J. OPERATOR THEORY
61:1(2009), 75–86

INFINITE SEQUENCES OF INNER FUNCTIONS AND SUBMODULES IN $H^2(\mathbb{D}^2)$

MICHIO SETO

Communicated by William Arveson

ABSTRACT. We deal with infinite sequences of inner functions $\{q_j\}_{j\geqslant 0}$ with the property that q_j *is divisible by* q_{j+1}. It is shown that these sequences have a close relation to the module structure of the Hardy space over the bidisk. Commutators, Hilbert–Schmidt norms and spectra of operators related to the module structure will be calculated exactly.

KEYWORDS: *Inner functions, Hardy space over the bidisk.*

MSC (2000): Primary 47B38; Secondary 47B35.

0. INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $H^2(z)$ denote the classical Hardy space over $\mathbb{D}$ with the variable z. $k_\lambda(z)$ will denote the reproducing kernel of $H^2(z)$ at a point λ in $\mathbb{D}$. The Hardy space over the bidisk $H^2 = H^2(\mathbb{D}^2)$ is the tensor product Hilbert space $H^2(z) \otimes H^2(w)$ with variables z and w. A closed subspace $\mathcal{M}$ of H^2 (respectively $H^2(z)$) is called a *submodule* if $\mathcal{M}$ is invariant under the action of the multiplication operator of any polynomial of z and w (respectively z).

In the classical Hardy space theory, Beurling proved that every submodule is characterized by an inner function. Beurling's theorem is one of the important theorems for the structure theory of a single operator acting on a Hilbert space (cf. [1], [7] and [11]). However, in the theory of the Hardy space over the bidisk, it is known that the structure of submodules is extremely complicated (cf. [5], [6] and [8]), so that we need good examples which help us to understand the structure of submodules in H^2.

In this paper, we will focus on computable submodules constructed below.

DEFINITION 0.1. An infinite sequence of functions $\{q_j\}_{j\geqslant 0}$ in $H^2(z)$ is called an *inner sequence* if $\{q_j\}_{j\geqslant 0}$ consists of inner functions having the following property:

(I) q_j is divisible by q_{j+1} for any j, that is, every (q_j/q_{j+1}) is inner.

First, we mention that inner sequences have already appeared in other previous works, for example, the theory of Jordan operators (cf. Chapter III in [4]) and Remark 3 in [2]. Next, we note that the condition (I) is equivalent to that $q_j(z)H^2(z)$ is contained in $q_{j+1}(z)H^2(z)$. Hence every inner sequence $\{q_j(z)\}_{j\geqslant 0}$ corresponds to a submodule $\mathcal{M}$ in H^2 as follows:

$$\mathcal{M} = \sum_{j=0}^{\infty} \bigoplus q_j(z)H^2(z)w^j.$$

Then the quotient module $\mathcal{N}$ of $\mathcal{M}$ is as follows:

$$\mathcal{N} = H^2/\mathcal{M} = \sum_{j=0}^{\infty} \bigoplus (H^2(z) \ominus q_j(z)H^2(z))w^j.$$

In this paper, the above submodule $\mathcal{M}$ is called the submodule arising from an inner sequence $\{q_j(z)\}_{j\geqslant 0}$.

Let P_j (respectively $P_j^{\perp}$) be the orthogonal projection from $H^2(z)$ onto $q_j(z)H^2(z)$ (respectively $H^2(z) \ominus q_j(z)H^2(z)$). Since $\{P_j\}_{j\geqslant 0}$ is a monotone increasing sequence of projections, P_j converges strongly to the projection P_∞ whose range is $q_\infty(z)H^2(z)$ for some inner function $q_\infty(z)$. Without loss of generality, we may assume that the first non-zero Taylor coefficient of $q_\infty(z)$ is positive. Let R_z (respectively R_w) denote the restriction of the Toeplitz operator T_z (respectively T_w) to a submodule $\mathcal{M}$. The quotient module $\mathcal{N} = H^2/\mathcal{M}$ is the orthogonal complement of a submodule $\mathcal{M}$ in H^2, and let S_z (respectively S_w) denote the compression of T_z (respectively T_w) to $\mathcal{N}$, that is, we set $S_z = P_{\mathcal{N}}T_z|_{\mathcal{N}}$ (respectively $S_w = P_{\mathcal{N}}T_w|_{\mathcal{N}}$) where $P_{\mathcal{N}}$ denotes the orthogonal projection from H^2 onto $\mathcal{N}$. $[A, B]$ denotes the commutator of operators A and B, that is, we set $[A, B] = AB - BA$.

This paper is a sequel to [10], and the purpose is to study submodules arising from inner sequences in detail. This paper has been divided into seven sections. Section 0 is the introduction. In Section 1, we calculate defect spaces and give a dimension formula. In Section 2, unitary equivalence of submodules will be discussed. In Section 3, we study commutators of S_z and S_w. Especially, the Hilbert–Schmidt norm of $[S_z^*, S_w]$ is calculated explicitly. In Section 4, we have complete descriptions of spectra of S_z and S_w. In Section 5, we show that the commutant of S_z and S_w is the weak closed algebra generated by S_z, S_w and the identity operator. In Section 6, we deal with Rudin's submodule as an example.

1. DEFECT SPACES

Defect spaces of operators are important objects in operator theory. In general, it is not easy to calculate defect spaces related to submodules in H^2. However, in our setting, we can calculate defect spaces of R_z and R_w.

