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CLASSIFICATION OF ACTIONS OF DUALS OF FINITE GROUPS ON THE AFD FACTOR OF TYPE II_1

TOSHIHIKO MASUDA

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ABSTRACT. We will show the uniqueness of outer coactions of finite groups on the AFD factor of type II_1 along the arguments by Connes, Jones and Ocneanu. Namely, we construct the infinite tensor product type action, adopt it as the model action, and prove that any outer coaction is conjugate to the model action.

KEYWORDS: *Coaction, action of group dual, cohomology vanishing, model action splitting.*

MSC (2000): 46L40.

1. INTRODUCTION

In the theory of operator algebras, the study of automorphisms is one of the most important topics. Especially, much progress has been made on classification of automorphisms and group actions on injective factors since fundamental works of A. Connes. In [2] and [3], A. Connes succeeded in classifying automorphisms of the approximately finite dimensional (AFD) factor of type II_1 up to outer conjugacy. The first generalization of Connes' results was made by V.F.R. Jones in [5], where he classified actions of finite groups on the AFD factor of type II_1. Soon after Jones' theory, A. Ocneanu classified actions of discrete amenable groups on the AFD factor of type II_1. One of the extension of their results is analysis (or classification) of actions of dual object of groups, i.e., coaction of groups, which will be useful to understand actions of compact groups. (See [9] on basic of coactions.)

In this paper, we give the classification theorem for outer coactions of finite groups. Here we have to remark that this follows indirectly from the works cited in above. In fact, the uniqueness of outer coactions of finite groups follows from [5], since every outer coaction of a finite group is dual to some outer (usual) action. Nowadays, this also follows from the general theory of Popa's classification of subfactors [11] (also see [13]). However in these approach one does not handle coactions directly. Hence in this paper, we present the direct approach for classification theorem of outer coactions of finite groups on the AFD factor of type II_1, which can be generalized to finite dimensional Kac algebras. Our argument is similar to Connes-Jones-Ocneanu theory. We construct the model action on the AFD factor of type II_1, prove several cohomology vanishing theorem, and compare a given action to the model action. The main technical tools in these arguments is the ultraproduct and central sequence algebras. Unfortunately, coactions do not necessary induce coactions on the central sequence algebra unlike the usual group action case. Hence we have to modify actions to apply the ultraproduct technique to handle coactions, and this is one of the important points in our theory.

Here we have another formulation to treat actions of duals of (finite) groups other than coactions due to Roberts in [14]. His approach is essentially equivalent to coactions. However it is convenient (at least for the author) to regard coactions as the Roberts type actions, which we often call actions of finite group duals. Hence in this paper we present our main theorem as the uniqueness of Roberts type actions of finite group duals.

This paper is organized as follows. In Section 2, we prepare notations used in this paper, and discuss the Roberts type actions. In Section 3, we construct the infinite tensor product type action, which we call the model action. In Section 4, we collect some technical lemmas, which is necessary to treat actions on the ultraproduct algebra in Section 6. In Section 5, we show three kinds of cohomology vanishing theorems, which are important tools for analysis of actions. The contents in Section 6 and Section 7 are central in this paper. We discuss actions on the ultraproducts algebra, and the central sequence algebra of the AFD factor of type II_1. By means of cohomology vanishing, we construct the piece of the model action, and complete classification. In the appendix, we present the Roberts type action approach for the (twisted) crossed product construction for actions of finite group duals.

After completion of this paper, the author and R. Tomatsu obtained a uniqueness of free actions of a discrete amenable Kac algebra on the AFD factor of type II_1 in [8], which contains the main result in this paper. However the approach is completely different from that of this paper. Namely, in [8], classification is done by the Evans-Kishimoto type intertwining argument without using model actions. The author thinks that it may be useful to present a model action splitting argument.

2. PRELIMINARIES AND NOTATIONS

2.1. NOTATIONS ON DUALS OF FINITE GROUPS. Throughout this paper, we always assume that G is a finite group. Let $\mathrm{Rep}(G)$ and $\mathrm{Irr}(G)$ be the collection of all finite dimensional unitary representations, and irreducible unitary representations of G respectively. We denote the trivial representation by $\mathbf{1}$. We fix representative elements of $\mathrm{Irr}(G)/\sim$, where $\sim$ means a usual unitary equivalence, denote by $\widehat{G}$ the set of representative elements, and assume $\mathbf{1} \in \widehat{G}$.

Let $\mathrm{d}\pi := \dim H_\pi$ be the dimension of $\pi \in \mathrm{Rep}(G)$. For $\pi, \rho \in \mathrm{Rep}(G)$, we denote the intertwiner space between σ and π by $(\sigma, \pi) := \{T \in B(H_\sigma, H_\pi) : T\sigma(g) = \pi(g)T, g \in G\}$. If σ is irreducible, (σ, π) becomes a Hilbert space with an inner product $\langle T, S\rangle 1 := S^*T$.

Let $\pi, \rho \in \widehat{G}$, and $\pi \otimes \rho \cong \bigoplus_{\sigma \in \widehat{G}} N_{\pi\rho}^\sigma \sigma$ be the irreducible decomposition, where $N_{\pi\rho}^\sigma$ is a multiplicity. Fix an orthonormal basis $\{T_{\pi,\rho}^{\sigma,e}\}_{e=1}^{N_{\pi\rho}^\sigma} \subset (\sigma, \pi \otimes \rho)$. Then we have $T_{\pi,\rho}^{\sigma,e*} T_{\pi,\rho}^{\xi,f} = \delta_{\sigma,\xi}\delta_{e,f} 1_\sigma$, and $\sum_{\sigma,e} T_{\pi,\rho}^{\sigma,e} T_{\pi,\rho}^{\sigma,e*} = 1_{\pi\otimes\rho}$, where $1_\sigma \in (\sigma, \sigma)$ is an identity. Hence we have $\pi(g) \otimes \rho(g) = \sum_{\sigma,e} T_{\pi,\rho}^{\sigma,e} \sigma(g) T_{\pi,\rho}^{\sigma,e*}$ especially.

Let $\{v(\pi)\}_{\pi \in \widehat{G}}$ with $v(\pi) \in M_{\mathrm{d}\pi}(\mathbb{C})$. Then $\sum_e T_{\pi,\rho}^{\sigma,e} v(\sigma) T_{\pi,\rho}^{\sigma,e*}$ does not depend on the choice of $\{T_{\pi,\rho}^{\sigma,e}\} \subset (\sigma, \pi \otimes \rho)$.

In a similar way, one can easily see

$$\sum_{\eta,a,b} (T_{\pi,\rho}^{\eta,a} \otimes 1_\sigma) T_{\eta,\sigma}^{\xi,b} v(\xi) T_{\eta,\sigma}^{\xi,b*} (T_{\pi,\rho}^{\eta,a*} \otimes 1_\sigma) = \sum_{\zeta,c,d} (1_\pi \otimes T_{\rho,\sigma}^{\zeta,c}) T_{\pi,\zeta}^{\xi,d} v(\xi) T_{\pi,\zeta}^{\xi,d*} (1_\pi \otimes T_{\rho,\sigma}^{\zeta,c*})$$

since both $\{(T_{\pi,\rho}^{\eta,a} \otimes 1_\sigma) T_{\eta,\sigma}^{\xi,b}\}_{\eta,a,b}$ and $\{(1_\pi \otimes T_{\rho,\sigma}^{\zeta,c}) T_{\pi,\zeta}^{\xi,d}\}_{\zeta,c,d}$ are orthonormal basis for $(\xi, \pi \otimes \rho \otimes \sigma)$.

REMARK 2.1. Assume that $\{v(\pi)\}_{\pi \in \widehat{G}}$, $v(\pi) \in A \otimes B(H_\pi)$, is given for some vector space A. We can extend $v(\pi)$ for a general $\pi \in \mathrm{Rep}(G)$ as follows. Let $\pi \cong \bigoplus_i \sigma^i$, $\sigma^i \in \widehat{G}$, be an irreducible decomposition, and fix $T^i \in (\sigma^i, \pi)$ with $T^{i*}T^j = \delta_{i,j}$ and $\sum_i T^i T^{i*} = 1$. Define $v(\pi) = \sum_i T^i v(\sigma^i) T^{i*} \in A \otimes B(H_\pi)$. Then $v(\pi)$ is well-defined, i.e., it is independent from the choice of $\{T^i\}$, and satisfies $v(\pi)T = Tv(\sigma)$ for $T \in (\sigma, \pi)$. In this notation, the contents of the previous paragraph is written as $v((\pi \otimes \rho) \otimes \sigma) = v(\pi \otimes (\rho \otimes \sigma))$, for example.

Let $\{e_i^\pi\}_{i=1}^{\mathrm{d}\pi}$ be an orthonormal basis for H_π, and fix it. Let us express $T_{\pi,\rho}^{\sigma,e} = (T_{\pi_i,\rho_k}^{\sigma_m,e})$ as a matrix form. Then we can write $T_{\pi,\rho}^{\sigma,e*} T_{\pi,\rho}^{\xi,f} = \delta_{\sigma,\xi}\delta_{e,f} 1_\sigma$ and $\sum_{\sigma,e} T_{\pi,\rho}^{\sigma,e} T_{\pi,\rho}^{\sigma,e*} = 1_{\pi\otimes\rho}$ by matrix coefficients as

$$\sum_{i,k} \overline{T_{\pi_i,\rho_k}^{\sigma_m,e}} T_{\pi_i,\rho_k}^{\xi_n,f} = \delta_{\sigma,\xi}\delta_{e,f}\delta_{m,n}, \quad \text{and} \quad \sum_{\sigma,m,e} T_{\pi_i,\rho_k}^{\sigma_m,e} \overline{T_{\pi_j,\rho_l}^{\sigma_m,e}} = \delta_{i,j}\delta_{k,l}.$$

Let $T_{\pi,\overline{\pi}}^{\mathbf{1}} \in (\mathbf{1}, \pi \otimes \overline{\pi})$ be an isometry given by $T_{\pi,\overline{\pi}}^{\mathbf{1}} 1 = \frac{1}{\sqrt{\mathrm{d}\pi}} \sum_i e_i^\pi \otimes e_i^{\overline{\pi}}$, and fix it. It is easy to see $T_{\pi_i,\overline{\pi}_j}^{\mathbf{1}} = \frac{\delta_{i,j}}{\sqrt{\mathrm{d}\pi}}$. Since $T_{\pi,\overline{\pi}}^{\mathbf{1}*} T_{\pi,\overline{\pi}}^{\rho,e} = \delta_{\mathbf{1},\rho}$, we have $\sum_k T_{\pi_k,\overline{\pi}_k}^{\rho_l,e} = \sqrt{\mathrm{d}\pi}\,\delta_{\mathbf{1},\rho}$.

Set $\widetilde{T}^{\rho,e}_{\overline{\pi},\sigma} := \frac{\sqrt{d\rho d\pi}}{\sqrt{d\sigma}}(1_{\overline{\pi}} \otimes T^{\sigma,e*}_{\pi,\rho})(T^{\mathbf{1}}_{\overline{\pi},\pi} \otimes 1_\rho) \in (\rho, \overline{\pi} \otimes \sigma)$. Then $\{\widetilde{T}^{\rho,e}_{\overline{\pi},\sigma}\}$ is an orthonormal basis for $(\rho, \overline{\pi} \otimes \sigma)$. It is easy to see $\widetilde{T}^{\rho_k,e}_{\overline{\pi}_i\sigma_m} = \sqrt{\frac{d\rho}{d\sigma}}\overline{T^{\sigma_m,e}_{\pi_i,\rho_k}}$. As a consequence we have, for example

$$\sum_{\sigma,m,n,e} T^{\sigma_m,e}_{\pi_i,\rho_k} v(\sigma)_{m,n} \overline{T^{\sigma_n,e}_{\pi_j,\rho_l}} = \sum_{\sigma,m,n,e} \frac{d\sigma}{d\rho} \overline{T^{\rho_k,e}_{\overline{\pi}_i,\sigma_m}} v(\sigma)_{m,n} T^{\rho_l,e}_{\overline{\pi}_j,\sigma_n}.$$

2.2. COACTIONS AND ROBERTS TYPE ACTIONS. Let A, B be von Neumann algebras. We denote the set of unital $*$-homomorphisms from A to B by $\mathrm{Mor}\,(A, B)$.

Let u_g be the (right) regular representation of G, and $R(G) := \{u_g\}''$ the group algebra. The coproduct Δ of $R(G)$ is given by $\Delta(u_g) = u_g \otimes u_g$.

For simplicity, we denote $1_M \otimes T \in M \otimes B(H_\pi, H_\rho)$, $T \in B(H_\pi, H_\rho)$, by T.

DEFINITION 2.2. (i) Let M be a von Neumann algebra. We say that $\alpha = \{\alpha_\pi\}_{\pi \in \mathrm{Rep}(G)}$ is an action of $\mathrm{Rep}(G)$ if $\alpha_\pi \in \mathrm{Mor}\,(M, M \otimes B(H_\pi))$, and the following hold:

(ia) $\alpha_{\mathbf{1}} = \mathrm{id}_M$.

(ib) $\alpha_\pi(x)T = T\alpha_\sigma(x)$ for any $T \in (\sigma, \pi)$.

(ic) $\alpha_\pi \otimes \mathrm{id}_\sigma \circ \alpha_\sigma = \alpha_{\pi\otimes\sigma}$.

(ii) We say $\alpha = \{\alpha_\pi\}_{\pi \in \mathrm{Irr}(G)}$ is an action of $\mathrm{Irr}(G)$ if $\alpha_\pi \in \mathrm{Mor}\,(M, M \otimes B(H_\pi))$ and the following holds:

(iia) $\alpha_{\mathbf{1}} = \mathrm{id}_M$.

(iib) $\alpha_\pi \otimes \mathrm{id}_\rho \circ \alpha_\rho(x)T = T\alpha_\sigma(x)$ for any $T \in (\sigma, \pi \otimes \rho)$.

(iii) We say $\alpha = \{\alpha_\pi\}_{\pi \in \widehat{G}}$ an action of $\widehat{G}$ if $\alpha_\pi \in \mathrm{Mor}\,(M, M \otimes B(H_\pi))$ and we have the following:

(iiia) $\alpha_{\mathbf{1}} = \mathrm{id}_M$.

(iiib) $\alpha_\pi \otimes \mathrm{id}_\rho \circ \alpha_\rho(x)T = T\alpha_\sigma(x)$ for any $T \in (\sigma, \pi \otimes \rho)$.

If an action α of $\widehat{G}$ is given, then it is a routine work to extend α to those of $\mathrm{Irr}(G)$ and $\mathrm{Rep}(G)$. Hence in this paper, we do not distinguish these notions. When M is properly infinite, it is not difficult to see Definition 2.2 is reduced to that of Roberts action.

We remark that α_π is automatically injective. Suppose $\alpha_\pi(x) = 0$. Then we have $0 = T^{\mathbf{1}*}_{\overline{\pi},\pi}\alpha_{\overline{\pi}} \otimes \mathrm{id}_\pi \circ \alpha_\pi(x)T^{\mathbf{1}}_{\overline{\pi},\pi} = T^{\mathbf{1}*}_{\overline{\pi},\pi}T^{\mathbf{1}}_{\overline{\pi},\pi}\alpha_{\mathbf{1}}(x) = x$.

Let $\{e^\pi_{ij}\}$ be a system of matrix units for $B(H_\pi)$. Then $\alpha_\pi(x)$ is decomposed as $\alpha_\pi(x) = \sum_{i,j} \alpha_\pi(x)_{ij} \otimes e^\pi_{ij}$. The $*$-homomorphism property of α_π implies $\sum_k \alpha_\pi(x)_{ik}\alpha_\pi(y)_{kj} = \alpha_\pi(xy)_{ij}$ and $(\alpha_\pi(x)_{ij})^* = \alpha(x^*)_{ji}$.

The group algebra $R(G)$ can be decomposed as $R(G) = \bigoplus_{\pi \in \widehat{G}} B(H_\pi)$, and $e^\pi_{ij} = \frac{d\pi}{|G|\sum_g \pi(g)_{ij}u_g}$ gives a matrix unit for $B(H_\pi)$. Then $\alpha \in \mathrm{Mor}\,(M, M \otimes R(G))$ can be decomposed as $\alpha(x) = \sum_\pi \alpha_\pi(x)_{ij}(x) \otimes e^\pi_{ij}$, and we get $\alpha_\pi \in \mathrm{Mor}\,(M,$

$M \otimes B(H_\pi))$. One can verify that α is a coaction, i.e., α is injective and satisfies $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$, if and only if $\{\alpha_\pi\}$ is an action of $\widehat{G}$ in the sense of Definition 2.2.

DEFINITION 2.3. Let α be an action of $\widehat{G}$ on M. The *fixed point algebra M^α* is defined as $M^\alpha := \{a \in M : \alpha_\pi(a) = a \otimes 1_\pi \text{ for any } \pi \in \widehat{G}\}$.

If $K \subset M^\alpha$, then we say α is trivial on K, and often write as $\alpha_\pi = \mathrm{id}$ on K. Let $K \subset M$ be a von Neumann subalgebra, on which α acts trivially. Then it is easily seen that $\alpha_\pi(K' \cap M) \subset (K' \cap M) \otimes B(H_\pi)$, and α is an action on $K' \cap M$. Note that even if we have $\alpha_\pi(K) \subset K \otimes B(H_\pi)$, α does not induce an action on $K' \cap M$ in general unlike the usual group action case.

Let α be an action of $\widehat{G}$ on M, and N be another von Neumann algebra. Then $\alpha'_\pi(x) := \sum\limits_{i,j} \alpha_\pi(x)_{ij} \otimes 1_N \otimes e^\pi_{ij}$ is an action of $\widehat{G}$ on $M \otimes N$, which we denote by $\alpha \otimes \mathrm{id}_N$ for simplicity.

2.3. CROSSED PRODUCT CONSTRUCTION BY ROBERTS TYPE ACTION.

Let α be a coaction of G on M. The crossed product $M \rtimes_\alpha \widehat{G}$ is defined as $\alpha(M) \vee \mathbb{C} \otimes \ell^\infty(G) \subset M \otimes B(\ell^2(G))$. We discuss the crossed product construction from the point of view of the Roberts type action. (Also see Appendix.)

We begin with the following definition.

DEFINITION 2.4. Let M be a von Neumann algebra. We say $\{U_\pi\}_{\pi \in \mathrm{Irr}(G)}$ is a (unitary) representation of $\mathrm{Irr}(G)$ in M if we have the following:

(i) $U_\pi \in U(M \otimes B(H_\pi))$, $U_{\mathbf{1}} = 1$.

(ii) Let $F_{\pi,\sigma} \in B(H_\pi \otimes H_\rho, H_\rho \otimes H_\pi)$ be a flip map. Set $U^{12}_\pi := U_\pi \otimes 1_\rho$, and $U^{13}_\rho := F_{\rho,\pi}(U_\rho \otimes 1_\pi)F_{\pi,\rho}$. Then $U^{12}_\pi U^{13}_\rho T = T U_\sigma$ for any $T \in (\sigma, \pi \otimes \rho)$.

