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FACTORIZATION AND REFLEXIVITY RESULTS FOR POLYNOMIALLY BOUNDED OPERATORS

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ABSTRACT. In this paper we prove some factorization and reflexivity results for polynomially bounded operators; in particular, we obtain the following theorem: Every polynomially bounded operator acting on a reflexive Banach space such that its spectrum contains the unit circle either has a non trivial hyperinvariant subspace or is reflexive.

KEYWORDS: *Hyperinvariant subspace, reflexivity, polynomially bounded operator, factorization.*

MSC (2000): 47A15.

INTRODUCTION

In 2003, Ambrozie and Müller have shown the following result: if T is a polynomially bounded operator on a Banach space such that its spectrum contains the unit circle, then T^* has a nontrivial invariant subspace (see [1]).

Invariant subspace results which, as this one, use ideas and tools related to the Brown technique or the theory of dual algebras, are often followed by results of factorization and reflexivity (these notions will be defined later). We recall some classical results illustrating this fact.

In [7], which is the beginning of the dual algebra theory, Brown proves the existence of nontrivial invariant subspaces for subnormal operators and some years later Olin and Thompson in [17] show the reflexivity of these operators.

Another important result in this field (see [10]), gives the existence of invariant subspaces for contractions on a Hilbert space with dominating spectrum. Then [5] shows the reflexivity for contractions under some similar hypothesis of dominating spectrum. Finally, in the article [11], it is proved that every contraction on a Hilbert space with its spectrum containing the unit circle has a nontrivial invariant subspace. An "associated" result of reflexivity is obtained in [12] where, among others, the reflexivity for contractions is proved under some factorization hypothesis (of type $(\mathbb{A}_{1,\aleph_0})$, see Section 1 for definition).

In fact, generally, the understanding and the improvement of the proofs of the results of invariant subspaces (in particular the results of factorization) lead to reflexivity theorems.

The aim of this paper is to give factorization and reflexivity results for polynomially bounded operators of Banach spaces (thus the result of [1] is no exception to the above rule). In particular, we prove that every polynomially bounded operator acting on a reflexive Banach space such that its spectrum contains the unit circle has a nontrivial hyperinvariant subspace or is reflexive (this is a generalization of a result from [12]).

Now, we recall some basic definitions. The next sections give the tools for the proof: some approximation sets (Section 1), a functional calculus (Section 2), the Ambrozie and Müller innovations (Section 3) and the notion of analytic invariant subspace and its use (Sections 4 and 5). The last section contains the proof of the theorem.

In all this paper X will be a complex Banach space. For $T \in \mathcal{L}(X)$ we say that a closed subspace $M \subset X$ is a nontrivial invariant subspace for T (respectively a nontrivial hyperinvariant subspace for T) if M is nontrivial and if $TM \subset M$ (respectively M is invariant for each operator commuting with T). We denote $\mathrm{Lat}(T)$ the lattice of invariant subspaces for T.

First, we define the notion of polynomially bounded operator.

DEFINITION 0.1. $T \in \mathcal{L}(X)$ is *polynomially bounded* if there exists $M > 0$ such that for all polynomials p we have $\|p(T) \leqslant M\|p\|_\infty$ (where $\|p\|_\infty := \{\sup |p(z)| : z \in \mathbb{D}\}$).

We denote by $PB_M(X)$ (or PB_M) the set of operators verifying this inequality and $PB(X) = PB := \bigcup\limits_{M \geqslant 0} PB_M$.

We will need the following class of operators:

DEFINITION 0.2. $T \in \mathcal{L}(X)$ is in the class $C_{\cdot 0}$ (or $T \in C_{\cdot 0}$) if for all $x^* \in X^*, T^{*n}x^* \to 0$.

$H^\infty(\mathbb{D})$ (or H^∞) is the space of bounded holomorphic functions in the unit disc $\mathbb{D}$, or, equivalently, the functions in $L^\infty(\mathbb{T})$ ($\mathbb{T}$ denotes the unit circle) with negative Fourier coefficients equal to zero.

$A(\mathbb{D})$ is the space of functions holomorphic in $\mathbb{D}$ and continuous on $\mathbb{T}$.

These two spaces are Banach spaces for the supremum norm ($\|f\|_\infty := \{\sup |f(z)| : z \in \mathbb{D}\}$).

We recall that H^∞ is the dual of a quotient space of $L^1(\mathbb{T}), L^1/H^1_0$ ($f \in H^1_0$ if $f \in L^1$ and $\widehat{f}(n) = 0$ for $n \leqslant 0$) and the duality is given by the formula:

$$\langle [f], h \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta}) h(\mathrm{e}^{\mathrm{i}\theta}) \mathrm{d}\theta \quad f \in L^1, h \in H^\infty.$$

This duality will be used in the definition of the functional calculus in Section 2.

The following definition is related to a result of Apostol.

DEFINITION 0.3. A subset $\Lambda \subset \mathbb{D}$ is called an *Apostol set* if the $\theta \in (-\pi, \pi]$ such that $\sup\{r \in [0,1) : re^{i\theta} \in \Lambda\} < 1$ is at most countable.

For $\varepsilon > 0, k \geqslant 1$ an integer and $T \in \mathcal{L}(X)$, we denote

$$\Lambda_{\varepsilon,k}(T) = \Lambda_{\varepsilon,k} = \{\lambda \in \mathbb{D} / \exists u \in X \text{ with } \|u\| = 1 \text{ and } \|(T-\lambda)u\| < \varepsilon(1-|\lambda|)^k\}.$$

The result of Apostol which will be used in Section 6 is the following (see [3], [13] or [4] for the proof).

THEOREM 0.4. *Let $T \in PB(X)$ such that $\sigma(T) \supset \mathbb{T}$. If there exist ε and k such that $\Lambda_{\varepsilon,k}$ is not an Apostol set, then T possesses a nontrivial hyperinvariant subspace.*

The last notion we define in this introduction is the reflexivity of an operator. It seems that Sarason was the first one to consider this notion in [19] where was proved the reflexivity of normal operators (the word reflexivity was introduced later by Halmos). Roughly speaking, it gives an idea of the richness of the lattice of invariant subspaces. To define it more precisely, we need some notations: given $T \in \mathcal{L}(X)$ we denote by $\mathrm{AlgLat}(T)$ the set of operators $A \in \mathcal{L}(X)$ such that $\mathrm{Lat}(T) \subset \mathrm{Lat}(A)$ and by $\mathcal{W}_T$ the closure for the weak operator topology of the algebra generated by T.

We always have $\mathcal{W}_T \subset \mathrm{AlgLat}(T)$ and we say that T is reflexive if $\mathcal{W}_T = \mathrm{AlgLat}(T)$.

1. APPROXIMATION SETS AND FACTORIZATION

In this section, B will denote a continuous bilinear application from $X \times Y$ to Z where X, Y, Z are complex Banach spaces.

The purpose of this section is to prove that under approximation assumption we have in fact a factorization result. The approximation sets defined below were introduced for a particular bilinear application on Hilbert space in [14]. This generalisation follows the ideas of [6] where some other approximation sets are defined (see also [18] where the proofs are given in the Banach space context).

DEFINITION 1.1. We say that B has the E_C^1 property if there exist $C \in (0,1)$, and two positive constants M and M' such that, for all $L \in Z$, all $z \in X$, all finite family $w_1, \dots, w_p \in Y$ and all $\delta > 0$, we can find $u \in X$ and $v \in Y$ verifying

$$\|B(u+z, v) - L\| \leqslant C\|L\|, \tag{1.1}$$

$$\|B(u, w_j)\| < \delta, \tag{1.2}$$

$$\|u\| \leqslant M\|L\|^{1/2}, \tag{1.3}$$

$$\|v\| \leqslant M'\|L\|^{1/2}. \tag{1.4}$$

REMARK 1.2. (i) If necessary, to avoid ambiguity, we will of course write $E^{\mathrm{l}}_{C,M,M'}$ for E^{l}_C.

(ii) We can also define some E^{r}_C property, changing the first two inequalities to

$$\|B(u, v+y) - L\| \leqslant C\|L\|, \tag{1.5}$$

$$\|B(x_j, v)\| < \delta, \tag{1.6}$$

where $y \in Y$ and $x_1, \ldots, x_p$ are vectors in X.

The first proposition shows that, in fact, the approximation property expressed by the definition extends to a finite family of vectors, at the expense of increasing the constant C.

