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ON OPERATOR ALGEBRAS DETERMINED BY A SEQUENCE OF OPERATOR NORMS

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ABSTRACT. We consider a family of operators determined by a sequence of operator norms. When the sequence of norms is determined by a single operator the natural question that arises is when the algebra properly contains the commutant of the operator. In this case the existence of invariant subspaces for the algebra is stronger than the existence of hyperinvariant subspaces for the operator.

KEYWORDS: *Operator algebras, operator norms, invariant subspace, spectral radius algebra.*

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1. INTRODUCTION

While the classical invariant subspace problem concerns itself with a single operator, the study of the relationship between operator algebras and their invariant subspaces was begun at least as far back as the 1960's. Of course the classical problem is also a question about the invariant subspaces of an operator algebra: is the weak closure of the algebra of polynomials in a given operator A non-transitive? The more general hyperinvariant subspace problem asks: for a given operator A on a Banach space X, is the (in general) non-commutative algebra $(A)'= \{T \in L(X) : AT = TA\}$ non-transitive? The extreme case is of course when $A = \lambda I$, a scalar multiple of the identity. In this case, and this case alone, $(A)' = L(X)$.

We consider a family of algebras determined by sequences of operator norms on $L(X)$, and following Lambert and Petrović ([12], Theorem 3.4) we state an invariant subspace theorem for such algebras using the principle of Lomonosov [15]. After discussing this general framework we consider three concrete examples of such algebras which have the common property that the sequence of norms determining the algebra is defined in terms of a single operator A in such a way that it contains its commutant $(A)'$. The significance of this fact from the point of view of invariant subspace theory is clear. In this case, an invariant subspace of the algebra is at least a hyperinvariant subspace for A. Thus it is fundamental to determine when on one hand the algebra properly contains $(A)'$ and on the other hand when it is not weakly dense in $L(X)$. In this situation an invariant subspace theorem is possible and is stronger than the existence of hyperinvariant subspaces.

In Hilbert space all three examples fit the following format. Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of invertible operators on X dependent on the given operator A and let $\{\|T\|_n\}$ be the sequence of operator norms on X defined by $\|T\|_n = \|D_n T D_n^{-1}\|$. We consider the unital algebra

$$\mathcal{B} = \Big\{ T \in L(X) : \sup_{n>0} \|D_n T D_n^{-1}\| < \infty \Big\}.$$

In all three examples the commutant of A is easily seen to be a subalgebra of $\mathcal{B}$. The extension to Banach space of the final example, the spectral radius algebra of [12] requires a more complicated construction.

In our first example we consider the sequence $D_n = I + nA$ such that, for each n, D_n is invertible. In this case the algebra $\mathcal{R}_A := \Big\{ T \in L(X) : \sup_n \|(I + nA)T(I + nA)^{-1}\| < \infty \Big\}$ will be maximal, $\mathcal{R}_A = L(X)$, if and only if A is invertible. If X is finite dimensional then $\mathcal{R}_A$ is minimal, $\mathcal{R}_A = (A)'$, if and only if A is nilpotent. For X infinite dimensional the situation is much more complicated. We give a class of compact quasinilpotent operators for which $\mathcal{R}_A = (A)'$ and another for which $\mathcal{R}_A$ is quite large.

Our next example is the algebra introduced by Deddens [7]. In this case A is invertible and $D_n = A^n$ for each n. Then $\mathcal{D}_A := \Big\{ T \in L(X) : \sup_n \|A^n T A^{-n}\| < \infty \Big\}$ contains $(A)'$, and it was shown ([19], [7]) that for X a Hilbert space, $\mathcal{D}_A$ is maximal, that is $\mathcal{D}_A = L(X)$, if and only if A is (similar to) a scalar multiple of a unitary operator. The question: when is $\mathcal{D}_A$ minimal, that is $\mathcal{D}_A = (A)'$?, is much more difficult. It was shown in [6] that if $A = \lambda I + N$ with N nilpotent then $\mathcal{D}_A = (A)'$ and that for X finite dimensional this is the only such case. Deddens [7] formulated a conjecture about the minimality of D_A, but Roth [17] disproved it in both directions. We give additional examples connected with the conjecture of Deddens, namely, examples of operators of the form $A = I + Q$ with Q quasinilpotent for which $\mathcal{D}_A$ properly contains $(A)'$.

Our final example is the extension to Banach space of the spectral radius algebras introduced in [12]. While the sequence of operator norms used for Deddens' algebra can be defined directly on $L(X)$ and this is the case for the spectral radius algebras on Hilbert space, as defined in [12], the extension to Banach space requires us to define a sequence of vector norms on X and then consider the induced operator norms on $L(X)$. The main issue in [12] was to extend

Lomonosov's famous result on non-transitivity of $(A)'$ when A is a compact operator [14]. The spectral radius algebra $\mathcal{B}_A$, determined by A, contains $(A)'$, and the main result of [12] is that for A compact $\mathcal{B}_A$ is non-transitive. Of course, for this theorem to actually extend Lomonosov it is necessary that $\mathcal{B}_A$ properly contain $(A)'$. They show that this is in fact the case for A compact with positive spectral radius. While it is far from clear that this approach will give new invariant subspace theorems for single operators it does give new results for operator algebras properly containing the commutant of an operator and thus a stronger result than the existence of hyperinvariant subspaces. This is reflected in the title of [12]: "Beyond Hyperinvariance".

Recently [3] the question: when is $\mathcal{B}_A$ maximal, that is $\mathcal{B}_A = L(X)$? was resolved for Hilbert space X. This is the case if and only if A is similar to a scalar multiple of an isometry. This result reflects strongly the connection between $\mathcal{D}_A$ and $\mathcal{B}_A$. In fact we see that for A invertible $\mathcal{D}_A \subseteq \mathcal{B}_A$.

These particular examples are considered within a general framework of algebras determined by a sequence of operator norms. This framework seems to be new, and many interesting problems relating to operator algebras and their invariant subspaces arise naturally. We hope that the results and examples presented here will stimulate further study of these issues.

2. ALGEBRAS GENERATED BY A SEQUENCE OF OPERATOR NORMS

Let X be a complex Banach space. $L(X)$ will denote the algebra of bounded linear operators on X. A norm $\|\cdot\|'$ on $L(X)$ is *admissible* if it is (topologically) equivalent to the initial vector induced operator norm on $L(X)$ and it is algebraic; that is:

(1) $\|A_1A_2\|' \leqslant \|A_1\|'\|A_2\|'$ for $A_1, A_2 \in L(X)$.
(2) $\|I\|' = 1$ for the identity operator $I \in L(X)$.

Let $\{\|T\|_m\}_{m=1}^{\infty}$ be a sequence of admissible norms on $L(X)$ and

$$\mathcal{B} = \mathcal{B}(\|T\|_m) = \Big\{B \in L(X) : \sup_{m>0} \|B\|_m < \infty\Big\}.$$

It is clear that $\mathcal{B}(\|T\|_m)$ is a unital subalgebra of $L(X)$. We will refer to it as the algebra *determined* by the sequence $\{\|T\|_m\}$ of operator norms. We show that with respect to the norm $|||B||| = \sup_m \|B\|_m$, $\mathcal{B}$ is a Banach algebra.

LEMMA 2.1. *Let Y be a Banach space and $\{\|\cdot\|_m\}$ a sequence of norms on Y, each equivalent to the initial norm $\|\cdot\|$. The subspace*

$$Y' = \Big\{y \in Y : \sup_m \|y\|_m < \infty\Big\}$$

is complete with respect to the norm

$$|||y||| = \sup_m \|y\|_m.$$

Proof. Let $\{y_n\} \subseteq Y'$ be a Cauchy sequence with respect to the norm $|||\cdot|||$. Then it is Cauchy with respect to each $\|\cdot\|_m$ and with respect to the initial norm $\|\cdot\|$. Thus there exists $y \in Y$ such that $\|y_n - y\| \to 0$, and hence $\|y_n - y\|_m \to 0$ for each m, as $n \to \infty$. Since $\{y_n\}$ is Cauchy with respect to $|||\cdot|||$, for any $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that $\|y_k - y_n\|_m < \varepsilon$ for $k, n \geqslant k_0$ and for all $m \in \mathbb{N}$. Therefore, $\sup_m \|y - y_n\|_m \leqslant \varepsilon$ for $n \geqslant k_0$. Thus $y - y_n \in Y'$ and $y \in Y'$. ∎

COROLLARY 2.2. *$\mathcal{B}(\|T\|_m)$ is complete with respect to the norm $|||T|||$.*

It is of interest to know when $\mathcal{B}$ is closed in $L(X)$. We recall the following well known consequence of the Open Mapping Theorem (see p. 274 of [4]).