THEOREM 1.1. *Let λ be a point in $\mathbb{D}$. If $\mathcal{M}$ is the submodule arising from an inner sequence $\{q_j(z)\}_{j\geqslant 0}$, then the following hold:*

(i) $\mathcal{M} \ominus (z-\lambda)\mathcal{M} = \sum_{j=0}^{\infty} \bigoplus \mathbb{C}k_\lambda(z)q_j(z)w^j$;

(ii) $\mathcal{M} \ominus w\mathcal{M} = q_0(z)H^2(z) \oplus \sum_{j=1}^{\infty} \bigoplus \{q_j(z)H^2(z) \ominus q_{j-1}(z)H^2(z)\}w^j$;

(iii) $\mathcal{M} \ominus ((z-\lambda)\mathcal{M} + w\mathcal{M}) = \mathbb{C}k_\lambda(z)q_0(z) \oplus \sum_j' \bigoplus \mathbb{C}k_\lambda(z)q_j(z)w^j$,

where the sum $\sum_j'$ is taken only over the subset of positive integers such that $k_\lambda(z)q_j(z)$ belongs to $q_j(z)H^2(z) \ominus q_{j-1}(z)H^2(z)$.

Proof. It is easy to check (i) and (ii). We shall show (iii). Let f be a function in $\mathcal{M} \ominus ((z-\lambda)\mathcal{M} + w\mathcal{M}) = (\mathcal{M} \ominus (z-\lambda)\mathcal{M}) \cap (\mathcal{M} \ominus w\mathcal{M})$. Taking Taylor expansions of f with respect to the variable w in $\mathcal{M} \ominus (z-\lambda)\mathcal{M}$ and $\mathcal{M} \ominus w\mathcal{M}$, we have

$$f = \sum_{j=0}^{\infty} c_j k_\lambda(z)q_j(z)w^j = q_0(z)g_0(z) \oplus \sum_{j=1}^{\infty} q_j(z)g_j(z)w^j,$$

where every c_j is some constant and every $g_j(z)$ is a function in $H^2(z)$ such that $q_j(z)g_j(z)$ is in $q_j(z)H^2(z) \ominus q_{j-1}(z)H^2(z)$ for any $j \geqslant 1$. Since the Taylor expansion of f is unique, we have $c_j k_\lambda(z) = g_j(z)$ for any $j \geqslant 0$, that is, $c_0 k_\lambda(z) = g_0(z)$ and $c_j k_\lambda(z)q_j(z)$ is in $q_j(z)H^2(z) \ominus q_{j-1}(z)H^2(z)$ for any $j \geqslant 1$.

Conversely, we set $f = c_0 k_\lambda(z)q_0(z) + \sum_j' c_j k_\lambda(z)q_j(z)w^j$. Then, trivially f belongs to $\mathcal{M} \ominus (z-\lambda)\mathcal{M}$ and $\mathcal{M} \ominus w\mathcal{M}$. This completes the proof. ∎

COROLLARY 1.2. *Let λ be a point in $\mathbb{D}$. If $\mathcal{M}$ is the submodule arising from an inner sequence $\{q_j(z)\}_{j\geqslant 0}$, then the following dimension formula holds:*

$$\dim(\mathcal{M} \ominus ((z-\lambda)\mathcal{M} + w\mathcal{M})) = 1 + |\{j \geqslant 1 : (q_{j-1}/q_j)(\lambda) = 0\}| < +\infty,$$

where $|A|$ denotes the cardinal number of a set A.

2. UNITARY EQUIVALENCE

Two submodules $\mathcal{M}_1$ and $\mathcal{M}_2$ are said to be unitarily equivalent if there exists a unitary module map U from $\mathcal{M}_1$ onto $\mathcal{M}_2$. In order to classify submodules, unitary equivalence is a natural equivalence relation. In the one variable case, all submodules are unitarily equivalent to $H^2(z)$ by Beurling's theorem. However, it is known that there exist many equivalence classes of submodules in H^2 (cf. [5] and [6]). In this section, we restrict unitary equivalence only to submodules arising from inner sequences; then the situation is simple.

THEOREM 2.1. *Let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ denote submodules arising from inner sequences $\{q_j(z)\}_{j\geqslant 0}$ and $\{\widetilde{q}_j(z)\}_{j\geqslant 0}$, respectively. Then $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are unitarily equivalent if and only if there exists a unimodular function $q = q(z)$ depending only on the variable z such that $\mathcal{M} = q\widetilde{\mathcal{M}}$. Moreover, for any $j \geqslant 0$, there exists a unimodular constant c_j such that $q = c_j q_j/\widetilde{q}_j$.*

Proof. If $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are unitarily equivalent, then there exists a unimodular function q such that the unitary module map from $\mathcal{M}$ onto $\widetilde{\mathcal{M}}$ is the multiplication operator of q and $\mathcal{M} = q\widetilde{\mathcal{M}}$ by the theorem of Agrawal–Clark–Douglas [2]. Since $\overline{q(z,w)}q_0(z)$ and $q(z,w)\widetilde{q}_0(z)$ are in H^2, $q(z,w)$ is w-analytic and conjugate w-analytic. Hence q depends only on the variable z. The converse is trivial.