PROPOSITION 1.3. *Suppose that B possesses the E^{l}_C property, and let $c \in (C, 1)$. Then, given $w_1, \ldots, w_p \in Y, z \in X, \delta > 0$ and $L_1, \ldots, L_N, N$ elements in Z, we can find $u \in X$ and $v_1, \ldots, v_N \in Y$ such that, for M and M' as in the definition,*

$$\|B(u+z, v_j) - L_j\| \leqslant c\|L_j\| \quad j = 1, \ldots, N\,, \tag{1.7}$$

$$\|B(u, w_j)\| < \delta \quad j = 1, \ldots, p\,, \tag{1.8}$$

$$\|u\| \leqslant M \sum_{k=1}^{N} (\|L_k\|)^{1/2}, \quad \|v_j\| \leqslant M'\|L_j\|^{1/2}. \tag{1.9}$$

Proof. We take $w_1, \ldots, w_p \in Y, z \in X, \delta > 0$ and $L_1, \ldots, L_N \in Z$. We fix $\delta_j > 0$ $(j = 1, \ldots, N)$ such that, for all $j, \delta_j \leqslant \delta/N$ and $C\|L_j\| + \sum_k \delta_k \leqslant c\|L_j\|$. We will show by induction that, for all $j \in \{1, \ldots, N\}$ there exists $u_j \in X$ and $v_j \in Y$ such that:

$$\begin{aligned}
&\|B(u_j + u_{j-1} + \cdots + u_1 + z), v_j - L_j\| \leqslant C\|L_j\|, \\
&\|B(u_j, w_l)\| < \delta \quad \text{for } l = 1, \ldots, p\,, \\
&\|B(u_j, v_l)\| < \delta_j \quad \text{for } l < p, \\
&\|u_j\| \leqslant M\|L_j\|^{1/2} \quad \text{and} \quad \|v_j\| \leqslant M'\|L_j\|^{1/2}.
\end{aligned}$$

For $j = 1$ we only use the definition of the E^{l}_C property. From the rank $j-1$ to the rank j, we apply once more the E^{l}_C property to $w_1, \ldots, w_p$, $v_1, \ldots, v_{j-1}$ (instead of $w_1, \ldots, w_p$), $u_{j-1} + \cdots + u_1 + z$ (instead of z), $\min(\delta_j, \delta)$ (instead of δ) and L_j. It is simple to check the four conditions needed. We denote $u = u_N + \cdots + u_1$. Then, for all $j \in \{1, \ldots, N\}$:

$$\begin{aligned}
\|B(u+z, v_j) - L_j\| &\leqslant \|B(u_j + \cdots + u_1 + z, v_j) - L_j\| + \sum_{k>j} \|B(u_k, v_j)\| \\
&\leqslant C\|L_j\| + \sum_{k>j} \delta_k \leqslant c\|L_j\|.
\end{aligned}$$

Besides, for all $j \in \{1, \ldots, p\}$

$$\|B(u, w_j)\| \leqslant \sum_k \|B(u_k, w_j)\| \leqslant \sum_k \delta_k \leqslant \delta.$$

Finally, we have

$$\|u\| \leqslant M \sum_{k=1}^{N} (\|L_k\|)^{1/2}. \quad \blacksquare$$

Now, we are going to define the factorization properties.

DEFINITION 1.4. Let $n, m \leqslant \aleph_0$. We say that B possesses the $(\mathbb{A}_n, m)$ *property* if for every $n \times m$ matrix $(L_{i,j})_{i \leqslant n, j \leqslant m}$, there exist $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_m \in Y$ such that:

$$L_{i,j} = B(x_i, y_j), \quad 1 \leqslant j \leqslant n, 1 \leqslant j \leqslant m.$$

We will write $(\mathbb{A}_n)$ instead of $(\mathbb{A}_{n,n})$. We say that B possesses the $(\mathbb{A}_n(r))$ *property* for $r > 0$ if for all $s > r$ and every matrix $(L_{i,j})_{i \leqslant n, j \leqslant m}$, there exist $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_m \in Y$ such that:

$$L_{i,j} = B(x_i, y_j), \quad 1 \leqslant j \leqslant n, 1 \leqslant j \leqslant m.$$

and

$$\|x_i\| \leqslant \Big(s \sum_{j=1}^{n} \|L_{i,j}\| \Big)^{1/2}, \quad i = 1, \ldots, n; \quad \|y_j\| \leqslant \Big(s \sum_{i=1}^{n} \|L_{i,j}\| \Big)^{1/2}, \quad j = 1, \ldots, n.$$

In [6] or [18] it is shown that, under some stronger approximation property than the E^1_C one, B has the $(A_{\aleph_0})$ property. Here, we will see that "E^1_C implies $(A_{1,\aleph_0})$", and even more.

We begin with a fundamental lemma which initializes the process. We take $\gamma \in (c, 1)$ (where c is given in the previous proposition).

LEMMA 1.5. *We suppose that B verifies the E^1_C property. Let $L_1, \ldots, L_N \in Z, x \in X, y_1, \ldots, y_N \in Y$, and $\rho_j > 0$ (for $j = 1, \ldots, N$) such that*

$$\|B(x, y_j) - L_j\| < \rho_j, \quad j = 1, \ldots, N.$$

Then there exists $\xi \in X$ and $\eta, \ldots, \eta_N \in Y$ such that

$$\|B(\xi, \eta_j) - L_j\| < \gamma\rho_j, \quad j = 1, \ldots, N$$

and for all $j \in 1, \ldots, N$:

$$\|\xi - x\| \leqslant M \sum_{k=1}^{N} \sqrt{\rho_k}, \quad \|\eta_j - y_j\| \leqslant M' \sqrt{\rho_j}.$$

Proof. We denote $A_j = L_j - B(x, y_j)$. We choose $\delta > 0$ such that $c\|A_j\| + \rho < \gamma\rho_j$ for all j and we use the previous proposition with $x, y_1, \ldots, y_N, \delta$ and $A_1, \ldots, A_N$: there exist $u \in X, v_1, \ldots, v_N \in Y$ such that:

$$\|L_j - B(x, y_j)) - B(u + x, v_j)\| \leqslant c\|A_j\|, \quad j = 1, \ldots, N,$$
$$\|B(u, y_j))\| < \delta \quad \text{for } j = 1, \ldots, p,$$
$$\|v_j\| \leqslant M'\|A_j\|^{1/2} \quad \text{and} \quad \|u\| \leqslant M \sum_{k=1}^{N} (\|a_k\|)^{1/2}.$$

We denote $\xi = u + x$ and $\eta_j = y_j + v_j$. Then we have $\|\xi - x\| \leqslant M \sum\limits_{k=1}^{N} \sqrt{\rho_k}$, $\|\eta_j - y_j\| \leqslant M' \sqrt{\rho_j}$ and

$$\|B(\xi,\eta_j) - L_j\| \leqslant \|L_j - B(x,y_j) - B(u+x,v_j)\| + \|B(u,y_j)\| < c\|A_j\| + \delta < \gamma\rho_j. \quad ∎$$

The following result gives an $(\mathbb{A}_{1,N})$ property (starting with arbitrary data).

PROPOSITION 1.6. *We suppose that B verifies the E^1_C property. Let $L_1, \ldots, L_N \in Z, x \in X, y_1, \ldots, y_N \in Y$, and $\rho_j > 0$ (for $j = 1, \ldots, N$) such that*

$$\|B(x,y_j)) - L_j\| < \rho_j, \quad j = 1, \ldots, N.$$

Then there exist $i \in X, v_1, \ldots, v_n \in Y$ such that for all $j \in \{1, \ldots, N\}$:

$$\|u - x\| \leqslant \frac{M \sum_{k=1}^{N} \sqrt{\rho_k}}{1 - \sqrt{\gamma}}, \quad \|y_j - v_j\| \leqslant \frac{M' \sqrt{\rho_j}}{1 - \sqrt{\gamma}} \quad \textit{and} \quad B(u,v_j) = L_j.$$

