ and norm (respectively, weak*) dense in $\mathbb{E} \cap B$. In particular, if $\mathbb{F}$ is a symmetric operator space with the Fatou norm such that $\mathbb{E} \cap \mathbb{F}$ is norm (respectively, weak*) dense in $\mathbb{E}$ and γ is $\sigma(\mathbb{F},\mathbb{F}^\times)$-continuous in $\mathbb{F}$, then the subspace* Dom$\delta\cap\mathbb{F}$ *is norm (respectively, weak*) dense in $\mathbb{E}$.*

Proof. We prove the case when γ is strongly continuous and outline the changes needed for weak* continuous group at the end of the proof.

Since the space Dom δ is invariant under γ, ([12], Lemma 1.1) , and the hypothesis $\gamma_t(B) \subseteq B$, we have $\gamma_t(\mathrm{Dom}\,\delta \cap B) \subseteq \mathrm{Dom}\,\delta \cap B$, $t \in \mathbb{R}$.

Let R_λ be the resolvent $R_\lambda = (\lambda - \delta)^{-1}$, then $R_\lambda(\xi) \in \mathrm{Dom}\,\delta$,

$$R_\lambda \xi = \int_0^\infty \mathrm{e}^{-\lambda t}\gamma_t(\xi)\,\mathrm{d}t \tag{2.12}$$

and
$$\|\cdot\|_{\mathbb{E}}\text{-}\lim_{\lambda\to\infty} \lambda R_\lambda(\xi) = \xi,$$
for every $\xi \in \mathbb{E}$, $\lambda > 0$, ([26], Lemma 1.3.2). If $\xi \in \mathbb{E} \cap B$, then $\lambda R_\lambda(\xi) \in \operatorname{Dom} \delta$ and $\lambda R_\lambda(\xi) \in B$. The last assertion follows from (2.12) and the elementary inequality
$$\|\lambda R_\lambda(\xi)\|_{\mathbb{F}} \leqslant \lambda \int_0^\infty E^{-t\lambda} \|\gamma_t(\xi)\|_{\mathbb{F}}\, \mathrm{d}t \leqslant \|\xi\|_{\mathbb{F}}, \quad \lambda > 0.$$
Therefore, for every $\xi \in \mathbb{E} \cap B$, the elements $\lambda R_\lambda(\xi)$ belong to $\operatorname{Dom} \delta \cap B$ and tend to ξ, as $\lambda \to \infty$ with respect to the norm of $\mathbb{E}$. This means that $\operatorname{Dom} \delta \cap B$ is norm dense in $\mathbb{E} \cap B$. For the second part, we note that, since $\mathbb{F}$ has the Fatou norm,
$$\begin{aligned}\tau(\eta\xi) &= \lim_{t\to 0} \tau(\eta\gamma_t(\xi)), \quad \eta \in \mathbb{F}^\times,\\ \|\xi\|_{\mathbb{F}} &= \sup_{\|\eta\|_{\mathbb{F}^\times} \leqslant 1} |\tau(\eta\xi)| \leqslant \liminf_{t\to 0} \|\gamma_t(\xi)\|_{\mathbb{F}}, \quad \xi \in \mathbb{F}.\end{aligned}$$
That means that the function $t \to \|\gamma_t(\xi)\|_{\mathbb{F}}$, $\xi \in \mathbb{F}$ is semi-continuous and, hence, measurable. The claim is proved.

For the weak* assertion, we have to apply Proposition 3.1.6 and Corollary 3.1.7 in [6], which are weak* variants of the results used above. ∎

Let us recall the notion of the left regular representation $\mathcal{M}_L$ of the algebra $\mathcal{M}$. That is, $\mathcal{M}_L$ is the algebra of all operators $\mathbf{x} \in \mathcal{L}(\mathbb{L}^2)$ given by the left multiplication $\mathbf{x}(\xi) = x\xi$ for every $\xi \in \mathbb{L}^2$, whenever $x \in \mathcal{M}$. The mapping $L : x \to \mathbf{x}$ is an $*$-isomorphism between the algebras $\mathcal{M}$ and $\mathcal{M}_L$. The trace τ_L is defined by $\tau_L(\mathbf{x}) = \tau(x)$, $x \in \mathcal{M}$. A symmetric operator space $\mathbb{E}_L = E(\mathcal{M}_L, \tau_L)$ consists of all operators $\mathbf{x}$ where $x \in \mathbb{E} = E(\mathcal{M}, \tau)$. The mapping L has a natural extension $\mathbb{E} \to \mathbb{E}_L$ which is an isometric isomorphism of Banach spaces $\mathbb{E}$ and $\mathbb{E}_L$, cf. Proposition 2.6 of [25].

3. LIPSCHITZ ESTIMATES

Throughout the section $\mathbf{a}$ and $\mathbf{b}$ are self-adjoint operators affiliated with $\mathcal{M}_L$, and $\mathbf{x} \in \mathcal{M}_L$. Let us note that the subspace
$$\mathcal{D}\mathbf{a} = \operatorname{Dom} \mathbf{a} \cap \mathbb{L}^1 \cap \mathbb{L}^\infty$$
is a core of the operator $\mathbf{a}$ for every $\mathbf{a}$ affiliated with $\mathcal{M}_L$, see Lemma 3.6 below. Since the operator $\mathbf{a}$ is affiliated with the algebra $\mathcal{M}_L$ so is the domain $\operatorname{Dom} \mathbf{a}$, i.e. for every unitary $u \in \mathcal{M}_L{}'$, $u(\operatorname{Dom} \mathbf{a}) \subseteq \operatorname{Dom} \mathbf{a}$. Consequently, the core $\mathcal{D}\mathbf{a}$ is also affiliated with $\mathcal{M}$.

We shall write $\mathbf{ax} - \mathbf{xb} \in \mathbb{E}_L$ for every $\mathbf{a}, \mathbf{b}$ affiliated with $\mathcal{M}_L$ and $\mathbf{x} \in \mathcal{M}_L$ provided $\mathbf{x}(\mathcal{D}\mathbf{b}) \subseteq \operatorname{Dom} \mathbf{a}$ and the operator $\mathbf{ax} - \mathbf{xb}$, initially defined on $\mathcal{D}\mathbf{b}$, is closable with closure, again denoted $\mathbf{ax} - \mathbf{xb}$, in $\mathbb{E}_L$. We may now formulate the main result of the section.

THEOREM 3.1. *Let $\mathbf{a}, \mathbf{b}$ be self-adjoint operators affiliated with $\mathcal{M}_L$, and $\mathbf{x} \in \mathcal{M}_L$. If $\mathbf{ax} - \mathbf{xb} \in \mathbb{L}_L^p$ and $f \in \mathfrak{F}(\mathbb{L}_L^p)$, for some $2 \leqslant p \leqslant \infty$, then $f(\mathbf{a})\mathbf{x} - \mathbf{x}f(\mathbf{b}) \in \mathbb{L}_L^p$ and*

$$f(\mathbf{a})\mathbf{x} - \mathbf{x}f(\mathbf{b}) = T^{\mathbf{a},\mathbf{b}}_{\psi_f,\mathbb{L}_L^p}(\mathbf{ax} - \mathbf{xb}).$$

In particular

$$\|f(\mathbf{a})\mathbf{x} - \mathbf{x}f(\mathbf{b})\|_{\mathbb{L}_L^p} \leqslant c_{f,p}\,\|\mathbf{ax} - \mathbf{xb}\|_{\mathbb{L}_L^p}.$$

Applying the theorem for more general symmetric operator spaces, we have

COROLLARY 3.2. *Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{x}$ be as in Theorem 3.1 and let E be a symmetric function space which is separable or dual to a separable symmetric function space with the Fatou property. If $\mathbf{ax} - \mathbf{xb} \in \mathbb{E}_L \cap \mathbb{L}_L^p$ and $f \in \mathfrak{F}(\mathbb{E}_L) \cap \mathfrak{F}(\mathbb{L}_L^p)$, for some $2 \leqslant p \leqslant \infty$, then*

$$f(\mathbf{a})\mathbf{x} - \mathbf{x}f(\mathbf{b}) = T^{\mathbf{a},\mathbf{b}}_{\psi_f,\mathbb{E}_L}(\mathbf{ax} - \mathbf{xb}).$$

In particular,

$$\|f(\mathbf{a})\mathbf{x} - \mathbf{x}f(\mathbf{b})\|_{\mathbb{E}_L} \leqslant c_{f,E}\,\|\mathbf{ax} - \mathbf{xb}\|_{\mathbb{E}_L}.$$

REMARK 3.3. Suppose that E is a separable r.i. function space with non-trivial Boyd indices (see, e.g. [21]). It follows from [15] and [25] that the assumptions above are satisfied for the absolute value function.

