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SAMUEL MULTIPLICITY FOR SEVERAL COMMUTING OPERATORS

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ABSTRACT. In this paper we show that the Samuel multiplicity of a lower semi-Fredholm tuple $T \in L(X)^n$ of commuting bounded operators on a complex Banach space X coincides with the generic dimension of the last cohomology groups $H^n(z-T, X)$ of its Koszul complex near $z = 0$. As applications we show that the algebraic and analytic Samuel multiplicities of T coincide and that the Samuel multiplicity is additive for closed invariant subspaces of the symmetric Fock space.

KEYWORDS: *Fredholm tuples, Samuel multiplicity, Koszul complex.*

MSC (2000): Primary 47A13; Secondary 47A53.

1. INTRODUCTION

Let $T \in L(X)^n$ be a commuting tuple of bounded linear operators on a complex Banach space X, and let $K^\bullet(T, X)$ be the cochain Koszul complex of T. The tuple T is said to be lower semi-Fredholm if the last cohomology group $H^n(T, X) = X/ \sum\limits_{i=1}^{n} T_i X$ of its Koszul complex is finite dimensional. In this case all the spaces $M_k(T) = \sum\limits_{|\alpha|=k} T^\alpha X \quad (k \in \mathbb{N})$ are finite codimensional, and the direct sum $\bigoplus\limits_{k\geqslant 0} M_k(T)/ M_{k+1}(T)$ can be turned in a canonical way into a graded finitely generated $\mathbb{C}[z]$-module. Hence by a classical result of Hilbert there is a polynomial $p \in \mathbb{Q}[x]$ of degree $\leqslant n$, the Hilbert–Samuel polynomial, with $\dim X/M_k(T) = p(k)$ for large k and such that its n-th order coefficient multiplied with $n!$

$$c(T) = n! \lim_{k\to\infty} \frac{\dim X/M_k(T)}{k^n}$$

defines a natural number, the so-called *Samuel multiplicity* of T.

On the other hand, for a given lower semi-Fredholm tuple $T \in L(X)^n$, there is an open polydisc U at $0 \in \mathbb{C}^n$ such that $\dim H^n(z-T, X) < \infty$ for all $z \in U$ and such that the last cohomology sheaf $\mathcal{H} = \mathcal{H}^n(z - T, \mathcal{O}_U^X)$ of the induced complex $K^\bullet(z - T, \mathcal{O}_U^X)$ of $\mathcal{O}_U$-modules is isomorphic to a quotient of a free module $\mathcal{O}_U^N$ on U. In particular, the stalk $\mathcal{H}_0$ is a Noetherian module over the local ring $\mathcal{O}_0$ of all convergent power series at $z = 0$, and hence possesses a Hilbert–Samuel polynomial $p_{\mathrm{an}} \in \mathbb{Q}[x]$ such that $\dim \mathcal{H}_0/\mathfrak{m}^k\mathcal{H}_0 = p_{\mathrm{an}}(k)$ for large k. Here $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_0$.

In a paper [4] from 1993 both versions of the Hilbert–Samuel polynomial were studied and it was conjectured that their leading coefficients and degrees should have a natural meaning in operator theory.

It is well known that, for T and U as above, the dimensions $\dim H^n(z - T, X)$ depend in an upper semicontinuous way on $z \in U$ and that the set of discontinuity points of this function is a proper analytic subset S of U. One can show that the stabilized value of this function, that is, its constant value on $U \setminus S$, coincides with the rank of the coherent sheaf $\mathcal{H} = \mathcal{H}^n(z - T, \mathcal{O}_U^X)$ at $z = 0$ (cf. Corollary 2.2 below). In the present paper we show that the Samuel multiplicity of a lower semi-Fredholm tuple $T \in L(X)^n$ on a complex Banach space X coincides with the stabilized dimension of the last cohomology group at $z = 0$

$$c(T) = \dim H^n(z - T, X) = \mathrm{rank}_0\, \mathcal{H} \quad (z \in U \setminus S).$$

This result was proved by Xiang Fang in the case of single operators [7] and in the case that $T = (M_{z_1}, \dots, M_{z_n})$ is given by the multiplication operators with the coordinate functions on a large class of functional Hilbert spaces [8]. In [5] the method of Xiang Fang was extended to prove the above formula for quotients of sufficiently regular Banach space tuples.

As elementary applications we prove that both versions of the Samuel multiplicity introduced by Douglas and Yan coincide for lower semi-Fredholm tuples on Banach spaces and deduce the additivity of the Samuel multiplicity in particular cases.

The Hilbert-space case of the above results, together with a number of interesting applications, is also proved, using a different method, in a paper of Xiang Fang [10], which the author received in the very last stage of preparing this note.

2. ANALYTICALLY PARAMETERIZED COMPLEXES

A bounded analytically parameterized complex of Banach spaces with only finite-dimensional cohomology groups is locally quasiisomorphic to a complex of finite-dimensional vector spaces (Remark 9.4.6 in [6]). Our first aim is to deduce a variant of this result which is still valid when only the last cohomology groups are supposed to be finite dimensional.

THEOREM 2.1. *Let X, Y be Banach spaces and let $\alpha \in \mathcal{O}(U, L(Y, X))$ be a holomorphic map on an open set $U \subset \mathbb{C}^n$ such that* $\dim X/\alpha(z_0)Y < \infty$ *for some point $z_0 \in U$. Then there are an open neighbourhood $V \subset U$ of z_0, a Banach space E and a* *finite-dimensional subspace $D \subset X$ as well as a holomorphic map $u \in \mathcal{O}(V, L(E, D))$ such that $u(0) = 0$ and such that all the following maps are well-defined vector-space isomorphisms:*

$$\frac{D}{\mathrm{Im}\, u(z)} \to \frac{X}{\mathrm{Im}\, \alpha(z)}, \quad [x] \mapsto [x] \quad (z \in V).$$

Proof. For simplicity, we assume that $z_0 = 0$. Define $T = \alpha(0) \in L(Y, X)$.