We show the last statement. Since $\mathcal{M} = q(z)\widetilde{\mathcal{M}}$, there exist inner functions $f_j(z)$ and $\widetilde{f}_j(z)$ in $H^2(z)$ such that

$$\begin{cases} \overline{q(z)}q_j(z)w^j = \widetilde{q}_j(z)\widetilde{f}_j(z)w^j & (j \geqslant 0), \\ q(z)\widetilde{q}_j(z)w^j = q_j(z)f_j(z)w^j & (j \geqslant 0). \end{cases}$$

It follows that

$$\widetilde{q}_j(z)\widetilde{f}_j(z) = \overline{q(z)}q_j(z) = \widetilde{q}_j(z)\overline{q_j(z)f_j(z)}q_j(z) = \widetilde{q}_j(z)\overline{f_j(z)}.$$

Hence we have that $f_j(z)$ and $\widetilde{f}_j(z)$ are constants. This completes the proof. ∎

3. COMMUTATORS

Let $\|A\|_2$ denote the Hilbert–Schmidt norm of an operator A. Following Yang's work ([12], [13], [14] and [15]), we set

$$\Sigma_0 = \|[R_w^*, R_w][R_z^*, R_z]\|_2^2 = \|[R_z^*, R_z][R_w^*, R_w]\|_2^2,$$
$$\Sigma_1 = \|[R_z^*, R_w]\|_2^2 = \|[R_w^*, R_z]\|_2^2.$$

In [13], Yang proved $\Sigma_0 = \Sigma_1 + 1$ under some mild condition.

The author and Yang showed the following in [10]:

THEOREM 3.1 ([10]). *If $\mathcal{M}$ is the submodule arising from an inner sequence $\{q_j(z)\}_{j\geqslant 0}$, then the following hold:*

(i) $\Sigma_0 = 1 + \sum\limits_{j\geqslant 0}(1 - |(q_j/q_{j+1})(0)|^2)$;

(ii) $\Sigma_1 = \sum\limits_{j\geqslant 0}(1 - |(q_j/q_{j+1})(0)|^2)$.

In this section, we study commutators of S_z and S_w on the quotient module arising from an inner sequence.

THEOREM 3.2. *Let $\mathcal{N}$ be the quotient module arising from an inner sequence $\{q_j(z)\}_{j\geqslant 0}$. Then, for any function $\sum\limits_{j\geqslant 0} g_j(z)w^j$ in $\mathcal{N}$, the following hold:*

(i) $[S_z^*, S_w]\Big(\sum\limits_{j\geqslant 0} g_j(z)w^j\Big) = \sum\limits_{j\geqslant 0}\{[T_z^*, P_{j+1}^{\perp}]g_j(z)\}w^{j+1}$;

(ii) $[S_w^*, S_z]\Big(\sum\limits_{j\geqslant 0} g_j(z)w^j\Big) = \sum\limits_{j\geqslant 1}\{(P_j^{\perp} - P_{j-1}^{\perp})T_z g_j(z)\}w^{j-1}$;

(iii) $(I_{\mathcal{N}} - S_z^* S_z)\Big(\sum\limits_{j\geqslant 0} g_j(z)w^j\Big) = \sum\limits_{j\geqslant 0}(T_z^* q_j(z) \otimes T_z^* q_j(z))g_j w^j$;

(iv) $(I_{\mathcal{N}} - S_z S_z^*)\Big(\sum\limits_{j\geqslant 0} g_j(z)w^j\Big) = \sum\limits_{j\geqslant 0}(P_j^{\perp}1 \otimes P_j^{\perp}1)g_j w^j$;

(v) $(I_{\mathcal{N}} - S_w^* S_w)\Big(\sum\limits_{j\geqslant 0} g_j(z)w^j\Big) = \sum\limits_{j\geqslant 0}(I - P_{j+1}^{\perp})g_j(z)w^j$;

(vi) $(I_{\mathcal{N}} - S_w S_w^*)\Big(\sum\limits_{j\geqslant 0} g_j(z)w^j\Big) = g_0(z)$.

Proof. We show only (i):

$$\begin{aligned}S_z^* S_w \sum_j g_j(z)w^j &= S_z^* \sum_j (P_{j+1}^{\perp} g_j(z))w^{j+1} = \sum_j S_z^*(P_{j+1}^{\perp} g_j(z))w^{j+1}\\ &= \sum_j P_{j+1}^{\perp} T_z^*(P_{j+1}^{\perp} g_j(z))w^{j+1} = \sum_j T_z^*(P_{j+1}^{\perp} g_j(z))w^{j+1},\end{aligned}$$

and

$$S_w S_z^* \sum_j g_j(z)w^j = S_w \sum_j S_z^* g_j(z)w^j = S_w \sum_j T_z^* g_j(z)w^j = \sum_j (P_{j+1}^{\perp} T_z^* g_j(z))w^{j+1}.$$

Therefore we have

$$(S_z^* S_w - S_w S_z^*)\sum_j g_j(z)w^j = \sum_j \{(T_z^* P_{j+1}^{\perp} - P_{j+1}^{\perp} T_z^*)g_j(z)\}w^{j+1}. \quad \blacksquare$$

COROLLARY 3.3. *Let $\mathcal{N}$ be the quotient module arising from an inner sequence $\{q_j(z)\}_{j\geqslant 0}$. Then S_w is a partial isometry.*

Proof. Since $I_{\mathcal{N}} - S_w^* S_w$ is a projection by (v) in Theorem 3.2, S_w is a partial isometry. $\blacksquare$

In order to calculate the Hilbert–Schmidt norm of $[S_z^*, S_w]$, we prove the following lemma:

LEMMA 3.4. *Let $\{q_j(z)\}_{j\geqslant 0}$ be an inner sequence. Then, for every $j \geqslant 1$,*

$$P_{j-1}q_j(z) = \overline{(q_{j-1}/q_j)(0)}\,q_{j-1}(z) = \langle q_j(z), q_{j-1}(z)\rangle q_{j-1}(z).$$