Proof. We will prove the proposition by induction. By the lemma, there exist x_0 in X and $y_{0,j} \in Y$ $(j = 1, \ldots, N)$ such that $\|x_0 - x\| \leqslant M \sum\limits_{p=1}^{N} \sqrt{\rho_p}$, $\|y_{0,j} - y_j\| \leqslant M' \sqrt{\rho_j}$ and $\|B(x_0, y_{0,j} - L_j\| < \gamma\rho_j$, for all j. We suppose that, for an integer k, we have x_l in X and $y_{l,j} \in Y (j = 1, \ldots, N$ and $l = k-1, k)$ such that $\|x_k - x_{k-1}\| \leqslant M \sum\limits_{p=1}^{N} \sqrt{\rho_p} \gamma^{(k-1)/2}$, $\|y_{k,j} - y_{k-1,j}\| \leqslant M' \sqrt{\rho_j} \gamma^{(k-1)/2}$ and $\|B(x_k, y_{k,j}) - L_j\| < \gamma^k \rho_j$. Then we apply the lemma (to x_k and $y_{k,j}$ instead of x and y_j), and there exist x_{k+1} in X and $y_{k+1,j} \in Y (j = 1, \ldots, N)$ such that

$$\|B(x_{k+1}, y_{k+1,j}) - L_j\| < \gamma^{k+1} \rho_j \quad j = 1, \ldots, N,$$

and for all j

$$\|x_{k+1} - x_k\| \leqslant M \sum_{p=1}^{N} \sqrt{\rho_p} \gamma^{k/2}, \quad \|y_{k+1,j} - y_{k,j}\| \leqslant M' \sqrt{\rho_j} \gamma^{k/2}.$$

The last two inequalities imply that $(x_k)_k$ and $(y_{k,j})_k$ (for $j = 1, \ldots, N$) are Cauchy sequences, so they converge. We denote respectively u and v_j their limits and we verify all the properties we want. ∎

The following result is a generalisation of the $(A_{1,\aleph_0})$ property; it is inspired by a similar result obtained for contractions on Hilbert space in [12]. We introduce a dense set which will be useful for a concrete subset later.

PROPOSITION 1.7. *We suppose that B verifies the E^1_C property. Let $\mathcal{N}$ be a dense subset of $Z, x \in X, (y_n)_{n \geqslant 1}$ a sequence in Y, some real numbers $\delta_n > 0$ such that $\sum\limits_{n \geqslant 1} n\sqrt{\delta_n} < \infty$, a sequence $(\varepsilon_n)_{n \geqslant 1}$ of real numbers strictly decreasing to 0 and a sequence $(L_n)_{n \geqslant 1}$ in Z such that, for all n, $\|L_n\| < \delta_n$.*

Then there exist $u \in X$, two sequences in Y, (v_n) and (z_n) and three constants C_1, C_2, C_3 such that, for all n:

$$L_n = B(u, v_n), \quad \|v_n\| \leqslant C_1 \sum_{n\geqslant 1} \sqrt{\delta_n}, \quad \|u - x\| \leqslant C_2 \sum_{n\geqslant 1} n\sqrt{\delta_n},$$
$$B(u, z_n) \in \mathcal{N}, \quad \|z_n - y_n\| \leqslant C_3\varepsilon_n.$$

Proof. We take a sequence of positive numbers $(a_n)_{n\geqslant 1}$ such that $\sum_{n=1}^{\infty} a_n \leqslant 1$ and for all n, we put $\alpha_n = \min\{\delta_n, a_n\varepsilon_1, \ldots, a_n\varepsilon_n\}$. Using the density of $\mathcal{N}$, we can find $K_1 \in \mathcal{N}$ such that $\|B(x, y_1) - K_1\| < \alpha_1$; furthermore, $\|B(x, 0) - K_1\| < \delta_1$. By the previous proposition, there exist $x_1 \in X, v_{1,1}, z_{1,1} \in Y$ such that:

$$B(x_1, v_{1,1}) = L_1, \quad B(x_1, z_{1,1}) = K_1, \quad \|x_1 - x\| \leqslant \frac{M\sqrt{\delta_1}}{1-\sqrt{\gamma}},$$
$$\|y_1 - z_{1,1}\| \leqslant \frac{M'\sqrt{\varepsilon_1 a_1}}{1-\sqrt{\gamma}}, \quad \|v_{1,1}\| \leqslant \frac{M'\sqrt{\delta_1}}{1-\sqrt{\gamma}}.$$

By induction, we will prove that for all $n \geqslant 1$ there exist $K_1, \ldots, K_n \in \mathcal{N}, x_{n-1}, x_n$ in X and $v_{j,k}, z_{j,k}$ in Y (for $j = n-1, n$ and $k = 1, \ldots, j$) such that:

$(\mathrm{A_n})$ $\quad B(x_j, v_{j,k}) = L_k \quad j = n-1, n,\ k = 1, \ldots, j;$

$(\mathrm{B_n})$ $\quad B(x_j, z_{j,k}) = K_k \quad j = n-1, n,\ k = 1, \ldots, j;$

$(\mathrm{C_n})$ $\quad \|x_n - x_{n-1}\| \leqslant \dfrac{nM\sqrt{\delta_n}}{1-\sqrt{\gamma}};$

$(\mathrm{D_n})$ $\quad \|z_{n,k} - z_{n-1,k}\| \leqslant \dfrac{M'\sqrt{\varepsilon_k a_n}}{1-\sqrt{\gamma}}, \quad k = 1, \ldots, n-1;$

$(\mathrm{E_n})$ $\quad \|y_n - z_{n,n}\| \leqslant \dfrac{M'\sqrt{\varepsilon_n a_n}}{1-\sqrt{\gamma}};$

$(\mathrm{F_n})$ $\quad \|v_{n,k} - v_{n-1,k}\| \leqslant \dfrac{M'\sqrt{\delta_n}}{1-\sqrt{\gamma}}, \quad k = 1, \ldots, n-1;$

$(\mathrm{G_n})$ $\quad \|v^*_{n,n}\| \leqslant \dfrac{M'\sqrt{\delta_n}}{1-\sqrt{\gamma}}.$

In the beginning of the proof, we have seen that all these properties are true for $n = 1$ (if we note $x_0 := x$ and see that $(D_1), (F_1)$ are empty). Let us suppose the properties are true for n. Since $\mathcal{N}$ is dense, there exists $K_{n+1} \in \mathcal{N}$ such that

$$\|B(x_n, z_{n,n}) - K_{n+1}\| \leqslant \alpha_{n+1}.$$

Furthermore,

$$\|B(x_n, 0) - L_{n+1} \leqslant \delta_{n+1}$$

and using (A_n) and (B_n), we have:

$$\|B(x_n, v_{n,k} - L_k\| \leqslant \alpha_{n+1}, \quad \|B(x_n, z_{n,k} - K_k\| \leqslant \alpha_{n+1} \quad k = 1, \ldots, n.$$

We apply the previous proposition and we obtain immediately the properties $(A_{n+1}), \ldots, (G_{n+1})$. Since $\sum n\sqrt{\delta_n}$ converges, using $(C_{n+p}), \ldots, (C_{n+1})$, we verify that $(x_n)_{n\geqslant 1}$ is a Cauchy sequence. Similarly, for any $k \leqslant 1, (v_{n,k})_{n\geqslant k}$ and $(z_{n,k})_{n\geqslant k}$ are Cauchy sequences. So, we note respectively u, v_k and z_k their limits. Then $u, (v_k)$ and (z_k) verify all the required properties. ∎

REMARK 1.8. In all the proofs, we can see that everything remains true under the weaker assumption that B is closed.

2. FUNCTIONAL CALCULUS

The functional calculus (on H^∞) will be very useful in the next sections, but right now we will already see the link between Section 1 and operator theory.

We take $T \in PB(X)$. It is easy to see that it means that the morphism $f \mapsto f(T)$ is continuous from $A(\mathbb{D})$ to $\mathcal{L}(X)$. In the beginning of this paragraph, we give conditions to extend this functional calculus (for T^*) to H^∞.

DEFINITION 2.1. Let $T \in PB(X)$. We say that T^* has a weak* (of w*) H^∞ functional calculus if there exists $\Phi_{T^*} : H^\infty(\mathbb{D}) \to \mathcal{L}(X^*)$ a continuous morphism with the following property: If $(f_n)_n$ tends to f in H^∞ for the weak* topology (of H^∞), then, for all $x^* \in X^*, \Phi_{T^*}(f_n)x^*$ tends to $\Phi_{T^*}(f_n)x^*$ for the weak* topology (in X^*).

REMARK 2.2. $\mathcal{L}(X^*)$ has a predual (see for example [20] page 125) and one can see that the above definition is equivalent to say that Φ_{T^*} is w*-continuous.

In the following, we will note $f(T^*)$ instead of $\Phi_{T^*(f)}$.