We first consider the particular case that the space Ker T has a direct complement in Y, that is, $Y = M \oplus \mathrm{Ker}\, T$ for some closed subspace $M \subset Y$. By hypothesis the space Im T possesses a finite-dimensional direct complement $D \subset X$. We denote by P_D and $P_{\mathrm{Im}\, T}$ the continuous projections of X onto D and Im T given by the decomposition $X = \mathrm{Im}\, T \oplus D$. For $z \in U$, the map $\alpha(z) : M \oplus \mathrm{Ker}\, T \to \mathrm{Im}\, T \oplus D$ possesses a matrix representation of the form

$$\alpha(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

with suitable analytic operator-valued functions a, b, c, d.

Since $a(0) = P_{\mathrm{Im}\, T} T | M$ is invertible and since $X = \mathrm{Im}\, \alpha(0) + D$, there is an open zero neighbourhood $V \subset U$ such that $a(z) \in L(M, \mathrm{Im}\, T)$ is invertible and $X = \mathrm{Im}\, \alpha(z) + D$ for every $z \in V$ (cf. Lemma 2.1.3 in [6]).

We claim that the holomorphic map

$$u : V \to L(\mathrm{Ker}\, T, D), \quad u(z) = d(z) - c(z)a(z)^{-1}b(z)$$

has the required properties. To check this, it suffices to prove that

$$\mathrm{Im}\, u(z) = D \cap \mathrm{Im}\, \alpha(z) \quad (z \in V).$$

For this purpose, the reader should observe that $u(z)x = P_D\alpha(z)x - P_D\alpha(z)y$, where $y \in M$ is the unique vector with $P_{\mathrm{Im}\, T}\alpha(z)y = P_{\mathrm{Im}\, T}\alpha(z)x$. Using the matrix-representation of $\alpha(z)$ chosen above, one finds that $y = a(z)^{-1}b(z)x$. As an immediate consequence we find that

$$u(z)x = P_D\alpha(z)x - \alpha(z)y + P_{\mathrm{Im}\, T}\alpha(z)y = \alpha(z)(x - y) \in D \cap \mathrm{Im}\, \alpha(z)$$

for every vector $x \in \mathrm{Ker}\, T$. Conversely, if $d \in D \cap \mathrm{Im}\, \alpha(z)$, then there are vectors $x \in \mathrm{Ker}\, T$, $y \in M$ with $d = \alpha(z)(x - y)$. Since $P_{\mathrm{Im}\, T}\alpha(z)(x - y) = 0$, it follows that $d = u(z)x$. Thus the proof is complete in the case where Ker $T = \mathrm{Ker}\, \alpha(0)$ is complemented in Y.

It remains to prove the assertion without this additional hypothesis. By shrinking U we can achieve that $\dim X/\mathrm{Im}\, \alpha(z) < \infty$ for all $z \in U$. In [12] (Section 1) it is shown that there is a family of continuous linear operators

$$\widetilde{\alpha}(z) : Z \to \ell^1(B) \quad (z \in U)$$

between a Banach space Z and some ℓ^1-space $\ell^1(B)$ such that the associated mapping $\widetilde{\alpha} : U \to L(Z, \ell^1(B))$ is holomorphic, all kernels $\mathrm{Ker}\, \widetilde{\alpha}(z) \subset Z$ $(z \in U)$

are complemented and such that there is a surjective continuous linear operator $\pi : \ell^1(B) \to X$ with

$$\overset{-1}{\pi}(\mathrm{Im}\,\alpha(z)) = \mathrm{Im}\,\widetilde{\alpha}(z) \quad (z \in U).$$

It follows that the induced maps

$$\widehat{\pi} : \frac{\ell^1(B)}{\mathrm{Im}\,\widetilde{\alpha}(z)} \to \frac{X}{\mathrm{Im}\,\alpha(z)} \quad (z \in U)$$

are well-defined vector-space isomorphisms. Applying the first part of the proof to the mapping $\widetilde{\alpha} \in \mathcal{O}(U, L(Z, \ell^1(B))$ we find a Banach space E, a finite-dimensional subspace $D \subset \ell^1(B)$ and a holomorphic map $u \in \mathcal{O}(V, L(E, D))$ on a suitable zero neighbourhood $V \subset U$ such that all the maps

$$\frac{D}{\mathrm{Im}\,u(z)} \to \frac{\ell^1(B)}{\mathrm{Im}\,\widetilde{\alpha}(z)}, \quad [x] \mapsto [x] \quad (z \in V)$$

are well-defined vector-space isomorphisms. Then $\pi(D) \subset X$ is a finite-dimensional subspace such that the map

$$V \to L(E, \pi(D)), \quad z \mapsto \pi u(z)$$

is holomorphic and the next induced mappings are vector-space isomorphisms:

$$\frac{\pi(D)}{\mathrm{Im}\,\pi u(z)} \to \frac{X}{\mathrm{Im}\,\alpha(z)}, \quad [x] \mapsto [x] \quad (z \in V). \quad \blacksquare$$

Let $\alpha \in \mathcal{O}(U, L(Y, X))$ and $z_0 \in U$ be given as in Theorem 2.1. Suppose first that Ker $\alpha(z_0)$ has a direct complement in Y. With the notations of the first part of the proof of Theorem 2.1, we define

$$V = \{z \in U : a(z) \text{ is invertible and } X = D + \mathrm{Im}\,\alpha(z)\}.$$

Then $V \subset U$ is an open neighbourhood of z_0. Exactly as in the first part of the proof of Theorem 2.1, one can show that the chosen mapping $u \in \mathcal{O}(V, L(E, D))$ has the property that

$$u\mathcal{O}(W, E) = \mathcal{O}(W, D) \cap \alpha\mathcal{O}(W, Y)$$

holds for each open set $W \subset V$. Using Lemma 2.1.5 from [6] one obtains the identity

$$\mathcal{O}_V^X = \mathcal{O}_V^D + \alpha\mathcal{O}_V^Y$$

for the associated analytic sheaves.