If we represent U_π and T as $U_\pi = (U_{\pi_{ij}})_{1 \leqslant i,j \leqslant \mathrm{d}\pi}$ and $T = (T^m_{i,k})^{1 \leqslant m \leqslant \mathrm{d}\sigma}_{1 \leqslant i \leqslant \mathrm{d}\pi, 1 \leqslant k \leqslant \mathrm{d}\rho}$ respectively by matrix elements, then Definition 2.4(ii) is written as $\sum\limits_{j,l} U_{\pi_{ij}} U_{\rho_{kl}} T^n_{j,k} = \sum\limits_m T^m_{i,k} U_{\sigma_{mn}}$.

LEMMA 2.5. *Let $\{U_\pi\}$ be a unitary representation of $\widehat{G}$. Then we have $U^*_{\pi_{ij}} = U_{\overline{\pi}_{ij}}$, and $[U_{\pi_{ij}}, U_{\rho_{kl}}] = 0$.*

Proof. Since we have $\sum\limits_{j,l} U_{\pi_{ij}} U_{\overline{\pi}_{kl}} T^{\mathbf{1}}_{\pi_j,\overline{\pi}_l} = T^{\mathbf{1}}_{\pi_i,\overline{\pi}_k} U_{\mathbf{1}}$, $\sum\limits_j U_{\pi_{ij}} U_{\overline{\pi}_{kj}} = \delta_{ik}$ holds. This implies $U_\pi {}^{\mathrm{t}}U_{\overline{\pi}} = 1$, and hence $U^*_\pi = {}^{\mathrm{t}}U_{\overline{\pi}}$. Thus we get $U^*_{\pi_{ij}} = U_{\overline{\pi}_{ij}}$.

We will verify the second statement. Since U_π is a representation, we have $U^{12}_\pi U^{13}_\rho = \sum\limits_{\sigma,e} T^{\sigma,e}_{\pi,\rho} U_\sigma T^{\sigma,e*}_{\pi,\rho}$. Then we get

$$F_{\pi,\rho}(U^{12}_\pi U^{13}_\rho)F_{\rho,\pi} = \sum_{i,j,k,l} U_{\pi_{ij}} U_{\rho_{kl}} \otimes e^\rho_{kl} \otimes e^\pi_{ij} = \sum_{\sigma,e} (F_{\pi,\rho} T^{\sigma,e}_{\pi,\rho}) U_\sigma (F_{\pi,\rho} T^{\sigma,e}_{\pi,\rho})^*.$$

On the other hand, $U_\rho^{12}U_\pi^{13}=\sum_{\sigma,e}F_{\pi,\rho}T_{\pi,\rho}^{\sigma,e}U_\sigma(F_{\pi,\rho}T_{\pi,\rho}^{\sigma,e})^*$ holds, since $\{F_{\pi,\rho}T_{\pi,\rho}^{\sigma,e}\}$ $\subset(\sigma,\rho\otimes\pi)$ is an orthonormal basis. (Note that we use $\pi\otimes\rho\sim\rho\otimes\pi$ here.) By comparing these, we get $[U_{\pi_{ij}},U_{\rho_{kl}}]=0$. ∎

Lemma 2.5 shows that $\{U_{\pi_{ij}}\}$ behave like matrix coefficients $\{\pi(g)_{ij}\}$. Let U_π be a representation of $\widehat{G}$, then it follows immediately that so is U_π^*, since $[U_{\pi_{ij}},U_{\rho_{kl}}]=0$.

REMARK 2.6. One can see that $\{U_{\overline{\pi}}^*\}$ is a conjugate representation of $\widehat{G}$, i.e., $(U_{\overline{\pi}}^*)^{12}(U_{\overline{\rho}}^*)^{13}\overline{T}=\overline{T}U_{\overline{\sigma}}^*$ for $T\in(\sigma,\pi\otimes\rho)$, without using the commutativity of $\widehat{G}$.

Let $\pi(g)_{ij}$ be a matrix coefficient for $\pi\in\mathrm{Rep}(G)$. We regard $\pi(g)_{ij}$ as an element π_{ij} in $\ell^\infty(G)$ and set $\lambda_{\pi_{ij}}:=1_M\otimes\pi_{ij}$. Then $\lambda_\pi=\sum_{i,j}\lambda_{\pi_{ij}}\otimes e_{ij}^\pi$ is the unitary representation of $\widehat{G}$ in the sense of Definition 2.4.

Since $\ell^\infty(G)=\bigvee\{\pi_{ij}\}$, we have $M\rtimes_\alpha\widehat{G}=\alpha(M)\vee\{\lambda_{\pi_{ij}}\}$. The relation of generators are $\sum_k\lambda_{\pi_{ik}}x\lambda_{\pi_{jk}}^*=\alpha_\pi(x)_{ij}$, or equivalently $\lambda_\pi(x\otimes 1_\pi)\lambda_\pi^*=\alpha_\pi(x)$. Here we identify $\alpha(x)$ and x as in the usual way. A unitary λ_π plays a roll of the implementing unitary in the usual crossed product construction. Hence we also call λ_π the implementing unitary in $M\rtimes_\alpha\widehat{G}$. We can expand $a\in M\rtimes_\alpha\widehat{G}$ as $\sum_{\pi,i,j}a_{\pi,i,j}\lambda_{\pi_{ij}}$, $a_{\pi,i,j}\in M$, uniquely.

DEFINITION 2.7. Let α be an action of $\widehat{G}$ on M. We say α is free if there exists no non-zero $a\in M\otimes B(H_\pi)$, $\mathbf{1}\neq\pi\in\widehat{G}$, so that $\alpha_\pi(x)a=a(x\otimes 1_\pi)$ for every $x\in M$. When α is an action of a factor M, then we also say α is outer if α is free.

In usual, freeness of a coaction α on a factor M is defined by the relative commutant condition $M'\cap M\rtimes_\alpha\widehat{G}=Z(M)$. We see that the usual definition and ours coincide in the following proposition.

PROPOSITION 2.8. *Let α be an action of $\widehat{G}$ on M. Then α is free if and only if $M'\cap M\rtimes_\alpha\widehat{G}=Z(M)$. Especially, $M\rtimes_\alpha\widehat{G}$ is a factor when α is free, and M is a factor.*

Proof. Let $a=\sum_{\pi,i,j}a_{\pi,i,j}\lambda_{\pi_{ij}}\in M\rtimes_\alpha\widehat{G}$. Set $a_\pi:=\sum_{i,j}a_{\pi,j,i}\otimes e_{ij}^\pi\in M\otimes B(H_\pi)$. Then it is easy to see $a\in M'\cap M\rtimes_\alpha\widehat{G}$ if and only if $(x\otimes 1_\pi)a_\pi=a_\pi\alpha_\pi(x)$ for any $x\in M$, $\pi\in\widehat{G}$. Then it is easily shown that α is free if and only if $M'\cap M\rtimes_\alpha\widehat{G}=Z(M)$. ∎

In the end of this subsection, we explain the dual action of G on the crossed product. Let α be an action of $\widehat{G}$ on M. Then the dual action $\widehat{\alpha}$ of G on $M\rtimes_\alpha\widehat{G}$ is given by $\widehat{\alpha}_g(a)=a$ for $a\in M$, and $\widehat{\alpha}_g\otimes\mathrm{id}_\pi(\lambda_\pi)=\lambda_\pi\pi(g)$, or equivalently

$\widehat{\alpha}_g(\lambda_{\pi_{ij}}) = \sum_k \lambda_{\pi_{ik}} \pi(g)_{kj}$. Then it is shown that $\widehat{\alpha}$ is an action of G, and the fixed point algebra is $(M \rtimes_\alpha \widehat{G})^{\widehat{\alpha}} = M$.

2.4. QUANTUM DOUBLE CONSTRUCTION FOR FINITE GROUP DUALS. In this subsection, we collect definitions and basic properties for quantum double construction (also known as the symmetric enveloping algebra [12], or the Longo-Rehren construction [6]) arising from actions of group duals. We will use them in Section 6. We refer Chapter 12.8, 15.5 of [4], or Appendix A of [7] for details of this topic.

For $\pi, \rho \in \mathrm{Rep}(G)$, let $\pi \widehat{\otimes} \rho$ a representation of $G \times G$ given by $\pi \widehat{\otimes} \rho(g, h) := \pi(g) \otimes \rho(h)$. Let α be an action of $\widehat{G} \times \widehat{G}$ on M. Set $P := M \rtimes_\alpha (\widehat{G} \times \widehat{G})$. Let $\lambda_{\pi \widehat{\otimes} \rho}$ be an implementing unitary for α.

LEMMA 2.9. *Set* $w_{\pi_{ij}} := \sum_k \lambda_{\pi_{ik} \widehat{\otimes} \overline{\pi}_{jk}}$. *Then* $w_\pi = (w_{\pi_{ij}})$ *is a unitary representation of* $\widehat{G}$.

Proof. Set $v_{\pi_{ij}} := \lambda_{\pi_{ij} \widehat{\otimes} \mathbf{1}}$, $u_{\pi_{ij}} := \lambda_{\mathbf{1} \widehat{\otimes} \overline{\pi}_{ji}}$. Obviously we have $w_{\pi_{ij}} = \sum_k v_{\pi_{ik}} u_{\pi_{kj}}$ and $[v_{\pi_{ij}}, u_{\rho_{kl}}] = 0$. Since $\{\overline{T^{\sigma,e}_{\pi,\rho}}\} \subset (\overline{\sigma}, \overline{\pi} \otimes \overline{\rho})$ is an orthonormal basis, $u_\pi = (u_{\pi_{ij}})_{ij}$ becomes a unitary representation of $\widehat{G}$ (also see Remark 2.6). Hence the following holds:

$$
\begin{aligned}
\sum_{j,l} w_{\pi_{ij}} w_{\rho_{kl}} T^{\sigma_m,e}_{\pi_j,\rho_l} &= \sum_{j,l,n,a} v_{\pi_{in}} v_{\rho_{ka}} u_{\pi_{nj}} u_{\rho_{aj}} T^{\sigma_m,e}_{\pi_j,\rho_l} \\
&= \sum_{\substack{j,l,n,a,\\ \xi,b,c,p\eta,d,f,q}} T^{\xi_b,p}_{\pi_i,\rho_k} v_{\xi_{bc}} \overline{T^{\xi_c,p}_{\pi_n,\rho_a}} T^{\eta_d,q}_{\pi_n,\rho_a} u_{\eta_{df}} \overline{T^{\eta_f,q}_{\pi_j,\rho_l}} T^{\sigma_m,e}_{\pi_j,\rho_l} \\
&= \sum_{\xi,b,c,p\eta,d,f,q} T^{\xi_b,p}_{\pi_i,\rho_k} \Big(\sum_{n,a} \overline{T^{\xi_c,p}_{\pi_n,\rho_a}} T^{\eta_d,q}_{\pi_n,\rho_a} \Big) v_{\xi_{bc}} u_{\eta_{df}} \Big(\sum_{j,l} \overline{T^{\eta_f,q}_{\pi_j,\rho_l}} T^{\sigma_m,e}_{\pi_j,\rho_l} \Big) \\
&= \sum_b T^{\sigma_b,e}_{\pi_i,\rho_k} \Big(\sum_c v_{\sigma_{bc}} u_{\sigma_{cm}} \Big) = \sum_b T^{\sigma_b,e}_{\pi_i,\rho_k} w_{\sigma_{bm}}. \quad \blacksquare
\end{aligned}
$$

DEFINITION 2.10. Set $N := M \vee \{w_{\pi_{ij}}\}$. We call $M \subset N$ is the *quantum double* for α.

REMARK 2.11. In the above definition, we consider an action of $\widehat{G} \times \widehat{G}$ on M directly. However, the usual quantum double construction is given as follows. Let M be a von Neumann algebra, and α be an action of $\widehat{G}$ on M. By the commutativity of $\widehat{G}$, $(\alpha_{\overline{\pi}})^{\mathrm{opp}}$ becomes an action of $\widehat{G}$ on M^{opp}. Hence we have an action of $\widehat{G} \times \widehat{G}$ on $M \otimes M^{\mathrm{opp}}$. The rest of construction is same as above.

We embed G into $G \times G$ by $g \to (g, g)$. Let $\beta := \widehat{\alpha}$ be the dual action of $G \times G$ on P. Then it is shown that $N = (M \rtimes_\alpha (\widehat{G} \times \widehat{G}))^G$.

For example, we have

$$\beta_{g,g}(w_{\pi_{ij}}) = \sum_k \beta_{g,g}(\lambda_{\pi_{ik}\widehat{\otimes}\overline{\pi}_{jk}}) = \sum_{k,l,m} \lambda_{\pi_{il}\widehat{\otimes}\overline{\pi}_{jm}} \pi(g)_{lk}\overline{\pi(g)_{mk}} = \sum_{l,m} \lambda_{\pi_{il}\widehat{\otimes}\overline{\pi}_{jl}} = w_{\pi_{ij}}.$$

If we expand $a \in P$ as $a = \sum a_{\pi_{ij},\rho_{kl}} \lambda_{\pi_{ij}\widehat{\otimes}\rho_{kl}}$, then $N = (M \rtimes_\alpha (\widehat{G} \times \widehat{G}))^G$ is verified in a similar way as above. We leave the proof to the reader. We remark that $a \in N$ can be expanded uniquely as $a = \sum_{\pi,i,j} a_{\pi,i,j} w_{\pi_{ij}}$, $a_{\pi,i,j} \in M$, and there exists the canonical conditional expectation $E : N \to M$ given by $E(a) = a_{\mathbf{1}}$.

2.5. MAIN RESULT.

DEFINITION 2.12. Let α be an action of $\widehat{G}$. We say $\{w_\pi\}_{\pi \in \widehat{G}}$ is a *(unitary) 1-cocycle* for α if $w_\pi \in U(M \otimes B(H_\pi))$, normalized as $w_{\mathbf{1}} = 1$, and the following holds:

$$(w_\pi \otimes 1_\rho)\alpha_\pi \otimes \mathrm{id}_\rho(w_\rho)T = Tw_\sigma, \quad T \in (\sigma, \pi \otimes \rho).$$

A 1-cocycle $\{w_\pi\}$ for α is called a *coboundary* if there exists a unitary $v \in U(M)$ such that $w_\pi = (v^* \otimes 1_\pi)\alpha_\pi(v)$.

If we extend v_ξ for $\xi \in \mathrm{Rep}(G)$ as in the remark in Section 2.1, then we have $(v_\xi \otimes 1_\eta)\alpha_\xi(v_\eta) = v_{\xi\otimes\eta}$. It is easy to see that $\mathrm{Ad}\, w_\pi \alpha_\pi$ is an action of $\widehat{G}$ for a 1-cocycle w_π.

DEFINITION 2.13. Let α and β be actions of $\widehat{G}$ on M.

(i) We say α and β are *conjugate* if there exists $\theta \in \mathrm{Aut}\,(M)$ with $\theta \otimes \mathrm{id}_\pi \circ \alpha_\pi \circ \theta^{-1} = \beta_\pi$ for every $\pi \in \widehat{G}$.

(ii) We say α and β are *cocycle conjugate* if there exists a 1-cocycle $\{w_\pi\}$ for α, and $\mathrm{Ad}\, w_\pi \alpha_\pi$ and β_π are conjugate.

Our main purpose is to show the following theorem by a traditional Connes-Jones-Ocneanu type approach, i.e., the model action splitting argument.

THEOREM 2.14. *Let $\mathcal{R}$ be the* AFD *factor of type* II_1. *Let α and β be outer actions of $\widehat{G}$ on $\mathcal{R}$. Then α and β are conjugate.*

3. MODEL ACTION

In this section, we construct an infinite tensor product type action of $\widehat{G}$ on $\mathcal{R}$, which we adopt as the model action.

It is easy to see the following lemma.

LEMMA 3.1. *Let M, N be von Neumann algebras, and U_π, V_π unitary representation of $\widehat{G}$ in M and N respectively. We regard U_π and V_π as representations of $\widehat{G}$ in $M \otimes N$ in the canonical way. Then $U_\pi V_\pi$ is also a representation of $\widehat{G}$.*

To construct the model action, we first construct (the canonical) unitary representation of $\mathrm{Irr}(G)$ on $M_{|G|}(\mathbb{C})$. Although we already discussed it in Section 2.3, we give a slightly different approach, which will be useful for our argument.

Let ϕ be the Haar functional for $R(G)$, i.e., $\phi(u_g) = |G|\delta_{e,g}$. For $v \in R(G)$, we denote by $v = \bigoplus v(\pi)$, $v(\pi) \in B(H_\pi)$, via the decomposition $R(G) \cong \bigoplus_{\pi\in\widehat{G}} B(H_\pi)$. Then we have $\phi(v) = \sum_\pi \mathrm{d}\pi \mathrm{Tr}_\pi(v(\pi))$, where Tr_π be the canonical (non-normalized) trace on $B(H_\pi)$. We regard $R(G)$ as a Hilbert space equipped with an inner product arising from ϕ, and denote by $\ell^2(\widehat{G})$. Namely, an inner product on $\ell^2(\widehat{G})$ is given by $\langle v, w\rangle = \sum_{\pi\in\widehat{G}} \mathrm{d}\pi\langle v(\pi), w(\pi)\rangle_\pi$ for $v = \bigoplus v(\pi)$, $w = \bigoplus w(\pi)$. Here $\langle v(\pi), w(\pi)\rangle_\pi = \mathrm{Tr}_\pi(w(\pi)^* v(\pi))$ It is easy to see that $\{\mathrm{d}\pi^{-1/2} e^\pi_{ij}\} \subset \ell^2(\widehat{G})$ forms an orthonormal basis with respect to this inner product.

Set $T^{\sigma,e}_{\rho,\pi_i} \in B(H_\sigma, H_\rho)$ by $(T^{\sigma,e}_{\rho,\pi_i})_{\rho_j,\sigma_k} = T^{\sigma_k,e}_{\rho_j,\pi_i}$.

LEMMA 3.2. *Define* $\lambda_{\pi_{ij}} \in B(\ell^2(\widehat{G})) = M_{|G|}(\mathbb{C})$ *by*

$$(\lambda_{\pi_{ij}} v)(\rho) := \sum_{\sigma,e} T^{\overline{\sigma},e}_{\overline{\rho},\pi_i} v(\sigma) T^{\overline{\sigma},e*}_{\overline{\rho},\pi_j}$$

and $\lambda_\pi := \sum_{i,j} \lambda_{\pi_{ij}} \otimes e^\pi_{ij} \in M_{|G|}(\mathbb{C}) \otimes B(H_\pi)$. *Then* $\{\lambda_\pi\}$ *is a unitary representation of* $\mathrm{Irr}(G)$ *on* $M_{|G|}(\mathbb{C})$.