If T^* has a w*-functional calculus, we consider, for $x \in X$ and $x^* \in X^*$, the linear form on $H^\infty x\Box x$ (or, if necessary, $x\overset{T^*}{\Box}x$) defined by $h \mapsto \langle x, h(T^*)x^*\rangle$. Obviously, it is w*-continuous, so it can be identified with an element of the predual of $H^\infty, L^1/H^1_0$ (also denoted by $x\Box x^*$). Then we have the identity:

$$\langle x\Box x^*, h\rangle = \langle x, h(T^*)x^*\rangle, \quad x \in X, x^* \in X^*, h \in H^\infty.$$

One sees that the map $(x, x^*) \mapsto x\Box x^*$ is bilinear, and we have the following definition:

DEFINITION 2.3. Let $T \in PB(X)$ such that T^* has a w*-functional calculus. Let $n, m \leqslant \aleph_0$. We say that T^* possesses the $(\mathbb{A}_{n,m})$ property if the bilinear map $\Box$ defined above has the same property, in the sense of the Definition 1.4.

By the definition of the functional calculus, it is obvious (but quite useful for the next lemmas) to see that if T^* has a w*-functional calculus and if $(f_n)_n$ is a sequence in $A(\mathbb{D})$ tending weak* to 0, then, for all $x \in X$ and $x^* \in X^*$, we have $\langle f_n(T)x, x^*\rangle \to 0$. In fact, by a result of Eschmeier (in [16]), we have the following theorem:

THEOREM 2.4. *Let $T \in PB(X)$. The operator T^* has a* w^**-functional calculus if and only if we have the following property: if $(f_n)_n$ is a sequence in $A(\mathbb{D})$ tending weak* to 0, then, for all $x \in X$ and $x^* \in X^*$, we have $\langle f_n(T)x, x^* \rangle \to 0$.*

Proof. It is easy to adapt the proof of Lemma 1.1 from [16] in our context. The Riesz representation on a Hilbert space is replaced by this simple result: if ϕ is a continuous bilinear form on $X \times X^*$ then there exists $A \in \mathcal{L}(X^*)$ such that, for $x \in X, x^* \in X^*$

$$\phi(x, x^*) = \langle x, Ax^* \rangle. \quad \blacksquare$$

The next corollary is very important in our context. It is a particular case of the previous theorem.

COROLLARY 2.5. *If $T \in PB_M(X) \cap C_{\cdot 0}$ then T^* has a* w^**-functional calculus.*

Proof. We take $x \in X, x^* \in X^*$ and $(f_n)_n$ a sequence in $A(\mathbb{D})$ tending weak* to 0. We can assume that $\|f\|_\infty \leqslant 1, \|x\| \leqslant 1$ and $\|x^*\| \leqslant 1$. Let $\varepsilon > 0$. We assume that $\varepsilon \leqslant 2M$. We take an integer k such that $\|T^{*k}x^*\| \leqslant \varepsilon/4M$. Using the Cauchy formula, and since $(f_n)_n$ converges pointwise to 0 and is bounded, we see that, for all $j \geqslant 0$, the sequence $(f_n^{(j)}(0))_n$ tends to 0. We write $f_n(z) = g_n(z) + z^k h_n(z)$ where $g_n(z) = \sum\limits_{j<k} (f_n^{(j)}(0)/j!)z^j$. Since $(f_n^{(j)}(0))_n$ tends to 0 there exists $N \in \mathbb{N}$ such that, for all $n \geqslant N$ and all $z \in \mathbb{D}, |g_n(z)| \leqslant \varepsilon/2M$. So, $\|h_n\|_\infty \leqslant \|f_n\|_\infty + \|g_n\|_\infty \leqslant 2$. Then, for all $n \geqslant N$,

$$\begin{aligned} |\langle f_n(T)x, x^* \rangle| &\leqslant |\langle g_n(T)x, x^* \rangle| + |\langle h_n(T)x, T^{*k}x^* \rangle| \\ &\leqslant M\|g_n\|_\infty + M\|h_n\|_\infty \|T^{*k}x^*\| \leqslant \varepsilon. \quad \blacksquare \end{aligned}$$

We end this section with three technical lemmas which will be useful when we will see the link between analytic invariant subspaces and reflexivity (Section 5). The notations and assumptions are the following:

(i) $T \in PB(X)$ such that T^* has a w^*-functional calculus;
(ii) $A \in \mathcal{L}(X)$;
(iii) $\mathcal{M} \in \mathrm{Lat}(T)$;
(iv) $\mathcal{N} \in \mathrm{Lat}(A)$;
(v) we note $S := T|_{\mathcal{M}}$ and $B := A|_{\mathcal{N}}$.

LEMMA 2.6. *S^* has a* w^**-functional calculus.*

Proof. Taking $f \in h^\infty, x^* \in \mathcal{M}^*$ and $f_n \in A(\mathbb{D})$ such that $f_n \xrightarrow{\mathrm{w}^*} f$, we define the functional calculus by:

$$f(S^*)x^* = \lim_{\mathrm{w}^*} f_n(S^*)x^*.$$

$f_n(S^*)$ is well defined because T is polynomially bounded, so $S \in PB(\mathcal{M})$. First we see that the definition is correct: if $g_n \in A(\mathbb{D})$ verifies $g_n \xrightarrow{\mathrm{w}^*} f$, then $h_n :=$

$f_n - g_n$ tends weak* to 0. We take $x \in \mathcal{M}$ and $x^* \in \mathcal{M}^*$. By the Hahn–Banach theorem, there exists $\phi \in X^*$ such that $\phi|_{\mathcal{M}^*} = x^*$ and $\|\phi\| = \|x^*\|$. Then

$$\langle x, h_n(T^*)x^* \rangle = \langle h_n(T)x, \phi \rangle = \langle x, h_n(T^*)\phi \rangle$$

where the right term tends to 0 since T^* has a w*-functional calculus. To verify all the properties of this functional calculus, we use again the equality:

$$\langle x, f(S^*)x^* \rangle = \langle x, f(T^*)\phi \rangle \quad \text{and} \quad \|\phi\| = \|x^*\|$$

(with x, x^*, f and ϕ as above). ∎

If x^{**} belongs to $\mathcal{M}^{**}$ we denote by $\widetilde{x^{**}}$ the element of X^{**} defined by

$$\langle x^*, \widetilde{x^{**}} \rangle = \langle x^*|_{\mathcal{M}}, x^{**} \rangle, \quad x^* \in X^*.$$

Using the Hahn–Banach theorem, one can see that the application $x^{**} \mapsto \widetilde{x^{**}}$ is a linear isometry. Then, except in some particular cases, we will assume that $\mathcal{M}^{**} \subset X^{**}$ and we will forget the notation $\widetilde{\ }$.

LEMMA 2.7. *For $h \in H^\infty, x^* \in X^*$ and $x^{**} \in \mathcal{M}^{**}$ we have $h(S^*)(x^*|_{\mathcal{M}}) = (h(T^*)x^*)|_{\mathcal{M}}$ and $h(S^*)^*x^{**} = h(T^*)^*x^{**}$. We have also $B^*(x^*|_{\mathcal{N}}) = (A^*x^*)|_{\mathcal{N}}$.*

Proof. Let $x \in \mathcal{M}$ and $h_n \in A(\mathbb{D})$ such that h_n tends to h for the weak* topology. Then we have the following equalities:

$$\begin{aligned}\langle x, h(S^*)(x^*|_{\mathcal{M}}) \rangle &= \lim_n \langle h_n(S)x, x^*|_{\mathcal{M}} \rangle = \lim_n \langle h_n(S)x, x^* \rangle = \lim_n \langle x, h_n(T^*)x^* \rangle \\ &= \langle x, h(T^*)x^* \rangle = \langle x, (h(T^*)x^*)|_{\mathcal{M}} \rangle.\end{aligned}$$

Then, for $x^* \in X^*$ and $x^{**} \in \mathcal{M}^{**}$ we have $\langle x^*|_{\mathcal{M}}, h(S^*)^*x^{**} \rangle = \langle h(T^*)x^*, \widetilde{x^{**}} \rangle$. Forgetting the $\widetilde{\ }$ this is exactly the equality $h(S^*)^*x^{**} = h(T^*)^*x^{**}$. The last equality is obtained in the same way. ∎

LEMMA 2.8. *For $x \in \mathcal{M}$ and $x^* \in X^*$ we have*

$$x \overset{T^*}{\Box} x^* = x \overset{S^*}{\Box} (x^*|_{\mathcal{M}}).$$

Proof. Using the previous lemma, for $h \in H^\infty$, we have:

$$\langle x \overset{T^*}{\Box} x^*, h \rangle = \langle x, h(T^*)x^* \rangle = \langle x, (h(T^*)x^*)|_{\mathcal{M}} \rangle = \langle x, h(S^*)(x^*|_{\mathcal{M}}) \rangle = \langle x \overset{S^*}{\Box} (x^*|_{\mathcal{M}}), h \rangle. \quad ∎$$

3. THE TOOLS OF THE AMBROZIE–MÜLLER THEOREM

We recall quickly some important results from the paper of Ambrozie and Müller which will be used in Section 6.