Proof. Since $q_{j-1}(z)/q_j(z)$ is analytic, we have the following that completes the proof:

$$\begin{aligned}P_{j-1}q_j(z) &= \sum_{k\geqslant 0}\langle q_j(z), q_{j-1}(z)z^k\rangle q_{j-1}(z)z^k = \sum_{k\geqslant 0}\langle q_j(z)/q_{j-1}(z), z^k\rangle q_{j-1}(z)z^k\\ &= \langle q_j(z)/q_{j-1}(z), 1\rangle q_{j-1}(z) = \langle q_j(z), q_{j-1}(z)\rangle q_{j-1}(z). \quad \blacksquare\end{aligned}$$

THEOREM 3.5. *Let $\mathcal{N}$ be the quotient module arising from an inner sequence $\{q_j(z)\}_{j\geqslant 0}$. Then*

$$\|[S_z^*, S_w]\|_2^2 = \sum_{j=0}^{\infty}(1-|q_{j+1}(0)|^2)(1-|(q_j/q_{j+1})(0)|^2).$$

Proof. By (i) in Theorem 3.2, $[S_z^*, S_w]$ can be identified with the following operator matrix acting on $H^2(z)\otimes l^2(\mathbb{Z}_{\geqslant 0})$:

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ (T_z^*P_1^\perp - P_1^\perp T_z^*)P_0^\perp & 0 & 0 & \cdots \\ 0 & (T_z^*P_2^\perp - P_2^\perp T_z^*)P_1^\perp & 0 & \cdots \\ 0 & 0 & (T_z^*P_3^\perp - P_3^\perp T_z^*)P_2^\perp & \ddots \\ 0 & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then it is easy to check that $(T_z^*P_{j+1}^\perp - P_{j+1}^\perp T_z^*)P_j^\perp = -P_{j+1}^\perp T_z^*(P_{j+1}-P_j)$. Hence we have

$$[S_z^*, S_w]^*[S_z^*, S_w] = \mathrm{diag}(P_{j+1}^\perp T_z^*(P_{j+1}-P_j))^*(P_{j+1}^\perp T_z^*(P_{j+1}-P_j)).$$

Therefore

$$\begin{aligned}\|[S_z^*, S_w]\|_2^2 &= \mathrm{tr}([S_z^*, S_w]^*[S_z^*, S_w]) = \sum_{j=0}^{\infty}\mathrm{tr}((P_{j+1}^\perp T_z^*(P_{j+1}-P_j))^*(P_{j+1}^\perp T_z^*(P_{j+1}-P_j))) \\ &= \sum_{j=0}^{\infty}\|P_{j+1}^\perp T_z^*(P_{j+1}-P_j)\|_2^2.\end{aligned}$$

We note that $P_{j+1}^\perp T_z^*(P_{j+1}-P_j)$ is a finite rank operator. In fact, for any function $f(z) = q_{j+1}(z)\sum c_n z^n$ in $(P_{j+1}-P_j)H^2(z) = q_{j+1}(z)H^2(z)\ominus q_j(z)H^2(z)$, we have

$$P_{j+1}^\perp T_z^* f = T_z^* c_0 q_{j+1}(z) = \langle f(z), q_{j+1}(z)\rangle T_z^* q_{j+1}(z) = (T_z^* q_{j+1}(z)\otimes q_{j+1}(z))f.$$

Let $\{e_k(z)\}$ be an orthonormal basis of $(P_{j+1}-P_j)H^2(z)$ in the case that $P_j \neq P_{j+1}$. Then we have

$$\begin{aligned}&\|P_{j+1}^\perp T_z^*(P_{j+1}-P_j)\|_2^2 \\ &\quad= \sum_k \|(T_z^* q_{j+1}(z)\otimes q_{j+1}(z))e_k(z)\|^2 = \sum_k |\langle e_k(z), q_{j+1}(z)\rangle|^2 \|T_z^* q_{j+1}(z)\|^2 \\ &\quad= (1-|q_{j+1}(0)|^2)\sum_k |\langle q_{j+1}(z), e_k(z)\rangle|^2 = (1-|q_{j+1}(0)|^2)\|(P_{j+1}-P_j)q_{j+1}(z)\|^2.\end{aligned}$$

By Lemma 3.4, we have $\|(P_{j+1}-P_j)q_{j+1}(z)\|^2 = 1-|(q_j/q_{j+1})(0)|^2$. We note that this formula holds in the case that $P_j = P_{j+1}$. Hence we have the following that concludes the proof:

$$\|[S_z^*, S_w]\|_2^2 = \sum_{j=0}^{\infty}(1-|q_{j+1}(0)|^2)(1-|(q_j/q_{j+1})(0)|^2). \qquad \blacksquare$$

REMARK 3.6. In Theorem 3.5, it may be worth remarking that the difference between Σ_1 and $\|[S_z^*, S_w]\|_2^2$ depends only on the first inner function $q_0(z)$ and the limit of $|q_j(0)|$ as follows:

$$\Sigma_1 - \|[S_z^*, S_w]\|_2^2 = \lim_{j\to\infty} |q_j(0)|^2 - |q_0(0)|^2.$$

4. SPECTRA OF S_z AND S_w

Let $\sigma(T)$ denote spectrum of an operator T on a Hilbert space, and $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{c}}(T)$ and $\sigma_{\mathrm{r}}(T)$ will denote the point spectrum, the continuous spectrum and the residual spectrum, respectively.

Spectra of S_z and S_w are described by the model theory of Sz.-Nagy and Foiaş (cf. [7] and [11]).