Proof. We freely use notations and results in Section 2.1. We first show $\lambda^{12}_\pi \lambda^{13}_\rho T^{\sigma,a}_{\pi,\rho} = T^{\sigma,a}_{\pi,\rho}\lambda_\sigma$, $T^{\sigma,a}_{\pi,\rho} = (T^{\sigma_m,a}_{\pi_i,\rho_k}) \in (\sigma, \pi\otimes\rho)$, equivalently $\lambda_{\pi_{ij}}\lambda_{\rho_{kl}} T^{\sigma_n,a}_{\pi_j,\rho_l} = \sum_m T^{\sigma_m,a}_{\pi_i,\rho_k}\lambda_{\sigma_{mn}}$. We have:

$$\begin{aligned}
\Big(\sum_{j,l} \lambda_{\pi_{ij}}\lambda_{\rho_{kl}} T^{\sigma_n,a}_{\pi_j,\rho_l} v\Big)(\xi) &= \sum_{j,l,\eta,e} T^{\overline{\eta},e}_{\overline{\xi},\pi_i} (\lambda_{\rho_{k,l}} T^{\sigma_n,a}_{\pi_j,\rho_l} v)(\eta) T^{\overline{\eta},e*}_{\overline{\xi},\pi_j} \\
&= \sum_{j,l,\eta,e,\zeta,f} T^{\overline{\eta},e}_{\overline{\xi},\pi_i} T^{\overline{\zeta},f}_{\overline{\eta},\rho_k} T^{\sigma_n,a}_{\pi_j,\rho_l} v(\zeta) T^{\overline{\zeta},f*}_{\overline{\eta},\rho_l} T^{\overline{\eta},e*}_{\overline{\xi},\pi_j} \\
&= \sum_{j,l,\eta,e,\zeta,f} T^{\eta,e}_{\pi_i,\rho_k} T^{\overline{\zeta},f}_{\overline{\xi},\eta} T^{\sigma_n,a}_{\pi_j,\rho_l} v(\zeta) T^{\overline{\zeta},f*}_{\overline{\xi},\eta} T^{\overline{\eta},e*}_{\pi_j,\rho_l} \\
&= \sum_{\eta,\zeta,e,f} T^{\eta,e}_{\pi_i,\rho_k} T^{\overline{\zeta},f}_{\overline{\xi},\eta} v(\zeta) T^{\overline{\zeta},f*}_{\overline{\xi},\eta} \Big(\sum_{j,l} T^{\sigma_n,a}_{\pi_j,\rho_l} T^{\eta*,e}_{\pi_j,\rho_l}\Big) \\
&= \sum_{\zeta,f} T^{\sigma,a}_{\pi_i,\rho_k} T^{\overline{\zeta},f}_{\overline{\xi},\sigma} v(\zeta) T^{\overline{\zeta},f*}_{\overline{\xi},\sigma_n} \\
&= \sum_{m,\zeta,f} T^{\sigma_m,a}_{\pi_i,\rho_k} T^{\overline{\zeta},f}_{\overline{\xi},\sigma_m} v(\zeta) T^{\overline{\overline{\zeta},f}*}_{\overline{\xi},\sigma_n} = \sum_m (T^{\sigma_m,a}_{\pi_i,\rho_k}\lambda_{\sigma_{m,n}} v)(\xi).
\end{aligned}$$

It is easy to see that $\sum_k \langle \lambda_{\pi_{ki}} v, \lambda_{\pi_{kj}} w\rangle = \delta_{i,j}$. Hence we have $\sum_k \lambda^*_{\pi_{ki}}\lambda_{\pi_{kj}} = \delta_{i,j}$ and consequently $\lambda^*_\pi\lambda_\pi = 1$. Thus it suffices to show $\lambda_\pi\lambda^*_\pi = 1$.

Here we have

$$\sum_k \lambda_{\pi_{ik}}\lambda_{\overline{\pi}_{jk}} = \sum_{k,\rho,l,m,e} T^{\rho_l,e}_{\pi_i,\overline{\pi}_j}\lambda_{\rho_{lm}}\overline{T^{\rho_m,e}_{\pi_k,\overline{\pi}_k}} = \sqrt{\mathrm{d}\pi}T^{\mathbf{1}}_{\pi_i,\overline{\pi}_j}\lambda_{\mathbf{1}} = \delta_{i,j}.$$

Hence we have $\lambda_\pi {}^{\mathrm{t}}\lambda_{\overline{\pi}} = 1$, and ${}^{\mathrm{t}}\lambda_{\overline{\pi}} = \lambda^*_\pi$. It follows that $\lambda^*_{\pi_{ij}} = \lambda_{\overline{\pi}_{ij}}$ and $\lambda_\pi\lambda^*_\pi = 1$. ∎

Let $E = \{e_{\pi_{ij},\rho_{kl}}\}$ be a system of matrix units for $B(\ell^2(\widehat{G})) \cong M_{|G|}(\mathbb{C})$, that is, $e_{\pi_{ij},\rho_{kl}}$ is a partial isometry which sends $\mathrm{d}\rho^{-1/2}e^\rho_{kl} \in \ell^2(\widehat{G})$ to $\mathrm{d}\pi^{-1/2}e^\pi_{ij}$. It is not difficult to see

$$\lambda_{\pi_{ij}} = \sum_{\rho,k,l,\sigma,m,n,e}\sqrt{\frac{\mathrm{d}\rho}{\mathrm{d}\sigma}}T^{\sigma_m,e}_{\pi_i,\rho_k}\overline{T^{\sigma_n,e}_{\pi_j,\rho_l}}e_{\sigma_{mn},\rho_{kl}}.$$

It follows that $\sqrt{\mathrm{d}\pi\mathrm{d}\rho}\lambda_{\pi_{ij}}e_{\mathbf{1},\mathbf{1}}\lambda_{\overline{\rho}_{kl}} = e_{\pi_{ij},\rho_{kl}}$ from the above expression of λ_π.

Let M be a von Neumann algebra, and $E = \{e_{\pi_{ij},\rho_{kl}}\} \subset M$ a system of matrix units for $B(\ell^2(\widehat{G}))$. Then we can construct a unitary representation λ_π of $\widehat{G}$ in E'' by the above formula. In this case, we call $\{\lambda_\pi\}$ a representation of $\widehat{G}$ associated with $E = \{e_{\pi_{ij},\rho_{kl}}\}$. When we have to specify E, we denote the unitary representation of $\widehat{G}$ associated with E by λ^E_π.

We define the product type action of $\widehat{G}$ on $\mathcal{R}$. Express $\mathcal{R} = \bigotimes_{n=1}^{\infty} K_n$, where K_n is a copy of $M_{|G|}(\mathbb{C})$. Let $\lambda^n_\pi := \lambda^{K_n}_\pi$ be a unitary representation of $\widehat{G}$ on K_n, and regard it as one on $\mathcal{R}$. Define $\widetilde{\lambda}^1_\pi := \lambda^1_\pi$, and $\widetilde{\lambda}^n_\pi = \widetilde{\lambda}^{n-1}_\pi\lambda^n_\pi$. Then $\widetilde{\lambda}^n_\pi$ is a representation of $\widehat{G}$ on $K_1 \otimes \cdots \otimes K_n$ by Lemma 3.1. Set $m^n_\pi(x) := \operatorname{Ad}\widetilde{\lambda}^n_\pi(x \otimes 1_\pi)$. Since $\widetilde{\lambda}^n_\pi$ is a unitary representation of $\widehat{G}$, m^n_π is indeed an action of $\widehat{G}$ on $\mathcal{R}$. If $x \in \bigotimes_{k=1}^{n-1} K_k$, then

$$\operatorname{Ad}\widetilde{\lambda}^n_\pi(x \otimes 1_\pi) = \operatorname{Ad}\widetilde{\lambda}^{n-1}_\pi\lambda^n_\pi(x \otimes 1_\pi) = \operatorname{Ad}\widetilde{\lambda}^{n-1}_\pi(x \otimes 1_\pi)$$

holds. Hence $\lim_{n\to\infty} m^n_\pi(x)$ exists for $x \in \bigcup_{n=1}^{\infty}\bigotimes_{k=1}^{n} K_k$, and so does $m_\pi(x) = \lim_{n\to\infty} m^n_\pi(x)$ for every $x \in \mathcal{R}$.

DEFINITION 3.3. We call $m = \{m_\pi\}$ the *model action* for $\widehat{G}$.

THEOREM 3.4. *The model action m is outer.*

Proof. Fix $\mathbf{1} \neq \pi \in \widehat{G}$. Assume there exists non-zero $a \in \mathcal{R} \otimes B(H_\pi)$ such that $m_\pi(x)a = a(x \otimes 1)$ holds for $x \in \mathcal{R}$. If $x \in \bigotimes_{k=1}^{n} K_n$, then $(x \otimes 1)\widetilde{\lambda}^{n*}_\pi a = \widetilde{\lambda}^{n*}_\pi a(x \otimes 1)$ holds. Hence a is expressed as $a = \widetilde{\lambda}^n_\pi b_{n+1}$, $b_{n+1} = \sum_{ij} b^{n+1}_{ij} \otimes e^\pi_{ij} \in \bigotimes_{k=n+1}^{\infty} K_k \otimes B(H_\pi)$. Since we assume $a \neq 0$, there exists $c \in \mathcal{R} \otimes B(H_\pi)$ with

$\tau\otimes\mathrm{Tr}_\pi(ca)\neq 0$. We may assume c is of the form $c_1\otimes e^\pi_{ij}$, $c_1\in\bigotimes_{k=1}^m K_k$ for some m. Then

$$\tau\otimes\mathrm{Tr}_\pi(ca)=\tau(c_1\lambda^{m+1}_{\pi_{ij}}b^{m+2}_{ji})=\tau(c_1\widetilde{\lambda}^{m+1}_{\pi_{ij}})\tau(b^{m+2}_{ji})=\sum_l\tau(c_1\widetilde{\lambda}^{m}_{\pi_{il}})\tau(\lambda^{m+1}_{\pi_{lj}})\tau(b^{m+2}_{ji})=0$$

holds, and this is a contradiction. Hence a must be 0, and m is an outer action. ∎

DEFINITION 3.5. Let $E=\{e_{\pi_{ij},\rho_{kl}}\}\subset M$ be a system of matrix units, and λ^E_π a representation of $\widehat{G}$ associated with E. Let α be an action of $\widehat{G}$ on M. We say $\{e_{\pi_{ij},\rho_{kl}}\}$ is an *α-equivariant system of matrix units* if $\alpha_\pi(x)=\mathrm{Ad}\,\lambda^E_\pi(x\otimes 1)$ for $x\in E$.

The following lemma is easily verified. We leave the proof to the reader.

LEMMA 3.6. *Let α be an action of $\widehat{G}$ on M.*

(i) *Let $E=\{e_{\pi_{ij},\rho_{kl}}\}$ be an α-equivariant system of matrix units. Then λ^{E*}_π is a 1-cocycle for α, and $\mathrm{Ad}\,\lambda^{E*}_\pi\alpha_\pi=\mathrm{id}$ on E. Hence $\mathrm{Ad}\,\lambda^{E*}_\pi\alpha_\pi$ induces an action on $E'\cap M$.*

(ii) *Let $M\supset K\cong M_{|G|}(\mathbb{C})$, and suppose α is trivial on K. Then λ^K_π is a 1-cocycle for α. It follows that $\mathrm{Ad}\,\lambda^K_\pi\alpha_\pi$ is an action on M.*

4. TECHNICAL RESULTS

In this section, we collect some technical lemmas, whose proofs can be found in [2], [5], [10]. In the following, M is a factor of type II_1, and τ is the unique normalized trace on M.

LEMMA 4.1 ([5], Lemma 3.2.7). *Let $f\in M$ be such that $\|f\|\leqslant 1$, $\|f^2-f\|_2<\delta$ and $\|f^*-f\|_2<\delta\leqslant\frac{1}{4}$. Then there exists a projection $p\in M$ such that $\|f-p\|_2<6\sqrt[4]{\delta}$ and $\tau(p)=\tau(f)$.*

LEMMA 4.2 ([5], Lemma 3.2.1). *Let $u\in M$ be such that $\|u^*u-1\|_2<\delta$. Then there exists a unitary $v\in M$ with $\|u-v\|_2<(3+\|u\|)\delta$.*

LEMMA 4.3 ([2], Proposition 1.1.3; [10], Proposition 7.1). *Let us fix a free ultrafilter ω over $\mathbb{N}$.*

(i) *Let $A\in M^\omega$ be a unitary (respectively projection). Then there exists a representing sequence $A=(a_n)$ consisting of unitaries (respectively projections).*

(ii) *Let $V\in M^\omega$ be a partial isometry with $V^*V=E$ and $VV^*=F$. Let $E=(e_n),F=(f_n)\in M^\omega$ be representing sequences consisting of projections such that e_n and f_n are equivalent for any n. Then there exist a representing sequence (v_n) for V such that $v_n^*v_n=e_n$, $v_nv_n^*=f_n$.*

(iii) *Let $\{E_{ij}\}_{1\leqslant i,j,\leqslant m}\subset M^\omega$ be a system of matrix units. Then there exists a representing sequence $E_{ij}=\{e^n_{ij}\}$ such that $\{e^n_{ij}\}_{1\leqslant i,j\leqslant m}$ is a system of matrix units for every n.*

5. COHOMOLOGY VANISHING

In this section, we mainly deal with actions of $\widehat{G}$ on factors of type II_1. However many parts of results in this section are valid for general factors (or von Neumann algebras).

We begin with the following lemma, which is known as the "push-down lemma" in subfactor theory ([4], Lemma 9.26).

LEMMA 5.1. *Let α be an action of $\widehat{G}$, and set $e := \frac{1}{|G|\sum_{\pi,i} d\pi\lambda_{\pi_{ii}}} \in M \rtimes_\alpha \widehat{G}$. For any $a \in M \rtimes_\alpha \widehat{G}$ there exists $b \in M$ such that $ae = be$.*

Proof. Let $a = \sum_{\rho,k,l} a_{\rho_{kl}}\lambda_{\rho_{kl}}, a_{\pi_{kl}} \in M$ be an expansion of a. Then

$$\begin{aligned}
|G|ae &= \sum_{\pi,i,\rho,k,l} d\pi a_{\rho_{kl}}\lambda_{\rho_{kl}}\lambda_{\pi_{ii}} = \sum_{\pi,i,\rho,k,l}\sum_{\sigma,m,n,e} d\pi a_{\rho_{kl}} T^{\sigma_m,e}_{\pi_i,\rho_k}\lambda_{\sigma_{mn}}\overline{T^{\sigma_n,e}_{\pi_i,\rho_l}} \\
&= \sum_{\rho,k,l,\sigma,m,n} d\sigma a_{\rho_{kl}}\Big(\sum_{\pi,i,e}\overline{T^{\pi_i,e}_{\sigma_m,\overline{\rho}_k}}T^{\pi_i,e}_{\sigma_n,\overline{\rho_l}}\Big)\lambda_{\sigma_{mn}} = \sum_{\rho,k,\sigma,m} d\sigma a_{\rho_{kk}}\lambda_{\sigma_{mm}} = \Big(\sum_{\rho,k} a_{\rho_{kk}}\Big)|G|e
\end{aligned}$$

holds. Set $b := \sum_{\rho,k} a_{\rho_{kk}}$, then we have $ae = be$ and $b \in M$. ∎

PROPOSITION 5.2. *Let α be an outer action of $\widehat{G}$. Then any 1-cocycle for α is a coboundary.*

Proof. Let $\{w_\pi\}$ be a 1-cocycle for α, and λ_π an implementing unitary in $M \rtimes_\alpha \widehat{G}$. It follows that $\{w_\pi\lambda_\pi\}$ is a representation of $\widehat{G}$. Set $e := |G|^{-1}\sum_{\pi,i} d\pi\lambda_{\pi_{ii}}$, and $f := |G|^{-1}\sum_{\pi,i} d\pi(w_\pi\lambda_\pi)_{ii} = |G|^{-1}\sum_{\pi,i,j} d\pi w_{\pi_{ij}}\lambda_{\pi_{ji}}$. Then e and f are projections in $M \rtimes_\alpha \widehat{G}$ with $E_M(e) = E_M(f) = |G|^{-1}$, where E_M is the canonical conditional expectation from $M \rtimes_\alpha \widehat{G}$ to M. Since $M \rtimes_\alpha \widehat{G}$ is a factor due to the outerness of α, there exists $v \in M \rtimes_\alpha \widehat{G}$ such that $vev^* = f$. By Lemma 5.1, we may assume $v \in M$. Since $vev^* = |G|^{-1}\sum_{\pi,i} d\pi v\lambda_{\pi_{ii}}v^* = |G|^{-1}\sum_{\pi,i,j} d\pi v\alpha_\pi(v^*)_{ij}\lambda_{\pi_{ji}}$, we have $w_{\pi_{ij}} = v\alpha_\pi(v^*)_{ij}$, and this implies $w_\pi = (v\otimes 1)\alpha_\pi(v^*)$. Especially v is a unitary. ∎

COROLLARY 5.3. *Let α be an outer action of $\widehat{G}$ on M. Then there exists an α-equivariant system of matrix units $\{e_{\pi_{ij},\rho_{kl}}\} \subset M$.*

Proof. We can choose a system of matrix units $F = \{f_{\pi_{ij},\rho_{kl}}\} \subset M^\alpha$, since M^α is of type II_1. Then λ^F_π is a 1-cocycle for α_π. By Proposition 5.2, there exists $v \in U(M)$ such that $\lambda^F_\pi = (v^*\otimes 1)\alpha_\pi(v)$. Define $E = \{e_{\pi_{ij},\rho_{kl}}\} := \{vf_{\pi_{ij},\rho_{kl}}v^*\}$. Then $\lambda^E_\pi = (v\otimes 1)\lambda^F_\pi(v^*\otimes 1)$ and

$$\alpha_\pi(e_{\pi_{ij},\rho_{kl}}) = \alpha_\pi(v)(f_{\pi_{ij},\rho_{kl}}\otimes 1)\alpha_\pi(v^*)$$

$$= (v \otimes 1)\lambda^F_\pi(f_{\pi_{ij},\rho_{kl}} \otimes 1)\lambda^{F*}_\pi(v^* \otimes 1) = \lambda^E_\pi(e_{\pi_{ij},\rho_{kl}} \otimes 1)\lambda^{E*}_\pi. \quad ∎$$

DEFINITION 5.4. Let $\alpha_\pi \in \mathrm{Mor}\,(M, M \otimes B(H_\pi))$, normalized as $\alpha_{\mathbf{1}} = \mathrm{id}_M$, and $U_{\pi,\rho} \in U(M \otimes B(H_\pi) \otimes B(H_\rho))$. We say $\{\alpha_\pi, U_{\pi,\rho}\}_{\pi,\rho\in\widehat{G}}$ is a cocycle twisted action of $\widehat{G}$ if we have the following:

(i) $U_{\pi,\mathbf{1}} = U_{\mathbf{1},\pi} = 1$.

(ii) $\alpha_\pi \otimes \mathrm{id}_\rho \circ \alpha_\rho(x)U_{\pi,\rho}T = U_{\pi,\rho}T\alpha_\sigma(x), T \in (\sigma, \pi \otimes \rho)$.

(ii) Let $\{T^{\sigma,a}_{\pi,\rho}\}_{a=1}^{N^\sigma_{\pi\rho}}$ be an orthonormal basis for $(\sigma, \pi \otimes \rho)$. Set $U^{\sigma,a}_{\pi,\rho} := U_{\pi,\rho}T^{\sigma,a}_{\pi,\rho}$. Then

$$\sum_{\eta,a,\xi,b}(\alpha_\pi \otimes \mathrm{id})(U^{\eta,a}_{\rho,\sigma})U^{\xi,b}_{\pi,\eta}T^{\xi,b*}_{\pi,\eta}(1_\pi \otimes T^{\eta,a*}_{\rho,\sigma}) = \sum_{\zeta,c,\xi,d}(U^{\zeta,c}_{\pi,\rho} \otimes 1_\sigma)U^{\xi,d}_{\zeta,\sigma}T^{\xi,d*}_{\zeta,\sigma}(T^{\zeta,c*}_{\pi,\rho} \otimes 1_\sigma)$$

holds. The unitary $U_{\pi,\rho}$ is called a 2-cocycle for α.