If Ker $\alpha(z_0)$ is not assumed to be complemented in Y, then after shrinking U, we can choose a lifting $\pi : \ell^1(B) \to X$ and a holomorphic map $\widetilde{\alpha} : U \to L(Z, \ell^1(B))$ with pointwise complemented kernels as in the proof of Theorem 2.1. As seen above, there are an open neighbourhood V of z_0, a finite-dimensional subspace $D \subset \ell^1(B)$ and a suitable holomorphic mapping $u \in \mathcal{O}(V, L(E, D))$ such that

$$u\mathcal{O}(W, E) = \mathcal{O}(W, D) \cap \widetilde{\alpha}\mathcal{O}(W, Z)$$

for each open set $W \subset V$ and such that $\mathcal{O}_V^{\ell^1(B)} = \mathcal{O}_V^D + \widetilde{\alpha}\mathcal{O}_V^Z$.

Using the explicit constructions of Z and $\widetilde{\alpha} \in \mathcal{O}(U, L(Z, \ell^1(B)))$ given by Kaballo in [12] one can easily check that in this case the identities

$$\pi u \mathcal{O}(W, E) = \mathcal{O}(W, \pi(D)) \cap \alpha \mathcal{O}(W, Y)$$

hold for every open set $W \subset V$ and that the equality

$$\mathcal{O}_V^X = \mathcal{O}_V^{\pi(D)} + \alpha \mathcal{O}_V^Y$$

remains true on the level of sheaves. Summarizing these observations we obtain the following result.

COROLLARY 2.2. *Let X, Y be Banach spaces and let $\alpha \in \mathcal{O}(U, L(Y, X))$ be a holomorphic map on an open set $U \subset \mathbb{C}^n$.*

(i) *For every point $z_0 \in U$ with $\dim X/\operatorname{Im} \alpha(z_0) < \infty$, there are an open neighbourhood V of z_0, a Banach space E, a finite-dimensional subspace $D \subset X$ and a holomorphic mapping $u \in \mathcal{O}(V, L(E, D))$ with $u(z_0) = 0$ such that*

$$u\mathcal{O}(W, E) = \mathcal{O}(W, D) \cap \alpha \mathcal{O}(W, Y)$$

for every open set $W \subset V$ and $\mathcal{O}_V^X = \mathcal{O}_V^D + \alpha \mathcal{O}_V^Y$.

(ii) *The subset $U_0 = \{z \in U : \dim X/\operatorname{Im} \alpha(z) < \infty\} \subset U$ is open, and the quotient sheaf $\mathcal{O}_{U_0}^X / \alpha \mathcal{O}_{U_0}^Y$ is a coherent analytic sheaf.*

Proof. Part (i) has already been proved. An elementary application of Lemma 2.1.5 from [6] shows that the set U_0 defined in part (ii) is open. But then part (i) implies that, locally on U_0, there are sheaf isomorphisms of the form

$$\frac{\mathcal{O}_V^D}{u\mathcal{O}_V^E} \xrightarrow{\sim} \frac{\mathcal{O}_V^X}{\alpha \mathcal{O}_V^Y}$$

with u as explained in part (i). A result of Markoe ([13], Proposition 5) shows that the quotient sheaf on the left is coherent. Hence the same is true for the quotient sheaf on the right. Since coherence of analytic sheaves is a local property, this observation completes the proof. ∎

Let $\alpha \in \mathcal{O}(U, L(Y, X))$ be a holomorphic map with Banach spaces X and Y. Suppose that U is a connected open neighbourhood of $0 \in \mathbb{C}^n$ and that

$$\dim X/\operatorname{Im} \alpha(0) < \infty.$$

After shrinking U we may suppose that there are a Banach space E, a finite-dimensional subspace $D \subset X$ and a holomorphic map $u \in \mathcal{O}(U, L(E, D))$ such that all maps

$$\frac{D}{\operatorname{Im} u(z)} \to \frac{X}{\operatorname{Im} \alpha(z)}, \quad [x] \mapsto [x] \quad (z \in U)$$

are well-defined isomorphisms and such that

$$u\mathcal{O}(U, E) = \mathcal{O}(U, D) \cap \alpha \mathcal{O}(U, Y).$$

Then $M = u\mathcal{O}(U,E) = \mathcal{O}(U,D) \cap \alpha\mathcal{O}(U,Y) \subset \mathcal{O}(U,D)$ is a $\mathbb{C}[z]$-submodule. Define

$$M_z = \{f(z) : f \in M\} \quad (z \in U).$$

Since $M_z = \operatorname{Im} u(z)$, we find that

$$\dim M_z + \dim X/\operatorname{Im} \alpha(z) = \dim D \quad (z \in U).$$

Define $\mathcal{O}_k(U,D) = \sum_{|\alpha|=k} z^\alpha \mathcal{O}(U,D)$ and denote by

$$T_k : \mathcal{O}(U,D) \to \mathcal{O}(U,D), \quad f \mapsto \sum_{|\alpha| \leqslant k} (f^{(\alpha)}(0)/\alpha!) z^\alpha$$

the linear map that associates with each analytic function on U its k-th Taylor polynomial.

COROLLARY 2.3. *With the hypotheses and notations from above, we can choose a proper analytic subset $S \subset U$ such that the following identity holds for all $z \in U \setminus S$:*

$$\dim X/\operatorname{Im} \alpha(z) = \min_{w \in U} \dim X/\operatorname{Im} \alpha(w) = \dim D - n! \lim_{k\to\infty} \dim \Big(\frac{M + \mathcal{O}_k(U,D)}{\mathcal{O}_k(U,D)}\Big)/k^n.$$

Proof. In [5] it was shown that there is a proper analytic set $S \subset U$ with

$$\dim M_z = \max_{w \in U} \dim M_w = n! \lim_{k\to\infty} \frac{\dim T_k(M)}{k^n}$$

for all $z \in U \setminus S$. To complete the proof it suffices to observe that the following maps are well-defined vector-space isomorphisms for $k \geqslant 1$:

$$\frac{M + \mathcal{O}_k(U,D)}{\mathcal{O}_k(U,D)} \to T_{k-1}(M), \quad [f] \mapsto T_{k-1}(f). \quad \blacksquare$$