Proof of Corollary 3.2. We only need to show that $T^{\mathbf{a},\mathbf{b}}_{\psi_f,\mathbb{E}_L}(\mathbf{y}) = T^{\mathbf{a},\mathbf{b}}_{\psi_f,\mathbb{L}_L^p}(\mathbf{y})$ for every $\mathbf{y} \in \mathbb{L}_L^p \cap \mathbb{E}_L$. First, it immediately follows from Definition 2.3, that, for any two symmetric spaces $\mathbb{E}_L$ and $\mathbb{F}_L$, and $f \in \mathfrak{F}(\mathbb{E}_L) \cap \mathfrak{F}(\mathbb{F}_L)$, we have

$$T^{\mathbf{a},\mathbf{b}}_{\psi_f,\mathbb{E}_L}(\mathbf{y}) = T^{\mathbf{a},\mathbf{b}}_{\psi_f,\mathbb{F}_L}(\mathbf{y}), \quad \mathbf{y} \in \mathbb{L}_L^1 \cap \mathbb{L}_L^\infty \subseteq \mathbb{E}_L \cap \mathbb{F}_L \cap \mathbb{L}_L^2. \tag{3.1}$$

Let us now note that, according to Lemma 2.4, applied to the spaces $\mathbb{E}_L$ and $\mathbb{F}_L$, together with (3.1), $f \in \mathfrak{F}(\mathbb{E}_L) \cap \mathfrak{F}(\mathbb{L}_L^p) = \mathfrak{F}(\mathbb{E}_L^\times) \cap \mathfrak{F}(\mathbb{L}_L^{p'})$, where p' is the conjugate exponent, and there exist bounded operators $T_1 : \mathbb{L}_L^{p'} \to \mathbb{L}_L^{p'}$, $T_2 : \mathbb{E}_L^\times \to \mathbb{E}_L^\times$ such that

$$T^{\mathbf{a},\mathbf{b}}_{\psi_f,\mathbb{E}_L}(\mathbf{y}) = T_1^*(\mathbf{y}),\ \ \mathbf{y} \in \mathbb{E}_L; \quad T^{\mathbf{a},\mathbf{b}}_{\psi_f,\mathbb{L}_L^p}(\mathbf{y}) = T_2^*(\mathbf{y}),\ \ \mathbf{y} \in \mathbb{L}_L^p;$$
$$T_1(\mathbf{y}) = T_2(\mathbf{y}),\ \ \mathbf{y} \in \mathbb{L}_L^1 \cap \mathbb{L}_L^\infty.$$

Hence, for every $\mathbf{y} \in \mathbb{L}_L^p \cap \mathbb{E}_L$ and $\mathbf{z} \in \mathbb{L}_L^1 \cap \mathbb{L}_L^\infty$, we obtain

$$\tau(\mathbf{z}T^{\mathbf{a},\mathbf{b}}_{\psi_f,\mathbb{E}_L}(\mathbf{y})) = \tau(\mathbf{y}T_1(\mathbf{z})) = \tau(\mathbf{y}T_2(\mathbf{z})) = \tau(\mathbf{z}T^{\mathbf{a},\mathbf{b}}_{\psi_f,\mathbb{L}_L^p}(\mathbf{y})).$$

Since $\mathbb{L}_L^1 \cap \mathbb{L}_L^\infty$ separates points in $\mathbb{L}_L^1 + \mathbb{L}_L^\infty$, it follows that

$$T^{\mathbf{a},\mathbf{b}}_{\psi_f,\mathbb{E}_L}(\mathbf{y}) = T^{\mathbf{a},\mathbf{b}}_{\psi_f,\mathbb{L}_L^p}(\mathbf{y}), \quad \mathbf{y} \in \mathbb{L}_L^p \cap \mathbb{E}_L. \qquad \blacksquare$$

If we take into account that $\Phi(\mathbb{E}_L) \subseteq \Phi(\mathbb{L}^2_L) = B_{\mathbb{C}}(\mathbb{R}^2)$, we obtain the result for L^p-spaces with $1 \leqslant p < 2$, namely

COROLLARY 3.4. *Let* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{x}$ *be as in Theorem* 3.1. *If* $\mathbf{ax} - \mathbf{xb} \in \mathbb{L}^p_L \cap \mathbb{L}^2_L$, *and* $f \in \mathfrak{F}(\mathbb{L}^p_L)$, $1 \leqslant p < 2$, *then*

$$f(\mathbf{a})\mathbf{x} - \mathbf{x}f(\mathbf{b}) = T^{\mathbf{a},\mathbf{b}}_{\psi_f,\mathbb{L}^p_L}(\mathbf{ax} - \mathbf{xb}).$$

In particular,

$$\|f(\mathbf{a})\mathbf{x} - \mathbf{x}f(\mathbf{b})\|_{\mathbb{L}^p_L} \leqslant c_{f,p}\, \|\mathbf{ax} - \mathbf{xb}\|_{\mathbb{L}^p_L}.$$

REMARKS 3.5. (i) In the special case when $x = \mathbf{1}$, Corollaries 3.2, 3.4 and Theorem 3.1 are reduced to inequalities

$$\|f(\mathbf{a}) - f(\mathbf{b})\|_{\mathbb{L}^p_L} \leqslant c_{f,p}\, \|\mathbf{a} - \mathbf{b}\|_{\mathbb{L}^p_L}, \quad 1 < p < \infty,$$

provided $\mathbf{a} - \mathbf{b} \in \mathbb{L}^p_L \cap \mathbb{L}^2_L$, and $f \in \mathfrak{F}(\mathbb{L}^p_L)$, where $\mathbf{a}, \mathbf{b}$ are arbitrary self-adjoint operators affiliated with $\mathcal{M}_L$, and $c_{f,p}$ is a constant depending of f and p only. This result extends Corollary 7.5 of [25] and Corollary 3.5 of [15].

(ii) If $\mathbf{a} = \mathbf{b}$ are as in Remark 3.5(i) above and $x \in \mathcal{M}_L$, then

$$\|[|\mathbf{a}|, \mathbf{x}]\|_{\mathbb{L}^p_L} \leqslant c_p\, \|[\mathbf{a}, \mathbf{x}]\|_{\mathbb{L}^p_L}, \quad 1 < p < \infty,$$

provided $[\mathbf{a}, \mathbf{x}] \in \mathbb{L}^p_L \cap \mathbb{L}^2_L$, where c_p is a constant depending on p only. This complements the result of Theorem 2.2 in [15] and provides a type II extension of (6.6) in [5].

(iii) If, in addition, $\mathbf{a} = \mathbf{b}$ has a bounded inverse, then for every symmetric operator space $\mathbb{E}_L$, we have

$$\|[|\mathbf{a}|^r, \mathbf{x}]\|_{\mathbb{E}_L} \leqslant c_{E,\mathbf{a},r}\, \|[\mathbf{a}, \mathbf{x}]\|_{\mathbb{E}_L}, \quad 0 < r \leqslant 1,$$

whenever $[\mathbf{a}, \mathbf{x}] \in \mathbb{E}_L \cap \mathbb{L}^p_L$, for some $2 \leqslant p \leqslant \infty$, where the constant $c_{E,\mathbf{a},r}$ does not depend on $\mathbf{x}$. This result extends similar inequalities for the case $E = L^\infty$, obtained earlier in Lemma 1.4 of [9] (see also [11], [29]) by different methods.

LEMMA 3.6. *If* $\mathbf{a}$ *is affiliated with* $\mathcal{M}_L$, *then the subspace* $\mathcal{D}\mathbf{a} = \mathrm{Dom}\, \mathbf{a} \cap \mathbb{L}^1 \cap \mathbb{L}^\infty$ *is a core of* $\mathbf{a}$.

Proof. Since $\mathbf{a}$ is affiliated with $\mathcal{M}_L$, it follows that $\mathrm{e}^{\mathrm{i}t\mathbf{a}} \in \mathcal{M}_L$ and the latter means that the group $\{\mathrm{e}^{\mathrm{i}t\mathbf{a}}\}_{t\in\mathbb{R}}$ may be extended from $\mathbb{L}^2$ to a group of contractions in every symmetric operator space $\mathbb{E}$. In particular, $\{\mathrm{e}^{\mathrm{i}t\mathbf{a}}\}_{t\in\mathbb{R}}$ is a group of contractions in $\mathbb{L}^1 \cap \mathbb{L}^\infty$. Applying Lemma 2.8 to the group $\{\mathrm{e}^{\mathrm{i}t\mathbf{a}}\}_{t\in\mathbb{R}}$, $\mathbb{E} = \mathbb{L}^2$ and $\mathbb{F} = \mathbb{L}^1 \cap \mathbb{L}^\infty$, we obtain that $\mathcal{D}\mathbf{a}$ is dense in $\mathbb{L}^2$. On the other hand, $\mathcal{D}\mathbf{a}$ is invariant with respect to $\mathrm{e}^{\mathrm{i}t\mathbf{a}}$, $t \in \mathbb{R}$. Thus, it follows from Theorem 1.9 of [12] that $\mathcal{D}\mathbf{a}$ is a core of $\mathbf{a}$. ∎

Proof of Theorem 3.1. Let $E^{\mathbf{a}}_n$ and $E^{\mathbf{b}}_n$, $n \geqslant 1$ be the spectral projections of the operators $\mathbf{a}, \mathbf{b}$ as in Lemma 2.5. Since $E^{\mathbf{b}}_n(\mathbb{L}^2) \subseteq \mathrm{Dom}\, \mathbf{b}$, we have that $E^{\mathbf{b}}_n(\mathbb{L}^1 \cap$

$\mathbb{L}^\infty) \subseteq \mathcal{D}\mathbf{b}$, for every $n \geqslant 1$. Hence, according to definition of $\mathbf{ax} - \mathbf{xb} \in \mathbb{L}^p_L$, we have

$$E^{\mathbf{a}}_n(\mathbf{ax} - \mathbf{xb})E^{\mathbf{b}}_n(\xi) = E^{\mathbf{a}}_n\mathbf{ax}E^{\mathbf{b}}_n(\xi) - E^{\mathbf{a}}_n\mathbf{xb}E^{\mathbf{b}}_n(\xi), \quad \xi \in \mathbb{L}^1 \cap \mathbb{L}^\infty, n \geqslant 1.$$

The operator on the right is bounded. Since $\mathbb{L}^1 \cap \mathbb{L}^\infty$ is dense in $\mathbb{L}^2$, we have that the operator on the left is also bounded and

$$E^{\mathbf{a}}_n(\mathbf{ax} - \mathbf{xb})E^{\mathbf{b}}_n = E^{\mathbf{a}}_n\mathbf{ax}E^{\mathbf{b}}_n - E^{\mathbf{a}}_n\mathbf{xb}E^{\mathbf{b}}_n, \quad n \geqslant 1. \tag{3.2}$$