We explain the meaning of Definition 5.4(iii). Note that $\{(1_\pi \otimes T^{\eta,a}_{\rho,\sigma})T^{\xi,b}_{\pi,\eta}\}_{\eta,a,b}$ and $\{(T^{\zeta,c}_{\pi,\rho} \otimes 1_\sigma)T^{\xi,d}_{\zeta,\sigma}\}_{\zeta,c,d}$ are both orthonormal bases for $(\xi, \pi \otimes \rho \otimes \sigma)$. Then $V_{(\zeta,c,d),(\eta,a,b)} = T^{\xi,d*}_{\zeta,\sigma}(T^{\zeta,c*}_{\pi,\rho} \otimes 1_\sigma)(1_\pi \otimes T^{\eta,a}_{\rho,\sigma})T^{\xi,b}_{\pi,\eta} \in (\xi,\xi) = \mathbb{C}$, and $V = \{V_{(\zeta,c,d),(\eta,a,b)}\}$ gives a unitary transformation between the above two orthonormal bases. From the condition (iii), we get

$$(\alpha_\pi \otimes \mathrm{id})(U^{\eta,a}_{\rho,\sigma})U^{\xi,b}_{\pi,\eta} = \sum_{\zeta,c,d}(U^{\zeta,c}_{\pi,\rho} \otimes 1_\sigma)U^{\xi,d}_{\zeta,\sigma}V_{(\zeta,c,d),(\eta,a,b)}.$$

This shows that the same unitary V gives the transformation between $(\alpha_\pi \otimes \mathrm{id})(U^{\eta,a}_{\rho,\sigma})U^{\xi,b}_{\pi,\eta}$ and $(U^{\zeta,c}_{\pi,\rho} \otimes 1_\sigma)U^{\xi,d}_{\zeta,\sigma}$.

Let $\{\alpha_\pi, U_{\pi,\sigma}\}$ be a cocycle twisted action of $\widehat{G}$, and extend it to that of $\mathrm{Rep}(G)$ as in the remark in Section 2.1. Then $(\alpha_\pi \otimes \mathrm{id}_\rho) \circ \alpha_\rho = \mathrm{Ad}\,(U_{\pi,\rho})\alpha_{\pi\otimes\rho}$, and $(U_{\pi,\rho} \otimes 1_\sigma)U_{\pi\otimes\rho,\sigma} = (\alpha_\pi \otimes \mathrm{id}_\rho \otimes \mathrm{id}_\sigma)(U_{\rho,\sigma})U_{\pi,\rho\otimes\sigma}$ from the above equalities.

DEFINITION 5.5. Let $\{\alpha_\pi, U_{\pi,\rho}\}$ be a cocycle twisted action of $\widehat{G}$. We say that $U_{\pi,\rho}$ is a *coboundary* if there exist unitaries $W_\pi \in M \otimes B(H_\pi)$, normalized as $W_{\mathbf{1}} = 1$, such that

$$W_\pi\alpha_\pi \otimes \mathrm{id}(W_\rho)U_{\pi,\rho}T = TW_\sigma, \quad T \in (\sigma, \pi \otimes \rho).$$

Define

$$(\partial_\alpha W)_{\pi,\rho} := \alpha_\pi \otimes \mathrm{id}_\rho(W_\rho)(W_\pi \otimes 1_\rho)W^*_{\pi\otimes\rho} = \sum_{\sigma,e}\alpha_\pi \otimes \mathrm{id}_\rho(W_\rho)(W_\pi \otimes 1_\rho)T^{\sigma,e}_{\pi,\rho}W^*_\sigma T^{\sigma,e*}_{\pi,\rho}.$$

Then the above condition is shown to be equivalent to $U_{\pi,\rho} = (\partial_\alpha W^*)_{\pi,\rho}$.

Let $\{\alpha, U_{\pi,\rho}\}$ be a cocycle twisted action, and assume $U_{\pi,\rho} = (\partial_\alpha W^*)_{\pi,\rho}$ for some $\{W_\pi\}$. Then $\mathrm{Ad}\,W_\pi\alpha_\pi$ becomes a genuine action of $\widehat{G}$.

REMARK 5.6. Here we make a useful remark on perturbations of cocycle twisted actions. Let α be an action of $\widehat{G}$, and $w_\pi \in U(M \otimes B(H_\pi))$. Then $\widetilde{\alpha}_\pi = \mathrm{Ad}\,w_\pi\alpha_\pi$ is a cocycle twisted action with a 2-cocycle $u(\pi,\rho) = \partial_{\widetilde{\alpha}}w(\pi,\rho)$. If there exists another unitary $\overline{w}_\pi$ such that $\partial_{\widetilde{\alpha}}\overline{w}^*(\pi,\rho) = u(\pi,\rho)$, then it is easy to verify that $\overline{w}_\pi w_\pi$ is a 1-cocycle for α.

In a similar way as in [5] and [15], we can prove the 2-cohomology vanishing theorem for cocycle twisted actions of $\widehat{G}$ as follows.

THEOREM 5.7. *Let $\{\alpha_\pi, U_{\pi,\rho}\}$ be a (not necessary outer) cocycle twisted action of $\widehat{G}$. Then $U_{\pi,\rho}$ is a coboundary.*

Proof. Fix a finite dimensional subfactor $K \subset M$, $K \cong M_{|G|}(\mathbb{C})$, and a system of matrix units $\{e_{ij}\}_{1\leqslant i,j\leqslant |G|}$ for K. Choose a unitary $u_\pi \in M\otimes B(H_\pi)$ with $\operatorname{Ad} u_\pi \alpha_\pi(e_{ij}) = e_{ij}\otimes 1$, and set $\widetilde{\alpha}_\pi := \operatorname{Ad} u_\pi\alpha_\pi$. Then $\widetilde{\alpha}_\pi(x) = x\otimes 1$, $x\in K$. Hence $\widetilde{\alpha}_\pi$ sends $K'\cap M$ into $(K'\cap M)\otimes B(H_\pi)$. Moreover if we define $\widetilde{U}_{\pi,\rho} := u_\pi^{12}\alpha_\pi(u_\rho)U_{\pi,\rho}u^*_{\pi\otimes\rho} = \sum\limits_{\sigma,a} u_\pi^{12}\alpha_\pi(u_\rho)U^{\sigma,a}_{\pi,\rho}u^*_\sigma T^{\sigma,a*}_{\pi\rho}$, then $\widetilde{U}_{\pi,\rho}\in (K'\cap M)\otimes B(H_\pi)\otimes B(H_\rho)$ and $\{\widetilde{\alpha}, \widetilde{U}_{\pi,\rho}\}$ is a cocycle twisted action of $\widehat{G}$ on $K'\cap M$. It is trivial that $\widetilde{U}_{\pi,\rho}$ is a coboundary if and only if so is $U_{\pi,\rho}$.

Hence we may assume α is of the form $\alpha_\pi = \alpha^0_\pi\otimes\mathrm{id}$ on $N\otimes B(\ell^2(\widehat{G}))$ and $U_{\pi,\rho} = \sum\limits_{\pi,i,j,\rho,k,l} U_{\pi_{ij},\rho_{kl}}\otimes 1\otimes e^\pi_{ij}\otimes e^\rho_{kl}\in N\otimes\mathbb{C}1\otimes B(H_\pi)\otimes B(H_\rho)$. Fix a system of matrix units $\{f_{\xi_{ab},\eta_{cd}}\}$ for $B(\ell^2(\widehat{G}))$.

In the rest of this section, we denote $U^{\sigma,a}_{\pi,\rho}$ and $T^{\sigma,a}_{\pi,\rho}$ by $U^\sigma_{\pi,\rho}$ and $T^\sigma_{\pi,\rho}$ respectively to simplify notations.

Define $w_{\pi_{ij}}$ as

$$w_{\pi_{ij}} := \sum_{\xi,a,b,\eta,c,d}\sqrt{\frac{\mathrm{d}\xi}{\mathrm{d}\eta}}\,T^{\eta_c}_{\pi_i\xi_a}U^{\eta_d^*}_{\pi_j\xi_b}\otimes f_{\eta_{cd},\xi_{ab}}\in N\otimes B(\ell^2(\widehat{G})).$$

We will see that $w_\pi\in U(M\otimes B(H_\pi))$, and $(\partial_\alpha w^*)_{\pi,\rho} = U_{\pi,\rho}$. At first we verify that w_π is a unitary. We will see that $\sum\limits_k w_{\pi_{ik}}w^*_{\pi_{jk}} = \delta_{i,j}$ holds as follows:

$$\begin{aligned}
&\sum_k w_{\pi_{ik}}w^*_{\pi_{jk}}\\
&= \sum_k\Big(\sum_{\rho,l,m,\sigma,s,t}\sqrt{\frac{\mathrm{d}\rho}{\mathrm{d}\sigma}}\,T^{\sigma_s}_{\pi_i\rho_l}U^{\sigma_t^*}_{\pi_k\rho_m}\otimes f_{\sigma_{st},\rho_{lm}}\Big)\Big(\sum_{\xi,a,b,\eta,c,d}\sqrt{\frac{\mathrm{d}\xi}{\mathrm{d}\eta}}\,\overline{T^{\eta_c}_{\pi_j\xi_a}}U^{\eta_d}_{\pi_k\xi_b}\otimes f_{\xi_{ab},\eta_{cd}}\Big)\\
&= \sum_{\rho,l,\sigma,s,t,\eta,c,d}\frac{\mathrm{d}\rho}{\sqrt{\mathrm{d}\sigma\mathrm{d}\eta}}T^{\sigma_s}_{\pi_i\rho_l}\overline{T^{\eta_c}_{\pi_j\rho_l}}\Big(\sum_{k,m}U^{\sigma_t^*}_{\pi_k\rho_m}U^{\eta_d}_{\pi_k\rho_m}\Big)\otimes f_{\sigma_{st},\eta_{cd}}\\
&= \sum_{\rho,m,\eta,c,d,s}\frac{\mathrm{d}\rho}{\mathrm{d}\eta}T^{\eta_s}_{\pi_i\rho_l}\overline{T^{\eta_c}_{\pi_j\rho_l}}\otimes f_{\eta_{sd},\eta_{cd}} = \sum_{\eta,c,d,s}\Big(\sum_{\rho,l}\overline{T^{\rho_l}_{\overline{\pi}_i\eta_s}}T^{\rho_l}_{\overline{\pi}_j\eta_c}\Big)\otimes f_{\eta_{sd},\eta_{cd}} = \delta_{i,j}\sum_{\eta,c,d}1\otimes f_{\eta_{cd},\eta_{cd}} = \delta_{i,j}.
\end{aligned}$$

In a similar way as above, $\sum\limits_k w^*_{\pi_{ki}}w_{\pi_{kj}} = \delta_{i,j}$ can be verified. Hence w_π is indeed a unitary.

We next show that $U_{\pi,\rho} = (\partial_\alpha w^*)_{\pi,\rho}$. It suffices to show $w_\pi \alpha_\pi(\omega_\rho) U^\sigma_{\pi,\rho} w^*_\sigma = T^\sigma_{\pi,\rho}$. This follows from the computation:

$$
\begin{aligned}
&(w_\pi \alpha_\pi(\omega_\rho) U^\sigma_{\pi,\rho} w^*_\sigma)^{\sigma_m}_{\pi_i,\rho_k} \\
&= \sum_{j,l,n,a} w_{\pi_{ij}} \alpha_\pi(w_{\rho_{kl}})_{jn} (U^{\sigma_a}_{\pi_n \rho_l} \otimes 1) w^*_{\sigma_{ma}} \\
&= \sum_{\substack{j,l,n,a,\xi,b,c,\\ \eta,d,e,\zeta,s,t,\phi,u,v}} \sqrt{\frac{\mathrm{d}\xi}{\mathrm{d}\eta}} \sqrt{\frac{\mathrm{d}\zeta}{\mathrm{d}\xi}} \sqrt{\frac{\mathrm{d}\zeta}{\mathrm{d}\phi}} T^{\eta_d}_{\pi_i \xi_b} U^{\eta_e *}_{\pi_j \xi_c} T^{\xi_b}_{\rho_k \zeta_s} \alpha_\pi(U^{\xi_c *}_{\rho_l \zeta_t})_{jn} U^{\sigma_a}_{\pi_n \rho_l} \overline{T^{\phi_u}_{\sigma_m, \zeta_s}} U^{\phi_v}_{\sigma_a \zeta_t} \otimes f_{\eta_{ed}, \phi_{uv}} \\
&= \sum_{\substack{l,n,a,\eta,d,e,\\ \zeta,s,t,\phi,u,v}} \frac{\mathrm{d}\zeta}{\sqrt{\mathrm{d}\eta \mathrm{d}\phi}} \Big(\sum_{j,\xi,b,c} \alpha_\pi(U^{\xi_c}_{\rho_l \zeta_t})_{nj} U^{\eta_e}_{\pi_j \xi_c} \overline{T^{\xi_b}_{\rho_k \zeta_s}}\, \overline{T^{\eta_d}_{\pi_i \xi_b}} \Big)^* U^{\sigma_a}_{\pi_n \rho_l} \overline{T^{\phi_u}_{\sigma_m, \zeta_s}} U^{\phi_v}_{\sigma_a \zeta_t} \otimes f_{\eta_{ed}, \phi_{uv}} \\
&= \sum_{\substack{l,n,a,\eta,d,e,\\ \zeta,s,t,\phi,u,v}} \frac{\mathrm{d}\zeta}{\sqrt{\mathrm{d}\eta \mathrm{d}\phi}} \Big(\sum_{\xi,b,c} U^{\xi_c}_{\pi_n \rho_l} U^{\eta_e}_{\xi_c \zeta_t} \overline{T^{\xi_b}_{\pi_i \rho_k}}\, \overline{T^{\eta_d}_{\xi_b \zeta_s}} \Big)^* U^{\sigma_a}_{\pi_n \rho_l} \overline{T^{\phi_u}_{\sigma_m, \zeta_s}} U^{\phi_v}_{\sigma_a \zeta_t} \otimes f_{\eta_{ed}, \phi_{uv}} \\
&= \sum_{\substack{a,\xi,b,c,\eta,d,e,\\ \zeta,s,t,\phi,u,v}} \frac{\mathrm{d}\zeta}{\sqrt{\mathrm{d}\eta \mathrm{d}\phi}} T^{\xi_b}_{\pi_i \rho_k} T^{\eta_d}_{\xi_b \zeta_s} U^{\eta_e *}_{\xi_c \zeta_t} \Big(\sum_{n,l} U^{\xi_c *}_{\pi_n \rho_l} U^{\sigma_a}_{\pi_n \rho_l} \Big) \overline{T^{\phi_u}_{\sigma_m, \zeta_s}} U^{\phi_v}_{\sigma_a \zeta_t} \otimes f_{\eta_{ed}, \phi_{uv}} \\
&= \sum_{\substack{b,\eta,d,e,\\ \zeta,s,\phi,u,v}} \frac{\mathrm{d}\zeta}{\sqrt{\mathrm{d}\eta \mathrm{d}\phi}} T^{\sigma_b}_{\pi_i \rho_k} T^{\eta_d}_{\sigma_b \zeta_s} \overline{T^{\phi_u}_{\sigma_m, \zeta_s}} \sum_{a,t} \Big(U^{\eta_e *}_{\sigma_a \zeta_t} U^{\phi_v}_{\sigma_a \zeta_t} \Big) \otimes f_{\eta_{ed}, \phi_{uv}} \\
&= \sum_{b,\eta,d,e,u,\zeta,s} \frac{\mathrm{d}\zeta}{\mathrm{d}\eta} T^{\sigma_b}_{\pi_i \rho_k} T^{\eta_d}_{\sigma_b \zeta_s} \overline{T^{\eta_u}_{\sigma_m, \zeta_s}} \otimes f_{\eta_{ed}, \eta_{ud}} = \sum_{b,\eta,d,e,u} T^{\sigma_b}_{\pi_i \rho_k} \Big(\sum_{\zeta,s} \overline{T^{\zeta_s}_{\bar{\sigma}_b \eta_d}} T^{\zeta_s}_{\sigma_m, \eta_u} \Big) \otimes f_{\eta_{ed}, \eta_{ud}} \\
&= T^{\sigma_m}_{\pi_i \rho_k} \sum_{\eta,d,e} 1 \otimes f_{\eta_{ed}, \eta_{ed}} = T^{\sigma_m}_{\pi_i \rho_k}.
\end{aligned}
$$

Hence we have $U_{\pi,\rho} = (\partial_\alpha w^*)_{\pi,\rho}$. ∎

We need another type of 2-cohomology vanishing theorem, which asserts that we can choose a coboundary close to 1 if a 2-cocycle is close to 1.

THEOREM 5.8. *Let $\{\alpha_\pi, U_{\pi,\rho}\}$ be a cocycle twisted outer action of $\widehat{G}$. If $\|U_{\pi,\rho} - 1\|_2 < \delta$ for sufficiently small enough δ, then there exist a unitary $W_\pi \in M \otimes B(H_\pi)$ such that $U_{\pi,\rho} = (\partial_\alpha W^*)_{\pi,\rho}$ and $\|W_\pi - 1\|_2 < f(\delta)$. Here $f(\delta)$ is a positive valued function, which depends only on G and is independent from α and $U_{\pi,\rho}$, with $\lim_{\delta \to 0} f(\delta) = 0$.*

Proof. Let $N := M \rtimes_{\alpha,U} \widehat{G}$ be a twisted crossed product, and λ_π be an implementing unitary. (See Appendix for the twisted crossed product construction.) Hence we have $\mathrm{Ad}\,\lambda_\pi(x \otimes 1_\pi) = \alpha_\pi(x)$ for $x \in M$, $\lambda^{12}_\pi \lambda^{13}_\rho T^\sigma_{\pi,\rho} = U_{\pi,\rho} T^\sigma_{\pi,\rho} \lambda_\sigma$ for $T^\sigma_{\pi,\rho} \in (\sigma, \pi \otimes \rho)$, and $M \rtimes_{\alpha,U} \widehat{G} = M \vee \{\lambda_{\pi_{ij}}\}$. By Theorem 5.7, there exist a unitary $w_\pi \in M \otimes B(H_\pi)$, $\pi \in \widehat{G}$ such that $U_{\pi,\rho} = (\partial_\alpha w^*)_{\pi,\rho}$. This implies

$\widetilde{\lambda}_\pi := w_\pi \lambda_\pi$ is a representation of $\widehat{G}$. Thus $e := |G|^{-1} \sum_{\pi,i} d\pi \widetilde{\lambda}_{\pi_{ii}}$ is a projection with $E_M(e) = \frac{1}{|G|}$.

(If one is not familiar to the twisted crossed product, he (or she) may treat it as follows. Let w_π be as above. Since $\mathrm{Ad}\, w_\pi \alpha_\pi$ is a usual action, we can construct a usual crossed product algebra $M \rtimes_{\mathrm{Ad}\, w\alpha} \widehat{G}$. Let $\widetilde{\lambda}_\pi$ be an implementing unitary, and set $\lambda_\pi := w_\pi^* \lambda_\pi$. Then it is easy to see that $\{\lambda_\pi\}$ behave like as the implementing unitary in $M \rtimes_{\alpha,U} \widehat{G}$. Hence $M \vee \{\lambda_\pi\}$ is identified with the twisted crossed product $M \rtimes_{\alpha,U} \widehat{G}$.)