LEMMA 2.3. *Suppose X is a Banach space with given norm $\|\cdot\|$ and that $\|\cdot\|'$ is another norm on X such that $\|\cdot\|$ is subordinate to $\|\cdot\|'$ (there exists $C > 0$ such that $\|x\| \leqslant C\|x\|'$ for all $x \in X$). Then X is complete with respect to $\|\cdot\|'$ if and only if the two norms are equivalent ($\|\cdot\|'$ is also subordinate to $\|\cdot\|$).*

We obtain the next theorem as an immediate consequence:

THEOREM 2.4. *The subalgebra $\mathcal{B}(\|T\|_m) \subseteq L(X)$ is closed in $L(X)$ if and only if there exists a constant $C > 0$ such that for each $T \in \mathcal{B}(\|T\|_m)$, $|||T||| \leqslant C\|T\|$.*

We now present an invariant subspace theorem for $\mathcal{B} = \mathcal{B}(\|T\|_m)$ which will be applied later. Let $\mathbb{Q}$ denote the class of quasinilpotent operators in $L(X)$, and consider the subset of $\mathcal{B}$

$$\mathbb{Q}(\mathcal{B}) = \mathbb{Q}(\mathcal{B}(\|T\|_m)) = \{T \in L(X) : \|T\|_m \to 0, m \to \infty\}.$$

LEMMA 2.5. *$\mathbb{Q}(\mathcal{B})$ is a two-sided ideal in $\mathcal{B}$ and $\mathbb{Q}(\mathcal{B}) \subseteq \mathbb{Q}$.*

Proof. If $T \in \mathbb{Q}(\mathcal{B})$ and $X \in \mathcal{B}$, then $\|TX\|_m \leqslant \|T\|_m\|X\|_m \to 0$ as $m \to \infty$, so $\mathbb{Q}(\mathcal{B})$ is a right ideal. The proof that it is a left ideal is similar.

Since for $T \in L(X)$ its spectral radius $r(T)$ satisfies $r(T) \leqslant \|T\|_m$ for each m, it follows that if $T \in \mathbb{Q}(\mathcal{B})$ then $r(T) = 0$. ∎

It is easily seen that $\mathbb{Q}(\mathcal{B})$ is contained in the radical of $\mathcal{B}$.

The idea of the proof of the next theorem is from [12], adapted to our framework.

THEOREM 2.6. *Suppose that the algebra $\mathcal{B} = \mathcal{B}(\|T\|_m)$ is such that:*

(i) *$\mathcal{B}$ contains a non-zero compact operator,*

(ii) *$\mathbb{Q}(\mathcal{B}) \neq \{0\}$.*

Then $\mathcal{B}$ has a non-trivial invariant subspace.

Proof. Let K be a non-zero compact operator in $\mathcal{B}$. We can assume that $QK = 0$ for every $Q \in \mathbb{Q}(\mathcal{B})$. Indeed if for some $Q \in \mathbb{Q}(\mathcal{B}), QK \neq 0$, then QK is compact quasinilpotent, and for any $B \in \mathcal{B}, BQK \in \mathbb{Q}$. If $\mathcal{B}$ is transitive, then by Lomonosov [15] there exists $B \in \mathcal{B}$ such that BQK has 1 as an eigenvalue, contradicting $BQK \in \mathbb{Q}$.

Let Q be a fixed non-zero operator in $\mathbb{Q}(\mathcal{B})$ and $T \in \mathcal{B}$. Then $QT \in \mathbb{Q}(\mathcal{B})$ and $QTK = 0$. Since $K \neq 0$, there is a non-zero vector z in the range of K. Clearly $QTz = 0$ so $Tz \in \mathrm{Ker} Q$ for all $T \in \mathcal{B}$. Now the closure of the subspace $\{Tz : T \in \mathcal{B}\}$ is invariant for $\mathcal{B}$ and is non-zero since $z \neq 0$ and $I \in \mathcal{B}$. It is not X since it is contained in the kernel of the non-zero operator Q. ∎

We saw before that $|||T||| = \sup\limits_m \|T\|_m$ is a natural norm on $\mathcal{B} = \mathcal{B}(\|T\|_m)$. It is of interest to consider the semi-norm $|T| = \lim \sup\limits_{m\to\infty} \|T\|_m$.

THEOREM 2.7. *If $\mathcal{B} = \mathcal{B}(\|T\|_m)$ is a transitive algebra and contains a non-zero compact operator, then $|T|$ is a norm on $\mathcal{B}$.*

Proof. To show that $|T|$ is a norm it suffices to show that if $|T| = 0$ then $T = 0$. This is equivalent to showing that $\mathbb{Q}(\mathcal{B}) = 0$. If $\mathbb{Q}(\mathcal{B}) \neq 0$ then by Theorem 2.6 $\mathcal{B}$ must be non-transitive. ∎

COROLLARY 2.8. *If $\mathcal{B}(\|T\|_m) = L(X)$ then $|T| = \lim \sup\limits_{m\to\infty} \|T\|_m$ is a norm on $L(X)$.*

We now consider a more concrete family of sequences of norms on $L(X)$. This family was studied in [2] from a completely different perspective and in [13] from a point of view similar to ours. Let $\{D_m\}$ be a sequence of bounded invertible operators on X. We define $\|T\|_m$ for $T \in L(X)$ by $\|T\|_m = \|D_m T D_m^{-1}\|$. Consider

$$\mathcal{B}(\|T\|_m) = \Big\{T \in L(X) : \sup_{m>0} \|T\|_m < \infty\Big\}.$$

For this family of sequences of norms it is easy to see when $\mathcal{B}(\|T\|_m)$ is $L(X)$.

THEOREM 2.9. *$\mathcal{B} = \mathcal{B}(\|T\|_m) = L(X)$ if and only if $\sup\limits_m \|D_m\|\|D_m^{-1}\| < \infty$.*

Proof. If $\sup\limits_m \|D_m\|\|D_m^{-1}\| = C$, then obviously for any $T \in L(X)$, $\|D_m T D_m^{-1}\| \leqslant C\|T\|$ and $\mathcal{B} = L(X)$.

If $\mathcal{B} = L(X)$, then for the sequence $G_m(T) = D_m T D_m^{-1}$ on $L(X)$ we have

$$\sup_m \|G_m(T)\| < \infty$$

for all $T \in L(X)$. By the Uniform Boundedness Principle, $\sup\limits_m \|G_m\| < \infty$. This means that there exists $C_1 > 0$ such that $\|D_m T D_m^{-1}\| \leqslant C_1\|T\|$ for all $T \in L(X)$.

In particular for any rank one operator $T = x \otimes f, x \in X, f \in X^*$, we have

$$\|D_m x\| \|D_m^{*-1} f\| \leqslant C_1 \|x\| \|f\|.$$

Since $x \in X$ and $f \in X^*$ are arbitrary, this implies

$$\|D_m\| \|D_m^{*-1}\| = \|D_m\| \|D_m^{-1}\| \leqslant C_1. \quad ∎$$

We now consider the sequence $D_m = I + mA$ for $A \in L(X)$. (Such sequences were considered in [13]).) We denote the corresponding algebra $\mathcal{B}$ by $\mathcal{R}_A$:

$$\mathcal{R}_A = \Big\{ T \in L(X) : \sup_{m>0} \|(I + mA)T(I + mA)^{-1}\| < \infty \Big\}.$$

Of course, we suppose that all the operators $I + mA$ are invertible. We exclude the trivial case $A = 0$.

THEOREM 2.10. *$\mathcal{R}_A = L(X)$ if and only if A is invertible.*

Proof. By the previous theorem, $\mathcal{R}_A = L(X)$ if and only if

$$\sup_m \{ \|(I + mA)\| \|(I + mA)^{-1}\| \} < \infty.$$

Since $m\|A\| - 1 \leqslant \|I + mA\| \leqslant 1 + m\|A\|$, this is equivalent to $\sup_m \{ m\|(I + mA)^{-1}\| \} < \infty$, or,

$$\sup_m \Big\| \Big(A + \frac{1}{m} I \Big)^{-1} \Big\| < \infty,$$

and this is equivalent to the invertibility of A. ∎

Since $(A)' \subseteq \mathcal{R}_A$ it is of interest to ask: when is $\mathcal{R}_A$ minimal? This is much more difficult to answer.

PROPOSITION 2.11. *If A is nilpotent then $\mathcal{R}_A = (A)'$.*

Proof. Suppose $A^l = 0$. Then $(I + mA)^{-1} = \sum_{k=0}^{l-1} m^k A^k (-1)^k$, and

$$(I + mA)T(I + mA)^{-1} = T - m[T, A] + m^2 S_2 + \cdots + m^l S_l$$

where $S_j = (-1)^j (TA^j - ATA^{j-1})$. If $T \in \mathcal{R}_A$, then $\{ \|(I + mA)T(I + mA)^{-1}\| \}$ is bounded and therefore

$$S_l = S_{l-1} = \cdots = S_2 = [T, A] = 0,$$

so $T \in (A)'$. ∎

For $\dim X < \infty$ it is easy to see that the converse holds as well. If $\mathcal{R}_A = (A)'$, then A is nilpotent. For X infinite dimensional the situation is more complicated. We describe a class of non-nilpotent quasinilpotent operators for which $\mathcal{R}_A = (A)'$ and another class of quasinilpotent operators for which $\mathcal{R}_A$ is quite large. We begin with the second class.

LEMMA 2.12. *Suppose $A \in L(X)$ such that $\{\|(I+mA)^{-1}\|\}$ is bounded. Then*

$$L(X)A \subseteq \mathcal{R}_A.$$

Proof. Suppose $T = SA$ for any $S \in L(X)$. Then

$$(I+mA)T(I+mA)^{-1} = T(I+mA)^{-1} + mAT(I+mA)^{-1}.$$

Since the first term is assumed to be bounded, $T \in \mathcal{R}_A$ if and only if $\{\|mAT(I+mA)^{-1}\|\}$ is bounded. But

$$\begin{aligned}\|mAT(I+mA)^{-1}\| &= \|ASmA(I+mA)^{-1}\| \\ &= \|AS[I-(I+mA)^{-1}]\| \leqslant \|AS\| + \|AS\|\|(I+mA)^{-1}\|. \quad \blacksquare\end{aligned}$$

LEMMA 2.13. *Suppose $A \in L(X)$ such that* $\mathrm{Ker}A = \{0\}$, *and suppose $\{\|(I+mA)^{-1}\|\}$ is bounded. Then $\mathcal{R}_A$ is a transitive algebra.*

Proof. Let M be any proper closed subspace of X and x_0 a non-zero vector in M. Since $Ax_0 \neq 0$, for any $y \notin M$ there exists $S \in L(X)$ such that $S(Ax_0) = y$. Since $L(X)A \subseteq \mathcal{B}$, it follows that M is not an invariant subspace for $\mathcal{B}$. ∎

An operator A on a Hilbert space H is *accretive* if $A + A^* \geqslant 0$. It is known (see, e.g., Inequality V.3.38 of [11]) that for A accretive, $\|(I+\lambda A)^{-1}\| \leqslant 1$ for all $\lambda > 0$.