THEOREM 4.1. *Let $\mathcal{N}$ be the quotient module arising from an inner sequence $\{q_j(z)\}_{j\geqslant 0}$. Then $\sigma(S_z) = \sigma(q_0(z))$, where $\sigma(q_0(z))$ is the spectrum of $q_0(z)$, that is, $\sigma(q_0(z))$ consists of all zero points of $q_0(z)$ in $\mathbb{D}$ and all points ζ on the unit circle $\partial\mathbb{D}$ such that $q_0(z)$ can not be extended analytically from $\mathbb{D}$ to ζ.*

Proof. Since S_z can be represented as a diagonal operator matrix on the infinite direct sum Hilbert space $\bigoplus\limits_{j=0}^{\infty} (H^2(z) \ominus q_j(z)H^2(z))$ and the minimal function of S_z is $q_0(z)$, we have the conclusion. ∎

In order to describe the spectrum of S_w, we need several lemmas. Let E_λ denote the right evaluation operator at a point λ in $\mathbb{D}$, that is, E_λ is defined as follows: $E_\lambda f(z,w) = f(z,\lambda)$ for any $f(z,w)$ in H^2. Then E_λ is a bounded linear operator from H^2 onto $H^2(z)$, and the adjoint operator is the multiplication operator of $k_\lambda(w)$ from $H^2(z)$ to H^2.

LEMMA 4.2 (Yang). *Let $\mathcal{N}$ be the quotient module arising from an inner sequence $\{q_j(z)\}_{j\geqslant 0}$. If there exists no non-negative integer k such that $q_j(z)H^2(z) = q_k(z)H^2(z)$ for any $j \geqslant k$, then $\sigma(S_w)$ is the whole closed unit disk $\overline{\mathbb{D}}$.*

Proof. Since $E_\lambda|_{\mathcal{M}\ominus w\mathcal{M}}$ can be identified with the characteristic function of S_w in the model theory of Sz.-Nagy and Foiaş, a point λ in $\mathbb{D}$ is in $\sigma(S_w)$ if and only if $E_\lambda|_{\mathcal{M}\ominus w\mathcal{M}}$ is not invertible as an operator from $\mathcal{M}\ominus w\mathcal{M}$ to $H^2(z)$. Taking a sequence of functions $\{g_j(z)w^j\}_{j\geqslant 0}$ in $\mathcal{M}\ominus w\mathcal{M}$ such that $\|g_j(z)\| = 1$ for every j (cf. (ii) in Theorem 1.1), we have that $\|E_\lambda g_j(z)w^j\| = |\lambda|^j$ converges to 0 as j tends to infinity for any λ in $\mathbb{D}$. It follows that $E_\lambda|_{\mathcal{M}\ominus w\mathcal{M}}$ is not invertible, which is equivalent to that λ is in $\sigma(S_w)$. This concludes the proof. ∎

LEMMA 4.3. *Let $\mathcal{N}$ denote the quotient module arising from an inner sequence $\{q_j(z)\}_{j\geqslant 0}$. Then the following assertions are equivalent:*

(i) $q_\infty(z) \neq 1$;

(ii) $\sigma_{\mathrm{p}}(S_w^*) = \mathbb{D}$;
(iii) S_w^* *has a non-zero eigenvalue.*

Proof. First, we suppose (i). Then there exists a non-zero function $f(z)$ in $H^2(z)$ such that $f(z)$ belongs to $P_j^{\perp}H^2(z)$ for every non-negative integer j, because $\{P_j^{\perp}\}_{j\geqslant 0}$ is a monotone decreasing sequence of orthogonal projections. Hence, for any λ in $\mathbb{D}$, we have that $f(z)k_\lambda(w)$ belongs to $\mathcal{N}$, which is an eigenfunction of S_w^*. This implies (ii). Trivially, (ii) implies (iii).

Next, we suppose (iii). If λ is a non-zero eigenvalue of S_w^*, then λ is in $\mathbb{D}$ and there exists a non-zero function $f(z)$ such that $f(z)k_{\overline{\lambda}}(w)$ is the eigenfunction of S_w^*. Hence $f(z)$ belongs to $P_j^{\perp}H^2(z)$ for every non-negative integer j. This implies (i). We complete the proof. ∎

REMARK 4.4. In the proof of Lemma 4.3, we showed the following:

$$\ker(S_w^* - \lambda) = \begin{cases} (H^2(z) \ominus q_\infty(z)H^2(z)) \otimes \mathbb{C}k_{\overline{\lambda}}(w) & (\lambda \neq 0), \\ (H^2(z) \ominus q_0(z)H^2(z)) \otimes \mathbb{C}1 & (\lambda = 0). \end{cases}$$

LEMMA 4.5. *Let $\mathcal{M}$ be the submodule arising from an inner sequence $\{q_j(z)\}_{j\geqslant 0}$. Then, for any non-zero λ in $\mathbb{D}$, $\mathcal{M} \cap \ker E_\lambda = (w-\lambda)\mathcal{M}$.*

Proof. Suppose that f is in $\mathcal{M} \cap \ker E_\lambda$. Then there exists a function $g = \sum_j g_j(z)w^j$ in H^2 such that

$$f = \sum_l q_l(z)f_l(z)w^l = (w-\lambda)\sum_j g_j(z)w^j = \sum_j g_j(z)w^{j+1} - \sum_j \lambda g_j(z)w^j.$$

Comparing vector coefficients with respect to the variable w, we have the following equations:

$$\begin{cases} q_0(z)f_0(z) = -\lambda g_0(z), \\ q_j(z)f_j(z) = g_{j-1}(z) - \lambda g_j(z) \quad (j \geqslant 1), \end{cases}$$