3. SAMUEL MULTIPLICITY

Let $T = (T_1, \ldots, T_n) \in L(X)^n$ be a commuting tuple of continuous linear operators on a Banach space X. In the following we apply the results of Section 2 to the case where $\alpha(z) = z - T \in L(X^n, X)$ is the last operator in the cochain Koszul complex $K^\bullet(z-T, X)$ of $z - T$, that is,

$$(z-T)(x_i)_{i=1}^n = \sum_{i=1}^n (z_i - T_i) x_i.$$

Suppose that $\dim H^n(T,X) < \infty$. The results proved so far imply that there exist an open neighbourhood U of $0 \in \mathbb{C}^n$, a Banach space E, a finite-dimensional subspace D of X and a holomorphic map $u \in \mathcal{O}(U, L(E,D))$ with $u(0) = 0$ such that the maps

$$\frac{D}{\operatorname{Im} u(z)} \to H^n(z-T,X), \quad [x] \mapsto [x] \quad (z \in U)$$

are well-defined isomorphisms and such that the identity

$$u\mathcal{O}(V,E) = \mathcal{O}(V,D) \cap \sum_{i=1}^{n}(z_i - T_i)\mathcal{O}(V,X)$$

holds for each open subset $V \subset U$. Our aim is to relate the limit occurring in Corollary 2.3 to the Samuel multiplicity of T

$$c(T) = n! \lim_{k\to\infty} \dim\Big(\frac{X}{\sum\limits_{|\alpha|=k} T^\alpha X}\Big)/k^n.$$

To establish this relation, we use an exactness result for analytically parametrized complexes of Banach spaces.

LEMMA 3.1. *Let X, Y, Z be Banach spaces and let $\Omega \subset \mathbb{C}^n$ be open. Suppose that $\alpha \in \mathcal{O}(\Omega, L(Y,X))$ and $\beta \in \mathcal{O}(\Omega, L(X,Z))$ are analytic maps with $\beta(z)\circ\alpha(z) = 0$ for all $z \in \Omega$ and $\mathrm{Ker}\beta(w) = \mathrm{Im}\alpha(w)$ for some point $w \in \Omega$. Then there is a real number $r_0 > 0$ with the following property. For each open polydisc $V = P_r(w)$ $(0 < r < r_0)$, each function $g = \sum\limits_{k\in\mathbb{N}^n} g_k z^k \in \mathcal{O}(V,X)$ with $\beta g = 0$ on V and any finite family $(f_k)_{|k|\leqslant N}$ of vectors in Y with*

$$\alpha(w)f_k = g_k - \sum_{0\neq\mu\leqslant k} \frac{\alpha^{(\mu)}(w)}{\mu!} f_{k-\mu}$$

for $|k| \leqslant N$, there is an analytic function $f \in \mathcal{O}(V,Y)$ with $g = \alpha f$ on V and $f^{(k)}(w)/k! = f_k$ for $|k| \leqslant N$.

Proof. The result forms a slight extension of Lemma 2.1.5 in [6] and can be proved in a similar way. For simplicity, let us suppose that $w = 0$.

Choose a real number $r_0 > 0$ as explained in the proof on Lemma 2.1.5 in [6]. Let $V = P_r(0)$ be an open polydisc with $0 < r < r_0$. Note that, for g and $(f_k)_{|k|\leqslant N}$ as in Lemma 3.1, the function $h \in \mathcal{O}(V,X)$ defined by

$$h(z) = g(z) - \alpha(z) \sum_{|k|\leqslant N} f_k z^k$$

possesses no non-zero Taylor coefficient of order $|k| \leqslant N$ and satisfies $\beta h \equiv 0$ on V.

The proof of Lemma 2.1.5 in [6] shows that there is a function $\widetilde{f} \in \mathcal{O}(V,Y)$ with $h = \alpha\widetilde{f}$ on V such that $\widetilde{f}$ possesses no non-zero Taylor coefficient of order $|k| \leqslant N$. But then $f = \widetilde{f} + \sum\limits_{|k|\leqslant N} f_k z^k$ has all the required properties. ∎

Let $V = P_r(0)$ be a polydisc as in the proof of Lemma 3.1. For a given analytic function $h \in \mathcal{O}(V,G)$ with values in some Banach space G, we shall denote by

$$h(z) = \sum_{k\in\mathbb{N}^n} h_k z^k$$

the Taylor expansion of h at $z = 0$. Note that, for given functions $f \in \mathcal{O}(V, Y)$ and $g \in \mathcal{O}(V, X)$, the relation $\alpha f = g$ holds if and only if the following identity holds for all $k \in \mathbb{N}$:

$$\alpha(0) f_k = g_k - \sum_{0 \neq \mu \leqslant k} \alpha_\mu f_{k-\mu}\,.$$

Let us return to the situation described at the beginning of Section 3. For a lower semi-Fredholm tuple $T \in L(X)^n$ and $u \in \mathcal{O}(U, L(E, D))$ as explained there, the condition $u(0) = 0$ implies that $X = \sum_{i=1}^{n} T_i X \oplus D$. In particular, the map

$$\alpha(z) : X^n \oplus D \to X, \quad ((x_i)_{i=1}^n, y) \mapsto \sum_{i=1}^{n} (z_i - T_i) x_i + y$$

is onto for $z = 0$. We apply Lemma 3.1 with $\beta \equiv 0$ and $\alpha \in \mathcal{O}(U, L(X^n \oplus D, X))$ to find an open polydisc $V = P_r(0) \subset U$ such that, for each function $g \in \mathcal{O}(V, X)$, there is a solution $f \in \mathcal{O}(V, X^n \oplus D)$ of the equation $\alpha f = g$ with finitely many prescribed Taylor coefficients as explained in Lemma 3.1.

For $k \in \mathbb{N}$, define $\mathcal{V}_k = \{p \in \mathbb{C}[z] : \deg(p) \leqslant k\}$ and recall the notation

$$M_k(T) = \sum_{|\alpha| = k} T^\alpha X.$$

We need the following elementary and well-known result.

LEMMA 3.2. *For $k \geqslant 1$, the following identity holds:*

$$\frac{X}{M_k(T)} = \bigvee \{p(T)x + M_k(T) : p \in \mathcal{V}_{k-1} \text{ and } x \in D\}.$$

Proof. For completeness sake, we indicate the elementary inductive proof.