The following follows immediately from Lemma 2.13.

THEOREM 2.14. *If A is an accretive operator on H with* $\mathrm{Ker}A = \{0\}$, *then $\mathcal{R}_A$ is a transitive algebra.*

COROLLARY 2.15. *If A is accretive, compact and quasinilpotent with* $\mathrm{Ker}A = \{0\}$, *then $(A)'$ is properly contained in $\mathcal{R}_A$.*

Proof. By Lomonosov's Theorem $(A)'$ is not transitive and by the previous theorem $\mathcal{R}_A$ is transitive. ∎

REMARK 2.16. (i) The best known example of such an operator is of course the Volterra operator defined on $L^2(0,1)$ by $(Vf)(x) = \int\limits_0^x f(t)\mathrm{d}t$.

(ii) If A is not compact, nothing is known about the transitive algebra $\mathcal{R}_A$. It is not necessary closed and its weak closure may very well be $L(X)$.

(iii) It is easy to see that for any $A \neq 0$ such that $\mathrm{Ker}A \neq \{0\}$, $\mathrm{Ker}A$ is a non-trivial invariant subspace for R_A.

Now we describe a large class of compact quasinilpotent operators A on a Hilbert space H for which $\mathcal{R}_A = (A)'$. This construction uses some ideas from [21], [17] and needs some results from complex analysis.

LEMMA 2.17. *Let $f(z)$ be an entire function such that*

$$\ln \max_{|z|=r} |f(z)| = o(r)(r \to +\infty).$$

If $\sup_{n\in\mathbb{N}} |f(n)| < \infty$, *then* $\sup_{x\in\mathbb{R}_+} |f(x)| < \infty$.

This lemma is a weak version of a theorem of Cartwright (see, e.g., Theorem 10.2.1 of [1]).

LEMMA 2.18. *Let $G(z)$ be an entire function with values in a Banach space Y. If* $\ln \max_{|z|=r} \|G(z)\| = o(r^{1/2})$ $(r \to +\infty)$, *and* $\sup_{n\in\mathbb{N}} \|G(n)\| < \infty$, *then $G(z)$ is a constant.*

Proof. Let h be an arbitrary functional from Y^* and $g(z) = h(G(z))$. Lemma 2.17 implies that $\sup_{x\in\mathbb{R}_+} |g(x)| < \infty$. Since $\ln \max_{|z|=r} |g(z)| = o(r^{1/2})$, a standard application of the Phragmen-Lindelöf theorem to the angle 2π (i.e. to the whole plane) shows that the entire function g is bounded and thus constant. Since $h \in Y^*$ is arbitrary, $G(z)$ is a constant vector. ∎

For a compact operator A on H, denote by $\{s_j(A)\}$ the sequence of its singular values, i.e. the sequence of all eigenvalues of the compact selfadjoint operator $(A^*A)^{1/2}$ (for details see, e.g., Chapter 6 of [9]).

THEOREM 2.19. *Let A be a compact quasinilpotent operator with*

$$\sum_{j=1}^{\infty} s_j^{1/2}(A) < \infty.$$

Then $\mathcal{R}_A = (A)'$.

Proof. Let $T \in \mathcal{R}_A$. This means that the operator valued entire function

$$G(z) = (I + zA)T(I + zA)^{-1}$$

satisfies $\sup_{n\in\mathbb{N}} \|G(n)\| < \infty$. The condition on the sequence $s_j(A)$ implies that

$$\ln \max_{|z|=r} \|(I + zA)^{-1}\| = o(r^{1/2})(r \to +\infty)$$

(see, e.g., Theorem X.2.2 of [9]). Hence, also $\ln \max_{|z|=r} \|G(z)\| = o(r^{1/2})$ $(r \to +\infty)$. It now follows from Lemma 2.19 (for the space $Y = L(H)$) that $G(z)$ is a constant. Taking $z = 0$ and $z = 1$ gives $(I + A)T(I + A)^{-1} = T$, or $TA = AT$. ∎

3. THE DEDDENS ALGEBRA

Here we consider another example where the sequence $\{D_m\}$ is determined by a single operator. Let $A \in L(X)$ be invertible and let $D_m = A^m$. We consider

$$\mathcal{D}_A = \Big\{ T \in L(X) : \sup_{m>0} \|A^m T A^{-m}\| < \infty \Big\}.$$

The question: when is $\mathcal{D}_A = L(X)$?, was resolved for X a Hilbert space in [19], [7]. $\mathcal{D}_A = L(X)$ if and only if $A = cU$ where U is (similar to) a unitary operator. This can be extended to the Banach space case in the following sense: $\mathcal{D}_A = L(X)$ if and only if $A = cU$ where U is an isometry in some norm $\|\cdot\|'$ on X which is equivalent to the initial norm. Indeed, the argument given in [7] shows that $\mathcal{D}_A = L(X)$ implies that $A = cU$ where $\sup\limits_{k\in\mathbb{Z}} \|U^k\| < \infty$. Define

$$\|x\|' = \sup_{k\in\mathbb{Z}} \|U^k x\|.$$

Then $\|Ux\|' = \|x\|'$ for all $x \in X$ (cf. Proposition 10 of [5]). Of course, the result in Hilbert space X is much stronger, since the famous result of Sz-Nagy [20] gives a norm $\|\cdot\|'$ in X which is also generated by an inner product.

We consider the minimality issue: when is $\mathcal{D}_A = (A)'$? The next proposition was proved in [6]. Another proof was proposed by Williams [21] (see also Theorem 0.4 of [17]). We give here a new proof of the proposition.

PROPOSITION 3.1. *Let $A = I + N$ where $N^{l+1} = 0$. Then $\mathcal{D}_A = (A)'$.*

Proof. Since $A^{-m} = (I+N)^{-m}$, we have that for $m > l$ and $T \in L(X)$,

$$A^m T A^{-m} = \sum_{k=0}^{l} \binom{m}{k} N^k T \sum_{j=0}^{l} \binom{-m}{j} N^j = T + mS_1 + m^2 S_2 + \cdots + m^{2l} S_{2l}$$

where $\{S_k\}_1^{2l}$ depend only on N and T (and not on m). In particular, if $[R, T] = RT - TR$,

$$S_1 = \Big[N - \frac{1}{2}N^2 + \frac{1}{3}N^3 + \cdots + (-1)^{l+1}\frac{N^l}{l}, T\Big].$$

Since for $T \in \mathcal{D}_A$, $\|A^m T A^{-m}\| \leqslant C$ for all m, all the operators S_k are 0. In particular, $S_1 = 0$. If

$$R = N - \frac{1}{2}N^2 + \frac{1}{3}N^3 + \cdots + (-1)^{l+1}\frac{N^l}{l},$$

then in fact $R = \ln(I+N)$ where $\ln(1+z)$ is defined to be the power series $\sum\limits_{k=0}^{\infty} (-1)^{k+1}\frac{z^k}{k}$ for $|z| < 1$. Hence $N = \exp(R)$ and $[R, T] = 0$ implies $[A, T] = 0$. ∎

COROLLARY 3.2. *For $A = I + N$ with N nilpotent, $\mathbb{Q}(\mathcal{D}_A) = \{0\}$.*

Proof. If $T \in \mathcal{D}_A$ then $AT = TA$, so, for all m,

$$\|A^m T A^{-m}\| = \|T\|. \quad ∎$$

Let X be a Banach space and X^* its dual space. Following [16], pp. 214–215, X^* will denote the set of functionals complex conjugated to the linear functionals on X. Hence for $x \in X, f \in X^*, \lambda \in \mathbb{C}$,

$$\lambda(x \otimes f) = (\lambda x \otimes f) = x \otimes (\overline{\lambda} f).$$

REMARK 3.3. (i) It was pointed out in [6] that if $\dim X < \infty$, then the converse of Proposition 3.1 holds as well. Here we give an alternate proof of this fact. If $A \in L(X)$ is invertible but not of the form $\lambda I + N$ with N nilpotent, it has at least two non-zero distinct eigenvalues. So suppose $x \in X$ and $f \in X^*$ are non-zero such that $Ax = \lambda x$, $A^* f = \overline{\mu} f$ with $|\lambda| \leqslant |\mu|, \lambda \neq \mu$. Then for $T = x \otimes f$,

$$\|A^m T A^{-m}\| = \|A^m x\| \|A^{*-m} f\| = \left|\frac{\lambda}{\mu}\right|^m \|x\| \|f\| \leqslant \|x\| \|f\|.$$

So $T \in \mathcal{D}_A$. On the other hand, $AT = Ax \otimes f = \lambda T$ but $TA = x \otimes A^* f = \mu T$, so $T \notin (A)'$.