It follows that $g_j(z)$ divisible by $q_j(z)$ for every j, that is, g belongs to $\mathcal{M}$. This concludes the proof. ∎

LEMMA 4.6. *Let $\mathcal{M}$ be the submodule arising from an inner sequence $\{q_j(z)\}_{j\geqslant 0}$. Then S_w has no non-zero eigenvalue, that is,* $\ker S_{w-\lambda} = \{0\}$ *for any non-zero λ.*

Proof. First we note that every eigenvalue of S_w is in $\mathbb{D}$, because S_w^n converges to 0 as n tends to infinity in the weak operator topology. If there exists a function g in $\mathcal{N}$ such that $S_w g = \lambda g$ for some non-zero λ in $\mathbb{D}$, then we have $S_{(w-\lambda)}g = 0$. It follows that $(w-\lambda)g$ belongs to $\mathcal{M}$. Since $(w-\lambda)g$ is an element in $\ker E_\lambda$, we have g is in $\mathcal{M}$ by Lemma 4.5. Therefore we have $g = 0$. This completes the proof. ∎

REMARK 4.7. We note that 0 is an eigenvalue of S_w in non-trivial cases. To see this, we shall recall Theorem 3.2(v) and Corollary 3.3. If 0 is not an eigenvalue of S_w, then S_w is an isometry. This implies that $P_j^\perp = P_{j+1}^\perp \neq 0$ for every j, that is, we have $\mathcal{N} = (H^2(z) \ominus q_0(z)H^2(z)) \otimes H^2(w)$, in which S_w is a unilateral shift.

THEOREM 4.8. *Let $\mathcal{N}$ be the quotient module arising from an inner sequence $\{q_j(z)\}_{j\geqslant 0}$.*

(i) *If $q_m(z) = 1$ for some finite m, then*

$$\sigma_{\mathrm{p}}(S_w) = \{0\},\ \sigma_{\mathrm{c}}(S_w) = \emptyset \text{ and } \sigma_{\mathrm{r}}(S_w) = \emptyset.$$

(ii) *If $q_\infty(z) = 1$ and $q_j(z) \neq 1$ for any j, then*

$$\sigma_{\mathrm{p}}(S_w) = \{0\},\ \sigma_{\mathrm{c}}(S_w) = \overline{\mathbb{D}} \setminus \{0\} \text{ and } \sigma_{\mathrm{r}}(S_w) = \emptyset.$$

(iii) *If $q_\infty(z) \neq 1$ and $q_j(z) \neq q_0(z)$ for some j, then*

$$\sigma_{\mathrm{p}}(S_w) = \{0\},\ \sigma_{\mathrm{c}}(S_w) = \partial\mathbb{D} \text{ and } \sigma_{\mathrm{r}}(S_w) = \mathbb{D} \setminus \{0\}.$$

(iv) *If $q_j(z) = q_0(z) \neq 1$ for any j, then*

$$\sigma_{\mathrm{p}}(S_w) = \emptyset,\ \sigma_{\mathrm{c}}(S_w) = \partial\mathbb{D} \text{ and } \sigma_{\mathrm{r}}(S_w) = \mathbb{D}.$$

Proof. Proof of (i): let m be the smallest integer such that $q_m(z) = 1$. Then we have $S_w^m = 0$, which implies that $\sigma(S_w) = \sigma_{\mathrm{p}}(S_w) = \{0\}$ by Remark 4.7.

Proof of (ii): By Lemmas 4.2, 4.6 and Remark 4.4, we have that $\sigma(S_w) = \overline{\mathbb{D}}$ and $\sigma_{\mathrm{p}}(S_w) = \{0\}$. Let λ be a non-zero point in the closed unit disk $\overline{\mathbb{D}}$. If λ was in $\sigma_{\mathrm{r}}(S_w)$, then $\overline{\lambda}$ would be in $\sigma_{\mathrm{p}}(S_w^*)$. Hence $\overline{\lambda}$ would be in $\mathbb{D}$. However, this is a contradiction by Lemma 4.3.

Proof of (iii): By Lemma 4.6 and Remark 4.7, we have $\sigma_{\mathrm{p}}(S_w)=\{0\}$. By Lemma 4.3, we have $\sigma_{\mathrm{p}}(S_w^*) = \mathbb{D}$. This implies that $\sigma(S_w) = \overline{\mathbb{D}}$ and $\mathbb{D} \setminus \{0\}$ is contained in $\sigma_{\mathrm{r}}(S_w)$ by Lemma 4.6. Let λ be a point on $\partial\mathbb{D}$. If λ was in $\sigma_{\mathrm{r}}(S_w)$, then $\overline{\lambda}$ would be an eigenvalue of S_w^*, that is, λ be in $\mathbb{D}$. This is a contradiction. Hence we have $\sigma_{\mathrm{c}}(S_w) = \partial\mathbb{D}$ and $\sigma_{\mathrm{r}}(S_w) = \mathbb{D} \setminus \{0\}$.