We claim that

$$\sigma(\mathbb{L}^p_L, \mathbb{L}^{p'}_L) - \lim_{n\to\infty} E^{\mathbf{a}}_n(\mathbf{ax} - \mathbf{xb})E^{\mathbf{b}}_n = \mathbf{ax} - \mathbf{xb} \in \mathbb{L}^p_L,$$

where p' is the conjugate exponent. Indeed, if $p < \infty$, then the space $\mathbb{L}^p$ is separable and this follows from Corollary 2.3 in [8]; if $p = \infty$ then this follows from Lemma 2.1. Since the operator $T^{\mathbf{a},\mathbf{b}}_{\psi_f, \mathbb{L}^p_L}$ is $\sigma(\mathbb{L}^p_L, \mathbb{L}^{p'}_L)$-continuous (this follows from Lemma 2.4) we may write that

$$\sigma(\mathbb{L}^p_L, \mathbb{L}^{p'}_L) - \lim_{n\to\infty} \mathbf{z}_n = \mathbf{z}, \quad \mathbf{z} \in \mathbb{L}^p_L, \tag{3.3}$$

where

$$\mathbf{z}_n := T^{\mathbf{a},\mathbf{b}}_{\psi_f, \mathbb{L}^p_L}(E^{\mathbf{a}}_n(\mathbf{ax} - \mathbf{xb})E^{\mathbf{b}}_n), \quad \mathbf{z} := T^{\mathbf{a},\mathbf{b}}_{\psi_f, \mathbb{L}^p_L}(\mathbf{ax} - \mathbf{xb}).$$

On the other hand, according to Corollary 2.7 and (3.2) we obtain

$$\mathbf{z}_n = E^{\mathbf{a}}_n f(\mathbf{a})\mathbf{x}E^{\mathbf{b}}_n - E^{\mathbf{a}}_n\mathbf{x}f(\mathbf{b})E^{\mathbf{b}}_n \in \mathcal{M}_L, \quad n \geqslant 1.$$

We next consider the bilinear form

$$\langle \mathbf{z}_n(\xi), \eta\rangle = \langle \mathbf{x}E^{\mathbf{b}}_n(\xi), E^{\mathbf{a}}_n\overline{f}(\mathbf{a})(\eta)\rangle - \langle E^{\mathbf{b}}_n f(\mathbf{b})(\xi), \mathbf{x}^* E^{\mathbf{a}}_n(\eta)\rangle,$$

for every $\xi \in \mathcal{D}f(\mathbf{b})$, $\eta \in \mathcal{D}\overline{f}(\mathbf{a})$, $n \geqslant 1$. By the definition of the sets $\mathcal{D}f(\mathbf{b})$ and $\mathcal{D}\overline{f}(\mathbf{a})$, we have $\xi\eta^* \in \mathbb{L}^1_L \cap \mathbb{L}^\infty_L \subseteq \mathbb{L}^{p'}$ for every $\xi \in \mathcal{D}f(\mathbf{b}), \eta \in \mathcal{D}\overline{f}(\mathbf{a})$. Therefore, it follows from (3.3) that

$$\langle \mathbf{z}_n(\xi), \eta\rangle = \tau_L(\mathbf{z}_n L(\xi\eta^*)) \to \tau_L(\mathbf{z}L(\xi\eta^*)) = \langle \mathbf{z}(\xi), \eta\rangle, \quad t \to \infty.$$

Hence, passing to the limit on the right, we obtain

$$\langle \mathbf{z}(\xi), \eta\rangle = \langle \mathbf{x}(\xi), \overline{f}(\mathbf{a})(\eta)\rangle - \langle f(\mathbf{b})(\xi), \mathbf{x}^*(\eta)\rangle, \quad \xi \in \mathcal{D}f(\mathbf{b}), \eta \in \mathcal{D}\overline{f}(\mathbf{a}). \tag{3.4}$$

For a fixed $\xi \in \mathcal{D}f(\mathbf{b})$, the linear form $\langle f(\mathbf{b})(\xi), \mathbf{x}^*(\eta)\rangle$ is continuous with respect to $\eta \in \mathbb{L}^2$. Furthermore, the linear form $\langle \mathbf{z}(\xi), \eta\rangle$ is also continuous with respect to $\eta \in \mathbb{L}^2$, since $2 \leqslant p \leqslant \infty$ and (2.1) yield

$$|\langle \mathbf{z}(\xi), \eta\rangle| = |\tau(z\xi\eta^*)| \leqslant \|z\xi\|_{\mathbb{L}^2} \|\eta\|_{\mathbb{L}^2} \leqslant \|z\|_{\mathbb{L}^p} \|\xi\|_{\mathbb{L}^1\cap\mathbb{L}^\infty} \|\eta\|_{\mathbb{L}^2}$$

and so

$$\|z\|_{\mathbb{L}^p} \|\xi\|_{\mathbb{L}^1\cap\mathbb{L}^\infty} < \infty.$$

Consequently, the linear form $\langle \mathbf{x}(\xi), \overline{f}(\mathbf{a})(\eta)\rangle$ is continuous with respect to $\eta \in \mathcal{D}\overline{f}(\mathbf{a})$. Since $\mathcal{D}\overline{f}(\mathbf{a})$ is a core of $\overline{f}(\mathbf{a})$, this implies that $\mathbf{x}(\xi) \in \operatorname{Dom} f(\mathbf{a})$ and therefore $\mathbf{x}(\mathcal{D}f(\mathbf{b})) \subseteq \operatorname{Dom} f(\mathbf{a})$. Now (3.4) becomes

$$\mathbf{z}(\xi) = f(\mathbf{a})\mathbf{x}(\xi) - \mathbf{x}f(\mathbf{b})(\xi), \quad \xi \in \mathcal{D}f(\mathbf{b}).$$

Since $\mathcal{D}f(\mathbf{b})$ is affiliated with $\mathcal{M}$, the closure of the operator on the right is $\mathbf{z}$, see Lemma 2.2. The proof is finished. ∎

4. COMMUTATOR ESTIMATES

In this section we consider the commutators $[\mathbf{A}, \mathbf{x}]$ and $[\mathbf{A}, f(\mathbf{x})]$ when $\mathbf{x} = \mathbf{x}^* \in \mathcal{M}_L$ and $\mathbf{A} = \mathbf{A}^*$ is a densely defined operator on $\mathbb{L}^2$ such that:

(i) $e^{it\mathbf{A}}\mathbf{x}e^{-it\mathbf{A}} \in \mathcal{M}_L$ and $\tau_L(e^{it\mathbf{A}}\mathbf{x}e^{-it\mathbf{A}}) = \tau_L(\mathbf{x})$, for all $\mathbf{x} \in \mathcal{M}_L$.

If $\mathbf{A}$ is affiliated with $\mathcal{M}_L$, then this assumption is satisfied automatically. Note that, according to the assumption, the group of isometries $\mathbf{x} \to e^{it\mathbf{A}}\mathbf{x}e^{-it\mathbf{A}}$, $\mathbf{x} \in \mathcal{M}_L$ induces naturally a group $\gamma : x \to \gamma_t(x) := L^{-1}(e^{it\mathbf{A}}L(x)e^{-it\mathbf{A}})$, $x \in \mathcal{M}$ of isometries of the original algebra $\mathcal{M}$, such that $\tau(\gamma_t(x)) = \tau(x)$, $t \in \mathbb{R}$. The group $\{\gamma_t\}_{t\in\mathbb{R}}$ is an $\mathbb{R}$-flow, i.e. an ultra-weakly continuous representation of $\mathbb{R}$ on $\mathcal{M}$ by trace preserving $*$-automorphisms of $\mathcal{M}$. This group extends uniquely to a group of $*$-automorphisms of any symmetric operator space, [24]. We also require that

(ii) $e^{it\mathbf{A}}(\xi) = \gamma_t(\xi)$, for every $\xi \in \mathbb{L}^1 \cap \mathbb{L}^\infty$.

The condition (ii) is used in the proof of Lemma 4.4 below; if $\mathbf{A}$ is affiliated with $\mathcal{M}_L$, then the assertion of that lemma is covered by that of Lemma 3.6. We note that (ii) implies that the group $\{e^{it\mathbf{A}}\}_{t\in\mathbb{R}}$ induces a group of contractions on every symmetric operator space. The latter is the only fact used in the proof of Lemma 4.4.

Following Proposition 3.2.55 in [6], we will write $[\mathbf{A}, \mathbf{x}] \in \mathbb{E}$ for $\mathbf{x} \in \mathcal{M}_L$ if and only if

(iii) There is a core $D \subseteq \mathbb{L}^1 \cap \mathbb{L}^\infty$ of $\mathbf{A}$ which is invariant under $e^{it\mathbf{A}}$ for every $t \in \mathbb{R}$, i.e. $e^{it\mathbf{A}}(D) \subseteq D$.

(iv) The operator $\mathbf{x}$ sends D into $\operatorname{Dom} \mathbf{A}$, i.e. $\mathbf{x}(D) \subseteq \operatorname{Dom} \mathbf{A}$ and the operator $\mathbf{A}\mathbf{x} - \mathbf{x}\mathbf{A}$ is closable up to an operator in $\mathbb{E}_L$.

In the case when $[\mathbf{A}, \mathbf{x}]$ happens to be bounded, the operator $\mathbf{x}$ preserves the whole domain $\operatorname{Dom} \mathbf{A}$, i.e. $\mathbf{x}(\operatorname{Dom} \mathbf{A}) \subseteq \operatorname{Dom} \mathbf{A}$. We give a short proof of this fact in Lemma 5.1 below.