3.1. ZENGER LEMMA. We recall a new Zenger lemma proved in [1]; but before, we need the following definition:

DEFINITION 3.1. Let $(X, \|\cdot\|)$ be a complex Banach space and $u_1, \dots, u_n \in X$. Let $L > 0$. We say that the vectors are *L-unconditional* if we have

$$\Big\| \sum_{j=1}^{n} \beta_j u_j \Big\| \leqslant L \Big\| \sum_{j=1}^{n} \gamma_j u_j \Big\|$$

whenever β_j, γ_j are complex numbers with $|\beta_j| \leqslant |\gamma_j|$ $(j = 1, \dots, n)$.

REMARK 3.2. Using the equivalence of the norms in finite dimensional spaces, one sees easily that every finite family of linearly independent vectors are L-unconditional for some L.

PROPOSITION 3.3. *Let $\|\cdot\|$ be a norm on $\mathbb{C}^*, \alpha_1, \dots, \alpha_n$ some positive numbers such that $\sum\limits_{j=1}^{n} \alpha_j = 1$, let $e_1, \dots, e_n$ be a basis $\mathbb{C}^n, L > 0$ such that the basis is L-unconditional, and let $s = \sum s_j e_j \in \mathbb{C}^n$. Then there exists $w = \sum w_j e_j \in \mathbb{C}^n$ and ψ a linear form such that $\|\psi\| \leqslant L\sqrt{2}, \|w - s\| \leqslant 1$ and $\psi(w_j e_j) = \alpha_j, j = 1, \dots, n$.*

We give now a dual version of this result.

PROPOSITION 3.4. *Let X be a complex Banach space, $s \in X, \phi_1, \dots, \phi_n \in X^*$ L-unconditional and $\alpha_1, \dots, \alpha_n$ some positive numbers such that $\sum\limits_{j=1}^{n} \alpha_j = 1$. Then there exist $\beta_1, \dots, \beta_n$ n complex numbers and $w \in X$ such that $\|w - s\| \leqslant 2, \Big\| \sum\limits_{j} \beta_j \phi_j \Big\| \leqslant L\sqrt{2}$ and $\langle w, \beta_j \phi_j \rangle = \alpha_j$ for all j.*

Proof. We note $E = \mathrm{vect}\{\phi_1, \dots, \phi_n\}$, E is isometric to $\mathcal{Q}^*$ where $\mathcal{Q} = X/{}^{\perp}E$. The dimension of $\mathcal{Q}$ is finite. So, there exist $\widetilde{e}_1, \dots, \widetilde{e}_n \in \mathcal{Q}$ such that $\langle \widetilde{e}_j, \phi_k \rangle = \delta_{jk}$, where δ_{jk} is the Kronecker symbol. We can verify (see [1] for the proof) that since $\phi_1, \dots, \phi_n \in X^*$ are L-unconditional, then $\widetilde{e}_1, \dots, \widetilde{e}_n$ are also L-unconditional. If we denote π the canonical surjection from X to $\mathcal{Q}$, then we can apply the Proposition 3.3: There exist $\widetilde{w} = \sum w_j \widetilde{e}_j \in \mathcal{Q}$ and $\psi = \sum \beta_j \phi_j \in E$ linear forms such that $\|\psi\| \leqslant L\sqrt{2}, \|\widetilde{w} - \pi(s)\| \leqslant 1$ and $\langle w_j \widetilde{e}_j, \psi \rangle = \alpha_j$, $(j = 1, \dots, n)$. But, $\langle w_j \widetilde{e}_j, \psi \rangle = \langle \widetilde{w}, \beta_j \phi_j \rangle$. We can find $w \in X$ such that $\pi(w) = \widetilde{w}$ and $\|w - s\| \leqslant 2$. By the definition of $\mathcal{Q}$, we have:

$$\langle w, \beta_j \phi_j \rangle = \langle \pi(w), \beta_j \phi_j \rangle = \alpha_j. \quad \blacksquare$$

3.2. APPROXIMATION IN L^1. Another innovation of [1] is the approximation of positive L^1 functions by some particular linear combination of Poisson kernels.

DEFINITION 3.5. For $\lambda = r\mathrm{e}^{\mathrm{i}\theta}$ we denote $I_\lambda = \{\mathrm{e}^{\mathrm{i}t} : |t - \theta| < 2(1 - r)\}$. A subset F of $\mathbb{D}$ is called *a-separated* if for all $\lambda, \mu \in F$ $(\lambda \neq \mu)$, we have $I_\lambda \cap I_\mu = \emptyset$.

PROPOSITION 3.6. *There exist $C \in (0,1)$ and $b > 0$ with the following property: if f is a positive function in L^1 and if Λ is an Apostol set, then for n large enough, there exist a finite and a-separated subset $F \subset \Lambda$ and positive numbers α_λ $(\lambda \in F)$ such that:*

(i) $|\lambda^n - 1| < 1/9$ *and* $|\lambda| \geqslant 3/4$ *for* $\lambda \in F$*;*

(ii) $\sum\limits_{\lambda \in F} \leqslant \|f\|_1$*;*

(iii) $\int\limits_{-\pi}^{\pi} \Big| \sum\limits_{\lambda \in F} \alpha_\lambda \lambda^n P_\lambda(t) - f(t) \Big| \mathrm{d}t \leqslant c_2 \int\limits_{-\pi}^{\pi} f(t)\mathrm{d}t.$

Furthermore, if $\Lambda = \Lambda_{\varepsilon,2}(T)$, with $0 < \varepsilon < 1/2Mb\pi$, for $T \in PB_M(X)$, then there exist vectors $u_\lambda \in X$, $2Mb$-unconditional such that $\|u_\lambda\| \leqslant 1$ and $\|(T-\lambda)u_\lambda\| < \varepsilon(1-|\lambda|)^2$ $(\lambda \in F)$.

4. ANALYTIC INVARIANT SUBSPACES

The notion of analytic invariant subspace is used in operator theory since [17]. In [8], Brown gives a definition of this notion in a general context; our definition is not exactly the same; in fact, we are trying to generalize to the Banach space case the definition of invariant analytic subspaces given in [12].

DEFINITION 4.1. Let $T \in \mathcal{L}(X)$ and $\mathcal{M} \in \mathrm{Lat}(T)$. We say that $\mathcal{M}$ is an *analytic invariant subspace* of T if there exists a non zero analytic function from $\mathbb{D}$ into $\mathcal{M}^*, e : \lambda \mapsto e_\lambda$ such that

$$(T|_{\mathcal{M}} - \lambda)^* e_\lambda = 0, \quad \lambda \in \mathbb{D}.$$

Furthermore if e satisfies

$$\bigvee_{\lambda \in \mathbb{D}} e_\lambda = \mathcal{M}^*$$

then $\mathcal{M}$ is called a *full analytic invariant subspace for T*.

REMARK 4.2. In the Hilbert space context we suppose that e_λ is a conjugate analytic function, since the duality is replaced by the inner product.

The following proposition is obvious.

PROPOSITION 4.3. *Let $T \in \mathcal{L}(X)$ and $\mathcal{M}$ be an analytic invariant subspace for T, with the associated map e, then there exists $(f_n)_{n \in \mathbb{N}}$ a sequence of linear forms in $\mathcal{M}^*$ such that e has an absolutely convergent power series expansion:*

$$e_\lambda = \sum_{n=0}^{\infty} \lambda^n f_n \quad \lambda \in \mathbb{D}. \tag{4.1}$$

Furthermore, the f_n satisfy:

$$f_n = \frac{1}{n!} \frac{d^n e}{d\lambda^n}\Big|_{\lambda=0}; \tag{4.2}$$

$$T^* f_0 = 0, \quad T^* f_n = f_{n-1} \quad n \geqslant 1; \tag{4.3}$$

(4.4) $$\limsup \|f_n\|^{1/n} \leqslant 1;$$

(4.5) $$\bigvee_{\lambda\in\mathbb{D}} e_\lambda = \bigvee_{n\in\mathbb{N}} f_n.$$

REMARK 4.4. (i) If there exists $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{M}^*$, where $\mathcal{M} \in \mathrm{Lat}(T)$, satisfying (4.3) and (4.4) we can define by (4.1) a map e which turns $\mathcal{M}$ into an analytic invariant subspace for T.