Proof of (iv): it is trivial by Remark 4.7. ∎

5. THE COMMUTANT OF S_z AND S_w

Let $\mathfrak{A}$ be the weak closed algebra generated by S_z, S_w and $I_{\mathcal{N}}$ on a quotient module $\mathcal{N}$, and $\mathfrak{A}'$ denotes the commutant of $\mathfrak{A}$ in the algebra of all bounded linear operators on $\mathcal{N}$. For a bounded analytic function φ, S_φ denotes the compression of the multiplication operator of φ to $\mathcal{N}$, that is, we set $S_\varphi f = P_{\mathcal{N}}\varphi f$ for any f in $\mathcal{N}$. Let H^∞ (respectively $H^\infty(z)$) denote the algebra of all bounded analytic functions on $\mathbb{D}^2$ (respectively $\mathbb{D}$). Then it is easy to check that $\mathfrak{A}$ is equal to the weak closure of the set $\{S_\varphi : \varphi \in H^\infty\}$. In [3], Amar and Menini gave an example in which Sarason's theorem ([9]) does not hold in $H^2(\mathbb{D}^2)$, that is, they gave an example of quotient module $\mathcal{N}$ such that $\mathfrak{A}' \neq \{S_\varphi : \varphi \in H^\infty\}$ on $\mathcal{N}$.

In this section we will show that $\mathfrak{A} = \mathfrak{A}'$ on quotient modules arising from inner sequences.

THEOREM 5.1. *Let $\mathcal{N}$ be the quotient module arising from an inner sequence $\{q_j(z)\}_{j\geqslant 0}$. Then $\mathfrak{A} = \mathfrak{A}'$. Moreover, for any element A in $\mathfrak{A}'$, there exists a sequence of bounded analytic functions $\{\varphi_j(z)\}_{j\geqslant 0}$ in $H^\infty(z)$ such that $A = \sum_{j\geqslant 0} S_{\varphi_j(z)} S_w^j$ in the weak operator topology.*

Proof. Let A be an operator in $\mathfrak{A}'$. Then A can be identified with an operator matrix (A_{ij}) acting on $\sum_{j\geqslant 0} \bigoplus \mathcal{N}_j$, where we set $\mathcal{N}_j = H^2(z) \ominus q_j(z) H^2(z)$. In this representation, every entry A_{ij} is an operator from $\mathcal{N}_j$ to $\mathcal{N}_i$. Since A commutes with S_z, A_{ij} intertwines $S_z^{(i)}$ and $S_z^{(j)}$ for any $i, j \geqslant 0$, where we set $S_z^{(k)} = P_k^\perp T_z|_{\mathcal{N}_k}$. Hence there exists a bounded analytic function $\varphi_{ij}(z)$ such that $A_{ij} = P_i^\perp T_{\varphi_{ij}(z)}|_{\mathcal{N}_j}$ by the commutant lifting theorem. Moreover, since A commutes with S_w, we have

$$\begin{cases} A_{ij} = 0 & (i < j), \\ P_{i+1}^\perp A_{ij} = A_{i+1,j+1} P_{j+1}^\perp & (i \geqslant j). \end{cases}$$

These equations imply that $P_{i+1}^\perp T_{\varphi_{ij}(z)}|_{\mathcal{N}_j} = P_{i+1}^\perp T_{\varphi_{i+1,j+1}(z)} P_{j+1}^\perp|_{\mathcal{N}_j}$ $(i \geqslant j)$, and we have

$$\varphi_{ij}(z) - \varphi_{i+1,j+1}(z) \in q_{i+1}(z) H^\infty(z) \quad (i \geqslant j).$$

Further we have $\varphi_{i0}(z) - \varphi_{i+k,k}(z) \in q_{i+k}(z) H^\infty(z)$ $(k \geqslant 0)$. Therefore we have

$$P_{i+k}^\perp T_{\varphi_{i0}(z)}|_{\mathcal{N}_k} = P_{i+k}^\perp T_{\varphi_{i+k,k}(z)}|_{\mathcal{N}_k} \quad (k \geqslant 0).$$

Setting $\varphi_i(z) = \varphi_{i0}(z)$, we have the following:

$$\begin{cases} A_{ij} = 0 & (i < j), \\ A_{i+k,k} = P_{i+k}^\perp T_{\varphi_i(z)}|_{\mathcal{N}_k} & (k \geqslant 0). \end{cases}$$

Then we have

$$\begin{pmatrix} P_0^\perp T_{\varphi_0(z)}|_{\mathcal{N}_0} & 0 & 0 & \cdots \\ P_1^\perp T_{\varphi_1(z)}|_{\mathcal{N}_0} & P_1^\perp T_{\varphi_0(z)}|_{\mathcal{N}_1} & 0 & \cdots \\ P_2^\perp T_{\varphi_2(z)}|_{\mathcal{N}_0} & P_2^\perp T_{\varphi_1(z)}|_{\mathcal{N}_1} & P_2^\perp T_{\varphi_0(z)}|_{\mathcal{N}_2} & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \cong \sum_{j=0}^{\infty} S_{\varphi_j(z)} S_w^j,$$

in the weak operator topology. This concludes the proof. ∎

6. EXAMPLE: RUDIN'S SUBMODULE

Let $\mathcal{M}$ be the submodule consisting of all functions in H^2 which have a zero of order greater than or equal to n at $(\alpha_n, 0) = (1 - n^{-3}, 0)$ for any positive integer n. This module was given by Rudin in [8], and he proved that this is not finitely generated. Let $b_n(z)$ denote the single Blaschke product which has a zero at α_n, that is, we set $b_n(z) = (\alpha_n - z)/(1 - \alpha_n z)$. Then Rudin's submodule arises from the inner sequence defined as follows (cf. [10]):

$$\begin{cases} q_0(z) = \prod_{n=1}^{\infty} b_n^n(z), \\ q_j(z) = q_{j-1}(z) / \prod_{n=j}^{\infty} b_n(z) \quad (j \geqslant 1). \end{cases}$$

LEMMA 6.1. *In Rudin's submodule,* $q_\infty(z) = 1$.