Set $f := |G|^{-1} \sum_{\pi,i} d\pi \lambda_{\pi_{ii}}$. We will show f is almost a projection, and apply Lemma 4.1. Set $\Lambda_\pi := \sum_i \lambda_{\pi_{ii}}$ and we investigate $\Lambda_\pi \Lambda_\rho - \sum_\sigma N_{\pi,\rho}^\sigma \Lambda_\sigma$ at first.

Since we have

$$
\begin{aligned}
\Lambda_\pi \Lambda_\rho &= \sum_{i,k} \lambda_{\pi_{ii}} \lambda_{\rho_{kk}} = \sum_{i,k,\sigma,m,n,a} U^{\sigma_m,a}_{\pi_i,\rho_k} \lambda_{\sigma_{mn}} \overline{T^{\sigma_n,a}_{\pi_i,\rho_k}} = \sum_{i,j,k,l,\sigma,m,n} U_{\pi_{ij},\rho_{kl}} T^{\sigma_m,a}_{\pi_j,\rho_l} \lambda_{\sigma_{mn}} \overline{T^{\sigma_n,a}_{\pi_i,\rho_k}} \\
&= \sum_{i,j,k,l,\sigma,m,n} (U_{\pi_{ij},\rho_{kl}} - \delta_{ij}\delta_{kl}) T^{\sigma_m,a}_{\pi_j,\rho_l} \lambda_{\sigma_{mn}} \overline{T^{\sigma_n,a}_{\pi_i,\rho_k}} + \sum_{i,k,\sigma,m,n} T^{\sigma_m,a}_{\pi_i,\rho_k} \lambda_{\sigma_{mn}} \overline{T^{\sigma_n,a}_{\pi_i,\rho_k}} \\
&= \sum_{i,j,k,l,\sigma,m,n} (U_{\pi_{ij},\rho_{kl}} - \delta_{ij}\delta_{kl}) T^{\sigma_m}_{\pi_j,\rho_l} \lambda_{\sigma_{mn}} \overline{T^{\sigma_n}_{\pi_i,\rho_k}} + \sum_\sigma N^\sigma_{\pi,\rho} \Lambda_\sigma,
\end{aligned}
$$

we get the following estimate:

$$
\begin{aligned}
\Big\| \Lambda_\pi \Lambda_\rho - \sum_\sigma N^\sigma_{\pi,\rho} \Lambda_\sigma \Big\|_2 &\leqslant \sum_{i,j,k,l} \Big\| (U_{\pi_{ij}\rho_{kl}} - \delta_{ij}\delta_{kl}) \sum_{\sigma,m,n,a} T^{\sigma_m,a}_{\pi_j\rho_l} \lambda_{\sigma_{mn}} \overline{T^{\sigma_n,a}_{\pi_i\rho_k}} \Big\|_2 \\
&\leqslant \sum_{i,j,k,l} \| U_{\pi_{ij}\rho_{kl}} - \delta_{ij}\delta_{kl} \|_2 \leqslant d\pi d\rho \sqrt{\sum_{i,j,k,l} \| U_{\pi_{ij}\rho_{kl}} - \delta_{ij}\delta_{kl} \|_2^2} \\
&\leqslant d\pi d\rho \sqrt{d\pi d\rho}\, \delta.
\end{aligned}
$$

Now we give the estimate of $\|f^2 - f\|_2$. Since

$$
\begin{aligned}
f^2 - f &= \frac{1}{|G|^2} \sum_{\pi,\rho} d\pi d\rho \Lambda_\pi \Lambda_\rho - \frac{1}{|G|} \sum_\sigma d\sigma \Lambda_\sigma \\
&= \frac{1}{|G|^2} \sum_{\pi,\rho,\sigma} d\pi d\rho (\Lambda_\pi \Lambda_\rho - N^\sigma_{\pi,\rho} \Lambda_\sigma) + \frac{1}{|G|^2} \sum_{\pi,\rho,\sigma} d\pi d\rho N^\sigma_{\pi,\rho} \Lambda_\sigma - \frac{1}{|G|} \sum_\sigma d\sigma \Lambda_\sigma \\
&= \frac{1}{|G|^2} \sum_{\pi,\rho,\sigma} d\pi d\rho (\Lambda_\pi \Lambda_\rho - N^\sigma_{\pi,\rho} \Lambda_\sigma)
\end{aligned}
$$

holds, we get

$$
\|f^2 - f\|_2 \leqslant \frac{1}{|G|^2} \sum_{\pi,\rho} d\pi d\rho \Big\| \Lambda_\pi \Lambda_\rho - \sum_\sigma N^\sigma_{\pi,\rho} \Lambda_\sigma \Big\|_2 \leqslant \frac{1}{|G|^2} \sum_{\pi,\rho} d\pi^2 d\rho^2 \sqrt{d\pi d\rho}\, \delta.
$$

Next we estimate $\|f^* - f\|_2$. Set $\widetilde{U}_{\pi_i,\overline{\pi}_j} = \sum_k U_{\pi_{ik},\overline{\pi}_{jk}}$. Then $\lambda^*_{\overline{\pi}_{ij}} = \sum_k \widetilde{U}^*_{\pi_k,\overline{\pi}_i}\lambda_{\pi_{kj}}$, (see Appendix), and we have $f^* = |G|^{-1} \sum_{\pi,i,j} \mathrm{d}\pi \widetilde{U}^*_{\pi_j,\overline{\pi}_i}\lambda_{\pi_{ji}}$.

Hence the following holds:

$$\|f^* - f\|_2 \leqslant \frac{1}{|G|}\sum_{\pi,i,j}\mathrm{d}\pi\|(\widetilde{U}^*_{\pi_j,\overline{\pi}_i} - \delta_{i,j})\lambda_{\pi_{ji}}\|_2 \leqslant \frac{1}{|G|}\sum_{\pi,i,j,k}\mathrm{d}\pi\|(U^*_{\pi_{jk},\overline{\pi}_{ik}} - \delta_{i,k}\delta_{j,k})\|_2$$
$$\leqslant \frac{1}{|G|}\sum_{\pi}\mathrm{d}\pi^2\sqrt{\mathrm{d}\pi}\sqrt{\sum_{i,j,k}\|(U^*_{\pi_{jk},\overline{\pi}_{ik}} - \delta_{i,k}\delta_{j,k})\|_2^2} \leqslant \frac{1}{|G|}\sum_{\pi}\mathrm{d}\pi^3\sqrt{\mathrm{d}\pi}\delta.$$

If we set $C := \max\left\{|G|^{-1}\sum_{\pi,\rho}\mathrm{d}\pi^2\mathrm{d}\rho^2\sqrt{\mathrm{d}\pi\mathrm{d}\rho}, |G|^{-1}\sum_{\pi}\mathrm{d}\pi^3\sqrt{\mathrm{d}\pi}\right\}$, then $\|f^2 - f\|_2 \leqslant C\delta$ and $\|f^* - f\|_2 \leqslant C\delta$. Here note that C is determined only by G.

By Lemma 4.1, there exists a projection $p \in M \rtimes_{\alpha,U} \widehat{G}$ such that $\|f - p\|_2 < 6\sqrt[4]{C\delta}$ and $\tau(p) = \frac{1}{|G|}$, provided $C\delta < \frac{1}{4}$. Then there exists a unitary $a \in M \rtimes_{\alpha,U} \widehat{G}$ such that $p = aea^*$, and by Lemma 5.1, there exists $u \in M$ with $p = ueu^*$. Note that u is not necessary a unitary, and we have $\|u\| \leqslant |G|$ since $u = |G|E_M(ae)$. Then $\|f - ueu^*\|_2 < 6\sqrt[4]{C\delta}$. By applying the canonical conditional expectation, we get $\|1 - uu^*\|_2 < 6|G|\sqrt[4]{C\delta}$. By Lemma 4.2, there exists a unitary $v \in M$ such that $\|u - v\|_2 < 6|G|(3 + |G|)\sqrt[4]{C\delta}$. Hence we have $\|f - vev^*\|_2 < f(\delta)$ for some positive valued function $f(\delta)$, which depends only on G, with $\lim_{\delta\to 0} f(\delta) = 0$. Set $\overline{w}_\pi := (v \otimes 1)w_\pi\alpha_\pi(v^*)$. Then $\overline{w}_\pi$ is a coboundary for $U_{\pi,\rho}$, and $\|\overline{w}_\pi - 1\|_2 < f(\delta)$ holds by looking at coefficients of $f - vev^* = |G|^{-1}\sum_{\pi,i,j}\mathrm{d}\pi(1 - \overline{w}_{\pi_{ij}})\lambda_{\pi_{ji}}$. ∎

6. ACTIONS AND ULTRA PRODUCT

Fix a free ultrafilter ω over $\mathbb{N}$. Then α^ω_π is an action of $\widehat{G}$ on M^ω, however α^ω_π does not preserve M_ω unless $\widehat{G}$ is a group, i.e., G is commutative.

Our first task in this section is to show the existence of an outer action of $\widehat{G} \times \widehat{G}$ on M_ω by modifying α^ω_π.

We first recall Ocneanu's central freedom lemma.

LEMMA 6.1. *Let $A \subset B \subset C$ be finite von Neumann algebras with $A \cong \mathcal{R}$. Then $(A' \cap B^\omega)' \cap C^\omega = A \vee (B' \cap C)^\omega$.*

See [4] for proof.

From now on, we assume $M \cong \mathcal{R}$. Set $M_1 := M \rtimes_\alpha \widehat{G}$. Let $\lambda_\pi \in M_1 \otimes B(H_\pi)$ be an implementing unitary for α.

LEMMA 6.2. *Let H be a finite dimensional Hilbert space. Then $\alpha \in \mathrm{Mor}\,(\mathcal{R}, \mathcal{R} \otimes B(H))$ is approximately inner in the following sense; there exists a sequence of unitary $\{u_n\} \subset \mathcal{R} \otimes B(H)$ such that $\lim_n \|\mathrm{Ad}\, u_n(x \otimes 1) - \alpha(x)\|_2 = 0$, $x \in \mathcal{R}$.*

Proof. Represent $\mathcal{R}=\bigotimes_{n=1}^{\infty} M_2(\mathbb{C})$, and set $L_n:=\bigotimes_{k=1}^{n} M_2(\mathbb{C})$. Let $\{e_{ij}^n\}$ be a system of matrix units for L_n. Then $\{e_{ij}^n\otimes 1\}$ and $\{\alpha(e_{ij}^n)\}$ are both systems of matrix units in $\mathcal{R}\otimes B(H)$. Hence there exists a unitary $u_n\in\mathcal{R}\otimes B(H)$ with $\alpha(e_{ij})=\operatorname{Ad}u_n(e_{ij}\otimes 1)$, and hence $\alpha(x)=\operatorname{Ad}u_n(x\otimes 1)$ for $x\in L_n$. Then it is easy to see that $\lim_n\|\alpha(x)-\operatorname{Ad}u_n(x\otimes 1)\|_2=0$ for $x\in\mathcal{R}$. ∎

By Lemma 6.2, there exists a unitary $U_\pi\in M^\omega\otimes B(H_\pi)$ such that $\alpha_\pi(x)=\operatorname{Ad}U_\pi(x\otimes 1)$ for $x\in M\subset M^\omega$, $\pi\in\widehat{G}$. Set $V_\pi:=U_\pi^*\lambda_\pi\in M_1^\omega$.

LEMMA 6.3. *Define* $\gamma_\pi^1(x):=\operatorname{Ad}V_\pi(x\otimes 1)$, $\gamma_\rho^2(x):=\operatorname{Ad}U_\rho^*(x\otimes 1_\rho)$. *Then* $\gamma_{\pi\widehat{\otimes}\rho}:=(\gamma_\pi^1\otimes 1_\rho)\circ\gamma_\rho^2$ *defines an outer cocycle twisted action of* $\widehat{G}\times\widehat{G}$ *on* M_ω.

Proof. Define $W_{\pi\widehat{\otimes}\rho}:=V_\pi^{12}U_\rho^{13*}\in M_1^\omega\otimes B(H_\pi)\otimes B(H_\rho)$. Then γ is a perturbation of the trivial action of $\widehat{G}\times\widehat{G}$ on M_1^ω by $W_{\pi\widehat{\otimes}\rho}$, and $w(\pi^1\widehat{\otimes}\rho^1,\pi^2\widehat{\otimes}\rho^2):=\partial_\gamma(W)(\pi^1\widehat{\otimes}\rho^1,\pi^2\widehat{\otimes}\rho^2)$ is a 2-cocycle for γ. Hence we only have to verify that γ preserves M_ω, outer on M_ω, and $w(\pi^1\widehat{\otimes}\rho^1,\pi^2\widehat{\otimes}\rho^2)\in M_\omega\otimes B(H_{\pi^1\widehat{\otimes}\rho^1})\otimes B(H_{\pi^2\widehat{\otimes}\rho^2})$.

We verify that $\gamma_\pi^i\in\operatorname{Mor}(M_\omega,M_\omega\otimes B(H_\pi))$, $i=1,2$. Indeed this follows from the computation below, where $x\in M, a\in M_\omega$ (note $\gamma_\pi^1(a)=\operatorname{Ad}U_\pi^*\alpha_\pi^\omega(a)$ for $a\in M_\omega$):

$$\begin{aligned}
&U_\pi^*(a\otimes 1)U_\pi(x\otimes 1)=U_\pi^*(a\otimes 1)\alpha_\pi(x)U_\pi=U_\pi^*\alpha_\pi(x)(a\otimes 1)U_\pi=(x\otimes 1)\operatorname{Ad}U_\pi^*(a\otimes 1),\\
&U_\pi^*\alpha_\pi^\omega(a)U_\pi(x\otimes 1)=U_\pi^*\alpha_\pi^\omega(a)\alpha_\pi(x)U_\pi=U_\pi^*\alpha_\pi^\omega(ax)U_\pi\\
&\qquad\qquad\qquad\qquad\quad =U_\pi^*\alpha_\pi^\omega(xa)U_\pi=(x\otimes 1)U_\pi^*\alpha_\pi^\omega(a)U_\pi.
\end{aligned}$$

Then it is trivial that $\gamma_{\pi\widehat{\otimes}\rho}\in\operatorname{Mor}(M_\omega,M_\omega\otimes B(H_\pi)\otimes B(H_\rho))$.

We next verify that $\gamma_{\pi\widehat{\otimes}\rho}$ is outer on M_ω. We divide to $\pi\neq\mathbf{1}$ case and $\pi=\mathbf{1}$ case. Fix $\mathbf{1}\neq\pi\in\widehat{G}$ and assume $\gamma_{\pi\widehat{\otimes}\rho}(x)a=a(x\otimes 1)$, $x\in M_\omega$, holds for some $a\in M_\omega\otimes B(H_\pi)\otimes B(H_\rho)$. On one hand, $b:=U_\rho^{13}V_\pi^{12*}a\in(M_\omega\otimes\mathbb{C})'\cap M_1^\omega\otimes B(H_\pi)\otimes B(H_\rho)=M\otimes B(H_\pi)\otimes B(H_\rho)$ by Lemma 6.1. On the other hand, it is easy to see $b=U_\rho^{13}V_\pi^{12*}a$ is of the form $\sum_{i,j,k,l,m,n}X_{i,j,m,n}^{k,l}\lambda_{\overline{\pi}_{m,n}}\otimes e_{ij}^\pi\otimes e_{kl}^\rho$, $X_{i,j,m,n}^{k,l}\in M^\omega$. Hence if $\pi\neq\mathbf{1}$, b must be 0, and consequently $a=0$.

Assume $\pi=\mathbf{1}$, and we verify that $\gamma_{\mathbf{1}\widehat{\otimes}\rho}=\gamma_\rho^2$ is outer for $\rho\neq\mathbf{1}$. Assume $\gamma_\rho^2(x)a=a(x\otimes 1)$. Then $b_\rho:=U_\rho a\in(M_\omega\otimes\mathbb{C})'\cap M^\omega\otimes B(H_\rho)=M\otimes B(H_\rho)$ by Lemma 6.1.

Then we have

$$\alpha_\rho(x)b_\rho=U_\rho(x\otimes 1)U_\rho^*b_\rho=U_\rho(x\otimes 1)a=U_\rho a(x\otimes 1)=b_\rho(x\otimes 1)$$

for $x\in M$. Since α is an outer action on M, b_ρ is 0 and hence so is a.

We will see $w(\pi^1\widehat{\otimes}\rho^1, \pi^2\widehat{\otimes}\rho^2) \in U(M_\omega \otimes B(H_{\pi^1\widehat{\otimes}\rho^1}) \otimes B(H_{\pi^2\widehat{\otimes}\rho^2}))$. Although we can prove this directly, we will show the statement for $w(\pi\widehat{\otimes}\mathbf{1}, \rho\widehat{\otimes}\mathbf{1})$, $w(\mathbf{1}\widehat{\otimes}\pi, \mathbf{1}\widehat{\otimes}\rho)$ and $w(\mathbf{1}\widehat{\otimes}\pi, \rho\widehat{\otimes}\mathbf{1})$ separately to abuse notations. Then we can obtain the desired result since

$$\begin{aligned}
&w(\pi^1\widehat{\otimes}\rho^1, \pi^2\widehat{\otimes}\rho^2)\\
&\quad = \gamma^1_{\pi_1} \otimes \mathrm{id}_{\rho_1} \otimes \mathrm{id}_{\pi_2} \otimes \mathrm{id}_{\rho_2}(w(\mathbf{1}\widehat{\otimes}\rho_1, \pi_2\widehat{\otimes}\mathbf{1}) \otimes 1_{\rho_2})\times\\
&\qquad\quad (w(\pi_1\widehat{\otimes}\mathbf{1}, \pi_2\widehat{\otimes}\mathbf{1}) \otimes 1_{\rho_1} \otimes 1_{\rho_2})\gamma^1_{\pi_1\otimes\pi_2} \otimes \mathrm{id}_{\rho_1} \otimes \mathrm{id}_{\rho_2}(w(\mathbf{1}\widehat{\otimes}\rho_1, \mathbf{1}\widehat{\otimes}\rho_2)).
\end{aligned}$$

(We identify $H_{\pi_1\widehat{\otimes}\rho_1} \otimes H_{\pi_2\widehat{\otimes}\rho_2}$ and $H_{\pi_1} \otimes H_{\pi_2} \otimes H_{\rho_1} \otimes H_{\rho_2}$ in the canonical way.)