For $k = 1$, the assertion obviously holds. Let $k \geqslant 2$ be given, and suppose that the assertion is true for $k - 1$. Given $x \in X$, choose polynomials $p_1, \ldots, p_s \in \mathcal{V}_{k-2}$ and vectors $x_\alpha \in X$ ($|\alpha| = k - 1$) such that

$$x - \sum_{|\alpha| = k-1} T^\alpha x_\alpha \in \sum_{i=1}^{s} p_i(T) D.$$

For each multiindex α with $|\alpha| = k - 1$, there are vectors $y_{\alpha,\nu} \in X$ ($\nu = 1, \ldots, n$) such that $x_\alpha - \sum_{\nu=1}^{n} T_\nu y_{\alpha,\nu} \in D$. The next observation completes the proof:

$$x - \sum_{|\alpha| = k-1} \sum_{\nu=1}^{n} T^\alpha T_\nu y_{\alpha,\nu} \in \sum_{i=1}^{s} p_i(T) D + \sum_{|\alpha| = k-1} T^\alpha D. \quad \blacksquare$$

According to Lemma 3.2 the linear maps

$$\tau_k : \frac{\mathcal{O}(V, D)}{\mathcal{O}_k(V, D)} \to \frac{X}{M_k(T)}, \quad [f] \mapsto \Big[\sum_{|\alpha| < k} T^\alpha f_\alpha \Big]$$

are surjective for $k \geqslant 1$. Here as before $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha z^\alpha$ is the Taylor expansion of f and we have used the notations introduced at the end of Section 2. Define

$$M = u\mathcal{O}(V,E) = \mathcal{O}(V,D) \cap \sum_{i=1}^{n} (z_i - T_i)\mathcal{O}(V,X).$$

The main result of this section is the following theorem.

THEOREM 3.3. *For every $k \geqslant 1$, the following sequence, where j_k denotes the inclusion map, is well defined and exact:*

$$0 \;\to\; \frac{M + \mathcal{O}_k(V,D)}{\mathcal{O}_k(V,D)} \;\overset{j_k}{\hookrightarrow}\; \frac{\mathcal{O}(V,D)}{\mathcal{O}_k(V,D)} \;\overset{\tau_k}{\to}\; \frac{X}{M_k(T)} \;\to\; 0.$$

The proof of Theorem 3.3 will be divided into several parts. Fix $k \geqslant 1$. Note first that the inclusion map j_k is obviously injective and that τ_k is surjective by Lemma 3.2. All that remains is to prove that

$$\operatorname{Ker} \tau_k = \operatorname{Im} j_k.$$

LEMMA 3.4. *Let $g \in M$ be given. Then we obtain, for all $k \geqslant 1$, that*

$$\sum_{|j|<k} T^j g_j \in \sum_{|\alpha|=k} T^\alpha X.$$

Proof. Let $g \in M = \mathcal{O}(V,D) \cap \sum_{i=1}^{n} (z_i - T_i)\mathcal{O}(V,X)$ be given. Choose functions $f_i = \sum_{j \in \mathbb{N}^n} f_{i,j} z^j \in \mathcal{O}(V,X)$ such that

$$g(z) = \sum_{i=1}^{n} (z_i - T_i) f_i(z) \quad (z \in V).$$

By comparing the Taylor coefficients of both sides, one easily obtains that $g_j = \Big(\sum_{i=1, e_i \leqslant j}^{n} f_{i,j-e_i}\Big) - \sum_{i=1}^{n} T_i f_{i,j}$ for all $j \in \mathbb{N}^n$. Here $e_i = (0,\ldots,0,1,0,\ldots,0)$ is the i-th canonical unit vector.

By induction on k we prove that

$$\sum_{|j|<k} T^j g_j = -\sum_{|j|=k-1} T^j \Big(\sum_{i=1}^{n} T_i f_{i,j}\Big) \in \sum_{|\alpha|=k} T^\alpha X$$

for all $k \geqslant 1$. For $k = 1$, the assertion follows by choosing $j = 0$ in the previous equations. Suppose that the assertion has been proved for some natural number $k \geqslant 1$. Then we find that

$$\begin{aligned}\sum_{|j|<k+1} T^j g_j &= \sum_{|j|<k} T^j g_j + \sum_{|j|=k} T^j g_j \\ &= -\sum_{|j|=k-1} T^j \Big(\sum_{i=1}^{n} T_i f_{i,j}\Big) + \sum_{|j|=k} T^j \Big(\sum_{\substack{i=1 \\ e_i \leqslant j}}^{n} f_{i,j-e_i}\Big) - \sum_{|j|=k} T^j \Big(\sum_{i=1}^{n} T_i f_{i,j}\Big)\end{aligned}$$

$$= - \sum_{|j|=k} T^j \Big(\sum_{i=1}^{n} T_i f_{i,j} \Big).$$

To verify the last equation, the reader should note that the map $(i, j) \mapsto (i, j + e_i)$ defines a bijection between the sets $\{(i, j) : |j| = k - 1,\ i = 1, \dots, n\}$ and $\{(i, j) : |j| = k \text{ and } i = 1, \dots, n \text{ with } e_i \leqslant j\}$. Thus the inductive proof is complete, and the assertion of Lemma 3.4 follows. ∎

As an immediate application of Lemma 3.4 we obtain that

$$\operatorname{Im} j_k \subset \operatorname{Ker} \tau_k \quad (k \geqslant 1).$$

We shall use Lemma 3.1 to prove the opposite inclusion. Recall that we have chosen the polydisc $V = P_r(0) \subset U$ in such a way that Lemma 3.1 can be applied with $\beta \equiv 0$ and $\alpha \in \mathcal{O}(V, L(X^n \oplus D))$ defined by

$$\alpha(z)((x_i)_{i=1}^n, y) = \sum_{i=1}^{n} (z_i - T_i) x_i + y.$$

Note that the only non-zero Taylor coefficients of α at $z = 0$ are given by

$$\alpha_0 : X^n \oplus D \to X, \quad ((x_i)_{i=1}^n, y) \mapsto - \sum_{i=1}^{n} T_i x_i + y;$$
$$\alpha_{e_j} : X^n \oplus D \to X, \quad ((x_i)_{i=1}^n, y) \mapsto x_j \quad (j = 1, \dots, n).$$