The main result of the section is

THEOREM 4.1. *Let $\mathbb{E}_L$ be a symmetric operator space with the Fatou norm. If $\mathbf{x} \in \mathcal{M}_L$ is a self-adjoint operator such that $[\mathbf{A}, \mathbf{x}] \in \mathbb{L}^p_L \cap \mathbb{E}_L$, for some $2 \leqslant p \leqslant \infty$,* *then for every complex-valued function f on $\mathbb{R}$ such that $f \in \mathfrak{F}(\mathbb{L}^p_L) \cap \mathfrak{F}(\mathbb{E}_L)$, we have $[\mathbf{A}, f(\mathbf{x})] \in \mathbb{L}^p_L \cap \mathbb{E}_L$ and*

$$\|[\mathbf{A}, f(\mathbf{x})]\|_{\mathbb{E}_L} \leqslant c_{f,E}\, \|[\mathbf{A}, \mathbf{x}]\|_{\mathbb{E}_L},$$

where

$$c_{f,E} = \sup_{\mathbf{x},\mathbf{x}'} \|T^{\mathbf{x},\mathbf{x}'}_{\psi_f,\mathbb{E}}\|_{\mathbb{E}_L \to \mathbb{E}_L}$$

and the latter maximum runs over all operators $\mathbf{x}$ and $\mathbf{x}'$ affiliated with $\mathcal{M}_L$.

Applying this result for the spaces $\mathbb{E}_L = \mathbb{L}^p_L$, $1 \leqslant p \leqslant \infty$ and taking into account that $\Phi(\mathbb{E}) \subseteq \Phi(\mathbb{L}^2_L) = B_{\mathbb{C}}(\mathbb{R}^2)$, we obtain

COROLLARY 4.2. *If $2 \leqslant p \leqslant \infty$ (respectively $1 \leqslant p < 2$), $f \in \mathfrak{F}(\mathbb{L}^p_L)$ and $\mathbf{x} \in \mathcal{M}_L$ such that $[\mathbf{A}, \mathbf{x}] \in \mathbb{L}^p_L$ (respectively $[\mathbf{A}, \mathbf{x}] \in \mathbb{L}^2_L \cap \mathbb{L}^p_L$), then*

$$\|[\mathbf{A}, f(\mathbf{x})]\|_{\mathbb{L}^p_L} \leqslant c_{f,p}\, \|[\mathbf{A}, \mathbf{x}]\|_{\mathbb{L}^p_L}.$$

REMARK 4.3. Corollary 4.2 extends Corollaries 6.9 and 7.5 of [24].

Before we proceed with Theorem 4.1, we prove two auxiliary lemmas. We firstly show that the assumptions (i) and (ii) guarantee that Dom $\mathbf{A} \cap \mathbb{L}^1 \cap \mathbb{L}^\infty$ is a core of $\mathbf{A}$. This is similar to the assertion of Lemma 3.6 of the previous section. Moreover, although the operator $\mathbf{A}$ may not be affiliated with $\mathcal{M}_L$, the closure of $\mathcal{D}\mathbf{A}$ in an appropriate weak topology is affiliated with $\mathcal{M}_L$. The exact statement follows.

LEMMA 4.4. *The subspace $\mathcal{D}\mathbf{A} :=$ Dom $\mathbf{A} \cap \mathbb{L}^1 \cap \mathbb{L}^\infty \subseteq \mathbb{L}^2$ is a core of the operator $\mathbf{A}$. Furthermore, $\mathcal{D}\mathbf{A} \cap \mathrm{B}_2$ is $\sigma(\mathbb{L}^1 \cap \mathbb{L}^\infty, \mathbb{L}^2 + \mathbb{L}^p)$-dense in $\mathbb{L}^1 \cap \mathbb{L}^\infty \cap \mathrm{B}_2$, for every $1 \leqslant p \leqslant 2$, where B_2 is the closed unit ball in $\mathbb{L}^2$. In particular, for every $\eta \in \mathbb{L}^1 \cap \mathbb{L}^\infty$ and every $1 \leqslant p \leqslant 2$, there is a net $\{\eta_\alpha\} \subseteq \mathcal{D}\mathbf{A}$ such that $\eta_\alpha \to \eta$ in the $\sigma(\mathbb{L}^1 \cap \mathbb{L}^\infty, \mathbb{L}^p)$-topology and $\sup_\alpha \|\eta_\alpha\|_{\mathbb{L}^2} \leqslant \|\eta\|_{\mathbb{L}^2}$.*

Proof. The first part is similar to the proof of Lemma 3.6, we leave details to the reader.

For the second part, we consider the group γ in the space $\mathbb{E} = \mathbb{L}^2 \cap \mathbb{L}^{p'}$, where p' is the conjugate exponent. If $\mathbb{E}_* = \mathbb{L}^2 + \mathbb{L}^p$, then $\mathbb{E} = (\mathbb{E}_*)^*$ and $\mathbb{E}^\times = \mathbb{E}_*$. According to Proposition 4.2 of [24], γ is weak* (= $\sigma(\mathbb{E}, \mathbb{E}^\times)$) continuous. Let δ be the generator of γ. According to Corollary 1.3.8 of [6], the set Dom $\mathbf{A}$ may be considered as a domain of generator of the weakly continuous group $\xi \to \mathrm{e}^{\mathrm{i}t\mathbf{A}}(\xi)$ in the space $\mathbb{L}^2$. Hence, from (ii) and $\mathbb{L}^2 \subseteq \mathbb{E}_*$, Dom $\delta \cap B^{(r)} \subseteq$ Dom $\mathbf{A} \cap B^{(r)}$, where $B^{(r)} = B_2 \cap B^{(r)}_{\mathbb{L}^1 \cap \mathbb{L}^\infty}$ and $B^{(r)}_{\mathbb{L}^1 \cap \mathbb{L}^\infty}$ is the ball of radius r in $\mathbb{L}^1 \cap \mathbb{L}^\infty$. Applying Lemma 2.8 to the space $\mathbb{E}$ and the set $B^{(r)}$, we obtain that the set Dom $\delta \cap B^{(r)} \subseteq$ Dom $\mathbf{A} \cap B^{(r)}$ is $\sigma(\mathbb{E}, \mathbb{E}_*)$-dense in $\mathbb{E} \cap B^{(r)}$. Taking the union over all $r > 0$, we obtain that $\mathcal{D}\mathbf{A} \cap B_2$ is $\sigma(\mathbb{E}, \mathbb{E}_*)$-dense in $\mathbb{L}^1 \cap \mathbb{L}^\infty \cap B_2$. ∎

The following lemma extends Theorem 7.3 of [24], where it is assumed that $[A, x] \in \mathcal{M} \cap \mathbb{E}$. We assume below only that $[\mathbf{A}, \mathbf{x}] \in \mathbb{E}_L$ and this relaxation is possible because the underlying Hilbert space $\mathbb{L}^2$ for the algebra $\mathcal{M}_L$ possesses an additional Banach structure of symmetric operator spaces, in particular $\mathbb{L}^1 \cap \mathbb{L}^\infty \subseteq \mathbb{L}^2$. We also use the assumption (iii) in a crucial way.

LEMMA 4.5. *Let E be a separable symmetric function space or $E = (E_*)^*$ where E_* is a separable symmetric function space with the Fatou property. If $\mathbf{x} \in \mathcal{M}_L$ and $[\mathbf{A}, \mathbf{x}] \in \mathbb{E}_L$, then for every $t \in \mathbb{R}$,*

$$\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}} - \mathbf{x} = \sigma(\mathbb{E}_L, \mathbb{E}_L^\times) - \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}s\mathbf{A}}[\mathbf{A}, \mathbf{x}]\mathrm{e}^{-\mathrm{i}s\mathbf{A}}\,\mathrm{d}s. \tag{4.1}$$

In particular,

$$\Big\| \frac{\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}} - \mathbf{x}}{\mathrm{i}t} \Big\|_{\mathbb{E}_L} \leqslant \|[\mathbf{A}, \mathbf{x}]\|_{\mathbb{E}_L}, \quad t \in \mathbb{R}$$

and

$$\sigma(\mathbb{E}_L, \mathbb{E}_L^\times) - \lim_{t \to 0} \frac{\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}} - \mathbf{x}}{\mathrm{i}t} = [\mathbf{A}, \mathbf{x}].$$

If the space E is separable, the integral in (4.1) *and the limit above may be taken in the norm of $\mathbb{E}_L$.*

Proof. The proof is similar to that of Theorem 7.3 of [24]. Since $[\mathbf{A}, \mathbf{x}] \in \mathbb{E}_L$, according to (iii)–(iv), there is a core $D \subseteq \mathbb{L}^1 \cap \mathbb{L}^\infty$ of $\mathbf{A}$ such that $\mathrm{e}^{\mathrm{i}t\mathbf{A}}(D) \subseteq D$, $t \in \mathbb{R}$ and $\mathbf{x}(D) \subseteq \mathrm{Dom}\,\mathbf{A}$. Fix $\xi \in D$ and consider the function $u(t) = \mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\xi)$, $t \in \mathbb{R}$ with values in $\mathbb{L}^2$. We have

$$\begin{aligned} \frac{u(t+s) - u(t)}{s} &= \mathrm{e}^{\mathrm{i}(t+s)\mathbf{A}}\mathbf{x}\frac{\mathrm{e}^{-\mathrm{i}s\mathbf{A}}(\mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\xi)) - \mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\xi)}{s} \\ &\quad + \mathrm{e}^{\mathrm{i}t\mathbf{A}}\frac{\mathrm{e}^{\mathrm{i}s\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\xi) - \mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\xi)}{s}, \quad t \in \mathbb{R},\ s \neq 0. \end{aligned}$$