(ii) (4.3) implies that $f_n \neq 0$ for n large enough.

We denote by $H(\mathbb{D})$ the space of holomorphic functions on $\mathbb{D}$ and by M_λ the operator of multiplication by λ, acting on $H(\mathbb{D}) : f(\lambda) \mapsto \lambda f(\lambda)$.

PROPOSITION 4.5. *Let $T \in \mathcal{L}(X)$ and $\mathcal{M}$ be an analytic invariant subspace for T, with the associated map e. We define the linear map $F : \mathcal{M} \to H(\mathbb{D})$ such that:*

$$(F(x))(\lambda) = \langle x, e_\lambda \rangle.$$

This application satisfies

$$F \circ (T|_{\mathcal{M}}) = \mathcal{M}_\lambda \circ F.$$

Furthermore, if $\mathcal{M}$ is full analytic, then F is one-to-one.

Now, we are going to show the reflexivity of polynomially bounded operators in some particular cases (it was shown in the Hilbert space context in [12]). We begin with a remark, which is a particular case of Lemma 2.7.

REMARK 4.6. Let $T \in PB(X)$ with a w*-functional calculus. We take $\mathcal{M} \in \mathrm{Lat}(T), \lambda \in \mathbb{D}$ and $x^* \in \mathcal{M}$ such that $(T|_{\mathcal{M}})^* x^* = \lambda x^*$. If $\phi \in X^*$ satisfies $\phi|_{\mathcal{M}} = x^*$, then, for all $h \in H^\infty$ we have

$$(h(T^*)\phi)|_{\mathcal{M}} = (h(\lambda)\phi)|_{\mathcal{M}}.$$

PROPOSITION 4.7. *Let $T \in PB(X)$ such that X is full analytic for T. Then T is reflexive.*

Proof. Let $e : \lambda \mapsto e_\lambda$ be an analytic map such that $X^* = \bigvee e_\lambda$. Since $T^* e_\lambda = \lambda e_\lambda$, we have $T \in C_{\cdot 0}$, then T^* possesses a w*-functional calculus. Let $A \in \mathrm{AlgLat}(T)$. If $\mathcal{M}_* \in \mathrm{Lat}(T^*)$ then ${}^\perp\mathcal{M}_* \in \mathrm{Lat}(T)$ and if $x \in {}^\perp\mathcal{M}_*$ and $x^* \in \mathcal{M}_*$, then:

$$\langle x, A^* x^* \rangle = \langle Ax, x^* \rangle = 0.$$

It means that $A^* x^* \in ({}^\perp\mathcal{M}_*)^\perp = \overline{\mathcal{M}_*}^{\mathrm{w}^*}$. Since $\mathbb{C}e_\lambda$ is weak* closed (it is a finite dimensional space), $A^* e_\lambda \in \mathbb{C}e_\lambda$, for $\lambda \in \mathbb{D} \setminus \Lambda$, where Λ denotes the set of λ such that $e_\lambda = 0$. So we can write,

$$A^* e_\lambda = h(\lambda) e_\lambda,$$

where h is some function defined pointwise for $\lambda \in \mathbb{D} \setminus \Lambda$. Clearly, $|h(\lambda)| \leqslant \|A\|$ for $\lambda \in \mathbb{D} \setminus \Lambda$. Furthermore,

(4.6) $$(F(Ay))(\lambda) = h(\lambda)(F(y))(\lambda), \quad \lambda \in \mathbb{D} \setminus \Lambda,$$

for $y \in X \setminus \{0\}$. Since $F(y) \neq 0$ for any such y, $\Lambda \cap \Lambda'$ is a countable set with no point of accumulation in $\mathbb{D}$, where Λ' denotes the zeros of $F(y)$. Furthermore (4.6) shows that h is a bounded holomorphic function on $\mathbb{D} \setminus \Lambda \cap \Lambda'$, so we can extend h on a function in H^∞. Since T^* has a w*-functional calculus, one can verify that, for $\lambda \in \mathbb{D} \setminus \Lambda \cap \Lambda'$,

$$h(T^*)e_\lambda = h(\lambda)e_\lambda = A^*e_\lambda.$$

So, $A^* = h(T^*)$ on $\bigvee_{\lambda \in \mathbb{D} \setminus \Lambda \cap \Lambda'} e_\lambda$. But,

$$\bigvee_{\lambda \in \mathbb{D}} e_\lambda = \bigvee_{\lambda \in \mathbb{D} \setminus \Lambda \cap \Lambda'} e_\lambda.$$

Indeed, if $\phi \in X^{**}$ satisfies $\langle e_\lambda, \phi \rangle = 0$ for $\lambda \in \mathbb{D} \setminus \Lambda \cap \Lambda'$ then the holomorphic function $\lambda \mapsto \langle e_\lambda, \phi \rangle$ is the zero function; and since X is full analytic, we have $\phi = 0$. Finally, we have shown that $A^* = h(T^*)$. Then, for $x \in X, x^* \in X^*$ and (h_n) a sequence in $A(\mathbb{D})$ tending weakly* to h, we have:

$$\langle Ax, x^* \rangle = \langle x, h(T^*)x^* \rangle = \lim_n \langle x, h_n(T^*)x^* \rangle = \lim_n \langle h_n(T)x, x^* \rangle.$$

We conclude that $A \in \mathcal{W}_T$. ∎

We denote by $CF(T)$ the set of all $x \in X$ such that $\bigvee_{n \geqslant 0} T^n x$ is full analytic for T.

PROPOSITION 4.8. *Let T be a polynomially bounded operator such that T^* has a w*-functional calculus. We suppose that $CF(T)$ is dense in X. Then T is reflexive.*

Proof. Let $A \in \mathrm{AlgLat}(T)$ and x_1 and x_2 in $CF(T)$; we denote $\mathcal{M}_i = \bigvee_{n \geqslant 0} T^n x_i$, $i = 1, 2$. We apply the previous proposition to $T_i := T|_{\mathcal{M}_i}$ and we can find h_1 and h_2 in H^∞ such that $h_i(T_i^*) = A_i^*$, where $A_i := A|_{\mathcal{M}_i}$ $(i = 1, 2)$.

In the first part of the proof, we will see that $h_1 = h_2$.

As it was said in Section 2, we assume that $\mathcal{M}_i^{**}$ is a subset of X^{**}; in particular, for $x^* \in X^*$ and $\phi_i \in \mathcal{M}_i^{**}$, we have (forgetting the ˜ defined in Section 2):

$$\langle x^*, \phi_i \rangle = \langle x^*|_{\mathcal{M}_i}, \phi_i \rangle.$$

If $\mathcal{M}_1^{**} \cap \mathcal{M}_2^{**} \neq \{0\}$, we take $y^{**} \neq 0$ in this intersection and we denote by e the analytic map associated to $\mathcal{M}_1$. Using the Hahn–Banach theorem, for each $\lambda \in \mathbb{D}$, we extend e_λ in $\varepsilon_\lambda \in X^*$. Then, using the Lemma 2.7 and the previous remark, we have:

$$\begin{aligned}
h_1(\lambda)\langle e_\lambda, y^{**} \rangle &= \langle h_1(\lambda)e_\lambda, y^{**} \rangle = \langle h_1(T_1^*)e_\lambda, y^{**} \rangle \\
&= \langle A_1^* e_\lambda, y^{**} \rangle = \langle A^* \varepsilon_\lambda, y^{**} \rangle \\
&= \langle A_2^*(\varepsilon_\lambda|_{\mathcal{M}_2}), y^{**} \rangle = \langle h_2(T_2^*)(\varepsilon_\lambda|_{\mathcal{M}_2}), y^{**} \rangle \\
&= \langle (h_2(T^*)\varepsilon_\lambda)|_{\mathcal{M}_2}, y^{**} \rangle = h_2(\lambda)\langle \varepsilon_\lambda|_{\mathcal{M}_2}, y^{**} \rangle \\
&= h_2(\lambda)\langle \varepsilon_\lambda, y^{**} \rangle = h_2(\lambda)\langle e_\lambda, y^{**} \rangle.
\end{aligned}$$

Since $\lambda \mapsto \langle y, e_\lambda \rangle$ is a nonzero analytic function, it follows that $h_1 = h_2$.