Let $(h_j)_{|j|<k} = ((f_{i,j})_{i=1}^n, g_j)_{|j|<k}$ be a family of vectors in $X^n \oplus D$. According to Lemma 3.1 and our choice of V, there is an analytic function $h = ((f_i)_{i=1}^n, G)$ in $\mathcal{O}(V, X^n \oplus D)$ with

$$\sum_{i=1}^{n} (z_i - T_i) f_i + G = \alpha h \equiv 0$$

on V such that the Taylor coefficients of h of order $|j| < k$ at $z = 0$ are precisely the given vectors h_j $(|j| < k)$ if the relation

$$g_j = \sum_{i=1}^{n} T_i f_{i,j} - \sum_{i=1, e_i \leqslant j}^{n} f_{i, j - e_i}$$

holds for all multiindices j with $|j| < k$. Hence to obtain the missing inclusions $\operatorname{Ker} \tau_k \subset \operatorname{Im} j_k$ $(k \geqslant 1)$ it is enough to prove the following result.

LEMMA 3.5. *Let $k \geqslant 1$. For every family $(g_j)_{|j|<k}$ of vectors in D with*

$$\sum_{|j|<k} T^j g_j \in M_k(T),$$

there are vectors $f_{i,j} \in X$ $(|j| < k,\ i = 1, \dots, n)$ such that, for $|j| < k$,

$$g_j = \sum_{i=1}^{n} T_i f_{i,j} - \sum_{i=1, e_i \leqslant j}^{n} f_{i, j - e_i}.$$

Proof. By hypothesis there is a family $(x_j)_{|j|=k}$ of vectors in X such that

$$\sum_{|j|<k} T^j g_j = \sum_{|j|=k} T^j x_j.$$

For $|j| = k-1$ and $i = 1, \ldots, n$, define

$$f_{i,j} = \frac{x_{j+e_i}}{n_{j+e_i}} \in X$$

where, for $\alpha \in \mathbb{N}^n$, we denote by n_α the number of indices $\nu \in \{1, \ldots, n\}$ with $\alpha_\nu \geqslant 1$. An elementary calculation shows that

$$\sum_{|j|<k} T^j g_j = \sum_{|j|=k} T^j \Big(\sum_{i=1, e_i \leqslant j}^{n} f_{i,j-e_i} \Big) = \sum_{|j|=k-1} T^j \Big(\sum_{i=1}^{n} T_i f_{i,j} \Big).$$

If $k = 1$, then the chosen vectors $f_{1,0}, \ldots, f_{n,0}$ have all properties required in the lemma.

Let us suppose that $k > 1$ and that, for some $0 < m < k$, a family of vectors $f_{i,j} \in X$ $(m \leqslant |j| < k,\ i = 1, \ldots, n)$ has been defined such that the relations

$$g_j = \sum_{i=1}^{n} T_i f_{i,j} - \sum_{i=1, e_i \leqslant j}^{n} f_{i,j-e_i}$$

hold for all $m < |j| < k$. For $|j| = m-1$ and $i = 1, \ldots, n$, define

$$f_{i,j} = \frac{\sum_{\nu=1}^{n} T_\nu f_{\nu,j+e_i} - g_{j+e_i}}{n_{j+e_i}}.$$

Then, for every multiindex j with $|j| = m$, we obtain that

$$\sum_{i=1}^{n} T_i f_{i,j} - \sum_{i=1, e_i \leqslant j}^{n} f_{i,j-e_i} = \sum_{i=1}^{n} T_i f_{i,j} - \sum_{i=1, e_i \leqslant j}^{n} \frac{\sum_{\nu=1}^{n} T_\nu f_{\nu,j} - g_j}{n_j} = g_j.$$

Thus by descending induction we have constructed a family of vectors $f_{i,j} \in X$ $(|j| < k,\ i = 1, \ldots, n)$ which satisfy the conditions required in the lemma for every multiindex j with $0 < |j| < k$.

Using repeatedly the above recursive definition we obtain the following identities which complete the proof of Lemma 3.5:

$$\begin{aligned}
\sum_{i=1}^{n} T_i f_{i,0} &= \sum_{i=1}^{n} T_i \Big(\sum_{\nu=1}^{n} T_\nu f_{\nu,e_i} - g_{e_i} \Big) = \sum_{|j|=2} T^j \Big(\sum_{i=1, e_i \leqslant j}^{n} f_{i,j-e_i} \Big) - \sum_{|j|=1} T^j g_j \\
&= \sum_{|j|=2} T^j \Big(\sum_{\nu=1}^{n} T_\nu f_{\nu,j} - g_j \Big) - \sum_{|j|=1} T^j g_j \\
&= \cdots \\
&= \sum_{|j|=k-1} T^j \Big(\sum_{\nu=1}^{n} T_\nu f_{\nu,j} \Big) - \sum_{0<|j|<k} T^j g_j = g_0. \quad \blacksquare
\end{aligned}$$

As explained before, Lemma 3.5 completes the proof of our main result (Theorem 3.3). As an elementary application we deduce that the Samuel multiplicity of a lower semi-Fredholm tuple $T \in L(X)^n$ on an arbitrary Banach space X calculates the stabilized dimension of the last cohomology groups $H^n(z-T,X)$ near $z=0$.

COROLLARY 3.6. *Let $T \in L(X)^n$ be a commuting tuple on a Banach space X with $\dim H^n(T,X) < \infty$. Then there is an open neighbourhood U of $0 \in \mathbb{C}^n$ such that*

$$\dim H^n(z-T,X) < \infty \quad (z \in U).$$

If U is connected, then there is a proper analytic subset $S \subset U$ satisfying the following inequality, for all $w \in S$ and $z \in U \setminus S$:

$$\dim H^n(w-T,X) > \dim H^n(z-T,X) = c(T).$$

Proof. By Lemma 2.1.3 in [6], the right essential resolvent set of T

$$\rho_e^r(T) = \{z \in \mathbb{C}^n : \dim H^n(z-T,X) < \infty\} \subset \mathbb{C}^n$$

is open. Suppose that $V \subset \rho_e^r(T)$ is a connected open zero neighbourhood. It was shown by Kaballo [12] (Satz 1.5) that the set

$$S = \Big\{w \in V : \dim H^n(w-T,X) > \min_{z\in V} \dim H^n(z-T,X)\Big\}$$

is an analytic subset of V. By its very definition, S is a proper subset of V. Since proper analytic subsets of connected open sets in $\mathbb{C}^n$ have $(2n)$-dimensional Lebesgue measure 0, the value of the minimum occurring above is independent of the choice of V. Let m denote this minimal value.