Since $\xi \in D$, we have $\mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\xi) \in D$ and $\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\xi) \in \mathrm{Dom}\,\mathbf{A}$. Hence,

$$\begin{aligned} u'(t) &= \lim_{s \to 0} \frac{u(t+s) - u(t)}{s} = \mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}(-\mathrm{i}\mathbf{A}\mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\xi)) + \mathrm{e}^{\mathrm{i}t\mathbf{A}}(\mathrm{i}\mathbf{A})(\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\xi)) \\ &= \mathrm{i}\,\mathrm{e}^{\mathrm{i}t\mathbf{A}}[\mathbf{A}, \mathbf{x}]\mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\xi), \quad t \in \mathbb{R},\ \xi \in D. \end{aligned}$$

Since $[\mathbf{A}, \mathbf{x}] \in \mathbb{E}_L$, the function

$$t \to \mathrm{e}^{\mathrm{i}t\mathbf{A}}[\mathbf{A}, \mathbf{x}]\mathrm{e}^{-\mathrm{i}t\mathbf{A}} \tag{4.2}$$

is continuous with respect to the $\sigma(\mathbb{E}_L, \mathbb{E}_L^\times)$-topology (cf. Proposition 4.2 of [24]). In particular, for any fixed $\xi \in D$ and $\eta \in \mathbb{L}^1 \cap \mathbb{L}^\infty$, the scalar function $t \to$

$\langle \mathrm{e}^{\mathrm{i}t\mathbf{A}}[\mathbf{A},\mathbf{x}]\mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\xi),\eta\rangle$ is continuous. Now, it follows from the Fundamental Theorem of Calculus that, for every fixed $t \in \mathbb{R}$,

$$\begin{aligned}(4.3)\quad &\langle \mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\xi),\eta\rangle - \langle \mathbf{x}(\xi),\eta\rangle \\ &= \langle u(t),\eta\rangle - \langle u(0),\eta\rangle = \mathrm{i}\int_0^t \langle \mathrm{e}^{\mathrm{i}s\mathbf{A}}[\mathbf{A},\mathbf{x}]\mathrm{e}^{-\mathrm{i}s\mathbf{A}}(\xi),\eta\rangle \mathrm{d}s, \quad \xi \in D, \eta \in \mathbb{L}^1 \cap \mathbb{L}^\infty.\end{aligned}$$

If the Riemannian sums are given by

$$S_n = \frac{\mathrm{i}}{n}\sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}\frac{kt}{n}\mathbf{A}}[\mathbf{A},\mathbf{x}]\mathrm{e}^{-\mathrm{i}\frac{kt}{n}\mathbf{A}} \in \mathbb{E}_L,$$

then $\langle S_n(\xi),\eta\rangle$ converges to the integral in (4.3). If E is dual to a separable symmetric space with the Fatou property, then, since the function given in (4.2) is $\sigma(\mathbb{E}_L,\mathbb{E}_L^\times)$-continuous and the unit ball of $\mathbb{E}$ is $\sigma(\mathbb{E}_L,\mathbb{E}_L^\times)$-complete (this ball is actually $\sigma(\mathbb{E}_L,\mathbb{E}_L^\times)$-compact according to Banach-Alaoglu theorem), the Riemannian sums S_n converge to

$$S = \sigma(\mathbb{E}_L,\mathbb{E}_L^\times) - \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}s\mathbf{A}}[\mathbf{A},\mathbf{x}]\mathrm{e}^{-\mathrm{i}s\mathbf{A}}\,\mathrm{d}s \in \mathbb{E}_L \tag{4.4}$$

in the $\sigma(\mathbb{E}_L,\mathbb{E}_L^\times)$-topology. If E is separable, then the function (4.2) is continuous in the norm of $\mathbb{E}_L$, see Corollary 4.3 of [24]. Therefore, the Riemannian sums S_n converge to the integral (4.4) in the norm of $\mathbb{E}_L$ and the integral itself may be understood as a Bochner integral of the function (4.2) in $\mathbb{E}_L$. In either case, it follows that

$$\langle S(\xi),\eta\rangle = \mathrm{i}\int_0^t \langle \mathrm{e}^{\mathrm{i}s\mathbf{A}}[\mathbf{A},\mathbf{x}]\mathrm{e}^{-\mathrm{i}s\mathbf{A}}(\xi),\eta\rangle\,\mathrm{d}s, \quad \xi \in D \subseteq \mathbb{L}^1 \cap \mathbb{L}^\infty,\ \eta \in \mathbb{L}^1 \cap \mathbb{L}^\infty.$$

Combining this with (4.3), we have

$$\langle S(\xi),\eta\rangle = \langle (\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}} - \mathbf{x})(\xi),\eta\rangle, \quad \xi \in D,\ \eta \in \mathbb{L}^1 \cap \mathbb{L}^\infty.$$

Since $\mathbb{L}^1 \cap \mathbb{L}^\infty$ separates points in $\mathbb{L}^1 + \mathbb{L}^\infty$, this implies

$$S(\xi) = \mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\xi) - \mathbf{x}(\xi), \quad \xi \in D.$$

The operator on the right is bounded, and since S is closed, we obtain that S is also bounded and S coincides with $\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}} - \mathbf{x}$. This establishes (4.1). The function (4.2) is $\sigma(\mathbb{E}_L,\mathbb{E}_L^\times)$-continuous, therefore it follows from (4.1) that

$$\frac{1}{\mathrm{i}t}\tau((\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}} - \mathbf{x} - \mathrm{i}t[\mathbf{A},\mathbf{x}])\mathbf{x}') = \frac{1}{\mathrm{i}t}\int_0^t \tau((\mathrm{e}^{\mathrm{i}s\mathbf{A}}[\mathbf{A},\mathbf{x}]\mathrm{e}^{-\mathrm{i}s\mathbf{A}} - [\mathbf{A},\mathbf{x}])\mathbf{x}')\mathrm{d}s \to 0 \quad \text{as } t \to 0$$

for every $\mathbf{x}' \in \mathbb{E}_L^\times$. We see that $(\mathrm{i}t)^{-1}(\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}} - \mathbf{x})$ converges to $[\mathbf{A},\mathbf{x}]$ with respect to the $\sigma(\mathbb{E}_L,\mathbb{E}_L^\times)$-topology. When E is separable, a similar argument shows that $(\mathrm{i}t)^{-1}(\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}} - \mathbf{x})$ converges to $[\mathbf{A}, \mathbf{x}]$ in the norm of $\mathbb{E}_L$. Further, for every $\mathbf{x}' \in \mathbb{E}_L^{\times}$,

$$|\tau((\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}} - \mathbf{x})\mathbf{x}')| = \Big| \int_0^t \tau(\mathrm{e}^{\mathrm{i}s\mathbf{A}}[\mathbf{A}, \mathbf{x}]\mathrm{e}^{-\mathrm{i}s\mathbf{A}}\mathbf{x}')\, \mathrm{d}s \Big| \leqslant |t|\, \|[\mathbf{A}, \mathbf{x}]\|_{\mathbb{E}_L}\, \|\mathbf{x}'\|_{\mathbb{E}_L^{\times}}.$$

The norm estimate in the lemma follows. ∎

Proof of Theorem 4.1. If follows from Lemma 4.5 that the operators

$$\mathbf{y}_t = \frac{\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}} - \mathbf{x}}{\mathrm{i}t}, \quad t \in \mathbb{R}$$

are uniformly bounded in the norms of $\mathbb{L}_L^p$ and $\mathbb{E}_L$, namely

$$\|\mathbf{y}_t\|_{\mathbb{L}_L^p} \leqslant \|[\mathbf{A}, \mathbf{x}]\|_{\mathbb{L}_L^p}, \quad \|\mathbf{y}_t\|_{\mathbb{E}_L} \leqslant \|[\mathbf{A}, \mathbf{x}]\|_{\mathbb{E}_L}, \quad t \in \mathbb{R}.$$

Applying Proposition 2.6 to $\mathbf{x}_t = \mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}}$, $\mathbf{x}$ and $\mathbf{1}$, we have

$$\mathbf{z}_t = \frac{\mathrm{e}^{\mathrm{i}t\mathbf{A}} f(\mathbf{x})\mathrm{e}^{-\mathrm{i}t\mathbf{A}} - f(\mathbf{x})}{\mathrm{i}t} = T_{\psi_f}^{\mathbf{x}_t, \mathbf{x}}\Big(\frac{\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}} - \mathbf{x}}{\mathrm{i}t}\Big)$$

are also uniformly bounded in $\mathbb{L}_L^p$ and $\mathbb{E}_L$ and

$$\|\mathbf{z}_t\|_{\mathbb{L}_L^p} \leqslant c_{f,p}\, \|[\mathbf{A}, \mathbf{x}]\|_{\mathbb{L}_L^p}, \quad \|\mathbf{z}_t\|_{\mathbb{E}_L} \leqslant c_{f,E}\, \|[\mathbf{A}, \mathbf{x}]\|_{\mathbb{E}_L}, \quad t \in \mathbb{R}.$$

Since the unit ball of $\mathbb{L}_L^p$ is $\sigma(\mathbb{L}_L^p, \mathbb{L}_L^{p'})$-compact, one can assume that there exists $\mathbf{z} \in \mathbb{L}_L^p$ such that

$$\|\mathbf{z}\|_{\mathbb{L}_L^p} \leqslant c_{f,p}\, \|[\mathbf{A}, \mathbf{x}]\|_{\mathbb{L}_L^p}, \quad \sigma(\mathbb{L}_L^p, \mathbb{L}_L^{p'}) - \lim_{t \to 0} \mathbf{z}_t = \mathbf{z}.$$