Proof. Since $q_0(0) = \prod_{n=1}^{\infty} \alpha_n^n$ converges, we have the following:

$$0 \geqslant \log q_j(0) = \sum_{n=j+1}^{\infty} (n-j) \log \alpha_n \geqslant \sum_{n=j+1}^{\infty} n \log \alpha_n \to 0 \quad (j \to +\infty),$$

which implies that $q_j(0)$ converges to 1 as j tends to infinity. Let $\{q_{j_k}(z)\}_k$ be a subsequence of $\{q_j(z)\}_j$ converging to some function $f(z)$ in the weak* topology. Then $q_{j_k}(0)$ converges to $f(0)$. Hence we have $f(0) = 1$. By the maximum principle, it follows that $f(z) = 1$. Since every $q_{j_k}(z)$ belongs to the weak* closed ideal $q_\infty(z)H^\infty(z)$, $f(z)$ is in $q_\infty(z)H^\infty(z)$. Therefore we have $q_\infty(z) = 1$. This concludes the proof. ∎

Regarding Rudin's submodule, the following were given by Clark (cf. [12]):

$$\sigma_{\mathrm{p}}(S_z) = \{\alpha_n : n \geqslant 1\},\ \sigma_{\mathrm{c}}(S_z) = \{1\},\ \sigma_{\mathrm{r}}(S_z) = \emptyset,$$
$$\Sigma_0 = 1 + \sum_{j=1}^{\infty} \Big(1 - \prod_{n=j}^{\infty} (1 - n^{-3})^2\Big),\ \Sigma_1 = \sum_{j=1}^{\infty} \Big(1 - \prod_{n=j}^{\infty} (1 - n^{-3})^2\Big).$$

We close this paper by adding the spectrum of S_w and the Hilbert–Schmidt norm of $[S_z^*, S_w]$ to this list as follows, by Theorems 3.5, 4.8 and Lemma 6.1:

$$\sigma_{\mathrm{p}}(S_w) = \{0\},\ \sigma_{\mathrm{c}}(S_w) = \overline{\mathbb{D}} \setminus \{0\},\ \sigma_{\mathrm{r}}(S_w) = \emptyset,$$
$$\|[S_z^*, S_w]\|_2^2 = \sum_{j=1}^{\infty} \Big(1 - \prod_{n=j}^{\infty} (1 - n^{-3})^{2(n-j)}\Big) \Big(1 - \prod_{n=j}^{\infty} (1 - n^{-3})^2\Big).$$

Acknowledgements. This paper was written during a visit at SUNY in Albany. The author expresses gratitude to R. Yang for very useful discussions and suggestions, and also thanks him and the members of the Department Mathematics and Statistics of SUNY at Albany for their kind hospitality. The author also wants to thank the referee for his comments. This research was supported by the Grant-in-Aid for Scientific Research, Japan Society for Promotion of Science.

REFERENCES

[1] J. AGLER, J.E. MCCARTHY, *Pick Interpolation and Hilbert Function Spaces*, Grad. Stud. Math., vol. 44, Amer. Math. Soc., Providence, RI 2002.

[2] O.P. AGRAWAL, D.N. CLARK, R.G. DOUGLAS, Invariant subspaces in the polydisk, *Pacific J. Math.* **121**(1986), 1–11.

[3] E. AMAR, C. MENINI, On an operator theory approach to the corona problem, *Bull. London Math. Soc.* **34**(2002), 369–373.

[4] H. BERCOVICI, *Operator Theory and Arithmetic in H^∞*, Math. Surveys Monographs, vol. 26, Amer. Math. Soc., Providence, RI 1988.

[5] X. CHEN, K. GUO, *Analytic Hilbert Modules*, Chapman and Hall/CRC Res. Notes Math., vol. 433, Chapman and Hall/CRC, Boca Raton, FL 2003.

[6] R.G. DOUGLAS, V.I. PAULSEN, *Hilbert Modules over Function Algebras*, Pitman Res. Notes Math. Ser., vol. 217, Longman Sci. Tech., Harlow; copublished in the United States with John Wiley and Sons, Inc., New York 1989.

[7] N.K. NIKOLSKI, *Operators, Functions, and Systems: an Easy Reading, Vol. 2, Model Operators and Systems*, Math. Surveys Monographs, vol. 93, Amer. Math. Soc., Providence, RI 2002.

[8] W. RUDIN, *Function Theory in Polydiscs*, Benjamin, New York 1969.

[9] D. SARASON, Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.* **127**(1967), 179–203.

[10] M. SETO, R. YANG, Inner sequence based invariant subspaces in $H^2(\mathbb{D}^2)$, *Proc. Amer. Math. Soc.* **135**(2007), 2519–2526.

[11] B. SZ.-NAGY, C. FOIAŞ, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam 1970.

[12] R. YANG, Operator theory in the Hardy space over the bidisk. III, *J. Funct. Anal.* **186**(2001), 521–545.

[13] R. YANG, Beurling's phenomenon in two variables, *Integral Equations Operator Theory* **48**(2004), 411–423.

[14] R. YANG, Hilbert–Schmidt submodules and issues of unitary equivalence, *J. Operator Theory* **53**(2005), 169–184.

[15] R. YANG, The core operator and congruent submodules, *J. Funct. Anal.* **228**(2005), 469–489.

MICHIO SETO, DEPARTMENT OF MATHEMATICS, SHIMANE UNIVERSITY, 1060 NISHI-KAWATSU, MATSUE, SHIMANE 690-8504, JAPAN

E-mail address: mseto@riko.shimane-u.ac.jp

Received May 9, 2006; revised November 2, 2006.