To apply our previous results, we choose V as an open polydisc with centre $0 \in \mathbb{C}^n$ such that Theorem 3.3 holds and such that Corollary 2.3 holds with V instead of U and with $\alpha(z) = z - T \in L(X^n, X)$. Since

$$\dim \frac{\mathcal{O}(V,D)}{\mathcal{O}_k(V,D)} = \dim \mathcal{V}_{k-1} \otimes D = \dim D \frac{k(k+1)\cdots(k+n-1)}{n!},$$

we find the following identities which complete the proof of Corollary 3.6:

$$c(T) = \dim D - n! \lim_{k\to\infty} \dim \Big(\frac{M + \mathcal{O}_k(V,D)}{\mathcal{O}_k(V,D)}\Big)/k^n = m. \quad \blacksquare$$

4. APPLICATIONS

Let $T \in L(X)^n$ be a commuting tuple of bounded operators on a Banach space X with $\dim H^n(T,X) < \infty$. Then there is a connected open neighbourhood U of $0 \in \mathbb{C}^n$ such that the quotient sheaf $\mathcal{H} = \mathcal{O}_U^X/(z-T)\mathcal{O}_U^{X^n}$ is a coherent analytic sheaf on U. The stalk $\mathcal{H}_0$ of $\mathcal{H}$ at $z=0$ is a Noetherian module over the local ring $\mathcal{O}_0$ of all convergent power series at $z=0$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_0$. It is well known that there is a polynomial $p_{\mathrm{an}} \in \mathbb{Q}[x]$ with $\deg(p) \leqslant n$ such that

$$\dim_{\mathbb{C}}\Big(\frac{\mathcal{H}_0}{\mathfrak{m}^k\mathcal{H}_0}\Big) = p_{\mathrm{an}}(k)$$

for sufficiently large natural numbers k and such that the limit

$$c_{\mathrm{an}}(T) = n!\lim_{k\to\infty}\Big(\dim\frac{\mathcal{H}_0}{\mathfrak{m}^k\mathcal{H}_0}\Big)/k^n$$

exists and defines a non-negative integer $c_{\mathrm{an}}(T)$. We call $c_{\mathrm{an}}(T)$ the analytic Samuel multiplicity of T. It was observed by Douglas and Yan [4] that the inequality $c_{\mathrm{an}}(T) \leqslant c(T)$ holds.

COROLLARY 4.1. *Let $T \in L(X)^n$ be a commuting tuple of bounded operators on a Banach space X with* $\dim H^n(T,X) < \infty$. *Then its Samuel multiplicity and analytic Samuel multiplicity coincide, that is, $c(T) = c_{\mathrm{an}}(T)$.*

Proof. Since, with the above notations, the sheaf $\mathcal{H} = \mathcal{O}_U^X/(z-T)\mathcal{O}_U^{X^n}$ is coherent, standard results from analytic geometry (Theorem 7.4 in [3]) show that the map

$$U \to \mathbb{N}, \quad z \mapsto c_{\mathrm{an}}(z-T) = n!\lim_{k\to\infty}\dim\Big(\frac{\mathcal{H}_z}{\mathfrak{m}_z^k\mathcal{H}_z}\Big)/k^n$$

is upper semicontinuous. We know that the points w in U with

$$\dim H^n(w-T,X) > \min_{z\in U}\dim H^n(z-T,X)$$

form a proper analytic subset S of U. Since $c_{\mathrm{an}}(z-T) = \dim H^n(z-T,X)$ for $z \in U\setminus S$ (Lemma 1.3 in [5]), it follows that

$$c_{\mathrm{an}}(\lambda-T) = \min_{z\in U}\dim H^n(z-T,X) = c(T) \quad (\lambda \in U\setminus S).$$

The mentioned upper semicontinuity of the function on the left implies that $c_{\mathrm{an}}(T) \geqslant c(T)$. Thus the proof is complete. ∎

Let us recall that the symmetric Fock space $\mathcal{H}(\mathbb{B}_n)$ over the open Euclidean unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ is the Hilbert space of analytic functions given by the reproducing kernel

$$K : \mathbb{B}_n \times \mathbb{B}_n \to \mathbb{C}, \quad (z,w) \mapsto (1-\langle z,w\rangle)^{-1}.$$

It can also be obtained by symmetrizing the full Fock space as was observed by Arveson in [1]. Let $\mathcal{D}$ be a Hilbert space. Then the Hilbertian tensor product $\mathcal{H}\otimes\mathcal{D}$ is in a canonical way unitarily equivalent to the Hilbert space of $\mathcal{D}$-valued analytic functions $\mathcal{H}(\mathbb{B}_n,\mathcal{D})$ on $\mathbb{B}_n$ given by the operator-valued reproducing kernel $K_{\mathcal{D}} : \mathbb{B}_n \times \mathbb{B}_n \to L(\mathcal{D})$, $(z,w) \mapsto K(z,w)1_{\mathcal{D}}$. Since the coordinate functions are multipliers of $\mathcal{H}(\mathbb{B}_n)$, one can consider the n-tuple $M_z = (M_{z_1},\dots,M_{z_n}) \in L(\mathcal{H}(\mathbb{B}_n,\mathcal{D}))^n$ consisting of the multiplication operators with the coordinate functions.