(ii) If $A \in L(X)$ is invertible, then $(A^l)' \subseteq \mathcal{D}_A$ for any $l \in \mathbb{N}$. For if $T \in (A^l)'$,

$$\sup_m \|A^m T A^{-m}\| = \max\{\|A^k T A^{-k}\| : 0 \leqslant k < l\} < \infty.$$

Since $(A)' \subseteq (A^l)'$, if there exists $l > 1$ such that (A^l) properly contains $(A)'$, then $\mathcal{D}_A$ properly contains $(A)'$.

It was conjectured in [7] that $\mathcal{D}_A = (A)'$ if and only if A is quasisimilar to an operator of the form $\alpha I + Q$ with $\alpha \neq 0$ and Q quasinilpotent. Roth [17] disproved this conjecture in both directions. We give here some additional examples of operators of the form $A = I + Q$ for which $\mathcal{D}_A \neq (A)'$.

LEMMA 3.4. *Suppose $A = \left[\begin{smallmatrix} C & 0 \\ 0 & B \end{smallmatrix}\right]$ is a non-scalar invertible operator on $X \oplus X$ such that $\sup_{n>0} \|B^n\| \|C^{-n}\| < \infty$. Then $\mathcal{D}_A \neq (A)'$.*

Proof. Let S be an arbitrary operator on X and $T = \left[\begin{smallmatrix} 0 & 0 \\ S & 0 \end{smallmatrix}\right]$. Since $A^n T A^{-n} = \left[\begin{smallmatrix} 0 & 0 \\ B^n S C^{-n} & 0 \end{smallmatrix}\right]$, the condition $\sup_{n>0} \|B^n\| \|C^{-n}\| < \infty$ implies that $T \in \mathcal{D}_A$. If $\mathcal{D}_A = (A)'$ this implies $BS = SC$. For $S = I$ this gives $C = B$, and since S is arbitrary, $(B)' = L(X)$. By Schur's lemma, $C = B = \lambda I$. ∎

Now suppose $X = H$ is a Hilbert space.

THEOREM 3.5. *Let R_1, R_2 be accretive operators, and*

$$A = \begin{bmatrix} I - 2R_1(I + R_1)^{-1} & 0 \\ 0 & I + R_2 \end{bmatrix}.$$

If A is a non-scalar operator, then $\mathcal{D}_A \neq (A)'$.

Proof. We mentioned in the previous section that for an accretive operator R

$$\|(I + \lambda R)^{-1}\| \leqslant 1$$

for all $\lambda > 0$. In particular, $\|(I + R_2)^{-1}\| \leqslant 1$. On the other hand, the Cayley transform $(I - R_1)(I + R_1)^{-1} = I - 2R_1(I + R_1)^{-1}$ of the accretive operator R_1 is a contraction, so we can apply Lemma 3.4. ∎

Now suppose that R_1 and R_2 are accretive and quasinilpotent. For example, take $R_1 = R_2 = V$ where V is the Volterra operator mentioned above. More generally one can consider R_j defined by $(R_j f)(x) = \int\limits_0^x k_j(x,t)f(t)\mathrm{d}t$ where $k_j(x,t), 0 \leqslant x,t \leqslant 1, j = 1,2$ are Hermitian positive square integrable functions. For such operators, A is of the form $I + Q$ where Q is quasinilpotent. By Theorem 3.5, $\mathcal{D}_A$ properly contains $(A)'$. The example of Roth [17] is the special case where $R_1 = 0, R_2 = V$.

A related question is: when does $\mathcal{D}_A = \mathcal{D}_{A^{-1}}$?

EXAMPLE 3.6. Let $A = \left[\begin{smallmatrix} I & 0 \\ 0 & \lambda I \end{smallmatrix}\right]$ where $0 \neq |\lambda| < 1$. Then for $T = \left[\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{smallmatrix}\right]$, $A^m T A^{-m} = \left[\begin{smallmatrix} T_{11} & \lambda^{-m} T_{12} \\ \lambda^m T_{21} & T_{22} \end{smallmatrix}\right]$. So

$$\mathcal{D}_A = \left\{ T = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix} \right\}, \quad \mathcal{D}_{A^{-1}} = \left\{ T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \right\}.$$

Also $(A)' = \left\{ T = \left[\begin{smallmatrix} T_{11} & 0 \\ 0 & T_{22} \end{smallmatrix}\right] \right\}$, and hence, $(A)' = \mathcal{D}_A \cap \mathcal{D}_{A^{-1}}$, $(A)' \neq \mathcal{D}_A$, $(A)' \neq \mathcal{D}_{A^{-1}}$, $\mathcal{D}_A \neq \mathcal{D}_{A^{-1}}$.

Since in general $(A)' \subseteq \mathcal{D}_A \cap \mathcal{D}_{A^{-1}}$, it follows that whenever $\mathcal{D}_A \neq \mathcal{D}_{A^{-1}}$ then either $(A)' \neq \mathcal{D}_A$ or $(A)' \neq \mathcal{D}_{A^{-1}}$, or both.

A similar example shows that, in general, for $A - \lambda I$ invertible, $\mathcal{D}_A \neq \mathcal{D}_{A-\lambda I}$, and when this is the case, either $(A)' \neq \mathcal{D}_A$ or $(A)' \neq \mathcal{D}_{A-\lambda I}$ or both.

We recall that Williams [21] proved the equality $(A)' = \mathcal{D}_A \cap \mathcal{D}_{A^{-1}}$ for $A = \alpha I + Q$ where $\alpha \neq 0$ and Q is quasinilpotent.

REMARK 3.7. (i) If A is normal invertible with polar decomposition $A = U|A|$, then (by [6]) $\mathcal{D}_A = \mathcal{D}_{|A|}$ is a nest algebra determined by completing the nest $\{E([0,a])H : a > 0\}$, where E is the spectral measure for $|A|$. Thus for A normal, $\mathcal{D}_A = (A)'$ if and only if $A = \lambda I$ for some $\lambda \in \mathbb{C}$. Also, Example 3.6 is an example of $\mathcal{D}_A$ for $A \neq \lambda I$ normal.

(ii) In order to apply Theorem 2.6 to $\mathcal{D}_A$ we must know when $\mathbb{Q}(\mathcal{D}_A) \neq \{0\}$. The answer to this question seems to be quite difficult even for $A = I + K$ with K compact (where $K \in \mathcal{D}_A$ and hence condition (i) of Theorem 2.6 holds).

(iii) Suppose A is invertible such that $Ax = \lambda x, A^* f = \mu f$ with $|\lambda| < |\mu|$. It is easy to see that in this case $T = x \otimes f \in \mathbb{Q}(\mathcal{D}_A)$.

4. OPERATOR NORMS INDUCED BY VECTOR NORMS

We now consider the situation where we are given a sequence $\{\|x\|_m\}$ of vector norms on the Banach space X which are equivalent to the given norm on X. Each such norm induces an operator norm on $L(X)$ equivalent to the given induced norm on $L(X)$. We will denote this operator norm by $\|T\|_m$ for $T \in L(X)$.

We consider the algebra

$$\mathcal{B} = \mathcal{B}(\|x\|_m) = \Big\{T \in L(X) : \sup_{m>0} \|T\|_m < \infty\Big\}.$$

In fact the sequence $\{\|D_m T D_m^{-1}\|\}$ of operator norms considered in the previous sections can be viewed in this way. They are simply the operator norms induced by the sequence $\{\|x\|_m = \|D_m x\|\}$ of vector norms on X. We note that in this case for $\{\|x\|_m\}$ to define a norm on X equivalent to the given norm it is enough that D_m be injective with closed range for each m. Thus, in particular the Deddens' algebra $\mathcal{D}_A$ can be considered in the more general case where A is such a non-invertible operator and the operator norms determining $\mathcal{D}_A$ are those induced by the sequence $\{\|x\|_m = \|A^m x\|\}$.

We are interested when $\mathcal{B} = \mathcal{B}(\|x\|_m) = L(X)$. For $v \in X^*$, we define

$$\|v\|_{-m} = \sup_m \{|v(x)| : x \in X, \|x\|_m = 1\}.$$

LEMMA 4.1. *For a vector $u \in X$ and a functional $v \in X^*$,*

$$u \otimes v \in \mathcal{B}(\|x\|_m) \Leftrightarrow \sup_m \|u\|_m \|v\|_{-m} < \infty$$

and

$$\lim_{m\to\infty} \|u \otimes v\|_m = 0 \Leftrightarrow \lim_{m\to\infty} \|u\|_m \|v\|_{-m} = 0.$$

Proof. For $x \in X$,

$$\|(u \otimes v)x\|_m = \|v(x)u\|_m = |v(x)|\|u\|_m$$

and

$$\|(u\otimes v)\|_m = \sup_m \{|v(x)|\|u\|_m : \|x\|_m = 1\} = \|u\|_m \sup_m \{|v(x)| : \|x\|_m = 1\} = \|u\|_m \|v\|_{-m}.$$

Both the equivalences now follow from the definition of $\mathcal{B}(\|x\|_m)$. ∎

THEOREM 4.2. *The following are equivalent:*

(i) $\mathcal{B}(\|x\|_m) = L(X)$.

(ii) $\mathcal{B}(\|x\|_m)$ *contains all rank-one operators from* $L(X)$.