We first verify $w(\pi\widehat{\otimes}\mathbf{1}, \rho\widehat{\otimes}\mathbf{1}) \in M^\omega \otimes B(H_\pi) \otimes B(H_\rho)$. This follows from the following computation. Here we extend V_π and U_π for $\pi \in \mathrm{Rep}(G)$ as in the remark in Section 2.1:

$$\begin{aligned}
w(\pi\widehat{\otimes}\mathbf{1}, \rho\widehat{\otimes}\mathbf{1}) &= V^{12}_\pi V^{13}_\rho V^*_{\pi\otimes\sigma} = \sum_{\sigma,a} U^{12*}_\pi \lambda^{12}_\pi U^{13*}_\rho \lambda^{13}_\rho T^{\sigma,a}_{\pi,\rho}\lambda^*_\sigma U_\sigma T^{\sigma,a*}_{\pi,\rho}\\
&= \sum_{\sigma,a} U^{12*}_\pi \alpha^\omega_\pi \otimes \mathrm{id}_\rho(U^*_\rho)\lambda^{12}_\pi \lambda^{13}_\rho T^{\sigma,a}_{\pi,\rho}\lambda^*_\sigma U_\sigma T^{\sigma,a*}_{\pi,\rho}\\
&= \sum_{\sigma,a} U^{12*}_\pi \alpha^\omega_\pi \otimes \mathrm{id}_\rho(U^*_\rho) T^{\sigma,a}_{\pi,\rho} U_\sigma T^{\sigma,a*}_{\pi,\rho}\\
&= U^{12*}_\pi \alpha^\omega_\pi \otimes \mathrm{id}_\rho(U^*_\rho) U_{\pi\otimes\rho} \in M^\omega \otimes B(H_\pi) \otimes B(H_\rho).
\end{aligned}$$

Let us examine whether $w(\pi\widehat{\otimes}\mathbf{1}, \rho\widehat{\otimes}\mathbf{1})$ commutes with $x \otimes 1_\pi \otimes 1_\rho \in M \otimes \mathbb{C}1_\pi \otimes \mathbb{C}1_\rho$. Note that $\mathrm{Ad}\, U_\pi(x) = \alpha_\pi(x)$ holds for $\pi \in \mathrm{Rep}(G)$, $x \in M$. Thus

$$\begin{aligned}
w(\pi\widehat{\otimes}\mathbf{1}, \pi\widehat{\otimes}\mathbf{1})(x \otimes 1_\pi \otimes 1_\rho) &= U^{12*}_\pi \alpha^\omega_\pi \otimes \mathrm{id}_\rho(U^*_\rho) U_{\pi\otimes\rho}(x \otimes 1_\pi \otimes 1_\rho)\\
&= U^{12*}_\pi \alpha^\omega_\pi \otimes \mathrm{id}_\rho(U^*_\rho)(\alpha_\pi \otimes \mathrm{id}_\rho) \circ \alpha_\rho(x) U_{\pi\otimes\rho}\\
&= U^{12*}_\pi \alpha^\omega_\pi \otimes \mathrm{id}_\rho(U^*_\rho \alpha_\rho(x)) U_{\pi\otimes\rho}\\
&= U^{12*}_\pi \alpha^\omega_\pi \otimes \mathrm{id}_\rho(x \otimes 1_\rho)\alpha_\pi \otimes \mathrm{id}_\rho(U^*_\rho) U_{\pi\otimes\rho}\\
&= (x \otimes 1_\pi \otimes 1_\rho) U^{12*}_\pi \alpha \otimes \mathrm{id}_\rho(U^*_\rho) U_{\pi\otimes\rho}\\
&= (x \otimes 1_\pi \otimes 1_\rho) w(\pi\widehat{\otimes}\mathbf{1}, \rho\widehat{\otimes}\mathbf{1})
\end{aligned}$$

holds, and $w(\pi\widehat{\otimes}\mathbf{1}, \rho\widehat{\otimes}\mathbf{1}) \in M_\omega \otimes B(H_\pi) \otimes B(H_\rho)$.

It is trivial that $w(\mathbf{1}\widehat{\otimes}\pi, \mathbf{1}\widehat{\otimes}\rho) = U^{12*}_\pi U^{13*}_\rho U_{\pi\otimes\rho} \in M^\omega \otimes B(H_\pi) \otimes B(H_\rho)$. Thus we only have to see that $[w(\mathbf{1}\widehat{\otimes}\pi, \mathbf{1}\widehat{\otimes}\rho), (x \otimes 1_\pi \otimes 1_\rho)] = 0$, $x \in M$. This follows from the following computation:

$$\begin{aligned}
w(\mathbf{1}\widehat{\otimes}\pi, \mathbf{1}\widehat{\otimes}\rho)(x \otimes 1_\pi \otimes 1_\rho) &= U^{12*}_\pi U^{13*}_\rho U_{\pi\otimes\rho}(x \otimes 1_\pi \otimes 1_\rho)\\
&= U^{12*}_\pi U^{13*}_\rho (\alpha_\pi \otimes \mathrm{id}_\rho) \circ \alpha_\rho(x) U_{\pi\otimes\rho}\\
&= U^{12*}_\pi U^{13*}_\rho (F_{\rho,\pi}\alpha_\rho \otimes \mathrm{id}_\pi(\alpha_\pi(x))F_{\pi,\rho}) U_{\pi\otimes\rho}
\end{aligned}$$

$$= (x \otimes 1_\pi \otimes 1_\rho) U_\pi^{12*} U_\rho^{13*} U_{\pi\otimes\rho} = (x \otimes 1_\pi \otimes 1_\rho) w(\mathbf{1}\widehat{\otimes}\pi, \mathbf{1}\widehat{\otimes}\rho),$$

where we used the commutativity of α_π and α_ρ in the third equality.

Finally we verify $w(\mathbf{1}\widehat{\otimes}\pi, \rho\widehat{\otimes}\mathbf{1}) \in M_\omega \otimes B(H_\rho) \otimes B(H_\pi)$. As in the above, we will see that $w(\mathbf{1}\widehat{\otimes}\pi, \rho\widehat{\otimes}\mathbf{1}) \in M^\omega \otimes B(H_\rho) \otimes B(H_\pi)$ and $[w(\mathbf{1}\widehat{\otimes}\pi, \rho\widehat{\otimes}\mathbf{1}), x \otimes 1_\rho \otimes 1_\pi] = 0$, $x \in M$ separately as follows:

$$\begin{aligned}
w(\mathbf{1}\widehat{\otimes}\pi, \rho\widehat{\otimes}\mathbf{1}) &= U_\pi^{13*} V_\rho^{12} U_\pi^{13} V_\rho^{12*} = U_\pi^{13*} U_\rho^{12*} \lambda_\rho^{12} U_\pi^{13} \lambda_\rho^{12*} U_\rho^{13} \\
&= U_\pi^{13*} U_\rho^{12*} \alpha_\rho^\omega \otimes \mathrm{id}_\pi(U_\pi) U_\rho^{12} \in M^\omega \otimes B(H_\rho) \otimes B(H_\pi); \\
w(\mathbf{1}\widehat{\otimes}\pi, \rho\widehat{\otimes}\mathbf{1})(x \otimes 1_\rho \otimes 1_\pi) &= U_\pi^{13*} U_\rho^{12*} \alpha_\rho^\omega \otimes \mathrm{id}_\pi(U_\pi) U_\rho^{12} (x \otimes 1_\rho \otimes 1_\pi) \\
&= U_\pi^{13*} U_\rho^{12*} (\alpha_\rho^\omega \otimes \mathrm{id}_\pi)(U_\pi)(\alpha_\rho(x) \otimes 1_\pi) U_\rho^{12} \\
&= U_\pi^{13*} U_\rho^{12*} (\alpha_\rho^\omega \otimes 1_\pi)(U_\pi(x \otimes 1_\pi)) U_\rho^{12} \\
&= U_\pi^{13*} U_\rho^{12*} (\alpha_\rho^\omega \otimes 1_\pi \circ \alpha_\pi(x))(\alpha_\rho \otimes \mathrm{id}_\pi(U_\pi)) U_\rho^{12} \\
&= (x \otimes 1_\pi \otimes 1_\rho) U_\pi^{13*} U_\rho^{12*} \alpha_\rho^\omega \otimes \mathrm{id}_\pi(U_\pi) U_\rho^{12} \\
&= (x \otimes 1_\rho \otimes 1_\pi) w(\mathbf{1}\widehat{\otimes}\pi, \rho\widehat{\otimes}\mathbf{1}). \quad \blacksquare
\end{aligned}$$

LEMMA 6.4. *We can choose $\overline{U}_\pi \in M^\omega$ so that $\mathrm{Ad}\,\overline{U}_\pi(x \otimes 1) = \alpha_\pi(x)$, $x \in M$, and $\overline{U}_\pi$ and $\overline{U}_\pi^* \lambda_\pi$ are both representations of $\widehat{G}$ with $[\overline{U}_{\pi_{ij}}, (\overline{U}_\rho^* \lambda_\rho)_{kl}] = 0$.*

Proof. By Lemma 6.3, $\gamma_{\pi\widehat{\otimes}\rho}$ defines a cocycle twisted action of $\widehat{G} \times \widehat{G}$ on M_ω. Hence by Theorem 5.7, $w(\cdot,\cdot) = \partial_\gamma(u)(\cdot,\cdot)$ for some $u_{\pi\widehat{\otimes}\rho} \in M_\omega \otimes B(H_{\pi\widehat{\otimes}\rho})$. Set $\widetilde{U}_\pi = U_\pi u^*_{\mathbf{1}\widehat{\otimes}\pi}$, and $\widetilde{V}_\pi = u_{\pi\widehat{\otimes}\mathbf{1}} V_\pi$. By the Remark 5.6, $u_{\pi\widehat{\otimes}\rho} V_\pi^{12} U_\rho^{13*}$ is a 1-cocycle for the trivial action of $\widehat{G} \otimes \widehat{G}$. This implies that $\widetilde{U}_\pi^*$ and $\widetilde{V}_\pi$ are representations of $\widehat{G}$ in M_1^ω with $[\widetilde{U}_{\pi_{ij}}, \widetilde{V}_{\rho_{kl}}] = 0$. Moreover we have

$$\mathrm{Ad}\,\widetilde{U}_\pi(x) = \mathrm{Ad}\,U_\pi u^*_{\mathbf{1}\widehat{\otimes}\pi}(x \otimes 1) = \mathrm{Ad}\,U_\pi(x \otimes 1) = \alpha_\pi(x), x \in M.$$

Put $c_\pi := \widetilde{V}_\pi \lambda_\pi^* \widetilde{U}_\pi = u_{\pi\widehat{\otimes}\mathbf{1}} U_\pi^* \lambda_\pi \lambda_\pi^* U_\pi u^*_{\mathbf{1}\widehat{\otimes}\pi} = u_{\pi\widehat{\otimes}\mathbf{1}} u^*_{\mathbf{1}\widehat{\otimes}\pi} \in M_\omega$.

Define $W_{\pi_{ij}} := \sum_k \widetilde{V}_{\pi_{ik}} \widetilde{U}_{\pi_{kj}}$, Set $P := M_\omega \vee \{W_{\pi_{ij}}\} (\subset M_1^\omega)$. Then $M_\omega \subset P$ is the quantum double for $\mathrm{Ad}\,u_\pi \gamma_\pi$. (Unitaries $\widetilde{V}_\pi$, $\widetilde{U}_\pi$, and W_π correspond to v_π, u_π, and w_π in the proof of Lemma 2.9.) We have the (unique) conditional expectation E from P on M_ω, and it satisfies $E(W_{\pi_{ij}}) = \delta_{\pi,\mathbf{1}}$.

We next prove $\sum_i \lambda_{\pi_{ii}} \in P$. Since $c_\pi = \widetilde{V}_\pi \lambda_\pi^* \widetilde{U}_\pi$, we have $\lambda_{\pi_{ij}} = \sum_{k,l} \widetilde{U}_{\pi_{ik}} c^*_{lk} \widetilde{V}_{\pi_{lj}}$. Note $\gamma_\pi^2(x) := \widetilde{U}_\pi^*(x \otimes 1)\widetilde{U}_\pi$ is an action on M_ω, and it follows $\widetilde{U}^*_{\pi_{ji}} x = \sum_k \gamma_\pi^2(x)_{ik} \widetilde{U}^*_{\pi_{jk}}$, $x \in M_\omega$.

Then we have

$$\sum \lambda_{\pi_{ii}} = \sum_{i,k,l} \widetilde{U}_{\pi_{ik}} c^*_{lk} \widetilde{V}_{\pi_{li}} = \sum_{i,k,l} \widetilde{U}^*_{\overline{\pi}_{ik}} c^*_{lk} \widetilde{V}_{\pi_{li}} = \sum_{i,j,k,l} \gamma_{\overline{\pi}}^2(c^*_{lk})_{kj} \widetilde{U}^*_{\overline{\pi}_{ij}} \widetilde{V}_{\pi_{li}}$$

$$= \sum_{i,j,k,l} \gamma^2_{\overline{\pi}}(c^*_{lk})_{kj}\widetilde{U}_{\pi_{ij}}\widetilde{V}_{\pi_{li}} = \sum_{j,k,l} \gamma^2_{\overline{\pi}}(c^*_{lk})_{kj}W_{\pi_{lj}},$$

and thus $\sum_i \lambda_{\pi_{ii}} \in P$.

Define $e := |G|^{-1} \sum_{\pi,i,j} d\pi W_{\pi_{ii}}$, and $f := |G|^{-1} \sum_{\pi,i,j} d\pi \lambda_{\pi_{ii}}$. Then e and f are projections in P with $E(e) = E(f) = \frac{1}{|G|}$. Hence there exists a unitary $z \in M_\omega$ such that $zfz^* = e$ by Lemma 5.1. (Though P is not a crossed product of M_ω by $\widehat{G}$, the proof of Lemma 5.1 works for $M_\omega \subset P$ since $\{W_\pi\}$ is a representation of $\widehat{G}$, and $a \in P$ can be expressed as $\sum_{\pi,i,j} a_{\pi,i,j}W_{\pi_{ij}}$.)

On one hand, we have

$$|G|zfz^* = \sum_{\pi,i} d\pi z\lambda_{\pi_{ii}}z^* = \sum_{\pi,i,j} d\pi z\alpha_\pi(z^*)_{ij}\lambda_{\pi_{ji}}.$$

On the other hand, since $\widetilde{V}_\pi = c_\pi \widetilde{U}^*_\pi \lambda_\pi$, we get

$$|G|e = \sum_{\pi,i,j} d\pi \widetilde{U}_{\pi_{ij}}\widetilde{V}_{\pi_{ji}} = \sum_{\pi,i,j,k,l} d\pi \widetilde{U}_{\pi_{ij}}c_{\pi_{jk}}\widetilde{U}^*_{\pi_{lk}}\lambda_{li}.$$

Since $z\alpha_\pi(z^*)_{ij}, \widetilde{U}_{\pi_{ij}}c_{\pi_{jk}}\widetilde{U}^*_{\pi_{lk}} \in M^\omega$, we have $z\alpha_\pi(z^*)_{il} = \sum_{j,k} \widetilde{U}_{\pi_{ij}}c_{\pi_{jk}}\widetilde{U}^*_{\pi_{lk}}$, and this implies $(z \otimes 1)\alpha^\omega_\pi(z^*) = \widetilde{U}_\pi c_\pi \widetilde{U}^*_\pi$.

Define $\overline{V}_\pi$ and $\overline{U}_\pi$ by $\overline{V}_\pi=(z^*\otimes 1)\widetilde{V}_\pi(z\otimes 1), \overline{U}_\pi=(z^*\otimes 1)\widetilde{U}_\pi(z\otimes 1)$. We have

$$\begin{aligned}\overline{V}_\pi\lambda^*_\pi\overline{U}_\pi &= (z^* \otimes 1)\widetilde{V}_\pi(z \otimes 1)\lambda^*_\pi(z^* \otimes 1)\widetilde{U}_\pi(z \otimes 1)\\ &= (z^* \otimes 1)\widetilde{V}_\pi\lambda_\pi\alpha_\pi(z)(z^* \otimes 1)\widetilde{U}_\pi(z \otimes 1)\\ &= (z^* \otimes 1)\widetilde{V}_\pi\lambda_\pi\widetilde{U}_\pi c^*_\pi\widetilde{U}^*_\pi\widetilde{U}_\pi(z \otimes 1) = 1,\end{aligned}$$

hence $\overline{V}_\pi = \overline{U}^*_\pi\lambda_\pi$. Since $z \in M_\omega$, $\mathrm{Ad}\,\overline{U}_\pi(x \otimes 1) = \alpha_\pi(x)$ holds for $x \in M \subset M^\omega$. It is clear that $\overline{U}_\pi$ and $\overline{V}_\pi$ are both representations of $\widehat{G}$ with $[\overline{V}_{\pi_{ij}}, \overline{U}_{\rho_{kl}}] = 0$. ∎

REMARK 6.5. To avoid using the commutativity of $\widehat{G}$, we should consider $\gamma^3_\pi(x) := \mathrm{Ad}U^*_{\overline{\pi}}(x \otimes 1)$ rather than $\gamma^2_\pi = \mathrm{Ad}\,U^*_\pi(x \otimes 1)$. By suitable inner perturbation, this γ^3_π is shown to be a "conjugate" action of $\widehat{G}$ in the sense $(\gamma^3_\pi \otimes \mathrm{id}_\rho) \circ \gamma^3_\rho(x)\overline{T} = \overline{T}\gamma^3_\sigma(x)$ for $T \in (\sigma, \pi \otimes \rho)$ without using the commutativity of $\widehat{G}$. (See Remark 2.6.)

COROLLARY 6.6. *Fix U_π as in Lemma 6.4. Then we have $\alpha^\omega_\rho(U_{\pi_{ij}})=\mathrm{Ad}\,U_\rho(U_{\pi_{ij}} \otimes 1_\rho)$.*

Proof. By Lemma 6.4, we have $U_\pi V_\pi = \lambda_\pi$. Since $[U_{\pi_{ij}}, V_{\rho_{kl}}] = 0$, we have

$$\alpha^\omega_\rho(U_{\pi_{ij}}) = \mathrm{Ad}\,\lambda_\pi(U_{\pi_{ij}} \otimes 1_\rho) = \mathrm{Ad}\,U_\rho V_\rho(U_{\pi_{ij}} \otimes 1_\rho) = \mathrm{Ad}\,U_\rho(U_{\pi_{ij}} \otimes 1_\rho). \quad ∎$$

LEMMA 6.7. *We choose U_π as in Lemma 6.4. There exists an α^ω-equivariant system of matrix units $E = \{E_{\pi_{ij},\rho_{kl}}\} \subset M^\omega$ such that $\lambda^E_\pi = U_\pi$ and $E_{\mathbf{1},\mathbf{1}} \in M_\omega$.*

Proof. Let γ_π^i and $\gamma_{\pi\widehat{\otimes}\rho}=(\gamma_\pi^1\otimes 1_\rho)\circ\gamma_\rho^2(x)$ be as in Lemma 6.3. By Lemma 6.4, γ is an outer action of $\widehat{G}\times\widehat{G}$ on M_ω. By Corollary 5.3, there exists a γ-equivariant system of matrix units $\{e_{(\xi_{ab}\widehat{\otimes}\pi_{ij}),(\eta_{cd}\widehat{\otimes}\rho_{kl})}\}$ in M_ω. Put $F_{\pi_{ij},\rho_{kl}}:=\sum_{\xi,a,b} e_{(\xi_{ab}\widehat{\otimes}\pi_{ij}),(\xi_{ab}\widehat{\otimes}\rho_{kl})}$. Then $\{F_{\pi_{ij},\rho_{kl}}\}$ is in $M_\omega^{\gamma^1}$, and becomes a γ^2-equivariant system of matrix units.

Set $\widetilde{F}_{i,j}^\pi := \sum_k F_{\pi_{ik},\pi_{jk}}$. Then it is easy to see that $\{\widetilde{F}_{i,j}^\pi\}$ forms a system of matrix units for $R(G)$. Namely we have $\widetilde{F}_{i,j}^\pi\widetilde{F}_{k,l}^\rho = \delta_{\pi,\rho}\delta_{j,k}\widetilde{F}_{i,l}^\pi$, $\widetilde{F}_{i,j}^{\pi *} = \widetilde{F}_{j,i}^\pi$ and $\sum_{\pi,i}\widetilde{F}_{i,i}^\pi = 1$. Since $F=\{F_{\pi_{ij},\rho_{kl}}\}$ is γ^2-equivariant, we have $\gamma_\pi^2(\widetilde{F}^1)_{i,j} = \sum_k \lambda^F_{\pi_{ik}}F_{\mathbf{1},\mathbf{1}}\lambda^F_{\overline{\pi}_{jk}} = \mathrm{d}\pi^{-1}\widetilde{F}_{i,j}^\pi$. Since $\gamma_\pi^2(x) = \mathrm{Ad}\, U_\pi^*(x\otimes 1_\pi)$ by definition and $U_{\pi_{ij}}^* = U_{\overline{\pi}_{ij}}$, we have $U_{\overline{\pi}_{ji}}x = \sum_k \gamma_\pi^2(x)_{ik}U_{\overline{\pi}_{jk}}$.