Since the space $\mathbb{E}_L$ has Fatou norm, we have that $\|\mathbf{x}\|_{\mathbb{E}_L} = \|\mathbf{x}\|_{\mathbb{E}_L{}^{\times\times}}$, for every $\mathbf{x} \in \mathbb{E}_L$. It follows from (2.2) that

$$\|\mathbf{z}\|_{\mathbb{E}_L} = \sup_{\mathbf{w} \in \mathbb{L}_L^1 \cap \mathbb{L}_L^\infty, \|\mathbf{w}\|_{\mathbb{E}_L^\times} \leqslant 1} \tau_L(\mathbf{z}\mathbf{w}).$$

Since $\mathbf{z}_t$ is uniformly bounded in $\mathbb{E}_L$, we have that

$$\tau_L(\mathbf{z}\mathbf{w}) = \lim_{t \to 0} \tau_L(\mathbf{z}_t\mathbf{w}) \leqslant \max_{t \in \mathbb{R}} \|\mathbf{z}_t\|_{\mathbb{E}_L}, \quad \mathbf{w} \in \mathbb{L}_L^1 \cap \mathbb{L}_L^\infty,\ \|\mathbf{w}\|_{\mathbb{E}_L^\times} \leqslant 1.$$

Hence, we obtain that $\mathbf{z} \in \mathbb{E}_L$ and

$$\|\mathbf{z}\|_{\mathbb{L}_L^p} \leqslant c_{f,p}\, \|[\mathbf{A}, \mathbf{x}]\|_{\mathbb{L}_L^p}, \quad \|\mathbf{z}\|_{\mathbb{E}_L} \leqslant c_{f,E}\, \|[\mathbf{A}, \mathbf{x}]\|_{\mathbb{E}_L}.$$

If now $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{L}^2$, then simple computation shows that

$$\begin{aligned}\Big\langle \frac{\mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\eta) - \eta}{-\mathrm{i}t}, f(\mathbf{x})(\xi) \Big\rangle =& \Big\langle \eta, \frac{\mathrm{e}^{\mathrm{i}t\mathbf{A}} f(\mathbf{x})(\xi) - f(\mathbf{x})(\xi)}{\mathrm{i}t} \Big\rangle \\ =& \langle \eta, \mathbf{z}_t(\xi) \rangle - \Big\langle \eta, \mathrm{e}^{\mathrm{i}t\mathbf{A}} f(\mathbf{x}) \frac{\mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\xi) - \xi}{\mathrm{i}t} \Big\rangle, \quad \xi, \eta \in \mathcal{D}\mathbf{A}.\end{aligned}$$

Since $(\mathrm{i}t)^{-1}(\mathrm{e}^{\mathrm{i}t\mathbf{A}}(\xi)-\xi)$ converges to $\mathbf{A}(\xi)$ in $\mathbb{L}^2$ and $\langle\eta,\mathbf{z}_t(\xi)\rangle=\tau(z_t\xi\eta^*)$ converges to $\langle\eta,\mathbf{z}(\xi)\rangle=\tau(z\xi\eta^*)$ whenever $\xi,\eta\in\mathcal{D}\mathbf{A}\subseteq\operatorname{Dom}\mathbf{A}\subseteq\mathbb{L}^1\cap\mathbb{L}^\infty$ ($\xi\eta^*\in\mathbb{L}^{p'}$), passing to the limit gives

$$\langle\mathbf{A}(\eta),f(\mathbf{x})(\xi)\rangle=\langle\eta,\mathbf{z}(\xi)\rangle+\langle\eta,f(\mathbf{x})\mathbf{A}(\xi)\rangle,\quad \xi,\eta\in\mathcal{D}\mathbf{A}. \tag{4.5}$$

For every fixed $\xi\in\mathcal{D}\mathbf{A}$, the right hand side is a bounded linear functional with respect to $\eta\in\mathbb{L}^2$. Indeed, since $f(\mathbf{x})\mathbf{A}(\xi)\in\mathbb{L}^2$, it is obvious that the last term on the right is continuous with respect to $\eta\in\mathbb{L}^2$ whereas the continuity of the first term on the right with respect to $\eta\in\mathbb{L}^2$ follows from the inequalities (see (2.1)),

$$|\langle\eta,\mathbf{z}(\xi)\rangle|=|\tau(\eta(z\xi)^*)|\leqslant\|\eta\|_{\mathbb{L}^2}\|z\xi\|_{\mathbb{L}^2}\leqslant\|\eta\|_{\mathbb{L}^2}\|z\|_{\mathbb{L}^p}\|\xi\|_{\mathbb{L}^1\cap\mathbb{L}^\infty},\quad \eta\in\mathbb{L}^2$$

and

$$\|z\|_{\mathbb{L}^p_L}\|\xi\|_{\mathbb{L}^1\cap\mathbb{L}^\infty}<\infty.$$

Hence the right hand side in (4.5) is continuous with respect to $\eta\in\mathbb{L}^2$. This implies that the form $\langle\mathbf{A}(\eta),f(\mathbf{x})(\xi)\rangle$ is continuous with respect to $\eta\in\mathcal{D}\mathbf{A}$ and so, since $\mathcal{D}\mathbf{A}$ is a core of $\mathbf{A}$, $f(\mathbf{x})(\xi)\in\operatorname{Dom}\mathbf{A}^*=\operatorname{Dom}\mathbf{A}$. Moreover, it now follows from (4.5)

$$(f(\mathbf{x})\mathbf{A}-\mathbf{A}f(\mathbf{x}))(\xi)=\mathbf{z}(\xi),\quad \xi\in\mathcal{D}\mathbf{A}.$$

Consider the operator $\mathbf{z}'=f(\mathbf{x})\mathbf{A}-\mathbf{A}f(\mathbf{x})$ with the domain $\operatorname{Dom}\mathbf{z}'=\mathcal{D}\mathbf{A}$. Let us show that $\mathbf{z}'^*=\mathbf{z}^*$. This will finish the proof of the theorem, because then we would have $\overline{\mathbf{z}}'=\mathbf{z}'^{**}=\mathbf{z}^{**}=\mathbf{z}$. Since $\mathbf{z}'\subseteq\mathbf{z}$, it is sufficient to show that $\operatorname{Dom}\mathbf{z}'^*\subseteq\operatorname{Dom}\mathbf{z}^*$. It follows from the definition of adjoint operator that $\xi\in\operatorname{Dom}\mathbf{z}'^*$ if and only if there is a constant $c(\xi)$ such that

$$|\tau((z^*\xi)\eta^*)|=|\tau(\xi(z\eta)^*)|=|\langle\xi,\mathbf{z}'(\eta)\rangle|\leqslant c(\xi)\,\|\eta\|_{\mathbb{L}^2},\quad \eta\in\mathcal{D}\mathbf{A}.$$

Since $z^*\in\mathbb{L}^p$ and $\xi\in\mathbb{L}^2$, it follows that $z^*\xi\in\mathbb{L}^q$, where $q^{-1}=2^{-1}+p^{-1}$ and $1\leqslant q\leqslant 2$, see (2.1). On the other hand, it follows from Lemma 4.4 that for every $\eta\in\mathbb{L}^1\cap\mathbb{L}^\infty$ there is a net $\{\eta_\alpha\}\subseteq\mathcal{D}\mathbf{A}$ such that $\eta_\alpha\to\eta$ in the $\sigma(\mathbb{L}^1\cap\mathbb{L}^\infty,\mathbb{L}^q)$-topology and $\sup_\alpha\|\eta_\alpha\|_{\mathbb{L}^2}\leqslant\|\eta\|_{\mathbb{L}^2}$. Hence, we obtain that

$$|\tau((z^*\xi)\eta^*)\leqslant c(\xi)\,\|\eta\|_{\mathbb{L}^2},\quad \eta\in\mathbb{L}^1\cap\mathbb{L}^\infty.$$

The latter means that $z^*\xi\in\mathbb{L}^2$ and, in particular, $\xi\in\operatorname{Dom}\mathbf{z}^*$. ∎

5. APPLICATION TO $\mathcal{L}(\mathcal{H})$.

Let us now go back to the original algebra $\mathcal{M}$. Consider a densely defined self-adjoint operator A which may be neither bounded nor affiliated with $\mathcal{M}$, such that $\mathrm{e}^{\mathrm{i}tA}x\mathrm{e}^{-\mathrm{i}tA}\in\mathcal{M}$ and $\tau(\mathrm{e}^{\mathrm{i}tA}x\mathrm{e}^{-\mathrm{i}tA})=\tau(x)$ whenever $x\in\mathcal{M}$ and $t\in\mathbb{R}$.

We show via the Stone theorem that we may construct an operator $\mathbf{A}$ which satisfies the assumptions (i)–(ii). Moreover, we will obtain that $[\mathbf{A},\mathbf{x}]\in\mathcal{M}_L$ whenever $[A,x]\in\mathcal{M}$ and $x\in\mathcal{M}$. We will also show using the technique of one-dimensional operators that, in the setting of $\mathcal{M} = \mathcal{L}(\mathcal{H})$, the converse to the last statement is valid, i.e. $[A, x] \in \mathbb{E}$ provided $[\mathbf{A}, \mathbf{x}] \in \mathbb{E}_L$.