(iii) *There exists $C > 0$ such that, for all $u \in X, v \in X^*$,*

$$\sup_m \|u\|_m \|v\|_{-m} \leqslant C\|u\|\|v\|.$$

Proof. That (i) implies (ii) is trivial. We first show (ii) implies (iii). By the previous lemma, $u \otimes v \in \mathcal{B}(\|x\|_m)$ means $\sup_m \|u\|_m \|v\|_{-m} < \infty$. Fix $v \in X^*$ and define $\varphi_m(u) = \|u\|_m \|v\|_{-m}$ for $u \in X$. (ii) implies that $\sup_m |\varphi_m(u)| < \infty$ for all $u \in X$. Thus by the Uniform Boundedness Principle for semi-additive functionals ([10], Theorem 2.5.4) there exists $C(v) > 0$ such that for all $m \in \mathbb{N}$, and $u \in X$,

$\varphi_m(u) \leqslant C(v)\|u\|$. Thus $\|u\|_m\|v\|_{-m} \leqslant C(v)\|u\|$. For $0 \neq u \in X$, define $\psi_{m,u}$ on X^* by

$$\psi_{m,u}(v) = \frac{\|u\|_m\|v\|_{-m}}{\|u\|}.$$

Then $\sup\limits_{m,u} \psi_{m,u} \leqslant C(v)$ for all $v \in X^*$ and, using uniform boundedness again, we obtain $C > 0$ such that $\sup\limits_{m,u} \psi_{m,u}(v) \leqslant C\|v\|$ for all $m \in \mathbb{N}, u \in X, v \in X^*$; that is,

$$\sup_m \|u\|_m\|v\|_{-m} \leqslant C\|u\|\|v\|.$$

We now show that (iii) implies (i). Let $T \in L(X)$, and let $x \in X$ be non-zero. By the definition of $\|v\|_{-m}$,

$$\|v\|_{-m} \geqslant \frac{|v(x)|}{\|x\|_m}$$

for all $v \in X^*$. Thus for $u = Tx$ (iii) implies that

$$\|Tx\|_m\frac{|v(x)|}{\|x\|_m} \leqslant C\|Tx\|\|v\|$$

for all $v \in X^*$. Using Hahn-Banach, we choose $v \in X^*$ such that $v(x) = \|x\|$, $\|v\| = 1$. Then,

$$\|Tx\|_m\frac{\|x\|}{\|x\|_m} \leqslant C\|Tx\|, \quad \text{or} \quad \|Tx\|_m\|x\| \leqslant C\|Tx\|\|x\|_m \leqslant C\|T\|\|x\|\|x\|_m.$$

Thus $\|Tx\|_m \leqslant C\|T\|\|x\|_m$ for all $m \in \mathbb{N}$ and $T \in \mathcal{B}(\|x\|_m)$. ∎

Fix $A \in L(X)$ and let $\{w_{mk}\}$ $(m = 1,2,\ldots; k = 0,1,\ldots)$ be an array of non-negative numbers such that for each m the following conditions hold:

(a) $w_{mk} > 0$ for at least one k;

(b) $\sum\limits_{k=0}^{\infty} w_{mk}^2\|A^k\|^2 < \infty$:

(c) either $w_{m0} > 0$ or A is injective with closed range.

Then

$$\|x\|_m = \Big[\sum_{n=0}^{\infty} w_{mk}^2\|A^k x\|^2\Big]^{1/2}$$

defines, for each m, a norm on X equivalent to the initial norm. We consider the subalgebra $\mathcal{B} = \mathcal{B}(\|x\|_m)$ of $L(X)$ determined by $\{\|x\|_m\}_1^\infty$. The Deddens algebra discussed above is the special case when A is invertible (or injective with closed range; see the beginning of this section) and $w_{mk} = \delta_{mk}$. The spectral radius algebra of [12], which will be studied here at length, also belongs to this framework. Since for these algebras $\mathcal{B}$ is determined by a single operator, we will use the notations $\mathcal{B}_A$ and $\mathbb{Q}(\mathcal{B}_A)$.

The proof of the next theorem is similar to that of Proposition 2.3 of [12].

THEOREM 4.3. *Let $B \in L(X)$ and $C > 0$ such that $\|B^n\| \leqslant C$ for all $n \in \mathbb{N}$ and $B \in (A)'$. If $T \in L(X)$ such that $AT = BTA$, then $T \in \mathcal{B}_A$. In particular, this holds for $B = \lambda I$, with $|\lambda| \leqslant 1$, and thus $(A)' \subseteq \mathcal{B}_A$.*

Proof. We have

$$\|Tx\|_m^2 = \sum_{k=0}^{\infty} (w_{mk}^2 \|A^k Tx\|)^2 = \sum_{k=0}^{\infty} (w_{mk} \|B^k TA^k x\|)^2 \leqslant C^2 \|T\|^2 \sum_{k=0}^{\infty} (w_{mk} \|A^k x\|)^2$$
$$= C^2 \|T\|^2 \|x\|_m^2.$$

So $\|T\|_m \leqslant C\|T\|$ for all $m \in \mathbb{N}$ and $T \in \mathcal{B}_A$. ∎

EXAMPLE 4.4. Let $r = r(A)$ denote the spectral radius of A and let $\{\varepsilon_m\}$ be a sequence of positive numbers. Then for $w_{mk} = \left(\frac{1}{r(A)+\varepsilon_m}\right)^k$, the series $\sum\limits_{k=0}^{\infty} [w_{mk} \|A^k\|]^2$ converges, and, for each $m \in \mathbb{N}$, the equation

$$\|x\|_m^2 = \sum_{k=0}^{\infty} [w_{mk} \|A^k x\|]^2$$

defines a norm on X which is equivalent to the given norm on X. Thus we can consider the algebra $\mathcal{B}_A \supseteq (A)'$ determined by the sequence of induced operator norms on $L(X)$. If $\varepsilon_m = \frac{1}{m}$, X is a Hilbert space, and we denote $\frac{1}{r(A)+\frac{1}{m}} = \frac{m}{1+mr(A)}$ by d_m, then

$$\|x\|_m^2 = \sum_{k=0}^{\infty} d_m^{2k} \|A^k x\|^2 = \sum_{k=0}^{\infty} d_m^{2k} (A^{*k} A^k x, x) = (R_m^2 x, x) = \|R_m x\|^2,$$

where $R_m = R_m(A) = \Big(\sum\limits_{k=0}^{\infty} d_m^{2k} A^{*k} A^k \Big)^{1/2} > 0$, and

$$\|T\|_m = \sup\{\|R_m Tx\| : \|R_m x\| = 1\} = \sup\{\|R_m T R_m^{-1} y\| : \|y\| = 1\} = \|R_m T R_m^{-1}\|.$$

Thus

$$\mathcal{B}_A = \Big\{ T \in L(X) : \sup_m \|R_m T R_m^{-1}\| < \infty \Big\}.$$

This is the spectral radius algebra studied in [12]. We note that in the Hilbert space case, $\mathcal{B}_A$ belongs to the class of algebras studied in Section 2 of this paper.

While formally $\mathcal{D}_A$ was defined for A invertible the meaning of the next proposition is clear.

PROPOSITION 4.5. *If A is injective with closed range, then $\mathcal{D}_A \subseteq \mathcal{B}_A$.*

Proof. If $T \in \mathcal{D}_A$, then $\|A^k Tx\| \leqslant C\|A^k x\|$ for all $k \geqslant 0$ and $x \in X$. Therefore

$$\|Tx\|_m^2 = \sum_{k=0}^{\infty} d_m^{2k} \|A^k Tx\|^2 \leqslant C^2 \sum_{k=0}^{\infty} d_m^{2k} \|A^k x\|^2 = C^2 \|x\|_m^2. \quad ∎$$

REMARK 4.6. (i) The main result of [7] is that for A normal and invertible, $\mathcal{D}_A$ is the nest algebra determined by the spectral measure of the positive factor in the polar decomposition of A. In particular, $(A)'$ is properly contained in $\mathcal{D}_A$ and therefore in $\mathcal{B}_A$. The main result of [3] is that for any non-scalar normal operator A, $(A)'$ is properly contained in $\mathcal{B}_A$.

(ii) It can be shown that in general neither of the inclusions $\mathbb{Q}(\mathcal{D}_A) \subseteq \mathbb{Q}(\mathcal{B}_A)$, $\mathbb{Q}(\mathcal{B}_A) \subseteq \mathbb{Q}(\mathcal{D}_A)$ hold.

In the next section we will study $\mathcal{B}_A$ in greater detail.

5. SPECTRAL RADIUS ALGEBRAS

We saw in the previous section that for $A \in L(X)$, and $d_m = \frac{m}{1+mr(A)}$, we have, for each $m \in \mathbb{N}$, that

$$\|x\|_m^2 = \sum_{k=0}^{\infty} d_m^{2k} \|A^k x\|^2 \geqslant \|x\|^2$$

defines a norm on X equivalent to the original norm. It is of interest to note that for each $A \neq 0$ there exists a non-zero vector $x \in X$ such that $\lim\limits_{m\to\infty} \|x\|_m = \infty$. This statement can be proved by an argument similar to that given in Proposition 3.11 of [12].

Each of these norms induces an operator norm $\|T\|_m$ on $L(X)$ equivalent to the initial induced norm and the *spectral radius algebra* determined by A is

$$\mathcal{B}_A = \Big\{ T \in L(X) : \sup_{m>0} \|T\|_m < \infty \Big\}.$$

By Corollary 2.2, $\mathcal{B}_A$ is a Banach algebra with respect to the norm

$$|||T||| = \sup_m \|T\|_m$$

for $T \in \mathcal{B}_A$, and as we saw, $(A)' \subseteq \mathcal{B}_A$.

We extend Proposition 1.5 of [12] to this framework.

PROPOSITION 5.1. $\|A\|_m \to r(A)$ *as* $m \to \infty$.