Define $E_{\pi_{ij},\rho_{kl}} := \sqrt{\mathrm{d}\pi\mathrm{d}\rho}\,U_{\pi_{ij}}F_{\mathbf{1},\mathbf{1}}U_{\overline{\rho}_{kl}}$. It is trivial that $E_{\mathbf{1},\mathbf{1}}\in M_\omega$.

We first prove $\lambda_\pi^E = U_\pi$:

$$
\begin{aligned}
\lambda^E_{\pi_{ij}} &= \sum_{\rho,k.l,\sigma,m,n,e}\sqrt{\frac{\mathrm{d}\rho}{\mathrm{d}\sigma}}\,T^{\sigma_m,e}_{\pi_i,\rho_k}\overline{T^{\sigma_n,e}_{\pi_j,\rho_l}}E_{\sigma_{mn},\rho_{kl}} = \sum_{\rho,k,l,\sigma,m,n,e}\mathrm{d}\rho\, T^{\sigma_m,e}_{\pi_i,\rho_k}U_{\sigma_{mn}}\overline{T^{\sigma_n,e}_{\pi_j,\rho_l}}F_{\mathbf{1},\mathbf{1}}U_{\overline{\rho}_{kl}}\\
&=\sum_{\rho,k,l}\mathrm{d}\rho U_{\pi_{ij}}U_{\rho_{kl}}F_{\mathbf{1},\mathbf{1}}U_{\overline{\rho}_{kl}} = \sum_{\rho,k,l,m}\mathrm{d}\rho U_{\pi_{ij}}\gamma^2_{\overline{\rho}}(F_{\mathbf{1},\mathbf{1}})_{lm}U_{\rho_{km}}U_{\overline{\rho}_{kl}}\\
&=\sum_{\rho,l,m}\mathrm{d}\rho U_{\pi_{ij}}\gamma^2_{\overline{\rho}}(F_{\mathbf{1},\mathbf{1}})_{lm}\Big(\sum_k U^*_{\overline{\rho}_{km}}U_{\overline{\rho}_{kl}}\Big) = \sum_{\rho,l}\mathrm{d}\rho U_{\pi_{ij}}\gamma^2_{\overline{\rho}}(F_{\mathbf{1},\mathbf{1}})_{ll} = \sum_{\rho,l}U_{\pi_{ij}}\widetilde{F}^\rho_{l,l} = U_{\pi_{ij}}.
\end{aligned}
$$

We next prove $E=\{E_{\pi_{ij},\rho_{kl}}\}$ is a system of matrix units.

If we set $\pi=\mathbf{1}$ in the above computation, we get $\sum_{\pi,i,j}E_{\pi_{ij},\pi_{ij}} = 1$. It is easy to see that $E^*_{\pi_{ij},\rho_{kl}} = E_{\rho_{kl},\pi_{ij}}$. Thus we only have to verify $E_{\pi_{ij},\rho_{kl}}E_{\sigma_{mn},\xi_{ab}} = \delta_{\rho,\sigma}\delta_{k,m}\delta_{l,n}E_{\pi_{ij},\xi_{ab}}$. At first we compute $F_{\mathbf{1},\mathbf{1}}U_{\pi_{ij}}U_{\rho_{kl}}F_{\mathbf{1},\mathbf{1}}$. Note that $F_{\mathbf{1},\mathbf{1}}\gamma_\sigma(F_{\mathbf{1},\mathbf{1}})_{m,n} = \mathrm{d}\sigma^{-1}\widetilde{F}^{\mathbf{1}}\widetilde{F}^\sigma_{m,n} = \delta_{\mathbf{1},\sigma}F_{\mathbf{1},\mathbf{1}}$.

Then

$$
\begin{aligned}
F_{\mathbf{1},\mathbf{1}}U_{\pi_{ij}}U_{\rho_{kl}}F_{\mathbf{1},\mathbf{1}} &= \sum_{\sigma,m,n}F_{\mathbf{1},\mathbf{1}}T^{\sigma_m}_{\pi_i,\rho_k}U_{\sigma_{mn}}\overline{T^{\sigma_n}_{\pi_j,\rho_l}}F_{\mathbf{1},\mathbf{1}} = \sum_{\sigma,m,n,a}T^{\sigma_m}_{\pi_i,\rho_k}\overline{T^{\sigma_n}_{\pi_j,\rho_l}}F_{\mathbf{1},\mathbf{1}}\gamma^2_{\overline{\sigma}}(F_{\mathbf{1},\mathbf{1}})_{m,a}U_{\sigma_{na}}\\
&=\sum_{\sigma,m,n,a}\mathrm{d}\sigma^{-1}T^{\sigma_m}_{\pi_i,\rho_k}\overline{T^{\sigma_n}_{\pi_j,\rho_l}}\widetilde{F}^{\mathbf{1}}\widetilde{F}^{\overline{\sigma}}_{m,a}U_{\sigma_{na}} = T^{\mathbf{1}}_{\pi_i,\rho_k}\overline{T^{\mathbf{1}}_{\pi_j,\rho_l}}\widetilde{F}^{\mathbf{1}} = \delta_{\pi,\overline{\rho}}\delta_{i,k}\delta_{j,l}\mathrm{d}\pi^{-1}F_{\mathbf{1},\mathbf{1}}
\end{aligned}
$$

holds, and hence we have

$$
\begin{aligned}
E_{\pi_{ij},\rho_{kl}}E_{\sigma_{mn},\xi_{ab}} &= \sqrt{\mathrm{d}\pi\mathrm{d}\rho\mathrm{d}\sigma\mathrm{d}\xi}\,U_{\pi_{ij}}F_{\mathbf{1},\mathbf{1}}U_{\overline{\rho}_{kl}}U_{\sigma_{mn}}F_{\mathbf{1},\mathbf{1}}U_{\overline{\xi}_{ab}}\\
&= \delta_{\rho,\sigma}\delta_{k,m}\delta_{l,n}\mathrm{d}\rho^{-1}\sqrt{\mathrm{d}\pi\mathrm{d}\rho\mathrm{d}\sigma\mathrm{d}\xi}\,U_{\pi_{ij}}F_{\mathbf{1},\mathbf{1}}U_{\overline{\xi}_{ab}} = \delta_{\rho,\sigma}\delta_{k,m}\delta_{l,n}E_{\pi_{ij},\xi_{ab}}.
\end{aligned}
$$

Finally we verify that $\{E_{\pi_{ij},\rho_{kl}}\}$ is an α^ω-equivariant system of matrix units. Since $F_{\mathbf{1},\mathbf{1}}\in M_\omega^{\gamma^1}$, $\alpha^\omega_\pi(F_{\mathbf{1},\mathbf{1}}) = \mathrm{Ad}\, U_\pi(F_{\mathbf{1},\mathbf{1}}\otimes 1)$ holds. Together with Corollary 6.6, we have

$$\begin{aligned}\alpha_\sigma^\omega(E_{\pi_{ij},\rho_{kl}}) &= \sqrt{\mathrm{d}\pi\mathrm{d}\rho}\,\alpha_\sigma^\omega(U_{\pi_{ij}})\alpha_\sigma^\omega(F_{\mathbf{1},\mathbf{1}})\alpha_\sigma^\omega(U_{\overline{\rho}_{kl}})\\ &= \sqrt{\mathrm{d}\pi\mathrm{d}\rho}\,\mathrm{Ad}\,U_\sigma(U_{\pi_{ij}}\otimes 1_\sigma)(F_{\mathbf{1},\mathbf{1}}\otimes 1_\sigma)(U_{\overline{\rho}_{kl}}\otimes 1_\sigma)\\ &= \mathrm{Ad}\,U_\sigma(E_{\pi_{ij},\rho_{kl}}\otimes 1_\sigma) = \mathrm{Ad}\,\lambda_\sigma^E(E_{\pi_{ij},\rho_{kl}}\otimes 1_\sigma). \quad \blacksquare\end{aligned}$$

REMARK 6.8. We can regard $\{\widetilde{F}^\pi_{i,j}\}$ as an analogue of Rohlin projections for γ^2. See [8] for more details on Rohlin projections for actions of group duals.

PROPOSITION 6.9. *Let $E = \{E_{\pi_{ij},\rho_{kl}}\} \subset M^\omega$ be an α^ω-equivariant system of matrix units. Then there exists a representing sequence of systems of matrix units $\{e^n_{\pi_{ij},\rho_{kl}}\}$ for $E_{\pi_{ij},\rho_{kl}}$, and 1-cocycles $\{u^n_\pi\}$ for α, $n = 1,2,3,\dots$, such that $(u^n_\pi) = 1$ in M^ω and each $\{e^n_{\pi_{ij},\rho_{kl}}\}$ is $\mathrm{Ad}\,u^n_\pi\alpha_\pi$-equivariant.*

Proof. Fix a representing sequence $\{e^n_{\pi_{ij},\rho_{kl}}\}$ for $E_{\pi_{ij},\rho_{kl}}$ consisting of systems of matrix units. Set $A_n := \{e^n_{\pi_{ij},\rho_{kl}}\}'' \subset M$, and λ^n_π the unitary representation of $\widehat{G}$ associated with A_n. Since $E_{\mathbf{1},\mathbf{1}}\otimes 1 = (E_{\mathbf{1},\mathbf{1}}\otimes 1)\lambda^{E*}_\pi\lambda^E_\pi(E_{\mathbf{1},\mathbf{1}}\otimes 1)$ and $\alpha_\pi(E_{\mathbf{1},\mathbf{1}}) = \mathrm{Ad}\,\lambda^E_\pi(E_{\mathbf{1},\mathbf{1}}\otimes 1) = \lambda^E_\pi(E_{\mathbf{1},\mathbf{1}}\otimes 1)(E_{\mathbf{1},\mathbf{1}}\otimes 1)\lambda^{E*}_\pi$, we can choose a representing sequence $\{v^n_\pi\}$ for $(E_{\mathbf{1},\mathbf{1}}\otimes 1)\lambda^*_\pi$ such that $v^n_\pi v^{n*}_\pi = e^n_{\mathbf{1},\mathbf{1}}\otimes 1$, and $v^{n*}_\pi v^n_\pi = \alpha_\pi(e^n_{\mathbf{1},\mathbf{1}})$ by Lemma 4.3. Set $w^n_\pi := \sum_{\pi,i,j}(e^n_{\pi_{ij},\mathbf{1}}\otimes 1)v^n_\pi\alpha_\pi(e^n_{\mathbf{1},\pi_{ij}})$. Then w^n_π is a unitary, and $\mathrm{Ad}\,w^n_\pi\alpha_\pi(e_{\sigma_{ij},\rho_{kl}}) = (e_{\sigma_{ij},\rho_{kl}}\otimes 1)$ holds. Define $\alpha^n_\pi := \mathrm{Ad}\,w^n_\pi\alpha_\pi$, and $U^n_{\pi,\rho} := (\partial_{\alpha^n}w^n)_{\pi,\rho}$. Since α^n_π is trivial on A_n, $U^n_{\pi,\rho} \in (A'_n\cap M)\otimes B(H_\pi)\otimes B(H_\rho)$ and $\{\alpha^n_\pi, U^n_{\pi,\rho}\}$ is a cocycle twisted action on $A'_n\cap M$.

We have $(w^n_\pi) = \sum_{\pi,i,j}(E_{\pi_{ij},\mathbf{1}}\otimes 1)(E_{\mathbf{1},\mathbf{1}}\otimes 1)\lambda^{E*}_\pi\alpha_\pi(E_{\mathbf{1},\pi_{ij}}) = \lambda^{E*}_\pi$, hence $U^n_{\pi,\rho}\to 1$ as $n\to\omega$. By Theorem 5.8, there exists $\overline{w}^n_\pi \in U((A'_n\cap M)\otimes B(H_\pi))$ with $U^n_{\pi,\rho} = (\partial_{\alpha^n}\overline{w}^{n*})_{\pi,\rho}$ and $\lim_{n\to\omega}\|\overline{w}^n_\pi - 1\|_2 = 0$. Set $u^n_\pi := \lambda^n_\pi\overline{w}^n_\pi w^n_\pi$. Then $(u^n_\pi) = (\lambda^n_\pi w^n_\pi) = \lambda^E_\pi\lambda^{E*}_\pi = 1$ in M^ω, and u^n_π is a 1-cocycle for α_π by Lemma 3.6 and the remark after Definition 5.5. It is trivial that $\mathrm{Ad}\,u^n_\pi\alpha_\pi = \mathrm{Ad}\,\lambda^n_\pi$ on A_n, and hence $\{e^n_{\pi_{ij},\rho_{kl}}\}$ is $\mathrm{Ad}\,u^n_\pi\alpha_\pi$-equivariant. $\blacksquare$

7. CLASSIFICATION

PROPOSITION 7.1. *Let α be an outer action on $\mathcal{R}$. Then α is conjugate to $\alpha\otimes\mathrm{id}_{\mathcal{R}}$.*

Proof. This follows from [1] since $\mathcal{R}'\cap(\mathcal{R}^\alpha)^\omega$ is noncommutative. $\blacksquare$

LEMMA 7.2. *Let $K\subset\mathcal{R}$ be a subfactor with $K\cong M_n(\mathbb{C})$, and $\{e_{ij}\}$ be a system of matrix units for K. If $\|[x,e_{ij}]\|_2 < \frac{\varepsilon}{n}$, then $\|E_{K'\cap\mathcal{R}}(x) - x\|_2 < \varepsilon$.*

Proof. Since $E_{K'\cap\mathcal{R}}(x) = \frac{1}{n}\sum_{i,j} e_{ij}xe_{ji}$, the following holds:

$$\|E_{K'\cap\mathcal{R}}(x) - x\|_2 \leqslant \frac{1}{n}\sum_{i,j}\|e_{ij}xe_{ji} - xe_{ij}e_{ji}\|_2 \leqslant \frac{1}{n}\sum_{i,j}\|[e_{ij}, x]\|_2 < \varepsilon. \quad \blacksquare$$

LEMMA 7.3. *For any $\varepsilon > 0$, $a_1, a_2, \dots, a_n \in \mathcal{R}$, there exist a 1-cocycle u_π for α_π, and an $\mathrm{Ad}\, u_\pi\alpha$-equivariant system of matrix units $E = \{e_{\pi_{ij},\rho_{kl}}\}$ such that $\|\alpha_\pi(a_i) - \mathrm{Ad}\,\lambda_\pi^E(a_i\otimes 1)\|_2 < \varepsilon$, $\|u_\pi - 1\|_1 < \varepsilon$ and $\|[e_{\mathbf{1},\mathbf{1}}, a_i]\|_2 < \varepsilon$.*

Proof. By Lemma 6.7 and Proposition 6.9, we have systems of matrix units $E_n = \{e^n_{\pi_{ij},\rho_{kl}}\}$ and 1-cocycles u^n_π for α_π such that $\{e^n_{\pi_{ij},\rho_{kl}}\}$ is $\mathrm{Ad}\,u^n_\pi\alpha_\pi$ equivariant, $\alpha_\pi(x) = \lim_{n\to\omega}\mathrm{Ad}\,\lambda_\pi^{E_n}(x\otimes 1)$, $\lim_{n\to\omega}\|u^n_\pi - 1\|_2 = 0$ and $\lim_{n\to\omega}\|[e^n_{\mathbf{1},\mathbf{1}}, x]\|_2 = 0$ for any $x \in M$. Put $E := E_n$ and $u_\pi := u^n_\pi$ for sufficiently large n. $\blacksquare$

Now we can prove the main theorem of this paper.

THEOREM 7.4. *Let α be an outer action of $\widehat{G}$ on $\mathcal{R}$. Then α is conjugate to the model action m.*

Proof. We use the notations in Section 3. Let $\{a_i\}_{i=1}^\infty$ be a strongly dense countable subset of the unit ball of $\mathcal{R}$. We fix a sequence $\{\varepsilon_n\}$ such that $0 < 9|G|^3\varepsilon_n \leqslant 2^{-n}$. Especially we have $\sum_n \varepsilon_n < \infty$. We will construct mutually commuting finite dimensional subfactors $K_n \cong M_{|G|}(\mathbb{C})$, unitary 1-cocycles v^n_π for α_π, a unitary 1-cocycle w^n_π for $\mathrm{Ad}\,\widetilde{\lambda}^{n-1*}_\pi v^{n-1}_\pi\alpha_\pi$ satisfying the following conditions inductively:

$$\begin{aligned}
&(1.n) \qquad v^n_\pi := \widetilde{\lambda}^{n-1}_\pi w^n_\pi \widetilde{\lambda}^{n-1*}_\pi v^{n-1}_\pi, n \geqslant 2;\\
&(2.n) \qquad \|w^n_\pi - 1\|_2 < \varepsilon_n;\\
&(3.n) \qquad \|[a_i, e^n_{\mathbf{1},\mathbf{1}}]\|_2 < \varepsilon_n, 1 \leqslant i \leqslant n;\\
&(4.n) \qquad \mathrm{Ad}\,v^n_\pi\alpha_\pi = \mathrm{Ad}\,\widetilde{\lambda}^n_\pi \text{ on } K_1 \vee \cdots \vee K_n;\\
&(5.n) \qquad \|\mathrm{Ad}\,v^n_\pi\alpha_\pi(a_i) - m^n_\pi(a_i)\|_2 < \varepsilon_n, 1 \leqslant i \leqslant n.
\end{aligned}$$

By Lemma 7.3, we get a unitary cocycle w^1_π for α_π, and an $\mathrm{Ad}\,w^1_\pi\alpha_\pi$-equivariant system of matrix units $\{e^1_{\pi_{ij},\rho_{kl}}\}$ such that $\|[a_1, e^1_{\mathbf{1},\mathbf{1}}]\|_2 < \varepsilon_1$, $\|w^1_\pi - 1\|_2 < \varepsilon_1$, and $\|\mathrm{Ad}\,w^1_\pi\alpha_\pi(a_1) - \mathrm{Ad}\,\lambda^1_\pi(a_1\otimes 1)\|_2 < \varepsilon_1$. Let K_1 be a finite dimensional subfactor generated by $\{e^1_{\pi_{ij},\rho_{kl}}\}$, and set $v^1_\pi := w^1_\pi$. Then we get the conditions (2.1), (3.1), (4.1) and (5.1).