If $\mathcal{M}_1^{**} \cap \mathcal{M}_2^{**} = \{0\}$, we choose $(u_n)_{n\geqslant 3}$ a sequence in $CF(T)$ such that $u_n \to x_1 + x_2$ (we take $n \geqslant 3$ to avoid confusion with the notation corresponding to x_1 and x_2). Using the previous proposition, we deduce that for each $n \geqslant 3$ there exists $g_n \in H^\infty$ such that $A_n^* = g_n(T_n^*)$ where $\mathcal{M}_n := \bigvee_{k\geqslant 0} T^k u_n$, $T_n := T|_{\mathcal{M}_n}$ and $A_n := A|_{\mathcal{M}_n}$; the proposition also gives $\|g_n\|_\infty \leqslant \|A_n^*\| \leqslant \|A\|$. By dropping down to a subsequence, we can suppose that (g_n) is weakly* convergent to a function $g \in H^\infty$. Since T^* has a w*-functional calculus, $(g_n(T^*))$ converges to $g(T^*)$ for the weak operator topology; furthermore, (u_n) converges to $x_1 + x_2$; then we have, for each $x^* \in X^*$,

$$\lim\langle x^*, g_n(T^*)^* u_n\rangle = \langle x^*, g(T^*)^*(x_1 + x_2)\rangle.$$

In other words,

$$\begin{aligned}
\langle x^*, g(T^*)^*(x_1 + x_2)\rangle &= \lim\langle u_n, g_n(T^*)x^*\rangle = \lim\langle u_n, (g_n(T^*)x^*)|_{\mathcal{M}_n}\rangle \\
&= \lim\langle u_n, g_n(T_n^*)(x^*|_{\mathcal{M}_n})\rangle = \lim\langle u_n, A_n^*(x^*|_{\mathcal{M}_n})\rangle \\
&= \lim\langle u_n, A^*x^*\rangle = \langle x_1 + x_2, A^*x^*\rangle \\
&= \langle x_1, A_1^*(x^*|_{\mathcal{M}_1})\rangle + \langle x_2, A_2^*(x^*|_{\mathcal{M}_2})\rangle \\
&= \langle x_1, h_1(T^*)x^*\rangle + \langle x_2, h_2(T^*)x^*\rangle.
\end{aligned}$$

Since $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$, we have $h_i(T^*)^* x_i = g(T^*)^* x_i$, $i = 1, 2$; using as previously the notation e_λ and ε_λ, we deduce that

$$h_1(\lambda)\langle x_1, e_\lambda\rangle = \langle x_1, h_1(T_1^*)e_\lambda\rangle = \langle x_1, h_1(T^*)\varepsilon_\lambda\rangle = \langle x_1, g(T_1^*)e_\lambda\rangle = g(\lambda)\langle x_1, e_\lambda\rangle.$$

So $h_1 = g$; we prove in the same way that $h_2 = g$. We have shown that there exists a function $h \in H^\infty$ such that, if x_1, x_2 are in $CF(T)$, then $A_i^* = h(T_i^*)$, $i = 1, 2$. If we take $x \in X$ and $x^* \in X^*$, there exists $x_n \in CF(T)$ such that $x_n \to x$ and we can verify that

$$\langle Ax, x^*\rangle = \lim\langle Ax_n, x^*\rangle = \lim\langle x_n, h(T^*)x^*\rangle = \langle x, h(T^*)x^*\rangle.$$

Then $A^* = h(T^*)$ and $A \in \mathcal{W}_T$. ∎

REMARK 4.9. In the two previous propositions, we can verify that T^* is also reflexive.

5. FACTORIZATION IN L^1/H_0^1 AND ANALYTIC INVARIANT SUBSPACES

In this section we will see that, under some assumptions of factorization in the quotient space L^1/H_0^1, polynomially bounded operators are reflexive. We denote by $[\mathrm{e}^{-int}]$ the class in L^1/H_0^1 of the function $t \mapsto \mathrm{e}^{-int}$ ($n \in \mathbb{N}$).

For $h \in H^\infty$, it follows that:

$$\langle h, [\mathrm{e}^{-int}]\rangle = \widehat{h}(n)$$

where $\widehat{h}(n)$ is the nth Fourier coefficient of h.

We denote by $\mathcal{N}_T$ the linear space algebraically spanned by the elements $[\mathrm{e}^{-\mathrm{i}nt}]$, for $n \in \mathbb{N}$; then $\mathcal{N}_T$ is dense in L^1/H^1_0.

PROPOSITION 5.1. *Let T be a polynomially bounded operator such that T^* has a* w**-functional calculus. We suppose that there exist vectors $x \in X$ and $(x_j^*)_{j\geqslant 0} \in X^*$ such that:*

$$x\Box x_j^* = [\mathrm{e}^{-\mathrm{i}jt}], \quad j \geqslant 0; \quad \limsup_{j\to\infty} \|x_j^*\|^{1/j} \leqslant 1.$$

Then $\bigvee\limits_{n\geqslant 0} T^n x$ is an analytic invariant subspace for T.

Proof. In this proof, we will write $\Box$ for $\overset{T^*}{\Box}$ and $\Diamond$ for $\overset{(T|_{\mathcal{M}})^*}{\Box}$. We denote $\mathcal{M} = \bigvee\limits_{n\geqslant 0} T^n x$. Since $\mathcal{M} \in \mathrm{Lat}(T)$, we have by the Lemma 2.8

$$x\Box x^* = x\Diamond(x^*|_{\mathcal{M}}), \quad x^* \in X^*.$$

But $x^* \mapsto x\Diamond x^*$ is one-to-one from $\mathcal{M}^*$ to L^1/H^1_0.

Indeed, suppose that $x\Diamond x^* = 0$ with $x^* \in \mathcal{M}^*$. Let $y \in \mathcal{M}$, then there exist polynomials (p_n) with $y = \lim p_n(T)x$. Then,

$$\langle y, x^*\rangle = \lim\langle x, p_n((T|_{\mathcal{M}})^*)x^*\rangle = \langle p_n, x\Diamond x^*\rangle = 0.$$

We deduce that $x^* = 0$.

Since $x\Diamond(x_0^*|_{\mathcal{M}}) \neq 0$, then $x_0^*|_{\mathcal{M}} \neq 0$ and we obtain:

$$Tx\Box x_0^* = 0; \quad Tx\Box x_j^* = [\mathrm{e}^{-\mathrm{i}(j-1)t}] = x\Box x_{j-1}^*, \quad j \geqslant 1.$$

Besides,

$$Tx\Box x_j^* = Tx\Diamond x_j^*|_{\mathcal{M}} = x\Diamond(T|_{\mathcal{M}})^* x_j^*|_{\mathcal{M}}.$$

It follows that $(x_j^*|_{\mathcal{M}})$ is nonzero and verifies (4.3) from Proposition 4.3.

Furthermore, $\|x^*|_{\mathcal{M}}\| \leqslant \|x^*\|$, so $\limsup \|x_j^*|_{\mathcal{M}}\|^{1/j} \leqslant 1$ and by the Remark 4.4, $\mathcal{M}$ is an analytic invariant subspace. ∎

PROPOSITION 5.2. *Let T be a polynomially bounded operator such that T^* has a* w**-functional calculus. We suppose that there exist a vector $x \in X$ and sequences (x_j^*) and (y_j^*) in X^* such that:*

$$x\Box x_j^* = [\mathrm{e}^{-\mathrm{i}jt}], \quad j \geqslant 0; \qquad \limsup \|x_j^*\|^{1/j} \leqslant 1;$$
$$x\Box y_j^* \in \mathcal{N}_T, \quad j \geqslant 0; \qquad \bigvee_{j\geqslant 0} y_j^* = X^*.$$

Then the cyclic subspace generated by x, $\bigvee\limits_{n\geqslant 0} T^n x$ is a full analytic invariant subspace for T.

Proof. As in the previous proof, we use the notations $\square$ and $\Diamond$. It follows from the previous proposition that $\mathcal{M} := \bigvee T^n x$ is an analytic invariant subspace for T. Furthermore, for all j, $x \square y_j^*$ can be written:

$$x \square y_j^* = \sum_{k=0}^{n_j} \alpha_k^j [\mathrm{e}^{-\mathrm{i}kt}] = x \square \Big(\sum_{k=0}^{n_j} \alpha_k^j x_k^* \Big).$$

Thus, since $x^* \mapsto x \Diamond x^*$ is one-to-one from $\mathcal{M}^*$ to L^1/H_0^1,

$$y_j^*|_{\mathcal{M}} = \Big(\sum_{k=0}^{n_k} \alpha_k^j x_k^* \Big)|_{\mathcal{M}}, \quad j \geqslant 0.$$

It follows that $\bigvee\limits_{j\geqslant 0} y_j^*|_{\mathcal{M}} \subset \bigvee\limits_{j\geqslant 0} x_j^*|_{\mathcal{M}}$, and then $\bigvee\limits_{j\geqslant 0} x_j^*|_{\mathcal{M}} = \mathcal{M}^*$.