Proof. We have

$$\|d_m A\|_m = \sup\{\|d_m Ax\|_m : \|x\|_m = 1\} = \sup\Big\{\Big(\sum_{k=0}^{\infty} d_m^{2(k+1)} \|A^{k+1}\|^2\Big)^{1/2} : \|x\|_m = 1\Big\}$$
$$= \sup\{(\|x\|_m^2 - \|x\|^2)^{1/2} : \|x\|_m = 1\} \leqslant 1.$$

Thus $\|A\|_m \leqslant \frac{1}{d_m}$ for all $m \in \mathbb{N}$, and $\lim \sup\limits_{m\to\infty} \|A\|_m \leqslant \lim \sup\limits_{m\to\infty} \frac{1}{d_m} = r(A)$. On the other hand, $r(A) \leqslant \|A\|_m$ for all $m \in \mathbb{N}$, and therefore

$$\lim \inf_{m\to\infty} \|A\|_m \geqslant r(A).$$

So $\|A\|_m \to r(A)$ as $m \to \infty$. ∎

We consider some examples.

EXAMPLE 5.2. (i) It was shown in [3] that for X a Hilbert space, $\mathcal{B}_A = L(X)$ if and only if $A = aU$ where U is (similar to) an isometry. The sufficiency extends immediately to Banach space and continues to hold when U is an isometry with respect to an equivalent norm on X. Thus if $A \in L(X)$ is such that for some positive integer j and $a \neq 0$, $A^j = aU$ where U is an isometry, then $A = a^{1/j}V$ where V is an isometry with respect to the equivalent norm

$$\|x\|' = \Big(\sum_{k=0}^{j-1} \|(a^{-1/j}A)^k x\|^2\Big)^{1/2},$$

so $\mathcal{B}_A = L(X)$. The situation is different for $a = 0$. Indeed suppose that for some $j > 1$, $A^j = 0$, $A^{j-1} \neq 0$. Then for $0 \neq x \in \operatorname{Ran} A^{j-1}$, $x = A^{j-1}y$ for some $y \in X$, $x \in \operatorname{Ker} A$, and $\|x\|_m = \|x\|$ for all $m \in \mathbb{N}$. Choose $f \in X^*$ such that $f(x) = 1$ and let $T = y \otimes f$. Then

$$\|Tx\|_m^2 = \sum_{k=0}^{j-1} m^{2k}\|A^k y\|^2\|x\|^2 \geqslant m^{2(j-1)}\|x\|^2 \to \infty$$

as $m \to \infty$. Thus $T \notin \mathcal{B}_A$ and $\mathcal{B}_A \neq L(X)$.

(ii) Let X be a Banach space with direct sum decomposition $X = Y \oplus Z$. Let A be an operator on X with matrix representation

$$A = \begin{bmatrix} \rho U & 0 \\ 0 & B \end{bmatrix},$$

where $\rho \neq 0, r(B) < |\rho|$, and U is an isometry. For simplicity, and without loss of generality, we assume that $\rho = 1$ and $\|y+z\|^2 = \|y\|^2 + \|z\|^2$ for $y \in Y, z \in Z$. Then

$$\|x\|_m^2 = \sum_{n=0}^{\infty} d_m^{2n}(\|y\|^2 + \|B^n z\|^2)$$

where $d_m = \frac{m}{m+1}$. Represent $T \in L(X)$ as the operator matrix

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}.$$

We show that $T \in \mathcal{B}_A$ if and only if $T_{12} = 0$, or equivalently, that Z is an invariant subspace for T.

Suppose $T \in \mathcal{B}_A$. Then there exists $C > 0$ such that for each $x \in X$ and $m \in \mathbb{N}$,

$$\|Tx\|_m \leqslant C\|x\|_m.$$

If $z \in Z$ and $\|z\| = 1$, we have

$$\sum_{n=0}^{\infty} d_m^{2n}(\|T_{12}z\|^2 + \|B^n T_{22}z\|^2) \leqslant C \sum_{n=0}^{\infty} d_m^{2n}\|B^n z\|^2.$$

Since $r(B) < 1$, the series $\sum_{n=0}^{\infty} \|B^n\|^2$ converges, so if $b = \sum_{n=0}^{\infty} \|B^n\|^2$, then

$$\|T_{12}z\|^2 \sum_{n=o}^{\infty} d_m^{2n} \leqslant Cb.$$

Since

$$\sum_{n=0}^{\infty} d_m^{2n} = \sum_{n=0}^{\infty} \left(\frac{m}{m+1}\right)^{2n} = \frac{(m+1)^2}{2m+1} \to \infty$$

as $m \to \infty$, it follows that $T_{12}z = 0$. Since $z \in Z$ was arbitrary, $T_{12} = 0$.

Now suppose $T_{12} = 0$. Then for $x = y + z$ with $y \in Y, z \in z$,

$$\sum_{n=0}^{\infty} d_m^{2n} \|A^n T x\|^2 = \sum_{n=0}^{\infty} d_m^{2n} (\|T_{11}y\|^2 + \|B^n(T_{21}y + T_{22}z)\|^2).$$

Obviously

$$\sum_{n=0}^{\infty} d_m^{2n} \|T_{11}y\|^2 \leqslant \|T_{11}\|^2 \sum_{n=0}^{\infty} d_m^{2n} \|y\|^2 \leqslant \|T_{11}\|^2 \|x\|_m^2,$$

and, since $d_m < 1$,

$$\begin{aligned}\sum_{n=0}^{\infty} d_m^{2n} \|B^n(T_{21}y+T_{22}z)\|^2 &\leqslant \sum_{n=0}^{\infty} \|B^n\|^2 \|T_{21}y+T_{22}z\|^2 \leqslant b(\|T_{21}\|^2+\|T_{22}\|^2)(\|y\|^2+\|z\|^2) \\ &= C_1\|x\|^2 \leqslant C_1 \sum_{n=0}^{\infty} d_m^{2n} \|A^n x\|^2 = C_1 \|x\|_m^2.\end{aligned}$$

Thus

$$\|Tx\|_m^2 = \sum_{n=0}^{\infty} d_m^{2n} \|A^n T x^2\| \leqslant (\|T_{11}\|^2 + C_1)\|x\|_m^2$$

and $T \in \mathcal{B}_A$.

A special case is Example 3.6, where $A = \left[\begin{smallmatrix} I & 0 \\ 0 & \lambda I \end{smallmatrix}\right]$, for $|\lambda| < 1$. In this case

$$\mathcal{D}_A = \mathcal{B}_A, \quad \mathcal{D}_{A^{-1}} = \mathcal{B}_{A^{-1}}.$$

We now turn to the non-minimality question: when does $\mathcal{B}_A$ properly contain $(A)'$?

THEOREM 5.3. *If $A \in L(X)$ is non-zero such that* $\operatorname{Ker} A \neq \{0\}$, *then $\mathcal{B}_A$ properly contains $(A)'$.*

Proof. Suppose $u \in X$ is a non-zero vector such that $Au = 0$, and let $f \in X^*$ such that $A^* f \neq 0$. We show that $F = u \otimes f \in \mathcal{B}_A$. Now

$$\|F\|_m = \sup_{x \neq 0} \frac{\sum_{n=0}^{\infty} d_m^{2n} \|A^n F x\|^2}{\sum_{n=0}^{\infty} d_m^{2n} \|A^n x\|^2}.$$

Since $A^nFx = f(x)A^nu = 0$ for $n \geqslant 1$, we have $\sum\limits_{n=0}^{\infty} d_m^{2n}\|A^nFx\|^2 = \|Fx\|^2 = (|f(x)|\|u\|)^2$. Since

$$\sum_{n=0}^{\infty} d_m^{2n}\|A^nx\|^2 \geqslant \|x\|^2, \quad \|F\|_m^2 \leqslant \sup_{x\neq 0} \frac{|f(x)|^2\|u\|^2}{\|x\|^2} = \|f\|^2\|u\|^2,$$

and $F \in \mathcal{B}_A$. However, $FA = u \otimes A^*f \neq 0$ and $AF = Au \otimes f = 0$, so $F \notin (A)'$. ∎

REMARK 5.4. Since $(A)' \subset \mathcal{D}_A \subset \mathcal{B}_A$ (Proposition 4.5) we obtain from Lemma 3.4 and Theorem 3.5 some examples of invertible A such that $\mathcal{B}_A \neq (A)'$. Here we give some examples of possibly non-invertible operators with this property. Let $A = \left[\begin{smallmatrix} C & 0 \\ 0 & B \end{smallmatrix}\right]$ be an operator on $X \oplus X$ and let $\alpha > 0$ be a number such that $\|B^kx\| \leqslant \alpha\|C^kx\|$ for $x \in X, k \in \mathbb{N}$. It is easy to check that the operator $T = \left[\begin{smallmatrix} 0 & 0 \\ I & 0 \end{smallmatrix}\right]$ belongs to the algebra $\mathcal{B}_A$. On the other hand, $T \in (A)'$ if and only if $B = C$. Hence for an arbitrary pair of different operators, B, C, which satisfy the inequality mentioned above, $\mathcal{B}_A \neq (A)'$. For example, we can choose C arbitrarily and take $B = GC$ where $G \neq I$ is any contraction commuting with C.

In fact, the only example known to us of a bounded operator A such that $\mathcal{B}_A = (A)'$ is $A = \lambda I$. It is easy to show that for X finite dimensional this is the only example.

6. COMPACTNESS AND $\mathcal{B}_A$

The results of this section strengthen slightly and extend to Banach space those of Section 4 of [12]. The main result is that for a compact operator K, $\mathcal{B}_K$ has a non-trivial invariant subspace. Since $(K)' \subseteq \mathcal{B}_K$, this is at least as strong as Lomonosov's theorem [14] and will give a stronger result when $\mathcal{B}_K$ properly contains $(K)'$. This will be a fundamental issue.