Suppose that we have done up to the n-th step. By $(4.n)$ it follows that $\mathrm{Ad}\,\widetilde{\lambda}^{n*}_\pi v^n_\pi\alpha_\pi = \mathrm{id}$ on $K_1\vee\cdots\vee K_n$. Hence $\mathrm{Ad}\,\widetilde{\lambda}^{n*}_\pi v^n_\pi\alpha_\pi$ induces an action of $\widehat{G}$ on $(K_1\vee\cdots\vee K_n)'\cap\mathcal{R}$. Decompose a_i as $a_i = \sum_i b_{ik}e_k$, $b_{ik} \in (K_1\vee\cdots\vee K_n)'\cap\mathcal{R}$, $e_k \in K_1\vee\cdots\vee K_n$. By Lemma 7.3, we get a unitary cocycle w^{n+1}_π for $\mathrm{Ad}\,\widetilde{\lambda}^{n*}_\pi v^n_\pi\alpha_\pi$, a $\mathrm{Ad}\,w^{n+1}_\pi\widetilde{\lambda}^{n*}_\pi v^n_\pi\alpha_\pi$ -equivariant system matrix units $K_{n+1} := \{e^{n+1}_{\pi_{ij},\rho_{kl}}\} \subset (K_1 \vee$

$\cdots \vee K_n)' \cap \mathcal{R}$, such that:

$$(a.n+1) \qquad \|[e^n_{\mathbf{1,1}}, b_{ik}]\|_2 < \delta_{n+1};$$
$$(b.n+1) \qquad \|w^{n+1}_\pi - 1\|_2 < \varepsilon_{n+1};$$
$$(c.n+1) \qquad \|\mathrm{Ad}\, w^{n+1}_\pi \widetilde{\lambda}^{n*}_\pi v^n_\pi \alpha_\pi(b_{ik}) - \mathrm{Ad}\, \lambda^{n+1}_\pi (b_{ik} \otimes 1)\|_2 < \delta_{n+1};$$

for sufficiently small $\delta_{n+1} > 0$. The condition $(b.n+1)$ is nothing but $(2.n+1)$. If we choose sufficiently enough small δ_{n+1}, then we get $(3.n+1)$ and

$$(c.n+1)' \qquad \|\mathrm{Ad}\, w^{n+1}_\pi \widetilde{\lambda}^{n*}_\pi v^n_\pi \alpha_\pi(a_i) - \mathrm{Ad}\, \lambda^{n+1}_\pi (a_i \otimes 1)\|_2 < \varepsilon_{n+1}, \quad 1 \leqslant i \leqslant n+1$$

from $(a.n+1)$ and $(c.n+1)$ respectively. Set $v^{n+1}_\pi := \widetilde{\lambda}^n_\pi w^{n+1}_\pi \widetilde{\lambda}^{n*}_\pi v^n_\pi$. Then we get $(1.n+1)$ and $(5.n+1)$. Since $\{e^{n+1}_{\pi_{ij},\rho_{kl}}\} \subset (K_1 \vee \cdots \vee K_n)' \cap \mathcal{R}$ is $\mathrm{Ad}\, w^{n+1}_\pi \widetilde{\lambda}^{n*}_\pi v^n_\pi \alpha_\pi$-equivariant, we get $(4.n+1)$, and K_{n+1} commutes with K_i, $1 \leqslant i \leqslant n$. Thus we complete induction.

We will show that $\{v^n_\pi\}$ is a Cauchy sequence:

$$\|v^{n+1}_\pi - v^n_\pi\|_2 = \|v^n_\pi \widetilde{\lambda}^n_\pi w^{n+1}_\pi \widetilde{\lambda}^{n*}_\pi - v^n_\pi\|_2 = \|\widetilde{\lambda}^n_\pi w^{n+1}_\pi \widetilde{\lambda}^{n*}_\pi - 1\|_2 < \varepsilon_{n+1}.$$

By the choice of ε_n, $\{v^n_\pi\}$ is Cauchy, and hence $\lim_{n\to\infty} v^n_\pi = v_\pi$ exists.

We will prove $\|[e^{n+1}_{\pi_{ij},\rho_{kl}}, a_i]\|_2 < \varepsilon_n$, $1 \leqslant i \leqslant n$. By $(5.n)$ and $(5.n+1)$, we get

$$\|\mathrm{Ad}\, v^{n*}_\pi \widetilde{\lambda}^n_\pi(a_i) - \mathrm{Ad}\, v^{n+1*}_\pi \widetilde{\lambda}^{n+1}_\pi(a_i)\|_2 < 2\varepsilon_n, 1 \leqslant i \leqslant n.$$

By the definition of v^n_π, we get $\|a_i \otimes 1 - \mathrm{Ad}\, w^{n+1*}_\pi \lambda^{n+1}_\pi(a_i \otimes 1)\|_2 < 2\varepsilon_n$. Then

$$\begin{aligned}
\|[a_i \otimes 1, \lambda^{n+1}_\pi]\|_2 &= \|[a_i \otimes 1, w^{n+1}_\pi w^{n+1*}_\pi \lambda^{n+1}_\pi]\|_2 \\
&\leqslant \|[a_i \otimes 1, w^{n+1}_\pi] w^{n+1*}_\pi \lambda^{n+1}_\pi\|_2 + \|w^{n+1}_\pi [a_i \otimes 1, w^{n+1*}_\pi \lambda^{n+1}_\pi]\|_2 \\
&\leqslant \|[a_i \otimes 1, w^{n+1}_\pi - 1]\|_2 + 2\varepsilon_n < 4\varepsilon_n.
\end{aligned}$$

Hence we get $\|[\lambda^{n+1}_{\pi_{ij}}, a_i]\|_2 < 4 \mathrm{d}\pi \varepsilon_n < 4|G|\varepsilon_n$ for $1 \leqslant i \leqslant n$. Then we have

$$\begin{aligned}
&\|[a_i, e^{n+1}_{\pi_{ij},\rho_{kl}}]\|_2 \\
&= \sqrt{\mathrm{d}\pi \mathrm{d}\rho}\, \|[a_i, \lambda^{n+1}_{\pi_{ij}} e^{n+1}_{\mathbf{1,1}} \lambda^{n+1}_{\overline{\rho}_{kl}}]\|_2 \\
&\leqslant |G| (\|[a_i, \lambda^{n+1}_{\pi_{ij}}] e^{n+1}_{\mathbf{1,1}}\|_2 + \|\lambda^{n+1}_{\pi_{ij}} [a_i, e^{n+1}_{\mathbf{1,1}}] \lambda^{n+1}_{\overline{\rho}_{kl}}\|_2 + \|\lambda^{n+1}_{\pi_{ij}} e^{n+1}_{\mathbf{1,1}} [a_i, \lambda^{n+1}_{\overline{\rho}_{kl}}]\|_2) \\
&< 9|G|^2 \varepsilon_n < \frac{1}{2^n |G|}.
\end{aligned}$$

This implies $\|E_{K'_{n+1} \cap \mathcal{R}}(a_i) - a_i\|_2 < \frac{1}{2^n}$ for $1 \leqslant i \leqslant n$ by Lemma 7.2. Set $K := \bigvee K_n (\cong \mathcal{R})$. By Lemma 2.3.6 of [2], $\mathcal{R} = K \vee K' \cap \mathcal{R} \cong K \otimes K' \cap \mathcal{R}$. By $(5.n)$, $\mathrm{Ad}\, v_\pi \alpha_\pi = m_\pi \otimes \mathrm{id}_{K' \cap \mathcal{R}}$. By Proposition 7.1 $m_\pi \cong m_\pi \otimes \mathrm{id}_\mathcal{R}$, and $\mathrm{Ad}\, v_\pi \alpha_\pi$ is conjugate to $m_\pi \otimes \mathrm{id}_{K \vee K' \cap \mathcal{R}} = m_\pi \otimes \mathrm{id}_\mathcal{R} \cong m_\pi$. By Proposition 5.2, α is conjugate to m. ∎

It is obvious that Theorem 2.14 follows immediately from Theorem 7.4.

REMARK 7.5. So far we treat only actions of $\widehat{G}$ for a finite group G. However we can generalize our theory to outer actions of finite dimensional Kac algebras. The difference between $\widehat{G}$ and general finite dimensional Kac algebras is the commutativity $\pi \otimes \rho \cong \rho \otimes \pi$. We do not use the commutativity of $\widehat{G}$ in proofs except Lemma 6.3. To generalize Lemma 6.3 to a finite dimensional Kac algebra $\mathcal{K}$, we should consider a (cocycle) action of $\mathcal{K} \otimes \mathcal{K}^{\mathrm{opp}}$ on M^ω as in the Remark 6.5. (Note $R(G)$ and $R(G)^{\mathrm{opp}}$ are essentially same Kac algebras due to cocommutativity of $R(G)$.)

APPENDIX A: TWISTED CROSSED PRODUCT CONSTRUCTION

Let $\{\alpha, U\}$ be a cocycle twisted action of $\widehat{G}$ on M. In this appendix, we give the definition of a twisted crossed product $M \rtimes_{\alpha,U} \widehat{G}$.

Let $H := L^2(M)$ be the standard Hilbert space. We identify $H \otimes \ell^2(\widehat{G})$ with $\left\{ \bigoplus_\pi v(\pi) : v(\pi) \in H \otimes B(H_\pi) \right\}$ as usual. Put $\langle v, w\rangle_\pi = \sum_{ij} \langle v_{ij}, w_{ij}\rangle$ for $v, w \in H \otimes B(H_\pi)$. Then the inner product is given by $\langle v, w\rangle = \sum_\pi \mathrm{d}\pi \langle v(\pi), \omega(\pi)\rangle_\pi$ for $v, w \in H \otimes \ell^2(\widehat{G})$.

We define an action α of M on $H \otimes \ell^2(\widehat{G})$, and $\lambda_{\pi_{ij}} \in B(H \otimes \ell^2(\widehat{G}))$ by

$$(\alpha(a)v)(\pi) = \alpha_\pi(a)v(\pi), \quad (\lambda_{\pi_{ij}} v)(\rho) := \sum_\sigma U^{\sigma,e}_{\rho,\pi_i} v(\sigma) T^{\sigma,e*}_{\rho,\pi_j}.$$

DEFINITION A.1. *Define* $M \rtimes_{\alpha,U} \widehat{G} := \alpha(M) \vee \{\lambda_{\pi_{ij}}\}$*, and call it the twisted crossed product of* M *by* $\{\alpha, U\}$.

LEMMA A.2. *Set* $\lambda_\pi = (\lambda_{\pi_{ij}}) \in B(H \otimes \ell^2(\widehat{G})) \otimes B(H_\pi)$*. Then* λ_π *is a unitary, and we have* $\lambda_\pi (a \otimes 1_\pi)\lambda_\pi^* = \alpha_\pi(a)$*, and* $\lambda_\pi^{12}\lambda_\rho^{13} T^{\sigma,e}_{\pi,\rho} = U^{\sigma,e}_{\pi,\rho}\lambda_\sigma$*. Set* $\widetilde{U}_{\pi_i,\overline{\pi}_j} := \sum_k U_{\pi_{ik},\overline{\pi}_{jk}}$*. Then we have* $\lambda^*_{\overline{\pi}_{ij}} = \sum_k U^*_{\pi_k,\overline{\pi}_i}\lambda_{\pi_{kj}}$*. Here we identify* $\alpha(a)$ *and* a*. We call* λ_π *an implementing unitary.*

To show Lemma A.2, we prepare the following lemma.

LEMMA A.3. *We have* $\sum_{k,l} \widetilde{U}^*_{\pi_k,\overline{\pi}_l} \alpha_\pi(\widetilde{U}_{\overline{\pi}_l,\pi_i})_{k,j} = \delta_{i,j}$.

Proof. Recall the following 2-cocycle condition (see a paragraph after Definition 5.4):

$$(\alpha_\pi \otimes \mathrm{id})(U^{\eta,a}_{\rho,\sigma})U^{\xi,b}_{\pi,\eta} = \sum_{\zeta,c,d} (U^{\zeta,c}_{\pi,\rho} \otimes 1_\sigma)U^{\xi,d}_{\zeta,\sigma} V_{(\zeta,c,d),(\eta_a,b)}.$$

We put $\overline{\rho} = \sigma = \pi$, $\eta = \mathbf{1}$ (hence $\xi = \pi$), and multiply $U^{\mathbf{1}}_{\pi,\overline{\pi}} \otimes 1_\pi$ from the left on both sides. Then we get the following:

$$(U^{\mathbf{1}*}_{\pi,\overline{\pi}} \otimes 1_\pi)\alpha_\pi \otimes \mathrm{id}(U^{\mathbf{1}}_{\overline{\pi},\pi}) = V_{\mathbf{1},\mathbf{1}} = (T^{\mathbf{1}*}_{\pi,\overline{\pi}} \otimes 1_\pi)(1_\pi \otimes T^{\mathbf{1}}_{\overline{\pi},\pi}) = \frac{1}{\mathrm{d}\pi}.$$

Since $U^{\mathbf{1}}_{\pi_i,\overline{\pi}_j} = \sum_k \frac{1}{\sqrt{\mathrm{d}\pi U_{\pi_{ik},\overline{\pi}_{jk}}}} = \frac{1}{\sqrt{\mathrm{d}\pi \widetilde{U}_{\pi_i,\overline{\pi}_j}}}$, we get the conclusion. ∎

Proof of Lemma A.2. It is easy to see α is an action of M on $H \otimes B(H_\pi)$. We verify that λ_π implements α_π. Then

$$\begin{aligned}(\lambda_{\pi_{ij}}\alpha(a)v)(\rho) &= \sum_{\sigma,e} U^{\sigma,e}_{\rho,\pi_i}(\alpha(a)v)(\sigma)T^{\sigma,e*}_{\rho,\pi_j} = \sum_{\sigma,e} U^{\sigma,e}_{\rho,\pi_i}\alpha_\sigma(a)v(\sigma)T^{\sigma,e*}_{\rho,\pi_j} \\ &= \sum_{\sigma,e,k} \alpha_\rho(\alpha_\pi(a)_{ik})U^{\sigma,e}_{\rho,\pi_k}v(\sigma)T^{\sigma,e*}_{\rho,\pi_j} = \sum_k (\alpha(\alpha_\pi(a)_{ik})\lambda_{\pi_{kj}}v)(\rho)\end{aligned}$$

holds. Therefore we have $\lambda_\pi(a \otimes 1_\pi) = \alpha_\pi(a)\lambda_\pi$ by identifying $\alpha(a)$ and a.

We next compute $\lambda^{12}_\pi \lambda^{13}_\sigma$ as follows:

$$\begin{aligned}(\lambda_{\pi_{ij}}\lambda_{\rho_{kl}}v)(\xi) &= \sum_{\sigma,a,\eta,b} U^{\sigma,a}_{\xi,\pi_i}U^{\eta,b}_{\sigma,\rho_k}v(\eta)T^{\eta,b*}_{\sigma,\rho_l}T^{\sigma,a*}_{\xi,\pi_j} \\ &= \sum_{\sigma,a,m,n,\eta,b} \alpha_\xi(U^{\sigma_m,a}_{\pi_i,\rho_k})U^{\eta,b}_{\xi,\sigma_m}v(\eta)T^{\eta,b*}_{\xi,\sigma_n}T^{\sigma_n,a*}_{\pi_j,\rho_l} \quad \text{(by 2-cocycle condition)} \\ &= \sum_{\sigma,m,n,a} (\alpha(U^{\sigma_m,a}_{\pi_i,\rho_k})\lambda_{\sigma_{m,n}}T^{\sigma_n,a}_{\pi_j,\rho_l}v)(\xi).\end{aligned}$$

Hence we have $\lambda^{12}_\pi \lambda^{13}_\rho T^{\sigma,a}_{\pi,\rho} = U^{\sigma,a}_{\pi,\rho}\lambda_\sigma$.

Finally, we verify that λ_π is a unitary. One can easily to see $\sum_k \lambda^*_{\pi_{ki}}\lambda_{\pi_{kj}} = \delta_{i,j}$ (hence $\lambda^*_\pi\lambda_\pi = 1$) from the definition of $\lambda_{\pi_{ij}}$ and $U^{\sigma,a*}_{\pi,\rho}U^{\xi,b}_{\pi,\rho} = \delta_{\sigma,\xi}\delta_{a,b}$. Hence we only have to see $\lambda_\pi\lambda^*_\pi = 1$.

To this end, we first show $\lambda^*_{\overline{\pi}_{ij}} = \sum_k \widetilde{U}^*_{\pi_k,\overline{\pi}_i}\lambda_{\pi_{kj}}$. Since $\sum_k \lambda_{\pi_{ik}}\lambda_{\overline{\pi}_{jk}} = \sum_{\rho,l,m,a} U^{\rho_l,a}_{\pi_i,\overline{\pi}_j}\lambda_{\rho_{lm}} T^{\rho_m,a}_{\pi_k,\overline{\pi}_k} = \widetilde{U}_{\pi_i,\overline{\pi}_j}$, we have $\lambda_\pi {}^{\mathrm{t}}\lambda_{\overline{\pi}} = (\widetilde{U}_{\pi_i,\overline{\pi}_j})_{i,j}$.

Then we get ${}^{\mathrm{t}}\lambda_{\overline{\pi}} = \lambda^*_\pi(\widetilde{U}_{\pi_i,\overline{\pi}_j})_{i,j}$. Comparing matrix elements of both sides, we get $\lambda^*_{\overline{\pi}_{ij}} = \sum_k \widetilde{U}^*_{\pi_k,\overline{\pi}_i}\lambda_{\pi_{kj}}$.

Then we get

$$\begin{aligned}\sum_k \lambda_{\pi_{ik}}\lambda^*_{\pi_{jk}} &= \sum_{k,l}\lambda_{\pi_{ik}}\widetilde{U}^*_{\overline{\pi}_l,\pi_j}\lambda_{\overline{\pi}_{lk}} = \sum_{k,l,m}\alpha_\pi(\widetilde{U}^*_{\overline{\pi}_l,\pi_j})_{im}\lambda_{\pi_{mk}}\lambda_{\overline{\pi}_{lk}} \\ &= \sum_{k,l,m,\xi,a,b}\alpha_\pi(\widetilde{U}^*_{\overline{\pi}_l,\pi_j})_{im}U^{\xi_a}_{\pi_m,\overline{\pi}_l}\lambda_{\xi_{ab}}T^{\xi_b}_{\pi_k,\overline{\pi}_k} = \sum_{k,l,m}\alpha_\pi(\widetilde{U}^*_{\overline{\pi}_l,\pi_j})_{im}\widetilde{U}_{\pi_m,\overline{\pi}_l} = \delta_{i,j}\end{aligned}$$

by Lemma 7, and λ_π is indeed a unitary. ∎

We construct a conditional expectation E from $M \rtimes_{\alpha,U} \widehat{G}$ onto M. Let P be a projection from $H \otimes \ell^2(\widehat{G})$ to $H \otimes B(H_{\mathbf{1}}) \cong H$, and set $E(x) := PxP^*$. Then E is indeed a conditional expectation from $M \rtimes_{\alpha,U} \widehat{G}$ onto M with $E(\lambda_{\pi_{ij}}) = \delta_{\mathbf{1},\pi}$. Then the following lemma can be easily verified as in the usual crossed product.

LEMMA A.4. *Every $a \in M \rtimes_{\alpha,U} \widehat{G}$ is expressed uniquely as $a = \sum_{\pi,i,j} a_{\pi,i,j}\lambda_{\pi_{ij}}$, $a_{\pi,i,j} \in M$.*

Here we only remark that a coefficient $a_{\pi,i,j}$ is given by $a_{\pi,i,j} = \mathrm{d}\pi E(a\lambda^*_{\pi_{ij}})$.

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TOSHIHIKO MASUDA, GRADUATE SCHOOL OF MATHEMATICS, KYUSHU UNIVERSITY, 6-10-1 HAKOZAKI, FUKUOKA, 812-8581, JAPAN
E-mail address: masuda@math.kyushu-u.ac.jp

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