As in the previous proof, we show that $(x_j^*|_{\mathcal{M}})$ verifies (4.3) from Proposition 4.3; we deduce that $\mathcal{M}$ is full analytic. ∎

6. THE MAIN RESULT

We begin this section with factorization and reflexivity results for some particular polynomially bounded operators; more precisely, we will suppose that $T \in C_{\cdot 0}$ and that $\Lambda_{\varepsilon,2}(T^*)$ are Apostol sets for ε small enough. It will imply that T^* has a w*-functional calculus.

We start with a simple lemma.

LEMMA 6.1. *Let $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and X be a complex Banach space with* $\mathrm{Dim}(X) \geqslant n$ *and $\phi_1, \dots, \phi_n$ in X^* and linearly independent. Then there exists $x \in X$ such that for $j \in \{1, \dots, n\}$,*

$$\phi_j(x) = \lambda_j.$$

Proof. We will see that the following map is onto:

$$\begin{array}{rcl} X & \to & \mathbb{C}^n \\ y & \mapsto & (\phi_1(y), \dots, \phi_n(y)). \end{array}$$

Assuming the contrary, there would exist $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ such that for all $y \in X$, $\sum\limits_{j=1}^{n} \alpha_j \phi_j(y) = 0$. So, the linear forms would not be independent. ∎

Using the previous lemma and the Proposition 3.4, we can adapt easily the proofs of Theorems 12, 13 and 14 of [2] (or their equivalent in [1]) to obtain the next two results.

THEOREM 6.2. *We suppose that $T \in PB_M(X) \cap C_{\cdot 0}$ and that $\Lambda_{\varepsilon,2}(T^*)$ are Apostol sets for all ε small enough. Let $y \in X$ and $f \geqslant 0$ in L^1 with $\|f\|_1 \leqslant 1$. Then for n large enough, there exist $x \in X$ and $x^* \in X^*$ such that $\|x\| \leqslant 2$, $\|x^*\| \leqslant 2Mb\sqrt{2}$ and*

$\|(T^n x + y)\Box x^* - [f]\| < c$, where b is a universal constant and $c \in (C, 1)$ (C is defined in Proposition 3.6*).*

REMARK 6.3. We see in the proof of Theorem 12 in [2], that ε depends on y in the following way: $\varepsilon\|y\| \leqslant 1$; so, as we want a property for all y (or all z, in the notation of the next proposition), we need to ask that $\Lambda_{\varepsilon,2}(T^*)$ are Apostol sets for all ε small enough.

We fix an integer m such that $c + \pi m^{-1} < 1$ and a constant $c' < 1$ with $1 - m^{-1}(1 - c - \pi m^{-1}) < c'$.

PROPOSITION 6.4. *We suppose that $T \in PB_M(X) \cap C_{\cdot 0}$ and that $\Lambda_{\varepsilon,2}(T^*)$ are Apostol sets for all ε small enough. Let $w_1^*, \dots, w_p^*$ be vectors in X, $z \in X$, $\delta > 0$ and $f \in L^1$. Then there exist $u \in X$ and $u^* \in X^*$ which verify:*

(i) *$\|(u + z)\Box u^* - [f]\| \leqslant c'\|f\|_1$;*

(ii) *$\|u\Box w_j^*\| < \delta$ pour $j = 1, \dots, p$;*

(iii) *$\|u\| \leqslant 2M\|f\|_1^{1/2}$ and $\|u^*\| \leqslant 2Mb\sqrt{2}\|f\|_1^{1/2}$.*

The previous proposition gives that the bilinear application $\Box$ verifies a property $E^1_{c'}$. Applying the results of Section 1 we have the following theorem:

THEOREM 6.5. *We suppose that $T \in PB_M(X) \cap C_{\cdot 0}$, that $\Lambda_{\varepsilon,2}(T^*)$ are Apostol sets for all ε small enough and that X^* is separable. Then T has the property $(\mathbb{A}_{1,\aleph_0})$ and T is reflexive.*

Proof. Take $x \in X$, $\delta > 0$, a sequence $(\varepsilon_n)_{n\geqslant 1}$ of reals strictly decreasing to 0 and a sequence $(x_n^*)_{n\geqslant 1}$ such that $X^* = \overline{\{x_n^*\}}$. For $n \geqslant 1$, we note

$$[h_n] = \frac{[\mathrm{e}^{-\mathrm{i}(n-1)t}]\delta^2}{n^6}.$$

Then we apply the Proposition 1.7 (with $\delta_n = \delta^2/n^6$) and deduce the existence of $y \in X$, two sequences in X^*, (y_n^*) and (z_n^*) and three constants C, C', C'' such that, for all n:

$$[h_n] = y\Box y_n^*, \ \ \|y_n^*\| \leqslant C\delta, \ \ \|y - x\| \leqslant C'\delta, \ \ y\Box z_n^* \in \mathcal{N}_T, \ \ \|z_n^* - x_n^*\| \leqslant C''\varepsilon_n,$$

(where $\mathcal{N}_T$ is defined in the beginning of the previous section).

Denoting $t_n^* = n^6 y_n^*/\delta^2$, we see that y, t_n^*, z_n^* verify the hypothesis of the Proposition 5.2. Thus, the space $\bigvee T^n y$ is full analytic for T. Since $\|y - x\| \leqslant C'\delta$, such y are dense in X. Then, by Proposition 4.8, T is reflexive. ∎

Now we can demonstrate the main result of this paper:

THEOREM 6.6. *Every polynomially bounded operator acting on a reflexive Banach space such that its spectrum contains the unit circle either has a nontrivial hyperinvariant subspace or is reflexive.*

Proof. First, suppose that X is separable. We use the following notations

$$X_1 = \{x \in X,\, T^n x \to 0\}, \quad Y_1 = \{x^* \in X^*,\, T^{*n} x \to 0\}.$$

Then we have one of the following cases:

(i) If X_1 (respectively Y_1) is non trivial, then X_1 (respectively ${}^{\perp}Y_1$) is a nontrivial hyperinvariant subspace for T.

(ii) If $X_1 = Y_1 = \{0\}$ then (Theorem 17 of [15]), T possesses a nontrivial hyperinvariant subspace or $T = \lambda I$ and thus is reflexive.

(iii) If $Y_1 = X^*$ et $\Lambda_{\varepsilon,2}(T^*)$ is not an Apostol set for one ε, then T possesses a nontrivial hyperinvariant subspace (this is the Apostol theorem).

(iv) If $Y_1 = X^*$ and $\Lambda_{\varepsilon,2}(T^*)$ are Apostol sets for all ε small enough, then we can apply the previous theorem and T is reflexive.

(v) If $X_1 = X$ we can apply one of the two previous points to $U = T^*$.

If X is not separable, we can suppose that $\operatorname{Ker} T = \{0\}$ and that $T \neq \lambda I$. Taking $x \in X$ we denote $\mathcal{M}_x := \bigvee_{n\geqslant 0} T^n x$; it is a non trivial invariant subspace for $x \neq 0$.

If for all $x \neq 0$ we have $T|_{\mathcal{M}_x} \in C_{\cdot 0}$ and $\Lambda_{\varepsilon,2}(T|_{\mathcal{M}_x}{}^*)$ is an Apostol set, then, following the proof of Proposition 6.5, $CF(T|_{\mathcal{M}_x})$ is dense. Thus $CF(T)$ is also dense and using the Proposition 4.8, we can say that T is reflexive. In the other case, there exists $x \neq 0$ such that $T|_{\mathcal{M}_x} \notin C_{\cdot 0}$ or $\Lambda_{\varepsilon,2}(T|^*_{\mathcal{M}_x})$ is not an Apostol set; then, we can use the points (i), (ii), or (iii) given in the first part of the proof. In the case (ii), we can have $T|_{\mathcal{M}_x} = \lambda I_{\mathcal{M}_x}$ and then $\operatorname{Ker}(T - \lambda)$ is a nontrivial hyperinvariant subspace. ∎

Following the results of [11], Brown and Chevreau had proven in [9] that every contraction on a Hilbert space with an isometric functional calculus on H^∞ is reflexive. The corresponding result in our context would be: every polynomially bounded operator with a functional calculus bounded below is reflexive. However, it seems difficult to adapt the proof of [9], even with the innovations of Ambrozie and Müller.

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