We recall that if $x \in \mathcal{M}$, then the symbol $[A, x]$ means that there is a core D of the operator A such that $\mathrm{e}^{\mathrm{i}tA}(D) \subseteq D$ for every $t \in \mathbb{R}$ and $x(D) \subseteq \mathrm{Dom}\, A$. The symbol $[A, x]$ stands for the closure of $Ax - xA$. As we mentioned earlier as soon as $[A, x]$ is bounded, the operator x preserves the whole domain $\mathrm{Dom}\, A$. The argument follows

LEMMA 5.1. *If $[A, x]$ is bounded, then $x(\mathrm{Dom}\, A) \subseteq \mathrm{Dom}\, A$.*

Proof. One has

$$\langle [A, x]\xi, \eta\rangle = \langle Ax\xi, \eta\rangle - \langle xA\xi, \eta\rangle, \quad \xi, \eta \in D.$$

For every fixed $\xi \in D$, both sides are bounded linear functionals with respect to $\eta \in \mathcal{H}$ which coincide for every $\eta \in D$ and so they coincide for every $\eta \in \mathcal{H}$, in particular,

$$\langle [A, x]\xi, \eta\rangle = \langle x\xi, A\eta\rangle - \langle A\xi, x^*\eta\rangle, \quad \xi \in D, \eta \in \mathrm{Dom}\, A.$$

Now, for every fixed $\eta \in \mathrm{Dom}\, A$, it follows that $\langle A\xi, x^*\eta\rangle$ is a bounded linear functional with respect to $\xi \in D$. The latter means that $x^*\eta \in \mathrm{Dom}\, A'^*$ where $A' = A|_D$ is the restriction of A onto D. Since D is a core of A, $A'^* = \overline{A'}^* = A^* = A$, hence $x^*\eta \in \mathrm{Dom}\, A$. We obtain that $x^*(\mathrm{Dom}\, A) \subseteq \mathrm{Dom}\, A$ and $[A, x^*] = -[A, x]^*$ is bounded. Applying the above argument again to the operator x^* gives the claim of the lemma. ∎

Consider the subspace

$$\mathrm{Dom}\, \mathbf{A} := \left\{\xi \in \mathbb{L}^2 : \|\circ\|_{\mathbb{L}^2} - \lim_{t\to 0} \frac{\mathrm{e}^{\mathrm{i}tA}\xi\mathrm{e}^{-\mathrm{i}tA} - \xi}{\mathrm{i}t} \text{ exists}\right\} \subseteq \mathbb{L}^2$$

and define the operator $\mathbf{A} : \mathrm{Dom}\, \mathbf{A} \to \mathbb{L}^2$ by

$$\mathbf{A}(\xi) := \|\circ\|_{\mathbb{L}^2} - \lim_{t\to 0} \frac{\mathrm{e}^{\mathrm{i}tA}\xi\mathrm{e}^{-\mathrm{i}tA} - \xi}{\mathrm{i}t}, \quad \xi \in \mathrm{Dom}\, \mathbf{A}.$$

According to the Stone theorem, $\mathbf{A}$ is a self-adjoint operator.

LEMMA 5.2. *If the operator $\mathbf{A}$ is as above, then $\mathrm{e}^{\mathrm{i}t\mathbf{A}}(\xi) = \gamma_t(\xi)$, where $\gamma(t) := L^{-1}(\mathrm{e}^{\mathrm{i}t\mathbf{A}}L(\xi)\mathrm{e}^{-\mathrm{i}t\mathbf{A}})$, for every $\xi \in \mathbb{L}^1 \cap \mathbb{L}^\infty$. If $\mathbf{x} \in \mathcal{M}_L$, then $\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}} \in \mathcal{M}_L$ and $\tau_L(\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}}) = \tau_L(\mathbf{x})$. If $x \in \mathcal{M}$ and $[A, x] \in \mathcal{M}$, then $[\mathbf{A}, \mathbf{x}] \in \mathcal{M}_L$ and $[\mathbf{A}, \mathbf{x}]$ is a multiplication operator by $[A, x]$.*

Proof. According to the Stone theorem, the unitary group $\mathrm{e}^{\mathrm{i}t\mathbf{A}}$ of the operator $\mathbf{A}$ is given by $\mathrm{e}^{\mathrm{i}t\mathbf{A}}(\xi) = \mathrm{e}^{\mathrm{i}tA}\xi\mathrm{e}^{-\mathrm{i}tA}$, $\xi \in \mathbb{L}^2$. Now, for every $\xi \in \mathbb{L}^2$, one has

$$\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}}(\xi) = \mathrm{e}^{\mathrm{i}tA}(x\mathrm{e}^{-\mathrm{i}tA}\xi\mathrm{e}^{\mathrm{i}tA})\mathrm{e}^{-\mathrm{i}tA} = \mathrm{e}^{\mathrm{i}tA}x\mathrm{e}^{-\mathrm{i}tA}\xi.$$

The latter means that the operator $\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}}$ is a left multiplication by $\mathrm{e}^{\mathrm{i}tA}x\mathrm{e}^{-\mathrm{i}tA}$ and so it belongs to $\mathcal{M}_L$; moreover $\gamma_t(x) = \mathrm{e}^{\mathrm{i}tA}x\mathrm{e}^{-\mathrm{i}tA}$ whenever $x \in \mathcal{M}$ and

$$\tau_L(\mathrm{e}^{\mathrm{i}t\mathbf{A}}\mathbf{x}\mathrm{e}^{-\mathrm{i}t\mathbf{A}}) = \tau(\mathrm{e}^{\mathrm{i}tA}x\mathrm{e}^{-\mathrm{i}tA}) = \tau(x) = \tau_L(\mathbf{x}), \quad \mathbf{x} \in \mathcal{M}_L.$$

Let now $x \in \mathcal{M}$ and $[A, x] \in \mathcal{M}$. Take $\xi \in D$, where

$$D = \{\xi \in \mathbb{L}^2 \cap \mathbb{L}^\infty : \xi(\mathrm{Dom}\ A) \subseteq \mathrm{Dom}\ A,\ [A, \xi] \in \mathbb{L}^2 \cap \mathbb{L}^\infty\}.$$

It follows from Theorem 7.3 of [24], that D is a core of $\mathbf{A}$ and $\mathbf{A}(\xi) = \mathrm{i}[A, \xi]$, $\xi \in D$. We show that $\mathbf{x}\xi \in D$. Indeed, from Lemma 5.1, $x(\mathrm{Dom}\ A) \subseteq \mathrm{Dom}\ A$, so $x\xi(\mathrm{Dom}\ A) \subseteq \mathrm{Dom}\ A$. Furthermore, one has that

$$Ax\xi(\eta) - x\xi A(\eta) = (Ax - xA)\xi(\eta) + x(A\xi - \xi A)(\eta), \quad \eta \in \mathrm{Dom}\ A.$$

Hence, the operator $Ax\xi - x\xi A$ is closable and for the closure one has

$$\mathbf{Ax}(\xi) = [A, x\xi] = [A, x]\xi + \mathbf{x}([A, \xi]) = [A, x]\xi + \mathbf{xA}(\xi) \in \mathbb{L}^2 \cap \mathbb{L}^\infty.$$

This means that the operator $\mathbf{x}(\xi) \in D$ and $[\mathbf{A}, \mathbf{x}](\xi) = [A, x]\xi$ for every $\xi \in D$. The lemma is completely proved. ∎

We shall now demonstrate how our results from the preceding section yield commutator estimates for the special case $\mathcal{M} = \mathcal{L}(\mathcal{H})$. In this special case, non-commutative symmetric spaces are symmetric ideals of compact operators [18]; in particular, the space $\mathbb{L}^p$, $1 \leqslant p \leqslant \infty$ is the Schatten-von Neumann ideal $\mathcal{C}^p$. We believe that the proof of the following theorem provides an additional insight into results of [5] and explains the additional technical obstacles in the type II setting.

THEOREM 5.3. *If $x \in \mathcal{L}(\mathcal{H})$ is self-adjoint and A is a self-adjoint operator on $\mathcal{H}$ such that $[A, x] \in \mathcal{C}^p$, $1 \leqslant p \leqslant \infty$, then for every function f such that $f \in \mathfrak{F}(\mathcal{C}^p)$,*

$$\|[A, f(x)]\|_{\mathcal{C}^p} \leqslant c_{f,p}\, \|[A, x]\|_{\mathcal{C}^p}. \tag{5.1}$$

Proof. Since $\mathcal{C}_L^p \cap \mathcal{C}_L^2 = \mathcal{C}_L^p$ whenever $1 \leqslant p \leqslant 2$, it follows from Corollary 4.2 and Lemma 5.1 that

$$\|[\mathbf{A}, f(\mathbf{x})]\|_{\mathcal{C}_L^p} \leqslant c_{f,p}\, \|[\mathbf{A}, \mathbf{x}]\|_{\mathcal{C}_L^p} = c_{f,p}\, \|[A, x]\|_{\mathcal{C}^p}. \tag{5.2}$$

It remains to show that (5.2) implies (5.1). To this end, we exploit the technique of one-dimensional operators. Note that $\mathrm{Dom}\ A \otimes \mathrm{Dom}\ A \subseteq \mathrm{Dom}\ \mathbf{A}$ where $\mathrm{Dom}\ A \otimes \mathrm{Dom}\ A$ is the algebraic tensor product consisting of all finite linear combinations of one dimensional operators $\xi \otimes \eta$, $\xi, \eta \in \mathrm{Dom}\ A$, given by

$$\xi \otimes \eta(\zeta) = \langle \zeta, \eta \rangle \xi, \quad \zeta \in \mathcal{H}.$$

Indeed, since $\|\xi \otimes \eta\|_{\mathcal{C}^2} = \|\xi\|_{\mathcal{H}} \, \|\eta\|_{\mathcal{H}}$, we have, for every $\xi, \eta \in \operatorname{Dom} A$,

$$\begin{aligned}&\frac{\mathrm{e}^{\mathrm{i}tA}\xi \otimes \eta \mathrm{e}^{-\mathrm{i}tA} - \xi \otimes \eta}{\mathrm{i}t} \\ &\qquad = \frac{\mathrm{e}^{\mathrm{i}tA}\xi - \xi}{\mathrm{i}t} \otimes \mathrm{e}^{\mathrm{i}tA}\eta + x\mathrm{i} \otimes \frac{\mathrm{e}^{\mathrm{i}tA}\eta - \eta}{\mathrm{i}t} \to A\xi \otimes \eta + \xi \otimes A\eta, \quad \text{as } t \to 0.\end{aligned}$$