LEMMA 6.1. *Suppose $K \in L(X)$ is a compact operator with $r(K) > 0$. Then there exists $f \in X^*$ such that* $\lim\limits_{m\to\infty} \|f\|_{-m} = 0$.

Proof. Let λ_0 be an eigenvalue of K such that $|\lambda_0| = r(K) = r$. Let u be a corresponding normalized eigenvector and f an eigenvector of K^* corresponding to its eigenvalue $\overline{\lambda}_0$. Then, for $F = u \otimes f$,

$$KF = Ku \otimes f = (\lambda_0 u) \otimes f = u \otimes (\overline{\lambda}_0 f) = u \otimes K^*f = FK,$$

so $F \in \mathcal{B}_K$. That is, $\sup\limits_m \|F\|_m < \infty$. This means that

$$\sup_m \sup_{x\neq 0} \frac{\sum_{n=0}^{\infty} d_m^{2n}\|K^nFx\|^2}{\sum_{n=0}^{\infty} d_m^{2n}\|K^nx\|^2} < \infty.$$

But $\sum\limits_{n=0}^{\infty} d_m^{2n}\|K^nFx\|^2 = \sum\limits_{n=0}^{\infty} d_m^{2n}|f(x)|^2 r^{2n} = \frac{|f(x)|^2}{1-(d_m r)^2}$. Since $\lim\limits_{m\to\infty}(1-d_m^2r^2)m = \frac{2}{r}$, we have

$$\lim_{m\to\infty}\|f\|_{-m}^2 = \lim_{m\to\infty}\sup_{x\neq 0}\frac{|f(x)|^2}{\|x\|_m^2} = 0. \quad ∎$$

LEMMA 6.2. *Let $K \in L(X)$ be a compact operator with $r(K) > 0$ and λ an eigenvalue of K such that $|\lambda| = r(K)$. If $f \in X^*$ such that $K^*f = \overline{\lambda}f$ and $w \in X$ such that $\sup\limits_m \|w\|_m < \infty$, then $w \otimes f \in \mathbb{Q}(\mathcal{B}_A)$.*

Proof. Let $F = w \otimes f$. Then

$$\|F\|_m^2 = \sup_{x\neq 0}\frac{\sum_{n=0}^{\infty} d_m^{2n}\|K^nFx\|^2}{\|x\|_m^2}.$$

Since $K^nFx = f(x)K^nw$, $\|K^nFx\| = |f(x)|\|K^nw\|$, and

$$\sum_{n=0}^{\infty} d_m^{2n}\|K^nFx\|^2 = |f(x)|^2\sum_{n=0}^{\infty} d_m^{2n}\|K^nx\|^2 = |f(x)|^2\|w\|_m^2.$$

Thus, as $m \to \infty$,

$$\|F\|_m^2 = \sup_{x\neq 0}\frac{|f(x)|^2}{\|x\|_m^2}\|w\|_m^2 = \|f\|_{-m}^2\|w\|_m^2 \to 0. \quad ∎$$

THEOREM 6.3. *Let K be a non-zero compact operator on X. Then $\mathcal{B}_K$ has a non-trivial invariant subspace.*

Proof. By Theorem 2.7 it suffices to show that $\mathbb{Q}(\mathcal{B}_K) \neq \{0\}$. If $r(K) = 0$, this follows from Proposition 5.1, so assume that $r(K) > 0$. Since the only limit point of the spectrum of K is 0, we can choose $0 < r_0 < r$ and a circle $\Gamma = \{\lambda : |\lambda| = r_0\}$ which contains no points of the spectrum of K. Let $P = -\frac{1}{2\pi i}\int\limits_{\Gamma}(K-\lambda I)^{-1}d\lambda$ be the corresponding Riesz projection with $\mathrm{Ran}P = N$. Then N is invariant for K, and for $K_0 = K|_N$ the spectrum of K_0 will be interior to Γ. Since $\dim X = \infty, N \neq \{0\}$. Let w be a unit vector in N. Then

$$\|K^nw\| = \|K_0^nw\| \leqslant \|K_0^n\|\|w\| = \|K_0^n\|,$$
$$\|w\|_m^2 = \sum_{n=0}^{\infty} d_m^{2n}\|K^nw\|^2 \leqslant \sum_{n=0}^{\infty} d_m^{2n}\|K_0^n\|^2 < \sum_{n=0}^{\infty}\frac{\|K_0^n\|^2}{r^{2n}},$$

since $d_m < \frac{1}{r}$ for all m. Since

$$\lim_{n\to\infty}\Big[\frac{\|K_0\|^2}{r^{2n}}\Big]^{1/n} = \frac{1}{r^2}\lim_{n\to\infty}[\|K_0^n\|^{1/n}]^2 = \frac{[r(K_0)]^2}{r^2} < 1,$$

the series converges and $\sup\limits_m \|w\|_m < \infty$. Thus for an eigenvalue λ of K with $|\lambda| = r$ and $f \in X^*$ such that $K^*f = \overline{\lambda}f$, $w \otimes f \in \mathbb{Q}(B_K)$ by Lemma 6.2. ∎

In order for this result to be stronger than that of Lomonosov we need that $(K)'$ be properly contained in $\mathcal{B}_K$. The next proposition is a little more general than the corresponding Proposition 4.3 of [12].

PROPOSITION 6.4. *If K is a non-zero compact operator on X such that K has an eigenvalue (possibly just zero), then $\mathcal{B}_K$ properly contains $(K)'$.*

Proof. If $\operatorname{Ker} K \neq \{0\}$ the result follows from Theorem 5.3. If $r(K) > 0$ let $F = w \otimes f$ be the operator constructed in Theorem 6.3. Then $F \in \mathcal{B}_K$, but $KF = K_0 w \otimes f$ and

$$FK = w \otimes K^* f = w \otimes \overline{\lambda} f = \lambda w \otimes f.$$

If $KF = FK$, then $K_0 w = \lambda w$. But $|\lambda| = r > r(K_0)$. So $F \notin (K)'$. ∎

REMARK 6.5. Let X be a Banach space for which there exists $A \in L(X)$ which is transitive (see e.g. [8]). It follows from Theorem 6.3 that for any compact operator $K \neq 0$, $A \notin \mathcal{B}_K$. Thus the analogue of Problem 4.7 of [12] is obviously solved in this case.

EXAMPLE 6.6. Let H be a Hilbert space with orthonormal basis $\{e_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ a sequence of complex numbers such that $w_n \to 0$ as $n \to \infty$. Let A be the weighted shift defined on H by $Ae_n = w_n e_{n+1}, n \geqslant 0$. Then [18] A is compact quasinilpotent, and since A is unitarily equivalent to the weighted shift with weights $\{|w_n|\}$, we can assume that $w_n \geqslant 0$ for all $n \in \mathbb{N}$.

In this case $R_m^2(A) = \sum\limits_{n=0}^{\infty} m^{2n} A^{*n} A^n$ is easily computed. It is the diagonal operator (with respect to the given basis)

$$R_m^2 = \operatorname{diag}[r_{mi}]_{i=0}^{\infty} \quad \text{where } r_{mi} = 1 + \sum_{n=1}^{\infty} m^{2n} w_i^2 \cdots w_{i+n-1}^2 \geqslant 1.$$

So $R_m = \operatorname{diag}[(r_{mi})^{1/2}]$ and $R_m^{-1} = \operatorname{diag}\Big[\frac{1}{(r_{mi})^{1/2}}\Big]$. For $T \in L(H)$ with matrix representation $T = [t_{ij}]_{i,j=0}^{\infty}$,

$$R_m T R_m^{-1} = \Big[\Big(\frac{r_{mi}}{r_{mj}}\Big)^{1/2} t_{ij}\Big].$$

Now

$$\frac{r_{mi}}{r_{mj}} = \frac{1 + \sum_{n=1}^{\infty} m^{2n} w_i^2 \cdots w_{i+n-1}^2}{1 + \sum_{n=1}^{\infty} m^{2n} w_j^2 \cdots w_{j+n-1}^2}.$$

If we write $\alpha_j = 1 + \sum\limits_{n=1}^{\infty} m^{2n} w_j^2 \cdots w_{j+n-1}^2$ then we obtain, for $i < j$,

$$\frac{r_{mi}}{r_{mj}} = \frac{1 + m^2 w_1^2 + \cdots + m^{2(j-i)} w_i^2 \cdots w_{j-1}^2 \alpha_j}{\alpha_j} \geqslant m^{2(j-i)} w_i^2 \cdots w_{j-1}^2.$$

Thus $\frac{r_{mi}}{r_{mj}} \to \infty$ as $m \to \infty$. If $t_{ij} \neq 0$ for some $i < j$ then $\left|\left(\frac{r_{mi}}{r_{mj}}\right)^{1/2} t_{ij}\right| \to \infty$ as $m \to \infty$. But

$$\left|\left(\frac{r_{mi}}{r_{mj}}\right)^{1/2} t_{ij}\right| = |(R_m T R_m^{-1} e_j, e_i)| \leqslant \|R_m T R_m^{-1}\|,$$

so if T is not lower triangular, $T \notin \mathcal{B}_A$. Thus every operator in $\mathcal{B}_A$ must be lower triangular.