Let us denote by $\mathrm{Lat}(M_z)$ the set of all closed subspaces of $\mathcal{H}(\mathbb{B}_n, \mathcal{D})$ that are invariant under M_{z_i} for $1 \leqslant i \leqslant n$. Fix a space $Y \in \mathrm{Lat}(M_z)$. Define $Z = \mathcal{H}(\mathbb{B}_n, \mathcal{D})/Y$ and denote by

$$S = M_z|Y\,, \quad R = M_z/Y$$

the restriction and the quotient tuple induced by M_z on Y and Z, respectively. Define $X = \mathcal{H}(\mathbb{B}_n, \mathcal{D})$ and $T = M_z \in L(X)^n$. It is well known that the augmented Koszul complex

$$K^\bullet(\lambda - T, X) \xrightarrow{\varepsilon_\lambda} \mathcal{D} \longrightarrow 0,$$

where ε_λ denotes the point evaluation $\varepsilon_\lambda(f) = f(\lambda)$, is exact for each point $\lambda \in \mathbb{B}_n$ ([11]).

Let us suppose in addition that $\dim \mathcal{D} < \infty$. In an effort to find natural situations in which the operator-theoretic Samuel multiplicity is additive, Xiang Fang proved in [9] (Theorem 15) that over the unit ball $\mathbb{B}_2 \subset \mathbb{C}^2$ the identity

$$c(S) + c(R) = c(T)$$

holds whenever $\dim H^2(S, Y) < \infty$.

Our formula for the Samuel multiplicity, together with results of Gleason–Richter–Sundberg [11], immediately implies that the same result remains true in three or more variables.

THEOREM 4.2. *Let $\mathcal{D}$ be a finite-dimensional Hilbert space and let $Y \neq \{0\}$ be a closed invariant subspace of the multiplication tuple $T = M_z$ on $\mathcal{H}(\mathbb{B}_n, \mathcal{D})$. Denote by $S = M_z|Y$ and $R = M_z/Y$ the restriction and quotient tuple induced by T. If $\dim H^n(S, Y) < \infty$, then*

$$c(S) + c(R) = c(T) = \dim \mathcal{D}.$$

Proof. With the notations fixed in the sections preceding the theorem, we obtain short exact sequences

$$0 \to K^\bullet(\lambda - S, Y) \to K^\bullet(\lambda - T, X) \to K^\bullet(\lambda - R, Z) \to 0$$

for $\lambda \in \mathbb{C}^n$ which induce long exact sequences of cohomology

$$\begin{array}{cccccccc}
0 & \longrightarrow & H^0(\lambda - S, Y) & \xrightarrow{j} & H^0(\lambda - T, X) & \xrightarrow{q} & H^0(\lambda - R, Z) & \\
 & \longrightarrow & H^1(\lambda - S, Y) & \xrightarrow{j} & \cdots & & \cdots & \\
 & & \cdots & & \cdots & & \cdots & \\
 & \longrightarrow & H^n(\lambda - S, Y) & \xrightarrow{j} & H^n(\lambda - T, X) & \xrightarrow{q} & H^n(\lambda - R, Z) & \longrightarrow 0.
\end{array}$$

The zero set $Z(Y) = \bigcap(Z(f) : f \in Y)$, where $Z(f)$ denotes the zero set of f, is a proper analytic subset of $\mathbb{B}_n$. By [11] (Theorem 3.4 and Corollary 3.5), for $\lambda \in \mathbb{B}_n \setminus Z(Y)$, at most the last three spaces in the above long exact sequence of cohomology can be non-zero. Hence, for $\lambda \in \mathbb{B}_n \setminus Z(Y)$, we have the short exact sequences

$$0 \to H^n(\lambda - S, Y) \to H^n(\lambda - T, X) \to H^n(\lambda - R, Z) \to 0.$$

Choose a connected open neighbourhood $V \subset \mathbb{B}_n$ of zero such that these three spaces are finite dimensional for every point $\lambda \in V$. Then we find that

$$\min_{\lambda \in V} \dim H^n(\lambda - S, Y) + \min_{\lambda \in V} \dim H^n(\lambda - R, Z) = \dim \mathcal{D}.$$

Now an application of Corollary 3.6 completes the proof. ∎

As in the paper of Fang [9], one can show that the additivity of the Samuel multiplicity implies its monotonicity in the following sense.

COROLLARY 4.3. *Suppose that $Y_1 \subset Y_2 \subset \mathcal{H}(\mathbb{B}_n, \mathcal{D})$ are closed invariant subspaces for M_z with $\dim Y_j / \sum\limits_{i=1}^{n} (z_i - M_{z_i}) Y_j < \infty$ for $j = 1, 2$. Then we obtain the inequalities*

$$c(M_z|Y_1) \leqslant c(M_z|Y_2) \leqslant \dim \mathcal{D}.$$

Proof. Let us denote by S_j the restrictions of M_z to Y_j and by R_j the corresponding quotient tuples on $Z_j = \mathcal{H}(\mathbb{B}_n, \mathcal{D})/Y_j$. Since there are canonical surjections

$$\frac{Z_1}{M_k(R_1)} \to \frac{Z_2}{M_k(R_2)}, \quad [z] \mapsto [z] \quad (k \geqslant 1),$$

it follows that $c(R_2) \leqslant c(R_1)$. But then Theorem 4.2 implies that $c(S_1) \leqslant c(S_2) \leqslant \dim \mathcal{D}$. ∎

It is elementary to check that, for a closed M_z-invariant subspace Y of the one-dimensional Bergman space $L^2_a(\mathbb{D})$ with finite index $\dim(Y/M_zY) < \infty$, the analogue of Theorem 4.2, that is, the formula

$$c(M_z|Y) + c\Big(M_z/Y\Big) = 1$$

holds if and only if $\dim(Y/M_zY) = 1$. Hence Theorem 4.2 fails for every M_z-invariant subspace $Y \subset L^2_a(\mathbb{D})$ with index $1 < \dim(Y/M_zY) < \infty$.

On the other hand, the results obtained in [11] (Theorem 3.4 and Corollary 3.5) can be used to show that, for a large class of M_z-invariant subspaces of the Hardy or Bergman spaces on the unit ball or polydisc in $\mathbb{C}^n$, the formula of Theorem 4.2 remains true. More precisely, for the formula in Theorem 4.2 to hold in these cases, it suffices that the bounded $\mathcal{D}$-valued analytic functions are dense in Y. If $\dim \mathcal{D} = 1$, then the existence of a single non-trivial bounded analytic function in Y is sufficient. It is an open question whether Theorem 4.2 is true in general for the Hardy space over the unit ball.

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