This means that $\xi \otimes \eta \in \operatorname{Dom} \mathbf{A}$. Moreover,

$$\mathbf{A}(\xi \otimes \eta) = A\xi \otimes \eta + \xi \otimes A\eta, \quad \xi, \eta \in \operatorname{Dom} A. \tag{5.3}$$

It follows from (5.2) and Lemma 5.1 that $f(\mathbf{x})(\operatorname{Dom} \mathbf{A}) \subseteq \operatorname{Dom} \mathbf{A}$. We show that $f(x)(\operatorname{Dom} A) \subseteq \operatorname{Dom} A$. To this end let us fix $\xi, \eta \in \operatorname{Dom} A$. We obtain that $f(\mathbf{x})(\xi \otimes \eta) = f(x)(\xi \otimes \eta) = (f(x)\xi) \otimes \eta \in \operatorname{Dom} \mathbf{A}$, that is

$$\| \circ \|_{\mathcal{C}^2} - \lim_{t \to 0} \frac{\mathrm{e}^{\mathrm{i}tA}(f(x)\xi) \otimes \eta \mathrm{e}^{-\mathrm{i}tA} - (f(x)\xi) \otimes \eta}{\mathrm{i}t} \quad \text{exists.}$$

Repeating the argument above gives

$$\begin{aligned}&\frac{\mathrm{e}^{\mathrm{i}tA} f(x)\xi - f(x)\xi}{\mathrm{i}t} \otimes \mathrm{e}^{\mathrm{i}tA}\eta + (f(x)\xi) \otimes \frac{\mathrm{e}^{\mathrm{i}tA}\eta - \eta}{\mathrm{i}t} \\ &\qquad = \frac{\mathrm{e}^{\mathrm{i}tA}(f(x)\xi) \otimes \eta \mathrm{e}^{-\mathrm{i}tA} - (f(x)\xi) \otimes \eta}{\mathrm{i}t} \to \mathbf{A} f(\mathbf{x})(\xi \otimes \eta), \quad \text{as } t \to 0.\end{aligned} \tag{5.4}$$

On the other hand, since $\eta \in \operatorname{Dom} A$, we have that

$$(f(x)\xi) \otimes \frac{\mathrm{e}^{\mathrm{i}tA}\eta - \eta}{\mathrm{i}t} \to (f(x)\xi) \otimes A\eta, \quad \text{as } t \to 0.$$

According to (5.3) and (5.4), this means that $(\mathrm{i}t)^{-1}(\mathrm{e}^{\mathrm{i}tA} f(x)\xi - f(x)\xi)$ converges, which in its turn implies that $f(x)\xi \in \operatorname{Dom} A$. Moreover, passing to the limit in (5.4) gives

$$(Af(x)\xi) \otimes \eta + (f(x)\xi) \otimes A\eta = \mathbf{A} f(\mathbf{x})(\xi \otimes \eta).$$

Combining the last identity with (5.3) we obtain that

$$\begin{aligned}&[\mathbf{A}, f(\mathbf{x})](\xi \otimes \eta) \\ &= \mathbf{A} f(\mathbf{x})(\xi \otimes \eta) - f(\mathbf{x})\mathbf{A}(\xi \otimes \eta) \\ &= (Af(x)\xi) \otimes \eta + (f(x)\xi) \otimes A\eta - (f(x)A\xi) \otimes \eta - (f(x)\xi) \otimes A\eta = ([A, f(x)]\xi) \otimes \eta.\end{aligned}$$

Consequently, the operator $[A, f(x)]$ coincides with that which yields $[\mathbf{A}, f(\mathbf{x})] \in \mathcal{C}^p_L$ as a left multiplication. The theorem is proved. ∎

Acknowledgements. The research is partially supported by ARC.

REFERENCES

[1] J. ARAZY, T.J. BARTON, Y. FRIEDMAN, Operator differentiable functions, *Integral Equations Operator Theory* **13**(1990), 462–487.

[2] M.S. BIRMAN, M.Z. SOLOMYAK, Double Stieltjes operator integrals [Russian], in *Spectral Theory and Wave Processes*, Probl. Math. Fiz., vol. 1, Izdat. Leningrad. Univ., Leningrad 1966, pp. 33–67.

[3] M.S. BIRMAN, M.Z. SOLOMYAK, Double Stieltjes operator integrals. II [Russian], in *Spectral Theory, Diffraction Problems*, Probl. Math. Fiz., vol. 2, Izdat. Leningrad. Univ., Leningrad 1967, pp. 26–60.

[4] M.S. BIRMAN, M.Z. SOLOMYAK, Double Stieltjes operator integrals. III [Russian], Prob. Maht. Fiz., vol. 6, Izdat. Leningrad. Univ., Leningrad 1973, pp. 27–53.

[5] M.S. BIRMAN, M.Z. SOLOMYAK, Operator integration, perturbations, and commutators, *J. Soviet Math.* **63**(1993), 129–148.

[6] O. BRATTELI, D.W. ROBINSON, *Operator Algebras and Quantum Statistical Mechanics. Vol. 1*, Springer-Verlag, New York 1979.

[7] A.L. CAREY, J. PHILLIPS, F.A. SUKOCHEV, On unbounded p-summable Fredholm modules, *Adv. Math.* **151**(2000), 140–163.

[8] V.I. CHILIN, F.A. SUKOCHEV, Weak convergence in non-commutative symmetric spaces, *J. Operator Theory* **31**(1994), 35–65.

[9] F. CIPRIANI, D. GUIDO, S. SCARLATTI, A remark on trace properties of K-cycles, *J. Operator Theory* **35**(1996), 179–189.

[10] A. CONNES, *Noncommutative Geometry*, Academic Press Inc., San Diego, CA 1994.

[11] A. CONNES, H. MOSCOVICI, Transgression du caractère de Chern et cohomologie cyclique, *C. R. Acad. Sci. Paris Sér. I Math.* **303**(1986), 913–918.

[12] E.B. DAVIES, *One-Parameter Semigroups*, London Math. Soc. Monographs, vol. 15, Academic Press Inc., London 1980.

[13] E.B. DAVIES, Lipschitz continuity of functions of operators in the Schatten classes, *J. London Math. Soc. (2)* **37**(1988), 148–157.

[14] P.G. DODDS, T.K. DODDS, B. DE PAGTER, Non-commutative Köthe duality, *Trans. Amer. Math. Soc.* **339**(1993), 717–750.

[15] P.G. DODDS, T.K. DODDS, B. DE PAGTER, F.A. SUKOCHEV, Lipschitz continuity of the absolute value and Riesz projections in symmetric operator spaces, *J. Funct. Anal.* **148**(1997), 28–69.

[16] N. DUNFORD, J.T. SCHWARTZ, *Linear Operators. Part* I, Wiley Classics Library, John Wiley & Sons Inc., New York 1988.

[17] T. FACK, H. KOSAKI, Generalized s-numbers of τ-measurable operators, *Pacific J. Math.* **123**(1986), 269–300.

[18] I.C. GOHBERG, M.G. KREĬN, *Vvedenie v teoriyu lineinykh nesamosopryazhennykh operatorov v gilbertovom prostranstve*, Izdat. "Nauka", Moscow 1965.

[19] S. GOLDSTEIN, Conditional expectation and stochastic integrals in noncommutative L^p-spaces, *Math. Proc. Cambridge Philos. Soc.* **110**(1991), 365–383.

[20] A. HARCHARRAS, Fourier analysis, Schur multipliers on S^p and non-commutative $\Lambda(p)$-sets, *Studia Math.* **137**(1999), 203–260.

[21] S.G. KREĬN, Y.Ī. PETUNĪN, E.M. SEMËNOV, *Interpolation of Linear Operators*, Transl. Math. Monographs, vol. 54, Amer. Math. Soc., Providence, R.I. 1982.

[22] E. NELSON, Notes on non-commutative integration, *J. Funct. Anal.* **15**(1974), 103–116.

[23] B. DE PAGTER, F.A. SUKOCHEV, Differentiation of operator functions in non-commutative L_p-spaces, *J. Funct. Anal.* **212**(2004), 28–75.

[24] B. DE PAGTER, F.A. SUKOCHEV, Commutator estimates and R-flows in non-commutative operator spaces, *Proc. Edinburgh Math. Soc. (2)* **50**(2007), 293–324.

[25] B. DE PAGTER, F.A. SUKOCHEV, H. WITVLIET, Double operator integrals, *J. Funct. Anal.* **192**(2002), 52–111.

[26] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci., vol. 44, Springer-Verlag, New York 1983.

[27] Ş. STRĂTILĂ, L. ZSIDÓ, *Lectures on von Neumann Algebras*, Editura Academiei, Bucharest 1979.

[28] M. TERP, L^p-spaces associated with von Neumann algebras, Copenhagen Univ., Copenhagen 1981.

[29] D. VOICULESCU, On the existence of quasicentral approximate units relative to normed ideals. I, *J. Funct. Anal.* **91**(1990), 1–36.

DENIS POTAPOV, SCHOOL OF INFORMATICS AND ENGINEERING, FACULTY OF SCIENCE AND ENGINEERING, FLINDERS UNIVERSITY OF SA, BEDFORD PARK, SA 5042, AUSTRALIA.

E-mail address: denis.potapov@flinders.edu.au

FYODOR SUKOCHEV, SCHOOL OF INFORMATICS AND ENGINEERING, FACULTY OF SCIENCE AND ENGINEERING, FLINDERS UNIVERSITY OF SA, BEDFORD PARK, SA 5042, AUSTRALIA.

E-mail address: sukochev@infoeng.flinders.edu.au

Received November 25, 2005.