On the other hand, assume in addition that $\{w_n\}$ is non-increasing. Then it is easily seen that for $j < i$, $\frac{r_{mi}}{r_{mj}} \leqslant 1$. Thus if T is lower triangular and Hilbert-Schmidt,

$$\|R_m T R_m^{-1}\| \leqslant \|R_m T R_m^{-1}\|_2 \leqslant \|T\|_2,$$

so $T \in \mathcal{B}_A$ (here $\|T\|_2$ is the Hilbert-Schmidt norm). In particular $\mathcal{B}_A$ contains all finite rank lower triangular operators, so its weak closure is the nest algebra of all lower triangular operators.

Also $\mathcal{B}_A$ properly contains $(A)'$, for if $T = e_1 \otimes e_0$, T is lower triangular and $T \in \mathcal{B}_A$. But $ATe_0 = w_1 e_2$ and $TAe_0 = 0$ so $T \notin (A)'$.

REMARK 6.7. Conjecture 4.6 of [12] asks if for any compact operator K, $\mathcal{B}_K$ properly contains $(K)'$. We have seen that this question reduces to the case where K is compact quasinilpotent with trivial kernel. [12] shows this to be true for the Volterra operator and we have added compact weighted shifts and some rather artificial examples in Remark 5.4. The answer to the general conjecture remains unknown.

7. ADDITIONAL REMARKS AND EXAMPLES

In Section 4 we introduced the family of sequences of vector norms of the type

$$\|x\|_m = \Big[\sum_{k=0}^{\infty} w_{mk}^2 \|A^k x\|^2\Big]^{1/2}$$

which includes the Deddens' algebra and the spectral radius algebra as special cases. In fact we can consider for each $1 \leqslant p < \infty$, the sequence

$$\|x\|_{m,p} = \Big[\sum_{k=0}^{\infty} w_{mk}^p \|A^k x\|^p\Big]^{1/p},$$

and for $p = \infty$

$$\|x\|_{m,\infty} = \sup_k w_{mk} \|A^k x\|.$$

These vector norms (for fixed p) induce a sequence of operator norms on $L(X)$ which determine a subalgebra $\mathcal{B}_{A,p}$ of $L(X)$. It is easy to check that for $w_{mk} = d_m^k$ all the results of Sections 5 and 6 which were obtained for the case $p = 2$ in fact hold for $1 \leqslant p \leqslant \infty$. For $1 \leqslant p < \infty$ we do not know if different algebras are obtained for different values of p. We give an example for which $\mathcal{B}_{A,\infty}$ is different from $\mathcal{B}_{A,2}$.

EXAMPLE 7.1. Consider the sequence of vector norms defined on X by
$$\|x\|_{m,\infty} = \sup_n d_m^n \|A^n x\|.$$
Let $\{\|T\|_{m,\infty}\}$ denote the sequence of induced operator norms, and let
$$\mathcal{B}_{A,\infty} = \Big\{ T \in L(X) : \sup_m \|T\|_{m,\infty} < \infty \Big\}.$$
Suppose $r(A) = 1$ and that there exists $C > 0$ such that for all $n \in \mathbb{N}$, $\|A^n\| \leqslant C$. Then
$$\|x\|_{m,\infty} = \sup_n \left[\frac{m}{m+1}\right]^n \|A^n x\| \leqslant C\|x\|.$$
Since $\|x\| \leqslant \|x\|_{m,\infty}$ for all $x \in X$, we have for $T \in L(X)$,
$$\|Tx\|_{m,\infty} \leqslant C\|Tx\| \leqslant C\|T\|\|x\|.$$
Thus $\sup_m \|T\|_{m,\infty} \leqslant C\|T\|$ and $\mathcal{B}_{A,\infty} = L(X)$. Now $A = \left[\begin{smallmatrix} I & 0 \\ 0 & \lambda I \end{smallmatrix}\right]$ with $|\lambda| < 1$ satisfies the requirements given above and we saw (Example 5.2(ii)) that $\mathcal{B}_A = \mathcal{B}_{A,2} \neq L(X)$.

A more interesting example of an operator that satisfies the mentioned conditions is $A = (I + V)^{-1}$, where V is the Volterra operator (see Remark 2.16(i)). Here also $(\mathcal{B}_A =)\ \mathcal{B}_{A,2} \neq L(X)\ (= \mathcal{B}_{A,\infty})$. Indeed, if $\mathcal{B}_A = L(X)$ then ([3], Theorem 2.7) A is similar to a constant multiple of an isometry. Since the spectrum of A is the set $\{1\}$, this implies that $A = I$ which is a contradiction.

Our final example is a compact weighted shift on the Banach space l^1 with the sequence of vector norms for $p = 1$.

EXAMPLE 7.2. Let $X = l^1$ with standard basis $\{e_n\}_{n=0}^{\infty}$. Every $T \in L(X)$ has a matrix representation $T = [t_{jk}]_{j,k=0}^{\infty}$, and the induced operator norm determined by the l^1 norm is
$$\|T\| = \sup_j \sum_{k=0}^{\infty} |t_{jk}|.$$
Let $\{w_n\}$ be a monotone sequence of positive numbers converging to zero, and consider the weighted shift $A \in L(X)$ defined by $Ae_n = w_n e_{n+1}$. Then for $x = \sum_{i=0}^{\infty} x_i e_i \in X$,
$$A^n x = \sum_{i=0}^{\infty} w_i \cdots w_{i+n-1} e_{i+n} = \sum_{i=0}^{\infty} w(i,n) e_{i+n},$$
where $w(i,n) = w_i \cdots w_{i+n-1}$. We consider the sequence of vector norms defined by
$$\|x\|_m = \sum_{n=0}^{\infty} d_m^n \|A^n x\| = \sum_{n=0}^{\infty} d_m^n \sum_{i=0}^{\infty} w(i,n)|x_i|.$$

For simplicity in this example we will write $\|\cdot\|_m$ for $\|\cdot\|_{m,1}$.

Suppose $T \in L(X)$ has a lower triangular matrix representation, $T=[t_{jk}]_{j,k=0}^{\infty}$, with $t_{jk} = 0$ for $j < k$. Then, for $x = \sum\limits_{i=0}^{\infty} x_i e_i$,

$$\|Tx\|_m = \sum_{n=0}^{\infty} d_m^n \|A^n Tx\| = \sum_{n=0}^{\infty} d_m^n \sum_{i=0}^{\infty} w(i,n) \Big| \sum_{k=0}^{\infty} t_{ik} x_k \Big|.$$

If we denote $w(i,n)x_i$ by $y_i^{(n)}$, this can be written as

$$\sum_{n=0}^{\infty} d_m^n \sum_{i=0}^{\infty} \Big| \sum_{k=0}^{\infty} t_{ik} \frac{w(i,n)}{w(k,n)} y_k^{(n)} \Big| \leqslant \sum_{n=0}^{\infty} d_m^n \|\widehat{T}\| \sum_{k=0}^{\infty} |y_k^{(n)}|$$

where $\widehat{T} = \Big[t_{ik} \frac{w(i,n)}{w(k,n)} \Big]_{i,k=0}^{\infty}$.

Since $w(i,n) \leqslant w(k,n)$, $\|\widehat{T}\| \leqslant \|T\|$, and

$$\|Tx\|_m \leqslant \|T\| \sum_{n=0}^{\infty} d_m^n \sum_{k=0}^{\infty} |y_k^{(n)}| = \|T\| \|x\|_m,$$

so that $T \in \mathcal{B}_{A,1} = \{T \in L(X) : \sup \|T\|_m < \infty\}$, where $\|T\|_m$ is the norm induced by the given norm $\|x\|_m$.

In the argument given above we used the monotonicity of the sequence $\{w_n\}$ but not the fact that it converges to zero. To show that operators which are not lower triangular are not in $\mathcal{B}_{A,1}$ we use the fact that $\{w_n\}$ converges to zero but not that it is monotone.

Suppose $T \in L(X)$ is not lower triangular. Then in its matrix representation $[t_{jk}]$, $t_{jk} \neq 0$ for some pair $\{j,k\}$ with $j < k$. Then $Te_k = \sum\limits_{i=0}^{\infty} t_{ik} e_i$ and $A^n Te_k = \sum\limits_{i=0}^{\infty} t_{ik} w(i,n) e_{i+n}$, so

$$\|Te_k\|_m = \sum_{n=0}^{\infty} d_m^n \sum_{i=0}^{\infty} |t_{ik}| w(i,n) \geqslant |t_{jk}| \sum_{n=0}^{\infty} d_m^n w(j,n),$$

$$\|e_k\|_m = \sum_{n=0}^{\infty} \|A^n e_k\| = \sum_{n=0}^{\infty} d_m^n w(k,n).$$

Since $w_n \to 0$ as $n \to \infty$, A is quasinilpotent and $d_m = m$. Therefore

$$\frac{\|Te_k\|_m}{\|e_k\|_m} \geqslant |t_{jk}| \frac{1 + \sum_{n=0}^{\infty} m^n w(j,n)}{1 + \sum_{n=0}^{\infty} m^n w(k,n)}.$$

Denote the denominator of the last expression by β_k. Then

$$\frac{\|Te_k\|_m}{\|e_k\|_m} \geqslant |t_{jk}| \frac{1 + mw_j + \cdots + m^{k-j} w_j \cdots w_{k-1} \beta_k}{\beta_k} \geqslant |t_{jk}| m^{k-j} w_j \cdots w_{k-1} \to \infty$$

as $m \to \infty$. Hence $T \notin \mathcal{B}_{A,1}$.

Thus $\mathcal{B}_{A,1}$ coincides with the set of all lower triangular operators from $L(l^1)$.

This algebra properly contains $(A)'$, since (as in Example 6.6) $T = e_1 \otimes e_0$ is lower triangular but does not commute with A.

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